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Is the output growth rate in NIPA a welfare measure?*

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Abstract

This paper shows that, for a general family of dynamic general equilibrium models, the rate of real output growth as measured by National Income and Product Accounts (NIPA) reflects changes in welfare in the precise sense of equivalent variation. The main argument is straightforward. In a two-sector dynamic general equilibrium model of heterogeneous households, recursive preferences, and quasi-concave technology, the Bellman equation provides a representation of household preferences over current consumption and investment. When applied to this representation of preferences, a Fisher-Shell true quantity index turns out to be equal to the Divisia index, closely approximated by the Fisher ideal chain index used in NIPA.

Keywords: Growth measurement, Dynamic General Equilibrium, Welfare, Quantity indexes, Equivalent variation, NIPA, Fisher-Shell index, Divisia index, Embodied technical change.

JEL classification numbers: C43, D91, O41, O47.

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1 Introduction

The dynamic general equilibrium approach, a fundamental pillar of modern macroeconomics, disciplines the analysis by designing models aimed at replicating the aggregate behavior of real economies as measured by their national account statistics.[1] Models are then used as artificial labs where policies are quantitatively evaluated by their effects on economic growth and welfare. In this framework, we study the welfare properties of the growth rate of real output as measured by National Income and Product Accounts (NIPA). The main contribution of this paper is to show that the class of chain indexes used by NIPA reflect changes in welfare when applied to two-sector (consumption and investment) dynamic general equilibrium economies with heterogeneous households and fairly general preferences and technology.

In a dynamic general equilibrium economy, preferences are defined on consumption streams, present and future. A fictitious statistical office following the methodology of NIPA in such an economy, however, only has access to current and past data. Unlike in static settings, such a statistical office has no straightforward way of using index number theory to define a money metric representation of preferences in this framework, i.e. a way to identify changes in welfare with changes in monetary units.[2] This paper shows that such a representation exits in a dynamic setting as well. The main argument is straightforward. The Bellman equation allows to circumvent the above mentioned problem providing a representation of household preferences over current consumption and investment. When a Fisher-Shell true quantity index is applied to this representation of preferences, it turns to be equal to a Divisia index defined on current consumption and current investment.[3] All information about changes in welfare is then contained in current (i.e., observable) changes in consumption and investment. While future consumption is not observable, observed investment growth measures the impact on overall

[1]To overcome the devastating effect of the Lucas (1976) critic, and following the seminal contributions of Kydland and Prescott (1982) and Long and Plosser (1983), dynamic general equilbrium became the main instrument to understand business cycle fluctuations and study monetary and fiscal policy –see the survey by Clarida et al. (1999), as well as long run growth and innovation policies –see Romer (1990), Aghion and Howitt (1992), Grossman and Helpman (1992) and Jones (1995).

[2]For economic index number theory, see Diewert (1993), Triplett (1992), and Fisher and Shell (1998), among many others. See McKenzie (1983) for a review of the notion of equivalent variation and money metric utility.

[3]Fisher and Shell (1968) introduces the Fisher-Shell index and discusses conditions under which is it applicable. The use of a Fisher-Shell index in this context was first suggested by Licandro et al. (2002). Duerneker et al. (2021) follow our approach for TFP measurement.

welfare of the induced changes in future consumption. From a welfare perspective, when measuring economic growth, investment also matters. In turn, a Divisia index is closely approximated by the Fisher ideal chain index used by NIPA.[4] For this reason, when a statistical office applies NIPA's methodology to this family of models, the resulting measure of real output growth is a money metric measure of welfare growth in the very precise sense of equivalent variation.

It is important to point out that the main result in this paper, that output growth in NIPA measures changes in welfare, does not require a representative household. The proof that a Fisher-Shell index is equal to a Divisia index holds true when agents have different preferences, wealth and income, even if equilibrium may differ from the equilibrium of the corresponding representative agent economy.[5] When a Fisher-Shell index is applied to a dynamic general equilibrium economy with heterogenous households, money is used as a common norm to evaluate welfare changes by implicitly adopting a utilitarian approach weighting each household proportionally to its own income. By design, National Accounts aim at measuring per capita income. They are then uninformative about issues related to income inequality, omitting a very important dimension of human welfare. Despite this limitation, this paper shows that NIPA delivers a welfare-based measure of output growth in economies with heterogenous households when a utilitarian welfare function is adopted. That is, the growth rate in NIPA represents the potential growth rate of welfare of all agents under an appropriate transfer scheme.

