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Salvador Barberà, Anke Gerber

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Salvador Barberà* Anke Gerber‡

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Abstract

We formalize, under the name of games of addition, the strategic interaction between agents that can play non-simultaneously by adding payoff relevant actions to those that any other players or themselves have already taken previously, but may also agree unanimously to stop adding them and collect the payoffs associated with the truncated sequence of moves. Our formalization differs from that of extensive form games in that the order of the agents' moves is not predetermined but emerges endogenously when applying an adapted version of a solution concept proposed by Dutta, Jackson and Le Breton (2004). We provide results regarding the properties of solutions to games of addition, and we also compare their corresponding equilibria with those we would obtain if using extensive form games and subgame perfection as alternative tools of analysis.

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1 Introduction

In some sequential games the order of play is naturally suggested by the nature of the phenomena under study. In other cases, this order may be imposed by a

*Barcelona GSE, Facultat d'Economia i Empresa, Edifici B, 08193 Bellaterra (Barcelona), Spain; e-mail: salvador.barbera@uab.cat. Salvador Barberà acknowledges financial support from the Spanish Agencia Estatal de Investigación (AEI), through the Severo Ochoa Programme for Centres of Excellence in R&D (Barcelona School of Economics CEX2019-000915-S), and from grant PID2020-116771GB-I00 funded by MCIN/AEI/10.13039/501100011033 and from the Generalitat de Catalunya, through grant 2021SGR00194.

‡Faculty of Business, Economics and Social Sciences, Department of Economics, Universität Hamburg, Von-Melle-Park 5, D-20146 Hamburg, Germany; e-mail: anke.gerber@uni-hamburg.de.

designer, maybe seeking some additional objectives. For example, in cases where the order of play may provide some players with an advantageous position, fixing a given order may reflect the designer's view that it is desirable to attribute the advantage to a specific player. However, we think that in many other cases the use of specific protocols imposes an unrealistic and not innocuous restriction in the analysis of interactions between agents who interact sequentially, because in fact an important part of their strategic behavior may consist in determining the order in which they decide to act. Examples abound. They include oligopolistic competition, agenda setting, nomination of candidates for some office, electoral competition, political debates.

In this paper we offer a framework for the analysis of strategic games played by individuals who must take decisions in a sequence of non-simultaneous actions, and whose opportunities to use their available actions, or not, after any history of the game, are not limited by any protocol establishing the order in which they must play, or forcing them to continue adding further actions to those that they have already used. We refer to the games that respond to our description as games of addition, and we study their equilibria following an adapted version of a solution concept proposed by Dutta, Jackson, and Le Breton (2004) for the specific class of agenda formation games, where individuals' preferences over agendas are determined by a voting rule that selects an alternative from each agenda.

The description of the strategic possibilities arising in our context is simpler than the one required to define extensive form games, because these must be very specific about the shape of game trees and the assignment of rights to move at any point in the history of the game. In our case, the definitional elements of the game are simply given by the set of players, the actions that are available for each player after each history of play which we call a "state", and the preferences of the players over all states.

The description is very flexible and allows to cover rather different applications. In some cases, actions are strictly personalized, like in private good economies (for example, actions may be the decision whether to address a given issue in a debate). In others, some or all actions may be undertaken by different players (for example, any agent may be able to nominate a candidate for election to a club). By contrast with our simple description of the basic game structure, the notion of equilibrium that we propose to use is more involved than subgame perfection, as it is required to endogenously determine the timing of the order of play by the different agents involved.

Sequential play by different agents can always be interpreted as a decision-making process during which the characteristics of the end-result are determined by the accumulation of actions taken by the agents involved. Yet, an important characteristic of those that we call games of addition is that the accumulation process can be stopped at any point if all players agree to, either implicitly or explicitly. Given this important characteristic, the payoffs at each possible state of development along the game become relevant: any of them may influence the decision to stop or to continue adding actions. Because of this, the equilibrium concept that we use involves a global view of the overall possibilities that are open to the players, including the consequences and the reasons for calling the addition process to a stop.

There is a related literature in industrial organization that studies the timing of actions in duopoly games (see, e.g., Hamilton and Slutsky (1990), Deneckere and Kovenock (1992), van Damme and Hurkens (2004)). Yet, different from our games of addition in these timing games either firms can also move simultaneously or they are forced to move sequentially by allowing one firm to take her action only in even time intervals and the other only in odd intervals. The latter is an extensive form that endogenizes the timing of actions but it still assumes an exogenous order of moves which may be a too strong assumption in many applications. Moreover, we will see that the equilibrium predictions for such extensive form games where players can decide to defer taking an action can be quite different from the equilibria in our games of addition.

There is also a small literature on endogenizing the order of moves in agenda setting and in electoral campaigns. We have already mentioned the seminal paper on agenda setting by Dutta, Jackson, and Le Breton (2004). Barberà and Gerber (2022) have studied a two-stage process, where in the first stage agents put issues on the agenda and in the second stage they take decisions on those issues. The agenda-setting stage is a special case of the class of games of addition we study in this paper. Kamada and Sugaya (2020) study an electoral campaign where candidates strategically decide when to announce their policies. Yet, different from the games studied in this paper candidates are not free to choose the time for their announcement. They have to wait for an opportunity which arrives stochastically according to a Poisson process. By contrast, Barberà and Gerber (2023) consider an electoral campaign with two candidates who can freely decide which issues to address and in what order. Again this model is a special case of the class of games we study in this paper.

An important general conclusion of our analysis is that our solution concept determines endogenously the order in which players will be interested to act at equilibrium, and the ensuing consequences on the games' equilibrium outcome. In many cases this order cannot be inferred from the specific application. Hence, modelling the strategic interaction as an extensive form game where the order of moves is fixed exogenously is not a reasonable alternative. This is one of the reasons why we expect that games of addition and their study through a generalization of the solution concept proposed by Dutta, Jackson, and Le Breton (2004) may become a welcome tool.

The paper is organized as follows. In Section 2 we introduce our general model of games of addition and our adapted version of equilibrium. In Section 3 we present some simple but clarifying examples and a general existence result for games with common sets of actions. Section 4 describes the connection between situations that are modelled by means of games of addition and a class of extensive form games that could be thought of as an alternative model of how the same players could interact. This also allows us to argue that, in general, the equilibria associated with these two different tools need not coincide. Section 5 provides some initial shots toward a better understanding of the games we are proposing. In particular, we provide a general existence result for the class of two-person zero-sum games and we prove that all equilibria are outcome equivalent under additional assumptions on the preferences of the players. This opens the way toward further extensions of the model and generalizations of results. Section 6 concludes. All proofs are in the appendix.

2 The Model

We consider the following class of games:

Definition 2.1 *A* ***(finite) game of addition*** *is defined by a four-tuple* $(N, A, \Sigma, (\succsim_i)_{i \in N})$ *that satisfies the following conditions:*

(i) $N = \{1, \ldots, n\}$ *with* $n \geq 2$ *is the set of players.*

(ii) A *is a nonempty finite set of actions.*

(iii) Σ *is a set of states with the following property. There is a number* $M \geq 1$ *such that each state* $\sigma \in \Sigma$ *is either empty* ($\sigma = \varnothing$) *or is a sequence* $\sigma = (s_1, \ldots, s_m)$ *with* $m \leq M$ *where* $s_k = (i_k, a_k)$ *with* $i_k \in N$ *and* $a_k \in A$ *for all* $k = 1, \ldots, m$.[1]

(iv) $\succsim_i$ *is a complete and transitive preference relation on* Σ *for all* $i = 1, \ldots, n$.

For a given state $\sigma = (s_1, \ldots, s_m) \in \Sigma$ we write, for short, $(i, a) \in \sigma$ whenever $(i, a) = s_k$ for some $k \in \{1, \ldots, m\}$, and $(i, a) \notin \sigma$ whenever $(i, a) \neq s_k$ for all $k = 1, \ldots, m$. For $\sigma = (s_1, \ldots, s_m) \in \Sigma$, where $0 \leq m \leq M - 1$, and $(i, a) \in N \times A$, let $\sigma' = (\sigma, (i, a)) = (s_1, \ldots, s_m, (i, a))$.

Notice that Σ includes information about what actions, if any, are available for each of the agents at each state $\sigma \in \Sigma$. Specifically, for each game of addition $(N, A, \Sigma, (\succsim_i)_{i \in N})$ and $\sigma \in \Sigma$ let $A^i(\sigma)$ be the set of feasible actions for player $i \in N$ at state $\sigma \in \Sigma$ which is given by

$$A^i(\sigma) = \{a | \, (\sigma, (i, a)) \in \Sigma\}.$$

Note that (iii) implies that $A^i(\sigma) = \emptyset$ for all σ of length M. That is, the game ends after at most M actions have been taken by the players. But it may end earlier, i.e. at a state with less than M actions, if either the sets of available actions for all players are empty or if no player wants to add a further action

[1] If $s_k = (i_k, a_k)$, then i_k indicates the player who added action a_k at state $(s_1, \ldots, s_{k-1})$ which is defined to be the empty state $\varnothing$ if $k = 1$.

at the given state. If $\sigma \in \Sigma$ is such that $A^i(\sigma) = \emptyset$ for all $i \in N$, σ is called a *terminal state*. In particular, any state σ of length M is a terminal state.

The above framework allows for a flexible specification of the characteristics of the actions available to players in different contexts. In that of agenda formation Dutta, Jackson, and Le Breton (2004) and Barberà and Gerber (2022) have considered the special case where for all i,

$$A^i(\varnothing) = A \text{ and } A^i(\sigma) = A \setminus \{a|\,(j,a) \in \sigma \text{ for some player } j\} \text{ for all } \sigma \neq \varnothing. \quad (1)$$

In this case there exists a common set of actions for all players and each action in the set can be taken at most once.

By contrast, in a model of electoral campaigns Barberà and Gerber (2023) have considered the special case with

$$A^i(\varnothing) = A^i \text{ and } A^i(\sigma) = A^i \setminus \{a|\,(i,a) \in \sigma\} \text{ for all } \sigma \neq \varnothing, \quad (2)$$

where $A^i \subseteq A$ is a set of possible actions of player i for all i. In this case each player has an individual set of actions and again any action can be taken at most once.

Our model has some limitations as a modelling tool. It does not allow for simultaneous moves by different players. Nor do we allow single players to take more than one action at a time, though we do not exclude their playing several times in a row. Yet, we think that these technical limitations do not detract from the wide applicability of our model.

We are now ready to introduce definitions and conditions that will lead us to define our proposed equilibrium concept. Let $\sigma \in \Sigma$. We say that $\sigma' \in \Sigma$ is a *continuation state* at σ if $\sigma = \varnothing$ or $\sigma = (s_1 \ldots, s_m) \in \Sigma$ for some $m \geq 1$ and $\sigma' = (\sigma, \ldots)$. Note that by definition σ is a continuation state at σ. By $C(\sigma)$ we denote the set of continuation states at $\sigma \in \Sigma$. We now define our equilibrium concept which is an adaptation of the concept introduced by Dutta, Jackson, and Le Breton (2004).

A *collection of sets of continuation states* is a family of subsets of $C(\sigma)$ for each $\sigma \in \Sigma$. A collection of sets of continuation states $(CE(\sigma))_{\sigma\in\Sigma}$ is an *equilibrium* *collection of sets of continuation states* if the following three conditions (E1)-(E3) are satisfied:

(E1) For all $\sigma \in \Sigma$, $CE(\sigma)$ is a nonempty subset of

$$\bigcup_{i=1}^{n} \bigcup_{a \in A^i(\sigma)} CE(\sigma, (i, a)) \cup \{\sigma\}.$$

(E2) For all $\sigma \in \Sigma$, $\sigma \in CE(\sigma)$ if and only if for all $i = 1, \ldots, n$,

$$\sigma \succsim_i \sigma' \text{ for all } \sigma' \in \bigcup_{a \in A^i(\sigma)} CE(\sigma, (i, a)).$$

Condition (E1) demands that any equilibrium continuation at a given state σ involves either stopping at σ or taking some action and then follow an equilibrium path from there. The latter can be interpreted as a consistency requirement: If a state is considered an equilibrium continuation at σ and it involves some player taking an action at σ, then it must still be an equilibrium continuation at the state reached after the player's action.

Condition (E2) demands that stopping at a state σ is an equilibrium if and only if no player can improve by taking an action at σ if the subsequent actions are taken according to some equilibrium continuation. Hence, for stopping not to be an equilibrium continuation at σ it is already sufficient that one player has an action that leads to some equilibrium continuation which the player strictly prefers over σ. Note that this implies some degree of optimism on the part of players because there could also be other equilibrium continuations resulting from a player's initial move that are worse than stopping at σ.

For the third condition, we first define a rationalizable state.[2]

Definition 2.2 *For $\sigma \in \Sigma$ the state $\sigma' = (\sigma, (i, a'), \ldots) \in \Sigma$ is* ***rationalizable*** *(relative to σ) if $\sigma' \in CE(\sigma, (i, a'))$ and the following conditions are satisfied:*

(i) If $CE(\sigma) \cap CE(\sigma, (j, a'')) \neq \emptyset$ for some (j, a'') with $j \neq i$, then there exists some $\sigma'' \in CE(\sigma)$ with $\sigma'' = (\sigma, (k, a''), \ldots)$ for some $k \neq i$ such that $\sigma' \succsim_i \sigma''$.[3]

[2] Rationalizability corresponds to *strong rationalizability* in Dutta, Jackson, and Le Breton (2004).

[3] k may but need not be equal to j.

(ii) For all $a'' \neq a'$ *with* $(\sigma, (i, a'')) \in \Sigma$ *there exists some* $\sigma'' \in CE(\sigma, (i, a''))$ *such that* $\sigma' \succsim_i \sigma''$.

(iii) If $\sigma \in CE(\sigma)$ *then also* $\sigma' \succsim_i \sigma$.

Accordingly, a continuation σ' at σ that is initiated by player i is rationalizable if it is an equilibrium continuation at the state reached after i's action and if three conditions are satisfied: (i) i weakly prefers σ' over some other equilibrium continuation at σ which is initiated by an action of some other player (if any). (ii) i does not prefer all equilibrium continuations that result from some other action taken by i at σ to σ'. (iii) If stopping at σ is an equilibrium continuation, then i weakly prefers σ' over stopping at σ. If any of the conditions (i)-(iii) were violated it would be better for i not to take the respective action.