Until the 90's the Bureau of Economic Analysis (BEA) featured a Laspeyres fixed-base quantity index to measure real GDP growth. The reason was that relative prices were then reasonably stable. As a consequence, the main components of output grew at a similar rate in real terms. In its theoretical counterpart, the Neoclassical growth model assumes that consumption and investment are the same good. The situation radically changed in the mid-80s. Following among others Gordon (1990), the BEA started deflating equipment investment by a constant quality price index, making the price of equipment investment to permanently decline relative to the price of non-durable consumption goods and services, and equipment investment to grow faster than non-durable consumption. The trend caused fixed-base quantity indexes quickly overstate the weight of durables relative to non-durables, the so-called substitution bias, forcing frequent revisions of the base year. Fixed-base quantity indexes were finally abandoned

[4]Triplett (1992) examines properties of the Fisher ideal index.

[5]The methodology suggested in this paper could then be used to understand the welfare properties of the output growth rate in heterogenous household economies like in Huggett (1993), Aiyagari (1994) and Krusell and Smith (1998), for example.

in favor of chain indexes; either the chain-Fisher ideal index, averaging the Laspeyres and Paasche indexes, or the chain-Laspeyres index.[6] In the framework of the dynamic general equilibrium approach, this paper provides a theoretical support for this change in methodology by showing that chain indexes, which approximate well a Divisia index, are welfare-based.[7]

The recognition that the price of durables was declining relative to the price of non-durables has also important implications for macroeconomics. Growth theory has been reformulated in the late 90's in order to replicate this fact. Greenwood et al. (1997) proposed a simple two-sector optimal growth model with investment specific technical change where productivity grows faster in the investment than in the consumption sector causing relative prices to change accordingly.[8] In this new class of models different components of output grow at different rates, raising the issue of aggregation as in actual data. Section 3 uses the general methodology suggested in this paper to measure output growth in the two-sector AK model proposed by Rebelo (1991).[9] The exercise illustrates that growth, as measured by the Divisia index, weighting changes in both consumption and investment, reflects indeed changes in welfare.

Our theoretical framework sheds also light on some open discussions in the literature. For example, the so-called Solow-Jorgenson controversy was revived by the differing interpretations found in Hulten (1992) and Greenwood et al. (1997). The controversy can be shown to boil down to the issue of the aggregation of consumption and investment when these are measured in different units and, more importantly, when its relative price has a trend. In our conceptual framework, it becomes clear that Greenwood et al. (1997) take a path that is more consistent with dynamic general equilibrium theory. However, implicitly these authors also adhere to a modern version of the paradigm that consump-

[6]National accounts in Europe measure real growth by the mean of a chain Laspeyres index following the Commission Decision 98/715/EC.

[7]Distortions affecting the relative price of capital goods are also relevant to development, as pointed out by Jones (1994) and Hsieh and Klenow (2007), among others. The methodology suggested in this paper could be extended to cross-country comparisons showing that differences in PPP adjusted NDP between countries measure differences in welfare in the sense of the Fisher-Shell index.

[8]The hypothesis that technical progress is embodied in capital goods was first formulated in Solow (1960). Since Greenwood et al. (1997), many other papers have followed. See Krusell (1998), Gort et al. (1999), Greenwood et al. (2000), Cummins and Violante (2002), Whelan (2003), Boucekkine et al. (2003, 2005) and Fisher (2006), among others.

[9]As shown by Felbermayr and Licandro (2005), Rebelo (1991) is the simplest two-sector general equilibrium model that replicates the permanent decline in the relative price of investment.

tion, and consequently its growth rate, is the relevant measure of real growth.[10] The results in this paper show that investment growth, encompassed by the Divisia index, also matters for welfare since it reflects utility gains associated with postponed consumption. This is particularly relevant in a world where technical change is embodied in durable goods, and hence where technical progress only materialize through the production of new capital, tangible or intangible.

As we shall discuss, our main result helps clarify that productivity growth, as measured using NIPA data, is an economic concept that embeds what is feasible with what is desired. When measuring output growth, and then technical progress, technology and preferences cannot be disentangled. In that sense it challenges the view that separates welfare from productivity measurements –see Whelan (2002, p.222) and Hulten (2001), among others. In this view, output quantity indexes are relevant for productivity measurement while consumer price indexes are relevant for welfare. This paper shows that the relevant deflator for measuring output growth and then technical progress is welfare-based, entailing that the same deflator is also relevant for making income comparisons.