The third condition on an equilibrium collection of sets of continuation states then is as follows:

(E3) Let $\sigma \in \Sigma$ and let $\sigma' \in CE(\sigma, (i, a))$ for some i and a such that $(\sigma, (i, a)) \in \Sigma$. If σ' is rationalizable, then $\sigma' \in CE(\sigma)$. Conversely, if $\sigma' = (\sigma, (i, a'), \ldots) \in CE(\sigma)$ and either $\sigma \in CE(\sigma)$ or $\sigma'' = (\sigma, (j, a''), \ldots) \in CE(\sigma)$ for some $(j, a'') \neq (i, a')$, then σ' is rationalizable. If $\sigma' = (\sigma, (i, a'), \ldots) \in CE(\sigma)$ and $\nexists j \neq i$ such that $\sigma'' = (\sigma, (j, a''), \ldots) \in CE(\sigma)$ for some $a'' \in A^j(\sigma)$, then $\sigma' \succsim_i \sigma$.

(E3) imposes three requirements on equilibrium continuations at any state σ: First, all rationalizable states must be equilibrium continuations at σ. Second, all equilibrium continuations $\sigma' \neq \sigma$ must be rationalizable except for the case where there is a unique equilibrium continuation at σ. Third, if an equilibrium continuation σ' at σ is initiated by an action of player i and there exists no equilibrium continuation initiated by an action of some other player j, then i must weakly prefer σ' over σ. This third condition rules out the case where a player continues from a state even though the continuation leads to a worse outcome than stopping at the current state and where the player's move cannot be justified by preventing any potentially worse outcomes resulting from continuations initiated by other players.

We focus attention on equilibrium continuations of the initial state:

Definition 2.3 *σ^* is an* ***equilibrium state*** *if there exists an equilibrium collection of sets of continuation states $(CE(\sigma))_{\sigma \in \Sigma}$ with $\sigma^* \in CE(\varnothing)$.*

3 Examples and a General Existence Result

3.1 Examples

We now present some simple examples that already allow us to illustrate different features that we may find when analyzing different games of addition through the use of our equilibrium concept. In Example 3.1 there is a unique equilibrium state. In Example 3.2 there are multiple equilibrium states and in Example 3.3 there are multiple equilibrium collections. Finally, there are also games for which there does not exist any equilibrium collection as we demonstrate in Example 3.4. In all examples there are two players and the sets of actions satisfy (2) with $A^1 = \{a\}$ and $A^2 = \{b\}$. Since each player has only one action we shortly write $(a), (b), (a, b), (b, a)$ for the states where player 1 takes action a, player 2 takes action b, first 1 takes action a and then 2 takes action b, and first 2 takes action b and then 1 takes action a.

Example 3.1 Let players' preferences over states be given by

$$(a) \succ_1 (b) \succ_1 (a, b) \sim_1 (b, a) \succ_1 \varnothing,$$
$$\varnothing \succ_2 (a, b) \sim_2 (b, a) \succ_2 (b) \succ_2 (a).$$

Then (E2) and (E3) imply that $CE(a) = \{(a, b)\}$ and $CE(b) = \{(b)\}$. Moreover, since $(a, b) \succ_1 \varnothing$ (E2) implies that $\varnothing \notin CE(\varnothing)$. Hence, $CE(\varnothing) \subseteq \{(a, b), (b)\}$.

$CE(\varnothing) = \{(b)\}$ is ruled out by (E3) because $\varnothing \succ_2 (b)$. $CE(\varnothing) = \{(a, b), (b)\}$ is ruled out by (E3) because (b) is not rationalizable if $(a, b) \in CE(\varnothing)$ since $(a, b) \succ_2 (b)$. Hence, $CE(\varnothing) = \{(a, b)\}$ which satisfies (E3).

We conclude that there exists a unique equilibrium collection and a unique equilibrium state (see Figure 1). ◇

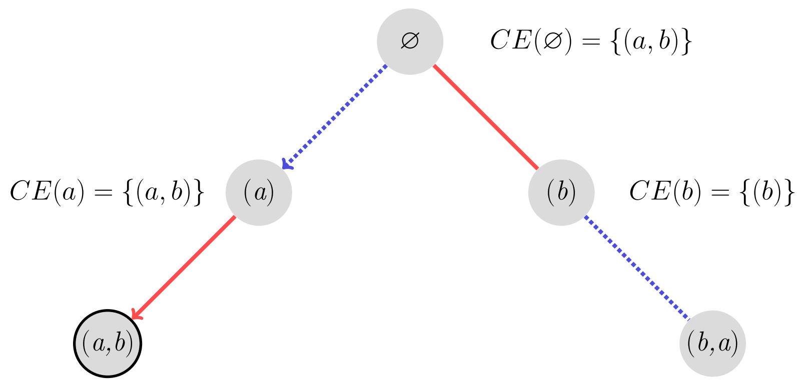

Figure 1: Equilibrium collection for the game in Example 3.1. Player 1 moves along the dotted lines and player 2 moves along the solid lines. Arrows denote the equilibrium path. The bordered node corresponds to the equilibrium continuation at $\varnothing$.

Example 3.2 Let players' preferences over states be given by

$$(a) \succ_1 (b) \succ_1 \varnothing \succ_1 (a,b) \sim_1 (b,a),$$
$$(b) \succ_2 (a) \succ_2 \varnothing \succ_2 (a,b) \sim_2 (b,a).$$

Then (E2) and (E3) imply that $CE(a) = \{(a)\}$ and $CE(b) = \{(b)\}$. Moreover, (E2) implies that $\varnothing \notin CE(\varnothing)$. Hence, $CE(\varnothing) \subseteq \{(a),(b)\}$.

$CE(\varnothing) = \{(a)\}$ is ruled out by (E3) because (b) is rationalizable if $(a) \in CE(\varnothing)$ since $(b) \succ_2 (a)$. $CE(\varnothing) = \{(b)\}$ is ruled out by (E3) because (a) is rationalizable if $(b) \in CE(\varnothing)$ since $(a) \succ_1 (b)$. Hence, $CE(\varnothing) = \{(a),(b)\}$ which satisfies (E3).

We conclude that there exists a unique equilibrium collection with multiple equilibrium continuations at $\varnothing$ (see Figure 2). $\diamond$

Example 3.3 Let players' preferences over states be given by

$$(b) \succ_1 (a) \succ_1 \varnothing \succ_1 (a,b) \sim_1 (b,a),$$
$$(a) \succ_2 (b) \succ_2 \varnothing \succ_2 (a,b) \sim_2 (b,a).$$

Then (E2) and (E3) imply that $CE(a) = \{(a)\}$ and $CE(b) = \{(b)\}$. Moreover, (E2) implies that $\varnothing \notin CE(\varnothing)$. Hence, $CE(\varnothing) \subseteq \{(a),(b)\}$.

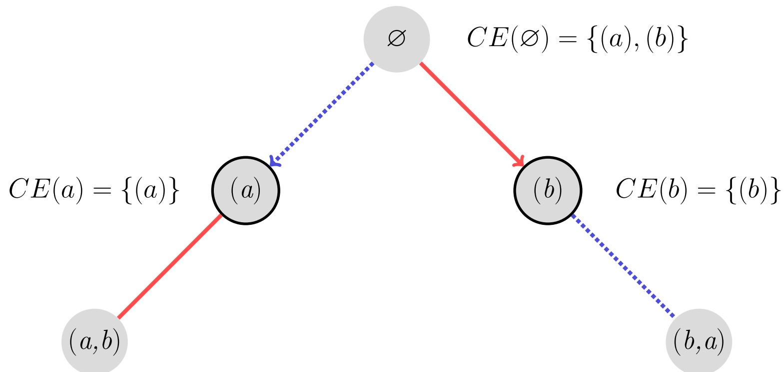

Figure 2: Equilibrium collection for the game in Example 3.2. Player 1 moves along the dotted lines and player 2 moves along the solid lines. Arrows denote the equilibrium path. The bordered nodes correspond to the equilibrium continuations at $\varnothing$.

$CE(\varnothing) = \{(a), (b)\}$ is ruled out by (E3) because (a) is not rationalizable if $(b) \in CE(\varnothing)$ since $(b) \succ_1 (a)$. Hence, $CE(\varnothing) = \{(a)\}$ or $CE(\varnothing) = \{(b)\}$ which both satisfy (E3).

We conclude that there exist multiple equilibrium collections that differ with respect to the unique equilibrium continuation at $\varnothing$ (see Figure 3). ⋄

Example 3.4 Let players' preferences over states be given by

$$\varnothing \succ_1 (a) \succ_1 (b) \succ_1 (a,b) \sim_1 (b,a),$$
$$(a) \succ_2 (b) \succ_2 \varnothing \succ_2 (a,b) \sim_2 (b,a).$$

Then (E2) and (E3) imply that $CE(a) = \{(a)\}$ and $CE(b) = \{(b)\}$. Moreover, (E2) implies that $\varnothing \notin CE(\varnothing)$. Hence, $CE(\varnothing) \subseteq \{(a), (b)\}$.

$CE(\varnothing) = \{(a)\}$ is ruled out by (E3) because $\varnothing \succ_1 (a)$. $CE(\varnothing) = \{(b)\}$ is ruled out by (E3) because (a) is rationalizable if $(b) \in CE(\varnothing)$ since $(a) \succ_1 (b)$. Finally, $CE(\varnothing) = \{(a), (b)\}$ is ruled out by (E3) because (b) is not rationalizable if $(a) \in CE(\varnothing)$ since $(a) \succ_2 (b)$.

Hence, $CE(\varnothing) = \emptyset$ which contradicts (E1). Therefore, there does not exist an equilibrium collection in this example. ⋄

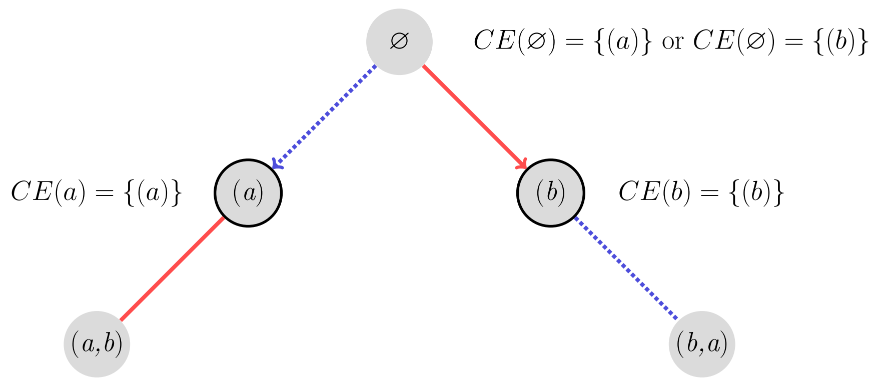

Figure 3: Equilibrium collections for the game in Example 3.3. Player 1 moves along the dotted lines and player 2 moves along the solid lines. Arrows denote the equilibrium path. Each bordered node corresponds to the unique equilibrium continuation at $\varnothing$ in one equilibrium collection.

3.2 Common sets of actions

We now provide a general existence result for an important class of games of addition. In these games, at all states the set of available actions is the same for all players. Moreover, players only care about the actions that are being taken and not about the identity of a player who takes a particular action. A prominent example is agenda setting.

Consider the following conditions on a game of addition $(N, A, \Sigma, (\succsim_i)_{i \in N})$.

(C1) For all $i \in N$,

$$A^i(\varnothing) = A$$

and

$$A^i(\sigma) = A \setminus \{a |\, (j, a) \in \sigma \text{ for some player } j\} \text{ for all } \sigma \in \Sigma \setminus \varnothing.$$

Under (C1) all players have the same feasible set of actions $A(\sigma) = A^i(\sigma)$ for all $\sigma \in \Sigma$ and each action can only be taken once. For $\sigma = ((i_1, a_1), \ldots, (i_m, a_m)) \in \Sigma$ with $m \geq 1$ let

$$\sigma|_A = (a_1, \ldots, a_m)$$

be the restriction of σ to actions in A. For $\sigma = \varnothing$ let $\sigma|_A = \varnothing$.

(C2) If $\sigma, \sigma' \in \Sigma$ are such that $\sigma|_A = \sigma'|_A$, then

$$\sigma \sim_i \sigma' \quad \text{for all } i \in N.$$

Under (C2) preferences over states only depend on the sequence of actions that have been taken but not on the identities of the players who have taken the actions.

Theorem 3.1 *Let $(N, A, \Sigma, (\succsim_i)_{i \in N})$ be a game of addition such that that (C1) and (C2) are satisfied. Then there exists an equilibrium collection of sets of continuation states $(CE(\sigma))_{\sigma \in \Sigma}$.*

4 Games of addition vs. extensive form games

In this section we demonstrate that the equilibrium predictions from the suggested analysis of games of addition will often differ substantially from those one can expect from the analysis of extensive forms where the order of play is given exogenously. Obviously, one cannot expect an extensive form game to deliver the same results than our suggested analysis unless one restricts attention to extensive form games that endow agents with the possibility of collectively stopping the play, by passing, at histories of any length.

We proceed in two steps. In the first step we argue that if we associate several extensive forms to each game of addition, there is no guarantee that all of them will lead to the same equilibrium outcomes that obtain for the given game of addition. In the second step we identify a very special and restricted set of games of addition that allow for the construction of a unique extensive form game such that the equivalence of equilibrium outcomes is guaranteed. The narrowness of this class leads us to argue that, in general, the analysis of games of addition is well differentiated from that of extensive form games.

Definition 4.1 *A* ***(finite) extensive form game with perfect information*** *is given by* $(N, A, H, P, (\succsim_i)_{i \in N})$ *that satisfies the following conditions:*[4]

(i) $N = \{1, \ldots, n\}$ *with* $n \geq 2$ *is the set of players.*

(ii) A *is a nonempty finite set of actions.*

(iii) H *is a set of histories that has the property that there exists some* $M \geq 1$ *such that each history* $h \in H$ *is either empty* ($h = \varnothing$) *or is a sequence* $h = (a_1, \ldots, a_m)$ *with* $m \leq M$ *where* $a_k \in A$ *for all* $k = 1, \ldots, m$. *A history* $h = (a_1, \ldots, a_m)$ *is* ***terminal*** *if there exist no* a_{m+1} *such that* $(a_1, \ldots, a_m, a_{m+1}) \in H$.

(iv) P *is a player function that assigns a player* $P(h) \in N$ *to every nonterminal history* h, *i.e.* $P(h)$ *is the player who takes an action after the history* h.

(v) $\succsim_i$ *is a complete and transitive preference relation on the set of terminal histories for all* $i = 1, \ldots, n$.

For any nonterminal history h the set of feasible actions of player $P(h)$ is given by

$$A(h) = \{a | \, (h, a) \in H\}.$$

Note that any history $h = (a_1, \ldots, a_m)$ defines a unique state

$$s = ((i_1, a_1), \ldots, (i_m, a_m))$$

with $i_1 = P(\varnothing)$ and $i_k = P(a_1, \ldots, a_{k-1})$ for $k = 2, \ldots, m$.

A *strategy* of player $i \in N$ in an extensive form game is a function α_i that assigns an action $\alpha_i(h) \in A(h)$ to each nonterminal history $h \in H$ for which $P(h) = i$.

The main difference between a game of addition and an extensive form game is that the latter specifies an order of moves while the former does not. Hence, any game of addition can be associated with several extensive form games, one for each possible order of moves. In the following we will describe this class of extensive form games and we will study the relation between equilibrium states

[4] See, for example, Osborne and Rubinstein (1994).

of a game of addition and subgame perfect Nash equilibria of the extensive form games associated with the game of addition.

4.1 Extensive form games with an option to pass

In view of our remarks at the beginning of the section, we propose to make an association between each given game of addition and a class of games in extensive form that allow agents to end the game at a truncated history if they unanimously prefer to do so, rather than continuing to play along undesirable paths. We do so by extending the players' sets of actions to always contain a "passing option". Informally, the consequences of a player's use of its passing option at a given history is that the action sets of all players remain the same at the continuation state that follows this pass action. In this subsection we formally analyze whether the introduction of this qualification may induce the equivalence between the equilibrium predictions of the given game of addition and those of the associated extensive form games. The answer is that, in general, including the pass option is not enough, since there will typically be orders of play for which the equilibrium outcomes of the associated extensive form will be different than those associated with the game of addition we start with.

Let us develop the reasoning formally. To this end let $G = (N, A, \Sigma, (\succsim_i)_{i \in N})$ be a game of addition. Consider an extensive form game with a set of players N and a set of actions $A' = A \cup \{pass\}$. Then inductively define a set of histories H^m, states $\sigma(h^m)$ for all $h^m \in H^m$ and a player function P' as follows.

Let $H^0 = \{\varnothing\}, \sigma(\varnothing) = \varnothing$ and let $P'(\varnothing)$ be some player i whose feasible set of actions at state $\varnothing$ is nonempty, i.e. $A^i(\varnothing) \neq \emptyset$.

Then assume that $H^k, \sigma(h^k)$ and $P'(h^k)$ have been defined for all $h^k \in H^k$ and for all $1 \leq k \leq m$. Let H^{m+1} be the set of all histories $h^{m+1} = (h^m, a_m)$, where $a_m \in A'$ is an action taken by player $P'(h^m)$. That is, a_m is a feasible action of $i = P'(h^m)$ at h^m, i.e. $a_m \in A^i(\sigma(h^m)) \cup \{pass\}$. Then define state $\sigma(h^{m+1})$ as

$$\sigma(h^{m+1}) = \begin{cases} \sigma(h^m) & \text{, if } h^{m+1} = (h^m, pass) \\ (\sigma(h^m), (i, a_m)), & \text{otherwise} \end{cases} .$$

Moreover, let $P'(h^{m+1})$ be a player whose feasible set of actions at $\sigma(h^{m+1})$ is nonempty and who has not played "*pass*" in h^m such that this pass action was only followed by pass actions of other players. Formally, $P'(h^{m+1})$ is a player i such that the following two conditions are satisfied:

(i) $A^i(\sigma(h^{m+1})) \neq \emptyset$.

(ii) If $h^{m+1} = (a_1, \ldots, a_k, \ldots, a_{m+1})$ is such that $P(a_1, \ldots, a_{k-1}) = i$ and $a_k = \mathit{pass}$, then $k < m+1$ and $a_l \neq \mathit{pass}$ for some l with $k < l \leq m+1$.[5]

If there exists no such player i, h^{m+1} is a terminal history. Continue in this manner and let M be the maximal k such that all histories in H^k are terminal.

Then define $H' = \bigcup_{m=0}^{M} H^m$ and define players' preferences $\succsim_i'$ on terminal histories in H' by

$$h \succsim_i' \hat{h} \iff \sigma(h) \succsim_i \sigma(\hat{h}).$$

This defines an extensive form game $\Gamma = (N, A', H', P', (\succsim_i')_{i \in N})$ which we call *extensive form game with an option to pass associated to G*. Notice that our construction may allow us to choose different players at the same history, and because of that we may construct different extensive form games with an option to pass associated to the same game of addition, each one with a specific player function.

The following example illustrates that the subgame perfect Nash equilibrium outcome may differ from the outcome in the equilibrium of the game of addition that we started with.

Example 4.1 Consider the game of addition with two players, 1 and 2, and set of actions $A = \{a, b, c, o\}$, where a and b are actions that can only be taken by player 1 and c and o are actions that can only be taken by player 2. Let the set of states be given by

$$\Sigma = \{\varnothing, (a), (o), (a, b), (a, c)\}$$

where we have again simplified the notation for states since each action corresponds to a unique player. The feasible sets of actions at the four states then

[5] Note that in particular this condition rules out that $P(h^{m+1}) = P(h^m) = i$ if $a_{m+1} = \mathit{pass}$, i.e. no player is selected again right after the player has chosen pass.

are

$$A^1(\varnothing) = \{a\}, A^2(\varnothing) = \{o\}, A^1(a) = \{b\}, A^2(a) = \{c\},$$
$$A^i(o) = A^i(a,b) = A^i(a,c) = \emptyset \text{ for } i = 1,2.$$

Let players' preferences on Σ be given by

$$(a,b) \succ_1 (a) \succ_1 \varnothing \succ_1 (o) \succ_1 (a,c),$$
$$(a,c) \succ_2 (a) \succ_2 \varnothing \succ_2 (o) \succ_2 (a,b).$$

The game and its continuation equilibria are illustrated in Figure 4. Consider state (a). Since $(a,b) \succ_1 (a)$, (E2) implies that $(a) \notin CE(a)$. Hence, $CE(a) \subseteq \{(a,b),(a,c)\}$. Suppose $(a,b) \in CE(a)$. Then (a,c) is rationalizable: To see this note that for (a,c) condition (i) in Definition 2 is satisfied since since $(a,c) \succ_2 (a,b)$. Moreover, (ii) is satisfied since c is the only available action for player 2 at state (a), and (iii) is satisfied since $(a) \notin CE(a)$. Hence, (a,c) is rationalizable and (E3) implies that $(a,c) \in CE(a)$. Similarly, $(a,c) \in CE(a)$ implies that (a,b) is rationalizable and hence $(a,b) \in CE(a)$ by (E3). Hence, $CE(a) = \{(a,b),(a,c)\}$ which satisfies (E1)-(E3).

Consider the initial state $\varnothing$. Since $(a,b) \in CE(a)$ and $(a,b) \succ_1 \varnothing$ (E2) implies that $\varnothing \notin CE(\varnothing)$. Hence, $CE(\varnothing) \subseteq \{(o),(a,b),(a,c)\}$. Suppose by way of contradiction that $(a,b) \notin CE(\varnothing)$. Then either (o) or (a,c) is in $CE(\varnothing)$. If $(o) \in CE(\varnothing)$, then (a,b) satisfies condition (i) in Definition 2 since $(a,b) \succ_1 (o)$. Moreover, (ii) is satisfied since a is the only available action for player 1 at state $\varnothing$, and (iii) is satisfied since $\varnothing \notin CE(\varnothing)$. Hence, (a,b) is rationalizable and (E3) implies that $(a,b) \in CE(\varnothing)$ which is a contradiction. If $CE(\varnothing) = \{(a,c)\}$, then (a,b) satisfies condition (i) in Definition 2 since there exists no continuation in $CE(\varnothing)$ that is initiated by player 2. Moreover, (ii) is satisfied since a is the only available action for player 1 at state $\varnothing$, and (iii) is satisfied since $\varnothing \notin CE(\varnothing)$. Hence, also in this case (a,b) is rationalizable and and (E3) implies that $(a,b) \in CE(\varnothing)$ which is a contradiction.

We therefore conclude that $(a,b) \in CE(\varnothing)$. In that case (o) is rationalizable: To see this note that (o) satisfies condition (i) in Definition 2 since $(o) \succ_2 (a,b)$. Moreover, (ii) is satisfied since o is the only available action for player 2 at state $\varnothing$, and (iii) is satisfied since $\varnothing \notin CE(\varnothing)$. Hence, (o) is rationalizable and and

(E3) implies that $(o) \in CE(\varnothing)$. Suppose by way of contradiction that also $(a,c) \in CE(\varnothing)$. Then (E3) implies that (a,c) is rationalizable which is not true: To see this not that $(o) \in CE(\varnothing)$ and (o) is the unique continuation in $CE(\varnothing)$ initiated by player 2. Therefore, condition (i) in Definition 2 requires that $(a,c) \succsim_1 (o)$ which is not true.

We conclude that $CE(\varnothing) = \{(o), (a,b)\}$ which satisfies (E1)-(E3).

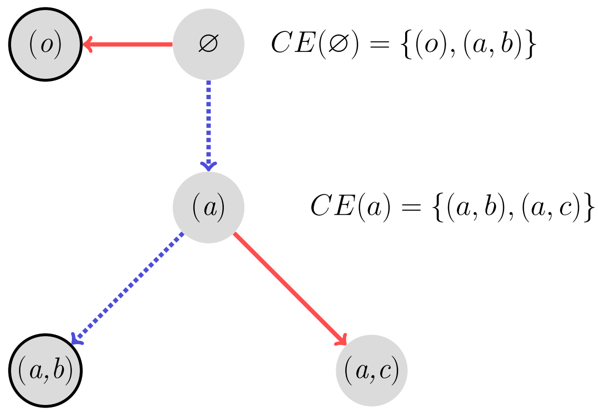

Figure 4: Equilibrium collection for the game in Example 4.1. Player 1 moves along the dotted lines and player 2 moves along the solid lines. Arrows denote the equilibrium path. The bordered nodes correspond to the equilibrium continuations at $\varnothing$.

Consider now all possible extensive form games with an option to pass associated to the given game of addition. If players take turns, then independent of the identity of the first mover this extensive form game has a unique subgame perfect Nash equilibrium with outcome $\varnothing$, i.e. in equilibrium no player takes an action. Figure 5 illustrates the subgame perfect Nash equilibrium for the case where player 1 is the first mover and players take turns. If instead the extensive form game is such that player 1 moves again after taking action a, then there is a unique subgame perfect Nash equilibrium with outcome (a,b) if player 1 is the first mover, and outcome (o) if player 2 is the first mover. Figure 6 illustrates the subgame perfect Nash equilibrium for the case where player 1 is the first mover and player 1 moves again after taking action a.

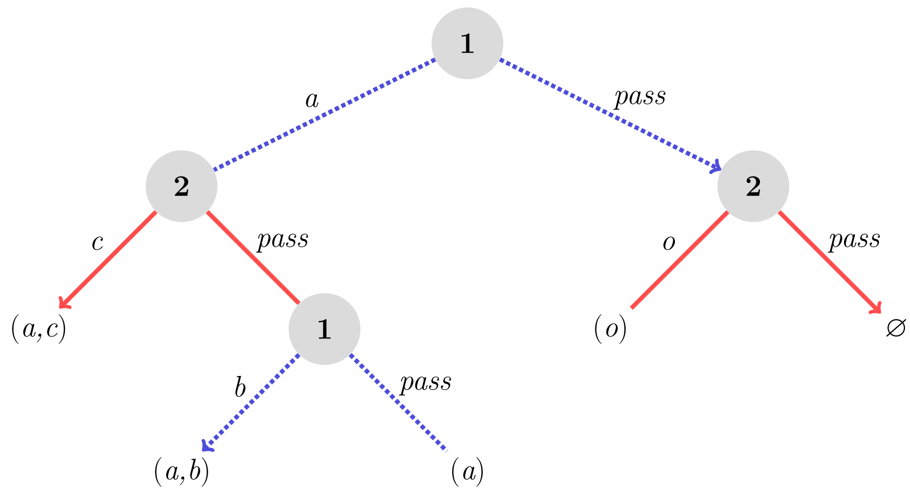

Figure 5: The game tree shows the subgame perfect Nash equilibrium for the extensive form games with an option to pass associated to the game in Example 4.1 if player 1 is the first mover and players take turns. Player 1 moves along the dotted lines and player 2 moves along the solid lines. Arrows denote the equilibrium path.