Observed trends in relative prices and different sectorial growth rates are critical for the literature on structural transformation since agriculture, manufacturing, and services grow at different rates during the development process.[11] Following our approach, Duernecker et al. (2021) show that using chain indexes more accurately reflects the effects of secular changes in relative prices, rendering the productivity slowdown compatible with balance growth.

Before closing this introduction, it is important to make clear that we are well aware of the recent debate on *beyond GDP*. In this paper we refer to welfare in a very narrow sense. We acknowledge the fact, as stated by Jones and Klenow (2016), that in practice "GDP is a flawed measure of economic welfare" since many relevant dimensions of people's welfare are not included in National Accounts.[12] The aim of this paper is different. It suggests

[10]Greenwood et al. (1997), in fact, is not a normative paper. It does perform the positive exercise of measuring the contribution of embodied technical change to US growth. However, in doing so, they measure output and its growth rate in units of consumption, de facto identifying real output growth with consumption growth. Cummins and Violante (2002) generalize the exercise and use standard NIPA methodology to the same objective, finding similar quantitative results. See also Greenwood and Jovanovic (2001).

[11]See Acemoglu and Guerrieri (2008), Duarte and Restuccia (2010), Herrendorf et al. (2013), and Ngai and Pissarides (2007), among many others.

[12]An extensive and highly informative discussion of the main issue is in the Stiglitz, Sen and Fitoussi (2009) report.

a methodological approach to use index number theory in dynamic general equilibrium models. In line with the Jones and Klenow (2016) observation that "GDP per person is an informative indicator of welfare," we show that, for a very general family of dynamic general equilibrium models, the rate of real output growth as measured by NIPA reflects changes in welfare in the precise sense of equivalent variation. More important, the methodological approach suggested in this paper could shed light on some issues raised in the beyond GDP debate, helping National Accounts to add those omitted dimensions for which a monetary valuation is possible.[13]

In the general framework of a two-sector dynamic general equilibrium economy with recursive preferences and quasi-concave technology, Section 2 proves the main result of this paper that the growth rate of output as measured by NIPA measures welfare gains in the precise sense of equivalent variation. It does first for a representative agent economy, and secondly for an economy with heterogeneous households. Section 3 applies this methodology to study the measurement of growth in an economy with embodied technical progress, allowing for a more intuitive interpretation of the main result. Section 4 discusses the implications for GDP measurement. Section 5 concludes and suggest future extensions.

2 Measuring output growth

Let us consider a two-sector non-stochastic perfectly competitive dynamic general equilibrium economy in continuous time. There are two goods, consumption and investment, and a quasi-concave technology transforming capital and labor into these two goods. Intertemporal preferences are recursive. Let us also assume that preferences and technology are such that an equilibrium path exists and is unique. In this economy, a fictitious statistical office uses a simple quantity index of changes in real output built out of observables at t. It does it in a way that is consistent with preferences and technology.

[13]For example, our approach could be applied to the "veil of ignorance" preferences suggested by Jones and Klenow (2016), measuring welfare gains by the mean of a money metric equivalent variation index instead of consumption equivalent. Even if the growth rate of GDP is not, and will likely never be, a comprehensive measure of welfare changes, this paper shows that it fundamentally embodies those welfare gains related to changes in observable market activities and non-market activities for which a monetary valuation is possible.

2.1 The Bellman equation under recursive preferences

For any date $t \geq 0$ and any consumption path $C : [0, \infty) \to \mathbb{R}_+$, let ${}_tC$ denote the restriction of C to $[t, \infty)$. Preferences of the representative household are represented by a recursive utility function U generated by the differential equation

$$\frac{d}{dt}U({}_tC) = -f(c_t, U({}_tC)). \tag{1}$$

The generating function f is assumed to be differentiable, with $f_1 > 0$ and $f_2 < 0$. Note that f_1 is the marginal utility from current consumption, lost when we move an infinitesimal period of time ahead, and so the negative sign in (1). In turn, $f_2 < 0$ is related to the implicit subjective discount rate.[14] For instance, the classical additively separable utility function is an important particular case of the general specification above in which

$$U({}_tC) = \int_t^\infty e^{-\rho(s-t)} u(c_s) ds$$

with $u'(c) > 0$, $u''(c) < 0$ and $\rho > 0$. Differentiating with respect to time t, we get

$$\frac{d}{dt}U({}_tC) = -u(c_t) + \rho U({}_tC).$$

Hence, in this case, $f(c, U) = u(c) - \rho U$ and indeed $f_1(c, U) = u'(c) > 0$ while $f_2(c, U) = -\rho < 0$. This illustrates the interpretation given above that f_1 is the marginal utility from current consumption, lost when we move an infinitesimal period of time ahead, and f_2 is the return to household assets, which value is represented by $U({}_tC)$ and the discount rate is ρ.