The example demonstrates that depending on the specification of the order of moves, the equilibrium predictions for extensive form games and games of addition can be very different. Moreover, the apparently natural order of moves where players take turns can produce equilibrium outcomes that never obtain in the underlying game of addition. ◇

4.2 Equivalence for a special case

In this section we identify a specific and narrow subclass of games of addition for which the associated extensive form game and the then unique associated extensive game with a passing option become equivalent formalizations and share the same equilibrium outcomes. This result and another equivalence result for two-player zero-sum games that we will present in section 5 indicate that only in extremely rare cases can we expect to identify a priori an extensive form game that will faithfully represent the strategic possibilities of players when the order of play in not directly constrained by additional information about the situations

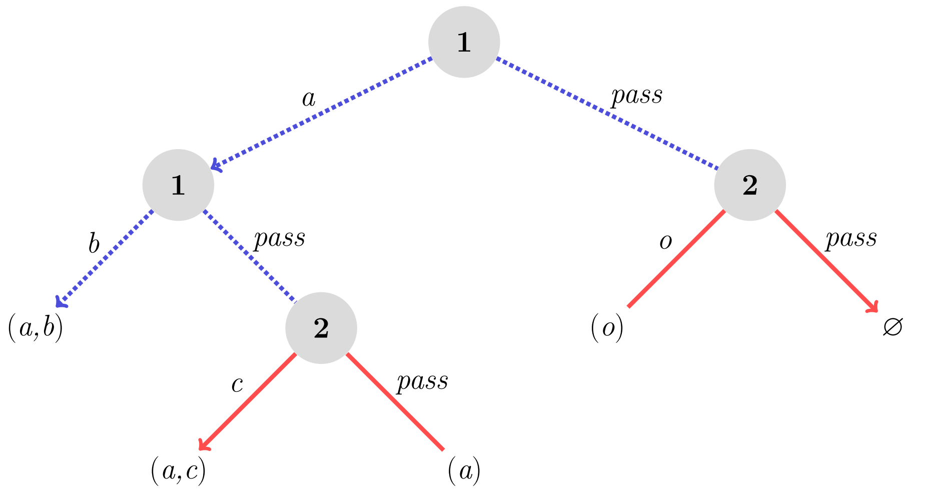

Figure 6: The game tree shows the subgame perfect Nash equilibrium for the extensive form games with an option to pass associated to the game in Example 4.1 if player 1 is the first mover and player 1 moves again after taking action a. Player 1 moves along the dotted lines and player 2 moves along the solid lines. Arrows denote the equilibrium path.

we want to model.

The subclass of games of addition has the property that all players have strict preferences on the set of states and for each state there is at most one player with a nonempty set of feasible actions. Formally, let $\mathcal{G}^U$ be the set of games of addition $G = (N, A, \Sigma, (\succsim_i)_{i \in N})$ such that the following conditions are satisfied:

(i) For all $\sigma \in \Sigma$, either there exists a unique $i \in N$ with $A^i(\sigma) \neq \emptyset$ or $A^j(\sigma) = \emptyset$ for all $j \in N$.

(ii) Players' preferences on Σ are strict, i.e. for all i, and for all $\sigma, \sigma' \in \Sigma$ with $\sigma \neq \sigma'$, it is true that $\sigma \succ_i \sigma'$ or $\sigma' \succ_i \sigma$.

It is then straightforward to show that for all $G \in \mathcal{G}^U$ there exists a unique equilibrium collection of sets of continuation states $(CE(\sigma))_{\sigma \in \Sigma}$. Moreover, $(CE(\sigma))_{\sigma \in \Sigma}$ has the property that for all $\sigma \in \Sigma$, there is a unique equilibrium continuation at σ, i.e. $CE(\sigma) = \{\sigma'\}$ for some $\sigma' \in \Sigma$. To see this first note that equilibrium conditions (E2) and (E3) imply that for all $\sigma \in \Sigma$,

$$CE(\sigma) = \{\sigma\} \iff \begin{array}{l} A^j(\sigma) = \emptyset \text{ for all } j \in N, \text{ or } \exists\, i \in N \text{ with } A^i(\sigma) \neq \emptyset \\ \text{and } \sigma \succ_i \sigma' \text{ for all } \sigma' \in \bigcup_{a \in A^i(\sigma)} CE(\sigma, (i,a)). \end{array}$$

Hence, in equilibrium the player i whose set of feasible actions is nonempty unilaterally ends the game if and only if any further action by i would yield a state that is worse for i than stopping at σ.

Suppose now that there exists some $i \in N$, some $a \in A^i(\sigma)$ and some $\sigma' \in CE(\sigma, (i,a))$ such that $\sigma' \succ_i \sigma$. Then equilibrium condition (E1) implies that

$$CE(\sigma) \subseteq \bigcup_{a \in A^i(\sigma)} CE(\sigma, (i,a)).$$

Moreover, (E3) implies that $CE(\sigma) = \{\sigma'\}$, where $\sigma' \in \bigcup_{a \in A^i(\sigma)} CE(\sigma, (i,a))$ is such that

$$\sigma' \succ_i \sigma'' \text{ for all } \sigma'' \in \bigcup_{a \in A^i(\sigma)} CE(\sigma, (i,a)).$$

We conclude that for all $\sigma \in \Sigma$, the equilibrium continuation at σ is unique. If σ is a nonterminal state, i.e. $A^i(\sigma) \neq \emptyset$ for some i, then $CE(\sigma) = \{\sigma'\}$, where

$$\sigma' \succ_i \sigma'' \text{ for all } \sigma'' \in \bigcup_{a \in A^i(\sigma)} CE(\sigma, (i,a)) \cup \{\sigma\} \text{ with } \sigma'' \neq \sigma'.$$

Let $G = (N, A, \Sigma, (\succsim_i)_{i \in N}) \in \mathcal{G}^U$ and let $\Gamma = (N, A', H', P', (\succsim'_i)_{i \in N})$ be the associated extensive form game with an option to pass defined in Section 4.1. Note that the player function P' in this case is uniquely determined since at each state there is at most one player with a nonempty feasible set of actions. Hence, for all $G \in \mathcal{G}^U$ there exists a unique extensive form game with an option to pass.

For any $G \in \mathcal{G}^U$ there exists a one-to-one correspondence between the set of states Σ in G and the set of histories H' in the associated extensive form game with an option to pass Γ. To see this, let $h = (a_1, \ldots, a_m) \in H'$ with $a_k \in A$ for $k = 1, \ldots, m-1$, and $a_m \in A' = A \cup \{pass\}$.[6] Then

[6] Note that in Γ there is no history $h = (a_1, \ldots, a_m)$ with $a_k = pass$ for some $k < m$ since a pass by some player immediately terminates the game.

$$\sigma(h) = \begin{cases} ((i_1, a_1), \ldots, (i_{m-1}, a_{m-1})), \text{ if } a_m = pass \\ ((i_1, a_1), \ldots, (i_m, a_m)) \qquad \text{, if } a_m \neq pass \end{cases}$$

is the corresponding state in G, where $i_1 = P'(\emptyset)$ and $i_k = P'(a_1, \ldots, a_{k-1})$ for $k = 2, \ldots, m,$.

The next theorem shows that for any $G \in \mathcal{G}^U$ the unique equilibrium collection of sets of continuation states is outcome equivalent to the unique subgame perfect Nash of the associated extensive form game Γ, i.e. both equilibrium concepts predict the same sequence of actions.

Theorem 4.1 *Let $G = (N, A, \Sigma, (\succsim_i)_{i \in N}) \in \mathcal{G}^U$ and let $\Gamma = (N, A', H', P', (\succsim'_i)_{i \in N})$ be the associated unique extensive form game with an option to pass. Then Γ has a unique subgame perfect Nash equilibrium α^* that is outcome equivalent to the unique equilibrium collection of sets of continuation states $(CE(\sigma))_{\sigma \in \Sigma}$ of G. That is, for all nonterminal histories $h \in H'$, if $\mathcal{O}_h \in H'$ is the outcome of α^* in the subgame that follows h, then $CE(\sigma(h)) = \{\sigma(\mathcal{O}_h)\}$.*

5 Two-Player Zero-Sum Games

In this section we study the special case of two-player zero-sum games of addition. That is, we consider the case where $n = 2$ and players' preferences satisfy the *zero-sum* condition that,

$$\text{for all } \sigma, \sigma' \in \Sigma, \quad \sigma \succsim_1 \sigma' \iff \sigma' \succsim_2 \sigma. \tag{3}$$

We first provide a general existence result for games of addition that have this form.[7]

Theorem 5.1 *Let $n = 2$ and let players' preferences be zero-sum. Then there exists an equilibrium collection of sets of continuation states $(CE(\sigma))_{\sigma \in \Sigma}$.*

In the rest of the section we will finesse the analysis by characterizing the equilibria when the zero-sum games of addition satisfy additional hypotheses. In

[7]Notice that the game in Example 3.1 is a zero-sum game while the games in Examples 3.2-3.4 are not. For the games in Examples 3.2 and 3.3 equilibria still exist, but not for the game in Example 3.4.

subsection 5.1 we study the case where players' preferences are order-independent and strict. There we first provide a characterization of the equilibrium collections of continuation states and prove that all of them share the same equilibrium outcomes. Then, we show that, unlike in the general case, in this particular one there exists an associated extensive form game whose subgame perfect equilibria are outcome equivalent to the equilibria of the game of addition. In subsection 5.2 we will consider the case where preferences over states also depend on the order in which actions have been taken.

5.1 Order independent preferences

Let $n = 2$ and let the sets of actions satisfy (2) with

$$A^i = \{a^i_1, \dots, a^i_{K^i}\} \text{ for } i = 1, 2,$$

where $K^i \geq 1$ is the number of actions for player $i \in \{1, 2\}$. For each $\sigma \in \Sigma$ and $i = 1, 2$, let $z^i(\sigma) \in \{0, 1\}^{K^i}$ be given by

$$z^i_k(\sigma) = \begin{cases} 1, \text{ if } (i, a^i_k) \in \sigma \\ 0, \text{ if } (i, a^i_k) \notin \sigma \end{cases}$$

for $k = 1, \dots, K^i$. Players' preferences are zero-sum and *order independent* if there exists some $p : \{0, 1\}^{K^1} \times \{0, 1\}^{K^2} \to \mathbb{R}$ such that for all $\sigma, \sigma' \in \Sigma$,

$$\sigma \succsim_1 \sigma' \iff p(z^1(\sigma), z^2(\sigma)) \geq p(z^1(\sigma'), z^2(\sigma')) \tag{4}$$

and

$$\sigma \succsim_2 \sigma' \iff p(z^1(\sigma), z^2(\sigma)) \leq p(z^1(\sigma'), z^2(\sigma')). \tag{5}$$

For the remainder of this subsection we will assume that players have order independent *strict preferences*, i.e. $p(z^1, z^2) \neq p(\hat{z}^1, \hat{z}^2)$ for all $(z^1, z^2), (\hat{z}^1, \hat{z}^2) \in \{0, 1\}^{K^1} \times \{0, 1\}^{K^2}$ with $(z^1, z^2) \neq (\hat{z}^1, \hat{z}^2)$. This is the generic case and it avoids dealing with multiple outcomes that are payoff equivalent.

The following theorem shows that all continuation equilibria at all states are outcome equivalent if preferences are zero-sum and order independent. Moreover, the theorem provides a characterization of the players' payoffs at the continuation equilibria at each state.

Theorem 5.2 *Let $n = 2$ and let players' preferences be zero-sum, order independent and strict.*

(i) All equilibrium collections of sets of continuation states are outcome equivalent, i.e. if $(CE(\sigma))_{\sigma\in\Sigma}$ and $\left(\widehat{CE}(\sigma)\right)_{\sigma\in\Sigma}$ are two equilibrium collections of sets of continuation states, then for all $\sigma \in \Sigma$, $\sigma' \in CE(\sigma)$ and $\sigma'' \in \widehat{CE}(\sigma)$ implies that $(z^1(\sigma'), z^2(\sigma')) = (z^1(\sigma''), z^2(\sigma''))$.

(ii) For $i = 1, 2$, let $e^{i,k} \in \{0,1\}^{K^i}$ be such that $e^{i,k}_k = 1$ and $e^{i,k}_l = 0$ for all $l \neq k$. For all $(z^1, z^2) \in \{0,1\}^{K^1} \times \{0,1\}^{K^2}$ let $p^*(z^1, z^2)$ be the unique payoff of player 1 in all continuation equilibria at $\sigma \in \Sigma$ where $z(\sigma) = (z^1, z^2)$. Then

$$p^*(z^1, z^2) =$$

$$\begin{cases} \min\limits_{k:z^2_k=0} p^*(z^1, z^2 + e^{2,k}), \ \textit{if}\ p(z^1, z^2) > \min\limits_{k:z^2_k=0} p^*(z^1, z^2 + e^{2,k}) \\ p(z^1, z^2), \ \textit{if}\ \min\limits_{k:z^2_k=0} p^*(z^1, z^2 + e^{2,k}) > p(z^1, z^2) > \max\limits_{k:z^1_k=0} p^*(z^1 + e^{1,k}, z^2) \\ \max\limits_{k:z^1_k=0} p^*(z^1 + e^{1,k}, z^2), \ \textit{if}\ \max\limits_{k:z^1_k=0} p^*(z^1 + e^{1,k}, z^2) > p(z^1, z^2) \end{cases}$$

where the minimum (maximum) over the empty set is defined to be ∞ ($-\infty$). In particular, it is true that

$$\min_{k:z^2_k=0} p^*(z^1, z^2 + e^{2,k}) \geq p^*(z^1, z^2) \geq \max_{k:z^1_k=0} p^*(z^1 + e^{1,k}, z^2).$$

In Section 4 we have seen that equilibrium outcomes for games of addition and for their associated extensive form games with an option to pass can be different, and that one cannot identify a priori an order of play for the extensive form that might eventually lead to their equality. Yet, for the class of two-player zero-sum games of addition considered in Theorem 5.2 equilibrium states are in fact outcome-equivalent to the subgame perfect Nash equilibria of a specific associated "move-or-pass" extensive form game, in which players take turns. Hence, in that particular case, imposing a priori this order of play would guarantee outcome equivalence, and both types of analysis would lead to identical predictions.

Move-or-pass extensive form game
Either player 1 or player 2 is the first mover and then players take turns. If it is player i's turn, i can either take an action from her set of actions A^i that i has not taken before or pass. In both cases the next move is by player $j \neq i$. The game ends if both players have taken all actions in their sets of actions, or if one player i has taken all actions in A^i and $j \neq i$ has passed in the last move, or if there are two passes in a row. Note that this move-or-pass game is an extensive form game associated to the above two-player game of addition as it was defined in section 4.1.

We then have the following equivalence between equilibrium states and subgame perfect Nash equilibria.

Theorem 5.3 *Let $n = 2$ and let players' preferences be zero-sum, order independent and strict. Then all subgame perfect Nash equilibria of the move-or-pass extensive form game are payoff equivalent. Moreover, the players' payoffs in a subgame perfect Nash equilibrium are independent of whether player 1 or player 2 is the first mover and if π^* is the unique payoff of player 1 in all subgame perfect Nash equilibria, then*

$$\pi^* = p^*(0\ldots0, 0\ldots0),$$

where $p^*(0\ldots0, 0\ldots0)$ is the unique payoff of player 1 in all equilibrium states.

The previous results on two-player zero-sum games with order independent preferences do not generalize to the case with more than two players. In particular, for more than two players equilibrium collections may not be outcome equivalent anymore as we show by the following example.

Example 5.1 Let there be three players and let the sets of actions satisfy (2) with $A^i = \{a\}$ for $i = 1, 2, 3$. For each $\sigma \in \Sigma$, and $i = 1, 2, 3$, let $z^i(\sigma) \in \{0, 1\}$ be given by

$$z^i(\sigma) = \begin{cases} 1, \text{ if } (i, a) \in \sigma \\ 0, \text{ if } (i, a) \notin \sigma \end{cases}.$$

Let $p(z) = (p^1(z), p^2(z), p^3(z))$ be the players' payoff at $z = (z^1, z^2, z^3) \in \{0, 1\}^3$ and let

$$
\begin{aligned}
p(0,0,0) &= (0.30, 0.30, 0.40), & p(1,0,0) &= (0.60, 0.20, 0.20),\\
p(0,1,0) &= (0.19, 0.62, 0.19), & p(0,0,1) &= (0.50, 0.25, 0.25),\\
p(1,1,0) &= (0.18, 0.19, 0.63), & p(1,0,1) &= (0.64, 0.18, 0.18),\\
p(0,1,1) &= (0.66, 0.17, 0.17), & p(1,1,1) &= (0.42, 0.16, 0.42).
\end{aligned}
$$

Moreover, assume that for all $\sigma, \sigma' \in \Sigma$,

$$\sigma \succsim_1 \sigma' \iff p^i(z^1(\sigma), z^2(\sigma), z^3(\sigma)) \geq p^i(z^1(\sigma'), z^2(\sigma'), z^3(\sigma')).$$

Note that the sum of the payoffs of the players is 1 at all states, i.e. this is a zero-sum game.

Figure 7 illustrates the equilibrium continuations. As we see there are two equilibrium states, where either only player 1 or only player 2 takes an action:

$$CE(\varnothing) = \{(1, a), (2, a)\}.$$

◇

5.2 Order dependent preferences

We now consider the case of order dependent preferences, where the players' preferences over states depend on the full sequence of actions taken by the players, not only on the set of actions taken by each player. We will show that in general with order dependent preferences equilibrium states are not unique even if there are only two players with zero-sum preferences. We also provide conditions on the players' preferences that guarantee uniqueness.

We again focus on the case where there are only two players, i.e. $n = 2$, and we assume that the sets of actions satisfy (2) with

$$A^i = \{a_1, \ldots, a_K\} \text{ for } i = 1, 2,$$

where $K \geq 1$ is the number of actions available for each player. We still assume that players' preferences are zero-sum, i.e. for all $\sigma, \sigma' \in \Sigma$,

$$\sigma \succsim_1 \sigma' \iff \sigma' \succsim_2 \sigma.$$

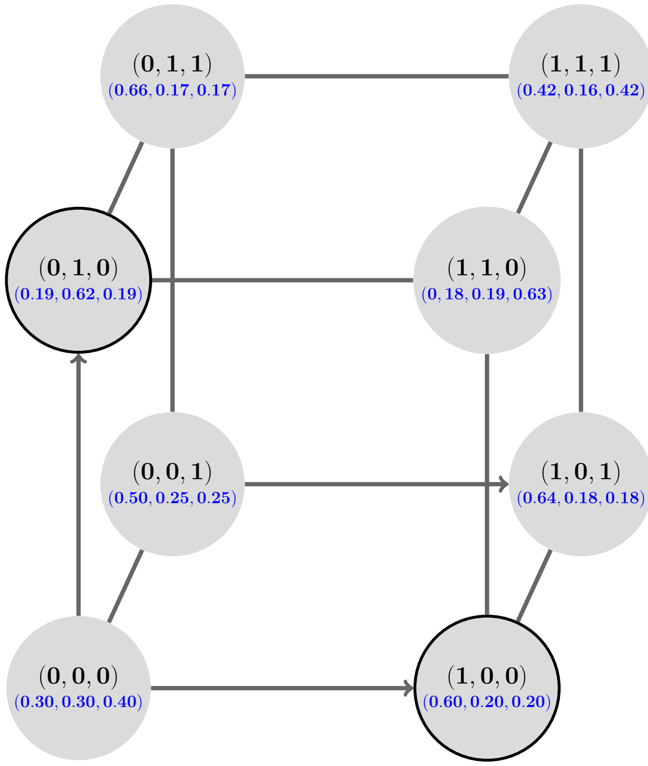

Figure 7: Equilibrium continuations in Example 5.1. At the top in each node is the vector z recording which players have taken an action at the given state and at the bottom are the corresponding payoffs of the players. Arrows denote the equilibrium path. If there is no outbound arrow at a node stopping at the corresponding state is the unique equilibrium continuation. The bordered nodes correspond to the equilibrium states at $\varnothing$.

Let $p(\sigma)$ for all $\sigma \in \Sigma$ be a utility function representing player 1's preferences $\succsim_1$.[8] This implies that player 2's preferences over Σ are represented by $1 - p(\cdot)$.

The following example demonstrates that with order dependent preferences there may be multiple equilibrium states which are not outcome equivalent even if the game is zero-sum.

[8] Since Σ is finite, there always exists a utility function representing $\succsim_1$.

Example 5.2 Let players' preferences over states be given by

$$\varnothing \succ_1 (a,b) \succ_1 (a) \succ_1 (b,a) \succ_1 (b),$$
$$(b) \succ_2 (b,a) \succ_2 (a) \succ_2 (a,b) \succ_2 \varnothing.$$

Then (E2) and (E3) imply that $CE(a) = \{(a)\}$ and $CE(b) = \{(b,a)\}$. Moreover, (E2) implies that $\varnothing \notin CE(\varnothing)$. Hence, $CE(\varnothing) \subseteq \{(a),(b,a)\}$.

$CE(\varnothing) = \{(a)\}$ is ruled out by (E3) because $\varnothing \succ_1 (a)$. $CE(\varnothing) = \{(b,a)\}$ is ruled out by (E3) because (a) is rationalizable if $(b,a) \in CE(\varnothing)$ since $(a) \succ_1 (b,a)$. Hence, $CE(\varnothing) = \{(a),(b,a)\}$ which satisfies (E3). Hence, there exist two equilibrium states which are not outcome equivalent. $\diamond$

The following theorem shows that under a specific assumption on the players' preferences there exists a unique continuation equilibrium at each state. The assumption is that for any two states which only differ with respect to the order in which the players sequentially take one action each, a player prefers the state where she is the last to take her action. This depicts situations where there is a second mover advantage, e.g. due to learning from the actions of others.

Theorem 5.4 *Let $n = 2$ and let players' preferences be zero-sum and strict, i.e. $p(\sigma) \neq p(\sigma')$ for all $\sigma \neq \sigma'$. Assume that*

$$p\left(\sigma,(2,a_l),(1,a_k),s_1,\ldots,s_m\right) > p\left(\sigma,(1,a_k),(2,a_l),s_1,\ldots,s_m\right)$$

for all $\left(\sigma,(2,a_l),(1,a_k),s_1,\ldots,s_m\right), \left(\sigma,(1,a_k),(2,a_l),s_1,\ldots,s_m\right) \in \Sigma$.

(i) There exists a unique equilibrium collection of sets of continuation states $\left(CE(\sigma)\right)_{\sigma\in\Sigma}$ and for all $\sigma \in \Sigma$, there is a unique continuation equilibrium in $CE(\sigma)$.

(ii) For all σ let $p^*(\sigma)$ be the utility of player 1 in the unique continuation equilibrium at σ. Then

$$p^*(\sigma) =$$

$$\begin{cases} \min_{k:(2,a_k)\notin\sigma} p^*(\sigma,(2,a_k)), \ \textit{if } p(\sigma) > \min_{k:(2,a_k)\notin\sigma} p^*(\sigma,(2,a_k)) \\ p(\sigma), \ \textit{if } \min_{k:(2,a_k)\notin\sigma} p^*(\sigma,(2,a_k)) > p(\sigma) > \max_{k:(1,a_k)\notin\sigma} p^*(\sigma,(1,a_k)) \\ \max_{k:(1,a_k)\notin\sigma} p^*(\sigma,(1,a_k)), \ \textit{if } \max_{k:(1,a_k)\notin\sigma} p^*(\sigma,(1,a_k)) > p(\sigma) \end{cases}$$

where the minimum (maximum) over the empty set is defined to be ∞ *(*$-\infty$*). In particular, it is true that*

$$\min_{k:(2,a_k)\notin\sigma} p^*(\sigma,(2,a_k)) \geq p^*(\sigma) \geq \max_{k:(1,a_k)\notin\sigma} p^*(\sigma,(1,a_k)).$$

(iii) For all $(\sigma,(2,a_l),(1,a_k),s_1,\ldots,s_m), (\sigma,(1,a_k),(2,a_l),s_1,\ldots,s_m) \in \Sigma$*,*

$$p^*(\sigma,(2,a_l),(1,a_k),s_1,\ldots,s_m) \geq p^*(\sigma,(1,a_k),(2,a_l),s_1,\ldots,s_m).$$

6 Conclusion

Although there are circumstances in which the order of play is either naturally predetermined or simply irrelevant, there are others in which order matters. Moreover, there are cases where players may decide whether to continue their interactions or just unanimously bring them to an end. We have offered a model of agents' interactions and a solution concept allowing for the order of play and the decisions to stop playing to be endogenized as part of the equilibrium, and we have proposed a number of existence and characterization results that clear the way to further work.

We claim that the analysis of strategic situations when the order of play does matter is often likely to be better served by using games of addition, rather than by the study of games in extensive form. To make this point as clearly as possible, we have studied a natural procedure to associate a family of extensive form games that differ in the order of play for each given game of addition. We have observed that, in general, each game of addition with well-defined equilibrium outcomes and order of plays is associated with extensive form ones whose equilibrium outcomes may widely differ. This casts doubts about the wisdom of selecting an a priori order of play when it is possible to determine it endogenously through an alternative approach. In all fairness, we have identified two special cases, and there may be others, where the equilibrium outcomes are guaranteed to be the same whether we start from a game of addition or from an extensive form game associated to it, but both correspond to very restrictive situations. Otherwise, the possibility of a mistaken identification of the appropriate order and its resulting equilibrium is wide open.

Our conclusion is that it may be fruitful to adopt the format of games of addition and its continuation equilibria to get conclusions that may differ from those that one reaches by imposing a specific order of moves. This can distort the understanding of the actual workings of processes of competition and negotiation. We hope that this first set of results may prompt further work in that direction.

Appendix

Proof of Theorem 3.1: Let $(N, A, \Sigma, (\succsim_i)_{i\in N})$ be a game of addition such that that (C1) and (C2) are satisfied. For $S \subset \Sigma$ let $S|_A = \left\{\sigma|_A \,\middle|\, \sigma \in S\right\}$. We will show that there exists an equilibrium collection of sets of continuation states $(CE(\sigma))_{\sigma\in\Sigma}$ such that for all $\sigma, \hat{\sigma} \in \Sigma$ with $\sigma|_A = \hat{\sigma}|_A$ it is true that $CE(\sigma)|_A = CE(\hat{\sigma})|_A$.

Let $K = \#A$ and let $\sigma = (s_1, \ldots, s_m) \in \Sigma$ and let $\hat{\sigma} \in \Sigma$ be such that $\sigma|_A = \hat{\sigma}|_A$. If $m = 0$, then $\sigma = \varnothing$. The proof is by induction over $K - m$.

If $K - m = 0$, then $A(\sigma) = A(\hat{\sigma}) = \emptyset$ and hence $CE(\sigma) = \{\sigma\}$ and $CE(\hat{\sigma}) = \{\hat{\sigma}\}$ fulfills (E1)-(E3) and satisfies $CE(\sigma)|_A = CE(\hat{\sigma})|_A$.

Assume that the claim has been proved for all m with $0 \le K - m \le L - 1$, where $1 \le L \le K$. Let $K - m = L$. Then by the induction hypothesis for all i, j, and for all $a \in A(\sigma)$, $CE(\sigma, (i, a)) \ne \emptyset, CE(\hat{\sigma}, (j, a)) \ne \emptyset$, and $CE(\sigma, (i, a))|_A = CE(\hat{\sigma}, (j, a))|_A$. Then there are two cases.

Case 1: For all i, $\sigma \succsim_i \sigma'$ for all $\sigma' \in \bigcup_{a\in A(\sigma)} CE(\sigma, (i, a))$.

In this case

$$CE(\sigma) = \{\sigma\} \cup \left\{\sigma' \,\middle|\, \exists\, i \text{ such that } \sigma' \in \bigcup_{a\in A(\sigma)} CE(\sigma, (i, a)) \text{ and } \sigma' \sim_i \sigma\right\}$$

fulfills (E1)-(E3). Since $\sigma|_A = \hat{\sigma}|_A$ and $CE(\sigma, (i, a))|_A = CE(\hat{\sigma}, (i, a))|_A$ for all i and for all $a \in A(\sigma)$ (C2) implies that

$$\hat{\sigma} \succsim_i \hat{\sigma}' \text{ for all } i \text{ and for all } \hat{\sigma}' \in \bigcup_{a\in A(\hat{\sigma})} CE(\hat{\sigma}, (i, a)).[9]$$

This implies

$$CE(\hat{\sigma}) = \{\hat{\sigma}\} \cup \left\{\hat{\sigma}' \,\middle|\, \exists\, i \text{ such that } \hat{\sigma}' \in \bigcup_{a\in A(\hat{\sigma})} CE(\hat{\sigma}, (i, a)) \text{ and } \hat{\sigma}' \sim_i \hat{\sigma}\right\}$$

which in turn implies $CE(\sigma)|_A = CE(\hat{\sigma})|_A$.

Case 2: $\exists i, a \in A(\sigma)$ and $\sigma' \in CE(\sigma, (i, a))$ such that $\sigma' \succ_i \sigma$. (C2) then implies that $\exists\, \hat{\sigma}' \in CE(\hat{\sigma}, (i, a))$ such that $\hat{\sigma}' \succ_i \hat{\sigma}$. Hence, $\sigma \notin CE(\sigma)$ and $\hat{\sigma} \notin CE(\hat{\sigma})$ by (E2).