The equilibrium can be characterized as a solution to a planner's problem. Each time t, the social planer chooses individual consumption c_t and per capita net investment $\dot{k}_t$ such that $(c_t, \dot{k}_t) \in \Gamma(k_t, a_t)$, where k_t is capital and a_t represents a vector of exogenous non-stochastic states (e.g., total factor productivity in the Solow model). We assume that, for every $k_t > 0$, there exists a unique consumption and investment path equilibrium $(c_s, \dot{k}_s)_{s \geq t}$ that maximizes $U({}_tC)$ subject to the technological constraint.

Suppose that in this abstract economy a statistical office wants to measure real output growth at time t. Changes in utility $U({}_tC)$ would be an impractical choice. Conceptually poor because utility (and changes in utility) can be altered by any monotone transformation of U. Impractical because it entails information about future consumption that

[14] Epstein (1987) explores conditions under which a generating function f represents a recursive utility function U. Becker and Boyd (1997, chapter 1) and Backus et al. (2004) motivate the study of general recursive preferences.

has not been observed yet. The statistical office is constrained to use only current consumption c_t, net investment $x_t = \dot{k}_t$, and capital stock k_t. Since the economy is recursive no past values are necessary, all is summarized in the state of the system. But then the statistical office needs a representation of preferences over the consumption and investment space. This is what the Bellman equation provides. The original problem is to maximize $U({}_tC)$ subject to $(c_s, \dot{k}_s) \in \Gamma(k_s, a_s)$ for all $s \geq t$, $k_t > 0$ given. The associated Bellman equation is

$$0 = \max_{(c,x)\in\Gamma(k_t,a_t)} f(c, v(k_t, a_t)) + v_1(k_t, a_t)x + v_2(k_t, a_t)\dot{a}_t. \tag{2}$$

The intuition behind this equation becomes clear if one notes that along an optimal path $v(k_t, a_t) = U({}_tC)$ so

$$\frac{dv(k_t, a_t)}{dt} = v_1(k_t, a_t)\dot{k}_t + v_2(k_t, a_t)\dot{a}_t = -f(c_t, v(k_t, a_t)).$$

With all past actions summarized in k_t, the objective function in (2) is giving us the preference relation over consumption and investment at time t.[15]

2.2 A Fisher-Shell true quantity index

As argued above, household preferences at time t can be seen as represented by the objective function in the Bellman equation

$$w_t(c, x) \doteq f(c, v(k_t, a_t)) + v_1(k_t, a_t)x + v_2(k_t, a_t)\dot{a}_t.$$

To save notation we are writing $w_t(c, x)$, but time enters only through the endogenous k_t and exogenous states a_t of the system. This function can then be seen as a representation of individual preferences over current consumption and net investment, the last summarizing postponed consumption. To the extent that states change along an equilibrium path, these "preferences" are time-dependent. This is precisely the building block of the true quantity index introduced by Fisher and Shell (1968). Since welfare comparisons must be done within the same preference map, the Fisher-Shell true quantity index proposes to fix not only prices but also preferences. In particular, it compares income today with the hypothetical level of income that would be necessary to attain the level of utility associated with tomorrow's income and prices with today's prices and

[15]The planner solves a standard recursive program in which the state variable summarizes at each time t all past information that could be relevant for today's decisions. For a brief exposition of recursive techniques in continuous time see Obstfeld (1992).