For all j let $Z^j(\sigma)$ be the set of all σ' that satisfy the following conditions:

[9] Note that $A(\sigma) = A(\hat{\sigma})$ since $\sigma|_A = \hat{\sigma}|_A$.

(1) $\sigma' \in CE(\sigma, (j, a))$ for some $a \in A(\sigma)$.

(2) For all $a' \neq a$ with $(\sigma, (j, a')) \in \Sigma$, $\exists\, \sigma'' \in CE(\sigma, (j, a'))$ such that $\sigma' \succsim_j \sigma''$.

Then $CE(\sigma) = \bigcup_{j=1}^{n} Z^j(\sigma)$ satisfies (E1)-(E3). Similarly, for all j define $Z^j(\hat{\sigma})$ and $CE(\hat{\sigma}) = \bigcup_{j=1}^{n} Z^j(\hat{\sigma})$. Using (C2) and the induction hypothesis we conclude that $\sigma' = (\sigma, (j, a)) \in CE(\sigma)$ if and only if $\hat{\sigma}' = (\hat{\sigma}, (j, a)) \in CE(\hat{\sigma})$. Hence, $CE(\sigma)|_A = CE(\hat{\sigma})|_A$.

This proves the theorem. □

Proof of Theorem 4.1: Let $G = (N, A, \Sigma, (\succsim_i)_{i \in N}) \in \mathcal{G}^U$ and let $\Gamma = (N, A', H', P', (\succsim_i')_{i \in N})$ be the associated unique extensive form game with an option to pass. We have already shown that there exists a unique equilibrium collection of sets of continuation states $(CE(\sigma))_{\sigma \in \Sigma}$ for G and for all $\sigma \in \Sigma$, $CE(\sigma)$ contains a unique equilibrium continuation. Moreover, for all nonterminal states $\sigma \in \Sigma$ the unique continuation equilibrium $\sigma' \in CE(\sigma)$ has the property that

$$\sigma' \succ_i \sigma'' \text{ for all } \sigma'' \in \bigcup_{a \in A^i(\sigma)} CE(\sigma, (i, a)) \cup \{\sigma\} \text{ with } \sigma'' \neq \sigma',$$

where $i \in N$ is the unique player with $A^i(\sigma) \neq \emptyset$. Since Γ is finite and players' preferences are strict, Γ has a unique subgame perfect Nash equilibrium α^*. It remains to prove that for all nonterminal histories $h \in H'$, if $\mathcal{O}_h \in H'$ is the outcome of α^* in the subgame that follows h, then $CE(\sigma(h)) = \{\sigma(\mathcal{O}_h)\}$.

The proof is by backwards induction. For each nonterminal history $h \in H'$ let $L(h)$ be the length of the longest history in the subgame that follows h. If $L(h) = 1$, then (h, a) is a terminal history for all feasible actions $a \in A(h)$ which implies that $CE(\sigma(h, a)) = \{\sigma(h, a)\}$ if $a \neq pass$.

Let $P'(h) = i$. Then the outcome of α^* in the subgame that follows h is $\mathcal{O}_h = (h, \alpha_i^*(h))$. We have

$$\begin{aligned}
\alpha_i^*(h) = pass \quad &\Longleftrightarrow \quad (h, pass) \succ_i' (h, a) \text{ for all } a \in A'(h), a \neq pass \\
&\Longleftrightarrow \quad \sigma(h) \succ_i \sigma(h, a) \text{ for all } a \in A'(h), a \neq pass \\
&\Longleftrightarrow \quad \sigma(h) \succ_i \sigma' \text{ for all } \sigma' \in \bigcup_{a \in A^i(\sigma(h))} CE(\sigma(h, a)) \\
&\Longleftrightarrow \quad CE(\sigma(h)) = \{\sigma(h)\} = \{\sigma(h, pass))\} = \{\sigma(\mathcal{O}_h)\}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\alpha_i^*(h) = a^* \neq pass \quad &\iff \quad (h, a^*) \succ_i' (h, a) \text{ for all } a \in A'(h), a \neq a^* \\
&\iff \quad \sigma(h, a^*) \succ_i \sigma(h, a) \text{ for all } a \in A'(h), a \neq a^* \\
&\iff \quad \sigma(h, a^*) \succ_i \sigma' \text{ for all} \\
&\qquad \sigma' \in \bigcup_{a \in A^i(\sigma(h)) \setminus \{a^*\}} CE(\sigma(h, a)) \cup \{\sigma(h)\} \\
&\iff \quad CE(\sigma(h)) = \{\sigma(h, a^*)\} = \{\sigma(\mathcal{O}_h)\}.
\end{aligned}$$

This proves the claim for $L(h) = 1$. Assume now the claim has been proved for all histories $h \in H'$ with $L(h) \leq m$, where $m \geq 1$. Let $h \in H'$ be such that $L(h) = m + 1$. Let $a \in A(h)$ with $a \neq pass$. Then by the induction hypothesis $CE(\sigma(h, a)) = \{\sigma(\mathcal{O}_{(h,a)})\}$. Let $P'(h) = i$. Then the outcome of α^* in the subgame that follows h is $\mathcal{O}_{(h, \alpha_i^*(h))}$. We have

$$\begin{aligned}
\alpha_i^*(h) = pass \quad &\iff \quad \mathcal{O}_{(h,pass)} \succ_i' \mathcal{O}_{(h,a)} \text{ for all } a \in A'(h), a \neq pass \\
&\iff \quad \sigma(h) \succ_i \sigma(\mathcal{O}_{(h,a)}) \text{ for all } a \in A'(h), a \neq pass \\
&\iff \quad \sigma(h) \succ_i \sigma' \text{ for all } \sigma' \in \bigcup_{a \in A^i(\sigma(h))} CE(\sigma(h, a)) \\
&\iff \quad CE(\sigma(h)) = \{\sigma(h)\} = \{\sigma(\mathcal{O}_{(h,pass)})\} = \{\sigma(\mathcal{O}_h)\}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\alpha_i^*(h) = a^* \neq pass \quad &\iff \quad \mathcal{O}_{(h,a^*)} \succ_i' \mathcal{O}_{(h,a)} \text{ for all } a \in A'(h), a \neq a^* \\
&\iff \quad \sigma(\mathcal{O}_{(h,a^*)}) \succ_i \sigma(\mathcal{O}_{(h,a)}) \text{ for all } a \in A'(h), a \neq a^* \\
&\iff \quad \sigma(\mathcal{O}_{(h,a^*)}) \succ_i \sigma' \text{ for all} \\
&\qquad \sigma' \in \bigcup_{a \in A^i(\sigma(h)) \setminus \{a^*\}} CE(\sigma(h, a)) \cup \{\sigma(h)\} \\
&\iff \quad CE(\sigma(h)) = \{\sigma(\mathcal{O}_{(h,a^*)})\} = \{\sigma(\mathcal{O}_h)\}.
\end{aligned}$$

This proves the theorem. □

Proof of Theorem 5.1: Let $n = 2$ and let preferences satisfy $\sigma \succsim_1 \sigma' \iff \sigma' \succsim_2 \sigma$ for all $\sigma, \sigma' \in \Sigma$. For all $\sigma \in \Sigma$ let $m(\sigma)$ be the length of σ and let $M(\sigma)$

be the maximum length of a continuation state at σ. We will prove by induction over $D(\sigma) = M(\sigma) - m(\sigma)$ that for all $\sigma \in \Sigma$ there exists $CE(\sigma)$ that satisfies (E1)-(E3).

If $\sigma \in \Sigma$ is such that $D(\sigma) = 0$, then $A^i(\sigma) = \emptyset$ for $i = 1, 2$, and $CE(\sigma) = \{\sigma\}$ satisfies (E1)-(E3).

Let $\bar{D} \geq 0$ and assume the claim has been proved for all $\sigma \in \Sigma$ with $D(\sigma) \leq \bar{D}$. Let $\sigma \in \Sigma$ be such that $D(\sigma) = \bar{D} + 1$. Then by the induction hypothesis for all $i = 1, 2$, and for all $a \in A^i(\sigma)$ there exists $CE(\sigma, (i, a))$ that satisfies (E1)-(E3). For $i = 1, 2$, let $Z^i(\sigma)$ be the set of all $\hat{\sigma}$ that satisfy the following conditions:

(1) $\hat{\sigma} \in CE(\sigma, (i, a))$ for some $a \in A^i(\sigma)$.

(2) For all $a' \neq a$ with $(\sigma, (i, a')) \in \Sigma$, $\exists\ \sigma' \in CE(\sigma, (i, a'))$ such that $\hat{\sigma} \succsim_i \sigma'$.

(3) If for $j = 1, 2$, $\sigma \succsim_j \sigma'$ for all $\sigma' \in \bigcup_{a'' \in A^j(\sigma)} CE(\sigma, (j, a''))$, then $\hat{\sigma} \succsim_i \sigma$.

Note that $Z^i(\sigma) = \emptyset$ implies that either $A^i(\sigma) = \emptyset$ or $A^i(\sigma) \neq \emptyset$ and

$$\sigma \succsim_j \sigma' \text{ for all } \ \sigma' \in \bigcup_{a'' \in A^j(\sigma)} CE(\sigma, (j, a'')) \text{ for } j = 1, 2. \tag{6}$$

Moreover, note that $A^i(\sigma) \neq \emptyset$ for at least one $i \in \{1, 2\}$ since $D(\sigma) \geq 1$. Hence, if $Z^i(\sigma) = \emptyset$ for $i = 1, 2$, then (6) holds and $CE(\sigma) = \{\sigma\}$ satisfies (E1)-(E3).

If $Z^i(\sigma) \neq \emptyset$ for some $i \in \{1, 2\}$ and $Z^j(\sigma) = \emptyset$ for $j \neq i$, then there are two cases: If (6) is satisfied, then $CE(\sigma) = Z^i(\sigma) \cup \{\sigma\}$ satisfies (E1)-(E3). If (6) is violated, then $CE(\sigma) = Z^i(\sigma)$ satisfies (E1)-(E3).

Finally, consider the case $Z^i(\sigma) \neq \emptyset$ for $i = 1, 2$. For $i = 1, 2$, let

$$\bar{Z}^i(\sigma) = \{\hat{\sigma} \in Z^i(\sigma) |\, \exists \tilde{\sigma} \in Z^j(\sigma) \text{ for } j \neq i \text{ with } \hat{\sigma} \succsim_i \tilde{\sigma}\}.$$

Note that for $i = 1, 2$, and $j \neq i$,

$$\begin{aligned} \hat{\sigma} \in \bar{Z}^i(\sigma) \quad &\Rightarrow \quad \exists\ \tilde{\sigma} \in Z^j(\sigma) \text{ s.t. } \hat{\sigma} \succsim_i \tilde{\sigma} \\ &\Rightarrow \quad \exists\ \tilde{\sigma} \in Z^j(\sigma) \text{ s.t. } \tilde{\sigma} \succsim_j \hat{\sigma} \\ &\Rightarrow \quad \exists\ \tilde{\sigma} \in \bar{Z}^j(\sigma). \end{aligned}$$

This implies that $\bar{Z}^1(\sigma) \neq \emptyset \iff \bar{Z}^2(\sigma) \neq \emptyset$ and that for all $i = 1, 2$, and $j \neq i$, and for all $\hat{\sigma} \in \bar{Z}^i(\sigma)$ there exists some $\tilde{\sigma} \in \bar{Z}^j(\sigma)$ with $\hat{\sigma} \succsim_i \tilde{\sigma}$.

If $\bar{Z}^i(\sigma) = \emptyset$ for $i = 1, 2$, and (6) holds, then for any $i \in \{1, 2\}$, $CE(\sigma) = Z^i(\sigma) \cup \{\sigma\}$ satisfies (E1)-(E3). If $\bar{Z}^i(\sigma) = \emptyset$ for $i = 1, 2$, and (6) is violated, then for any $i \in \{1, 2\}$, $CE(\sigma) = Z^i(\sigma)$ satisfies (E1)-(E3).

If $\bar{Z}^i(\sigma) \neq \emptyset$ for $i = 1, 2$, and (6) holds, then $CE(\sigma) = \bar{Z}^1(\sigma) \cup \bar{Z}^2(\sigma) \cup \{\sigma\}$ satisfies (E1)-(E3). If $\bar{Z}^i(\sigma) \neq \emptyset$ for $i = 1, 2$, and (6) is violated, then $CE(\sigma) = \bar{Z}^1(\sigma) \cup \bar{Z}^2(\sigma)$ satisfies (E1)-(E3).

This proves the claim for $\sigma \in \Sigma$ with $D(\sigma) = \bar{D} + 1$ and concludes the proof of the theorem. □

Proof of Theorem 5.2: Let $(CE(\sigma))_{\sigma\in\Sigma}$ be an equilibrium collection of sets of continuation states and let σ be a state with $(z^1(\sigma), z^2(\sigma)) = (z^1, z^2)$. The proof is by induction over the sum of the actions that have not been taken by the players which we denote by L, i.e. $L = \#\{(i, k) |\, z^i_k = 0\}$. Note that $0 \leq L \leq K^1 + K^2$.

If σ is such that $L = 0$, then $(z^1, z^2) = (1 \dots 1, 1 \dots 1)$ and by (E1) $CE(\sigma) = \{\sigma\}$ which implies $p^*(1 \dots 1, 1 \dots 1) = p(1 \dots 1, 1 \dots 1)$. This proves the claim for $L = 0$.

Let $1 \leq N \leq K^1 + K^2$ and assume that the claim has been proved for all L with $0 \leq L \leq N - 1$. Let σ be such that $L = N$. From the induction hypothesis we then know that for all k with $z^i_k = 0$ all continuation equilibria in $CE(\sigma, (i, a^i_k))$ are outcome equivalent. Moreover, if l is such that $z^1_l = 0$, then by the induction hypothesis

$$\min_{l':z^2_{l'}=0} p^*(z^1 + e^{1,l}, z^2 + e^{2,l'}) \geq p^*(z^1 + e^{1,l}, z^2) \geq \max_{l'\neq l:z^1_{l'}=0} p^*(z^1 + e^{1,l} + e^{1,l'}, z^2), \quad (7)$$

and if k is such that $z^2_k = 0$, then

$$\min_{k'\neq k:z^2_{k'}=0} p^*(z^1, z^2 + e^{2,k} + e^{2,k'}) \geq p^*(z^1, z^2 + e^{2,k}) \geq \max_{k':z^1_{k'}=0} p^*(z^1 + e^{1,k'}, z^2 + e^{2,k}), \quad (8)$$

where the minimum (maximum) over the empty set is defined to be ∞ $(-\infty)$. (7) and (8) imply that for all k and l such that $z^1_l = 0$ and $z^2_k = 0$,

$$p^*(z^1, z^2 + e^{2,k}) \geq p^*(z^1 + e^{1,l}, z^2 + e^{2,k}) \geq p^*(z^1 + e^{1,l}, z^2). \quad (9)$$

(9) implies that

$$\min_{k:z_k^2=0} p^*(z^1, z^2+e^{2,k}) \geq \max_{k:z_k^1=0} p^*(z^1+e^{1,k}, z^2). \tag{10}$$

From (E1) we know that $CE(\sigma) \subseteq \bigcup_{(i,a_k^i):z_k^i=0} CE(\sigma,(i,a_k^i)) \cup \{\sigma\}$. By (E2) $\sigma \in CE(\sigma)$ if and only if

$$\min_{k:z_k^2=0} p^*(z^1, z^2+e^{2,k}) > p(z^1,z^2) > \max_{k:z_k^1=0} p^*(z^1+e^{1,k}, z^2). \tag{11}$$

Assume that (11) is satisfied which implies $\sigma \in CE(\sigma)$. Suppose by way of contradiction that there exist some (i, a_k^i) with $z_k^i = 0$ such that $\sigma' \in CE(\sigma) \cap CE(\sigma,(i,a_k^i))$. Then σ' is not rationalizable which contradicts (E3). Hence, if (11) is satisfied, then $CE(\sigma) = \{\sigma\}$ and

$$p^*(z^1,z^2) = p(z^1,z^2).$$

Assume now that

$$p(z^1,z^2) > \min_{k:z_k^2=0} p^*(z^1, z^2+e^{2,k}) \geq \max_{k:z_k^1=0} p^*(z^1+e^{1,k}, z^2).$$

Then (11) is violated and (E2) implies that $\sigma \notin CE(\sigma)$. Suppose $\sigma' \in CE(\sigma) \cap CE(\sigma,(1,a_k^1))$ for some k with $z_k^1 = 0$. Then there must exist some l with $z_l^2 = 0$ and $CE(\sigma) \cap CE(\sigma,(2,a_l^2)) \neq \emptyset$ because otherwise (E3) would imply that $p^*(z^1+e^{1,k},z^2) \geq p(z^1,z^2)$ which is not true. (E3) then implies that both $\sigma' \in CE(\sigma) \cap CE(\sigma,(1,a_k^1))$ and $\sigma'' \in CE(\sigma) \cap CE(\sigma,(2,a_l^2))$ are rationalizable and hence

$$p^*(z^1+e^{1,k},z^2) \geq \min_{h:z_h^2=0} p^*(z^1,z^2+e^{2,h}) \geq \max_{h:z_h^1=0} p^*(z^1+e^{1,h},z^2) \geq p^*(z^1,z^2+e^{2,l}).$$

This implies that

$$p^*(z^1+e^{1,k},z^2) = \max_{h:z_h^1=0} p^*(z^1+e^{1,h},z^2) = \min_{h:z_h^2=0} p^*(z^1,z^2+e^{2,h}) = p^*(z^1,z^2+e^{2,l}).$$

Hence, if there exists some k and $\sigma' \in CE(\sigma) \cap CE(\sigma,(1,a_k^1))$, then given our assumption that $p(z^1,z^2) \neq p(\hat{z}^1,\hat{z}^2)$ for all $(z^1,z^2) \neq (\hat{z}^1,\hat{z}^2)$, all continuation equilibria in $CE(\sigma)$ are outcome equivalent and $p^*(z^1,z^2) = \min_{l:z_l^2=0} p^*(z^1,z^2+e^{2,l})$.

Suppose now that $CE(\sigma) \cap CE(\sigma, (1, a_k^1)) = \emptyset$ for all k with $z_k^1 = 0$. (E1) then implies that

$$CE(\sigma) \subseteq \bigcup_{(2,a_l^2)} CE(\sigma, (2, a_l^2)).$$

Let $\sigma' \in CE(\sigma) \cap CE(\sigma, (2, a_k^2))$ for some k with $z_k^2 = 0$. Suppose by way of contradiction that $p^*(z^1, z^2 + e^{2,k}) > \min_{l:z_l^2=0} p^*(z^1, z^2 + e^{2,l})$ and let $p^*(z^1, z^2 + e^{2,h}) = \min_{l:z_l^2=0} p^*(z^1, z^2 + e^{2,l})$. Then $\sigma'' \in CE(\sigma, (2, a_h^2))$ is rationalizable and (E3) implies that $\sigma'' \in CE(\sigma)$. But then σ' is not rationalizable and (E3) implies that $\sigma' \notin CE(\sigma)$ which is a contradiction. Hence, for all k with $CE(\sigma) \cap CE(\sigma, (2, a_k^2)) \neq \emptyset$ it is true that $p^*(z^1, z^2 + e^{2,k}) = \min_{l:z_l^2=0} p^*(z^1, z^2 + e^{2,l})$. We therefore again conclude that all continuation equilibria in $CE(\sigma)$ are outcome equivalent and $p^*(z^1, z^2) = \min_{l:z_l^2=0} p^*(z^1, z^2 + e^{2,l})$.

Finally, assume that

$$\min_{k:z_k^2=0} p^*(z^1, z^2 + e^{2,k}) \geq \max_{k:z_k^1=0} p^*(z^1 + e^{1,k}, z^2) > p(z^1, z^2).$$

Then analogously to the previous case ones shows that all continuation equilibria in $CE(\sigma)$ are outcome equivalent and

$$p^*(z^1, z^2) = \max_{k:z_k^1=0} p^*(z^1 + e^{1,k}, z^2).$$

This proves the claim for $L = N$ and concludes the proof of the theorem. □

Proof of Theorem 5.3: Let H be the set of all feasible histories in the extensive form game, i.e. the set of all sequences of actions by the players (including pass) that are feasible in the given extensive form. We shortly write $a_k^i \in h$ if player i has taken action a_k^i in h. For any $h \in H$ and $i = 1, 2$, let $Z^i(h) \in \{0, 1\}^{K^i}$ be given by

$$Z_k^i(h) = \begin{cases} 1, \text{ if } a_k^i \in h \\ 0, \text{ if } a_k^i \notin h \end{cases}$$

for $k = 1, \ldots, K^i$. Let $Z(h) = (Z^1(h), Z^2(h))$.

Fix a subgame perfect Nash equilibrium of the extensive form game[10] and let $\pi(h)$ be player 1's equilibrium payoff in the subgame starting at history $h \in H$

[10]Existence is guaranteed since the game is finite.

if the last element of h is an action of a player or if h is the empty history $\varnothing$. Moreover, for $i = 1, 2$, let $\mathring{\pi}_i(h)$ be player 1's equilibrium payoff in the subgame starting at $h \in H$ if the last element of h is a pass of player $j \neq i$. For all $z = (z^1, z^2) \in \{0,1\}^{K^1} \times \{0,1\}^{K^2}$ let $p^*(z)$ be the unique payoff of player 1 in all continuation equilibria at any state σ with $z(\sigma) = z$ (see Theorem 5.2).

Let $h \in H$ be a history that is either empty or such that the last element of h is an action of a player. Let $z = Z(h)$. We will show that

$$\pi(h) = p^*(z). \tag{12}$$

The proof is by induction over the sum of the feasible actions of both players at h which we denote by L, i.e. $L = \#\{(i,k) |\, Z^i_k(h) = 0\}$. Note that $0 \leq L \leq K^1 + K^2$.

The claim is obviously true for $L = 0$ and $L = 1$. Let $2 \leq M \leq K^1 + K^2$ and assume that the claim has been proved for all L with $0 \leq L \leq M - 1$. Let the history h be such that $L = M$ and let $z = Z(h)$. Let 0^i denote a pass by player i. From the induction hypothesis we know that for all k with $z^1_k = 0$,

$$\pi(h, a^1_k) = p^*(z^1 + e^{1,k}, z^2),$$

and for all k with $z^2_k = 0$,

$$\pi(h, a^2_k) = p^*(z^1, z^2 + e^{2,k}).$$

If player 1 is moving at h, then

$$\pi(h) = \max\left\{\mathring{\pi}_2(h, 0^1), \max_{k:z^1_k=0} \pi(h, a^1_k)\right\},$$

where

$$\begin{aligned}
\mathring{\pi}_2(h, 0^1) &= \begin{cases} p(z), \text{ if } \min\limits_{k:z^2_k=0} \pi(h, 0^1, a^2_k) > p(z) \\[2ex] \min\limits_{k:z^2_k=0} \pi(h, 0^1, a^2_k), \text{ otherwise} \end{cases} \\[4ex]
&= \begin{cases} p(z), \text{ if } \min\limits_{k:z^2_k=0} p^*(z^1, z^2 + e^{2,k}) > p(z) \\[2ex] \min\limits_{k:z^2_k=0} p^*(z^1, z^2 + e^{2,k}), \text{ otherwise} \end{cases},
\end{aligned} \tag{13}$$

where the last equality follows from the induction hypothesis.

If player 2 is moving at h, then

$$\pi(h) = \min\left\{\mathring{\pi}_1(h, 0^2), \min_{k:z_k^2=0} \pi(h, a_k^2)\right\},$$

where

$$\mathring{\pi}_1(h, 0^2) = \begin{cases} p(z), \text{ if } p(z) > \max_{k:z_k^1=0} \pi(h, 0^2, a_k^1) \\ \\ \max_{k:z_k^1=0} \pi(h, 0^2, a_k^1), \text{ otherwise} \end{cases}$$

$$= \begin{cases} p(z), \text{ if } p(z) > \max_{k:z_k^1=0} p^*(z^1 + e^{1,k}, z^2) \\ \\ \max_{k:z_k^1=0} p^*(z^1 + e^{1,k}, z^2), \text{ otherwise} \end{cases}, \qquad (14)$$

where again the last equality follows from the induction hypothesis.

There are three cases.

Case 1: $p(z) > \min_{k:z_k^2=0} p^*(z^1, z^2 + e^{2,k})$

Then (13) implies that

$$\mathring{\pi}_2(h, 0^1) = \min_{k:z_k^2=0} p^*(z^1, z^2 + e^{2,k}),$$

and (14) implies that

$$\mathring{\pi}_1(h, 0^2) = p(z).$$

Hence, if player 1 is moving at h, then

$$\pi(h) = \max\left\{\min_{k:z_k^2=0} p^*(z^1, z^2 + e^{2,k}), \max_{k:z_k^1=0} p^*(z^1 + e^{1,k}, z^2)\right\} = \min_{k:z_k^2=0} p^*(z^1, z^2+e^{2,k}),$$

and if player 2 is moving at h, then

$$\pi(h) = \min \left\{ p(z), \min_{k:z^2_k=0} p^*(z^1, z^2 + e^{2,k}) \right\} = \min_{k:z^2_k=0} p^*(z^1, z^2 + e^{2,k}).$$

Hence, independent of whether player 1 or player 2 is moving at h,

$$\pi(h) = \min_{k:z^2_k=0} p^*(z^1, z^2 + e^{2,k}) = p^*(z^1, z^2)$$

by Theorem 5.2. This proves the claim for this case.

Case 2: $\min_{k:z^2_k=0} p^*(z^1, z^2 + e^{2,k}) > p(z) > \max_{k:z^1_k=0} p^*(z^1 + e^{1,k}, z^2)$

Then (13) and (14) imply that

$$\mathring{\pi}_2(h, 0^1) = \mathring{\pi}_1(h, 0^2) = p(z).$$

Hence, if player 1 is moving at h, then

$$\pi(h) = \max \left\{ p(z), \max_{k:z^1_k=0} p^*(z^1 + e^{1,k}, z^2) \right\} = p(z),$$

and if player 2 is moving at h, then

$$\pi(h) = \min \left\{ p(z), \min_{k:z^2_k=0} p^*(z^1, z^2 + e^{2,k}) \right\} = p(z).$$

Hence, independent of whether player 1 or player 2 is moving at h,

$$\pi(h) = p(z) = p^*(z^1, z^2)$$

by Theorem 5.2. This proves the claim for this case.

Case 3: $\max_{k:z^1_k=0} p^*(z^1 + e^{1,k}, z^2) > p(z)$

Then (13) implies that

$$\mathring{\pi}_2(h, 0^1) = p(z),$$

and (14) implies that

$$\mathring{\pi}_1(h, 0^2) = \max_{k:z^1_k=0} p^*(z^1 + e^{1,k}, z^2).$$

Hence, if player 1 is moving at h, then

$$\pi(h) = \max\left\{p(z), \max_{k:z_k^1=0} p^*(z^1 + e^{1,k}, z^2)\right\} = \max_{k:z_k^1=0} p^*(z^1 + e^{1,k}, z^2),$$

and if player 2 is moving at h, then

$$\pi(h) = \min\left\{\max_{k:z_k^1=0} p^*(z^1 + e^{1,k}, z^2), \min_{k:z_k^2=0} p^*(z^1, z^2 + e^{2,k})\right\} = \max_{k:z_k^1=0} p^*(z^1+e^{1,k}, z^2).$$

Hence, independent of whether player 1 or player 2 is moving at h,

$$\pi(h) = \max_{k:z_k^1=0} p^*(z^1 + e^{1,k}, z^2) = p^*(z^1, z^2)$$

by Theorem 5.2. This proves the claim for this case.

This proves (12). We conclude that player 1's payoff π^* in a subgame perfect Nash equilibrium is uniquely determined and satisfies

$$\pi^* = \pi(\varnothing) = p^*(0\ldots0, 0\ldots0),$$

where $\varnothing$ is the empty history with $Z(\varnothing) = (0\ldots0, 0\ldots0)$. This proves the theorem. □

Proof of Theorem 5.4: Define $\Sigma^0 = \{\varnothing\}$ and for $1 \le m \le M$ define $\Sigma^m = \{\sigma \in \Sigma | \, \sigma = (s_1, \ldots, s_m)\}$. Let $(CE(\sigma))_{\sigma\in\Sigma}$ be an equilibrium collection of sets of continuation states and let $\sigma \in \Sigma$ be an arbitrary state. Then $\sigma \in \Sigma^m$ for some $0 \le m \le 2K$. The proof is by induction over L, where $L = \#\{(i, a_k) | \, (i, a_k) \notin \sigma\}$. Note that $L = 2K - m$ if $\sigma \in \Sigma^m$.

If σ is such that $L = 0$, then $A^i(\sigma) = \emptyset$ for $i = 1, 2$. (E1) then implies that $CE(\sigma) = \{\sigma\}$ and $p^*(\sigma) = p(\sigma)$. This proves the claim for $L = 0$.

Let $1 \le N \le 2K$ and assume that the claim has been proved for all L with $0 \le L \le N - 1$. Let σ be such that $L = N$. From the induction hypothesis we then know that for all (i, a_k) with $(i, a_k) \notin \sigma$ there is a unique continuation equilibrium in $CE(\sigma, (i, a_k))$. Moreover, if k is such that $(1, a_k) \notin \sigma$, then by the induction hypothesis

$$\min_{l:(2,a_l)\notin\sigma} p^*(\sigma, (1, a_k), (2, a_l)) \ge p^*(\sigma, (1, a_k)) \ge \max_{l:(1,a_l)\notin\sigma} p^*(\sigma, (1, a_k), (1, a_l)), \quad (15)$$

and if k is such that $(2, a_k) \notin \sigma$, then

$$\min_{l:(2,a_l)\notin\sigma} p^*(\sigma, (2, a_k), (2, a_l)) \geq p^*(\sigma, (2, a_k)) \geq \max_{l:(1,a_l)\notin\sigma} p^*(\sigma, (2, a_k), (1, a_l)), \quad (16)$$

where the minimum (maximum) over the empty set is defined to be ∞ $(-\infty)$. (15) and (16) imply that for all k and l such that $(1, a_k) \notin \sigma$ and $(2, a_l) \notin \sigma$,

$$p^*(\sigma, (2, a_l)) \geq p^*(\sigma, (2, a_l), (1, a_k)) \text{ and } p^*(\sigma, (1, a_k), (2, a_l)) \geq p^*(\sigma, (1, a_k)). \quad (17)$$

From the induction hypothesis it follows that

$$p^*(\sigma, (2, a_l), (1, a_k)) \geq p^*(\sigma, (1, a_k), (2, a_l)),$$

which together with (17) implies that

$$\min_{k:(2,a_k)\notin\sigma} p^*(\sigma, (2, a_k)) \geq \max_{k:(1,a_k)\notin\sigma} p^*(\sigma, (1, a_k)). \quad (18)$$

From (E1) we know that $CE(\sigma) \subseteq \bigcup_{(i,a_k)\notin\sigma} CE(\sigma, (i, a_k)) \cup \{\sigma\}$. By (E2) $\sigma \in CE(\sigma)$ if and only if

$$\min_{k:(2,a_k)\notin\sigma} p^*(\sigma, (2, a_k)) > p(\sigma) > \max_{k:(1,a_k)\notin\sigma} p^*(\sigma, (1, a_k)). \quad (19)$$

Assume that (19) is satisfied which implies $\sigma \in CE(\sigma)$. Suppose by way of contradiction that there exist some $(i, a_k) \notin \sigma$ such that there exists $\sigma' \in CE(\sigma) \cap CE(\sigma, (i, a_k))$. Then σ' is not rationalizable which contradicts (E3).

Hence, if (19) is satisfied, then $CE(\sigma) = \{\sigma\}$ and

$$p^*(\sigma) = p(\sigma).$$

Assume now that

$$p(\sigma) > \min_{k:(2,a_k)\notin\sigma} p^*(\sigma, (2, a_k)) > \max_{k:(1,a_k)\notin\sigma} p^*(\sigma, (1, a_k)).$$

Then (19) is violated and (E2) implies that $\sigma \notin CE(\sigma)$. Suppose by way of contradiction that $\exists\ \sigma' \in CE(\sigma) \cap CE(\sigma, (1, a_k))$ for some k with $(1, a_k) \notin \sigma$. Then there must exist some l with $CE(\sigma) \cap CE(\sigma, (2, a_l)) \neq \emptyset$ because otherwise (E3) would imply that $p^*(\sigma, (1, a_k)) \geq p(\sigma)$ which is not true. (E3) then implies that both $\sigma' \in CE(\sigma) \cap CE(\sigma, (1, a_k))$ and $\sigma'' \in CE(\sigma) \cap CE(\sigma, (2, a_l))$ are rationalizable and hence

$$p^*(\sigma, (1, a_k)) \geq \min_{h:(2,a_h)\notin\sigma} p^*(\sigma, (2, a_h)) > \max_{h:(1,a_h)\notin\sigma} p^*(\sigma, (1, a_h)) \geq p^*(\sigma, (2, a_l))$$

which is impossible. Hence, $CE(\sigma) \cap CE(\sigma, (1, a_k)) = \emptyset$ for all k with $(1, a_k) \notin \sigma$. (E1) then implies that

$$CE(\sigma) \subseteq \bigcup_{(2,a_l)\notin\sigma} CE(\sigma, (2, a_l)).$$

Let

$$\hat{l} = \operatorname*{arg\,min}_{l:(2,a_l)\notin\sigma} p^*(\sigma, (2, a_l)). \tag{20}$$

Note that $\hat{l}$ is well defined since by assumption players' preferences over states are strict. Suppose by way of contradiction that $CE(\sigma) \cap CE(\sigma, (2, a_{\hat{l}})) = \emptyset$ and let $\sigma' \in CE(\sigma, (2, a_{\hat{l}}))$. Then σ' is rationalizable and hence (E3) implies that $\sigma' \in CE(\sigma)$ which is a contradiction. Hence, there exists some $\sigma' \in CE(\sigma) \cap CE(\sigma, (2, a_{\hat{l}}))$. By the induction hypothesis $CE(\sigma, (2, a_{\hat{l}}))$ has a unique element and hence σ' is unique.

Suppose by way of contradiction that $CE(\sigma) \cap CE(\sigma, (2, a_l)) \neq \emptyset$ for some $l \neq \hat{l}$. Let $\bar{l} \neq \hat{l}$ be such that there exists some $\sigma' \in CE(\sigma) \cap CE(\sigma, (2, a_{\bar{l}}))$. Then from (E3) we conclude that σ' is rationalizable which implies that $p^*(\sigma, (2, a_{\bar{l}})) \geq p^*(\sigma, (2, a_{\hat{l}}))$ which is a contradiction to (20) since players' preferences over states are strict. Hence, $CE(\sigma) \cap CE(\sigma, (2, a_l)) = \emptyset$ for all $l \neq \hat{l}$ with $(2, a_l) \notin \sigma$. This implies that $\sigma' \in CE(\sigma) \cap CE(\sigma, (2, a_{\hat{l}}))$ is the unique continuation equilibrium in $CE(\sigma)$ and

$$p^*(\sigma) = \min_{k:(2,a_k)\notin\sigma} p^*(\sigma, (2, a_k)).$$

Finally, assume that

$$\min_{k:(2,a_k)\notin\sigma} p^*(\sigma, (2, a_k)) > \max_{k:(1,a_k)\notin\sigma} p^*(\sigma, (1, a_k)) > p(\sigma).$$

Then analogously to the previous case ones shows that there is a unique continuation equilibrium in $CE(\sigma)$ and

$$p^*(\sigma) = \max_{k:(1,a_k)\notin\sigma} p^*(\sigma, (1, a_k)).$$

This proves (i) and (ii) for $L = N$.

To prove (iii) let

$$\begin{aligned} \sigma^{BA} &= (\sigma, (2, a_l), (1, a_k), s_1, \ldots, s_m) \in \Sigma^{2K-N} \\ \text{and} \quad \sigma^{AB} &= (\sigma, (1, a_k), (2, a_l), s_1, \ldots, s_m) \in \Sigma^{2K-N}. \end{aligned}$$

Then by assumption $p(\sigma^{BA}) \geq p(\sigma^{AB})$ and we have to prove that $p^*(\sigma^{BA}) \geq p^*(\sigma^{AB})$. Since we have already proved (ii) for $L = N$ we have the following cases:

Case 1: $p(\sigma^{BA}) > \min_{l':(2,a_{l'}) \notin \sigma^{BA}} p^*(\sigma^{BA}, (2, a_{l'}))$

Then $p^*(\sigma^{BA}) = \min_{l':(2,a_{l'}) \notin \sigma^{BA}} p^*(\sigma^{BA}, (2, a_{l'}))$ and the induction hypothesis for $L = N - 1$ implies that

$$p^*(\sigma^{BA}) = \min_{l':(2,a_{l'}) \notin \sigma^{BA}} p^*(\sigma^{BA}, (2, a_{l'})) \geq \min_{l':(2,a_{l'}) \notin \sigma^{AB}} p^*(\sigma^{AB}, (2, a_{l'})) \geq p^*(\sigma^{AB}).$$

Case 2: $\min_{l':(B,l') \notin \sigma^{BA}} p^*(\sigma^{BA}, (2, a_{l'})) > p(\sigma^{BA}) > \max_{l':(1,a_{l'}) \notin \sigma^{BA}} p^*(\sigma^{BA}, (1, a_{l'}))$

Then $p^*(\sigma^{BA}) = p(\sigma^{BA})$.

Case 2a: $p(\sigma^{AB}) > \min_{l':(2,a_{l'}) \notin \sigma^{BA}} p^*(\sigma^{AB}, (2, a_{l'}))$

Then $p^*(\sigma^{AB}) = \min_{l':(2,a_{l'}) \notin \sigma^{AB}} p^*(\sigma^{AB}, (2, a_{l'}))$ and

$$p^*(\sigma^{BA}) = p(\sigma^{BA}) \geq p(\sigma^{AB}) > p^*(\sigma^{AB}).$$

Case 2b: $\min_{l':(2,a_{l'}) \notin \sigma^{BA}} p^*(\sigma^{AB}, (2, a_{l'})) > p(\sigma^{AB}) > \max_{l':(1,a_{l'}) \notin \sigma^{AB}} p^*(\sigma^{AB}, (1, a_{l'}))$

Then $p^*(\sigma^{AB}) = p(\sigma^{AB})$ which implies

$$p^*(\sigma^{BA}) = p(\sigma^{BA}) \geq p(\sigma^{AB}) = p^*(\sigma^{AB}).$$

Case 2c: $\max_{l':(1,a_{l'}) \notin \sigma^{AB}} p^*(\sigma^{AB}, (1, a_{l'})) > p(\sigma^{AB})$

Then $p^*(\sigma^{AB}) = \max_{l':(1,a_{l'})\notin\sigma^{AB}} p^*(\sigma^{AB},(1,a_{l'}))$ and the induction hypothesis for $L = N-1$ implies that

$$\begin{aligned} p^*(\sigma^{BA}) = p(\sigma^{BA}) &> \max_{l':(1,a_{l'})\notin\sigma^{BA}} p^*(\sigma^{BA},(1,a_{l'})) \\ &\geq \max_{l':(1,a_{l'})\notin\sigma^{AB}} p^*(\sigma^{AB},(1,a_{l'})) = p^*(\sigma^{AB}). \end{aligned}$$

Case 3: $\max_{l':(1,a_{l'})\notin\sigma^{BA}} p^*(\sigma^{BA},(1,a_{l'})) > p(\sigma^{BA})$

Then $p^*(\sigma^{BA}) = \max_{l':(1,a_{l'})\notin\sigma^{BA}} p^*(\sigma^{BA},(1,a_{l'}))$.

Case 3a: $p(\sigma^{AB}) > \min_{l':(2,a_{l'})\notin\sigma^{BA}} p^*(\sigma^{AB},(2,a_{l'}))$

Then $p^*(\sigma^{AB}) = \min_{l':(2,a_{l'})\notin\sigma^{AB}} p^*(\sigma^{AB},(2,a_{l'}))$ and

$$p^*(\sigma^{BA}) = \max_{l':(1,a_{l'})\notin\sigma^{BA}} p^*(\sigma^{BA},(1,a_{l'})) > p(\sigma^{BA}) \geq p(\sigma^{AB}) > p^*(\sigma^{AB}).$$

Case 3b: $\min_{l':(2,a_{l'})\notin\sigma^{BA}} p^*(\sigma^{AB},(2,a_{l'})) > p(\sigma^{AB}) > \max_{l':(1,a_{l'})\notin\sigma^{AB}} p^*(\sigma^{AB},(1,a_{l'}))$

Then $p^*(\sigma^{AB}) = p(\sigma^{AB})$ which implies

$$p^*(\sigma^{BA}) = \max_{l':(1,a_{l'})\notin\sigma^{BA}} p^*(\sigma^{BA},(1,a_{l'})) > p(\sigma^{BA}) \geq p(\sigma^{AB}) = p^*(\sigma^{AB}).$$

Case 3c: $\max_{l':(1,a_{l'})\notin\sigma^{AB}} p^*(\sigma^{AB},(1,a_{l'})) > p(\sigma^{AB})$

Then $p^*(\sigma^{AB}) = \max_{l':(1,a_{l'})\notin\sigma^{AB}} p^*(\sigma^{AB},(1,a_{l'}))$ and the induction hypothesis for $L = N-1$ implies that

$$\begin{aligned} p^*(\sigma^{BA}) &= \max_{l':(1,a_{l'})\notin\sigma^{BA}} p^*(\sigma^{BA},(1,a_{l'})) \\ &\geq \max_{l':(1,a_{l'})\notin\sigma^{AB}} p^*(\sigma^{AB},(1,a_{l'})) = p^*(\sigma^{AB}). \end{aligned}$$

Hence, in all cases it is true that $p^*(\sigma^{BA}) \geq p^*(\sigma^{AB})$ which proves (iii) for $L = N$ and concludes the proof of the theorem. □

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