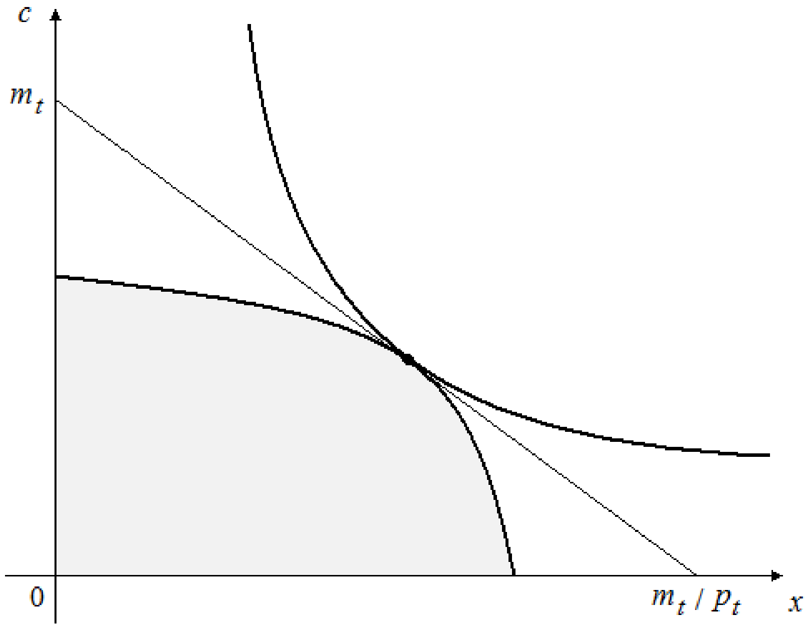

Figure 1: The production possibilities frontier and competitive prices

today's preferences as evaluated by $w_t(c, x)$. In a sense, we are fixing preferences because we are "asking" the agent today. Hence, the idea is identical to the notion of equivalent variation in a static setting.

Without loss of generality, we choose the consumption good as numeraire. Under standard assumptions optimal choices will lie in the boundary of $\Gamma(k_t, a_t)$ so that there is a well-defined equilibrium price of investment $p_t > 0$ relative to consumption (fig. 1). Equilibrium nominal net income at time t is then $m_t \doteq c_t + p_t x_t$. Hence, the technological constrain in (2) can be replaced by $c + p_t x \leq m_t$. As a consequence, the indirect utility function can be defined as

$$u_t(m_t, p_t) \doteq \max_{c + p_t x \leq m_t} w_t(c, x)$$

and the expenditure function as

$$e_t(u_t, p_t) \doteq \min_{w_t(c,x) \geq u_t} c + p_t x.$$

When comparing time t with time $t + h$, for some $h > 0$ arbitrarily small, the fictitious statistical office would like to design a true quantity index of welfare change using standard index number theory. Since preferences $w_t(c, x)$ are changing over time, the statistical office uses a Fisher-Shell true quantity index, which fixes not only prices but also preferences. The reason is that comparisons must be done within the same

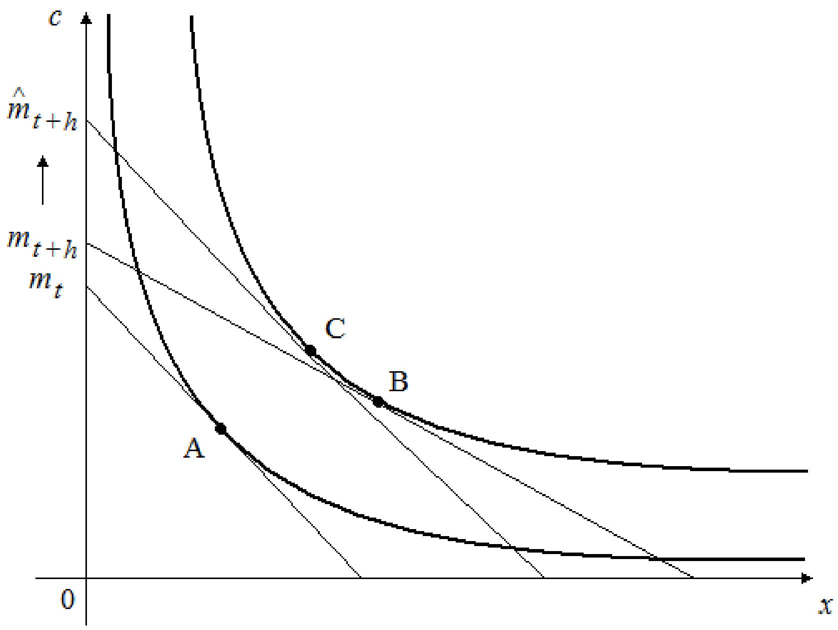

Figure 2: The Fisher-Shell true quantity index

preference map. To be precise, the Fisher-Shell index compares income today m_t with the hypothetical level of income tomorrow $\hat{m}_{t+h}$ that would be necessary to attain the level of utility $u_t(m_{t+h}, p_{t+h})$ associated with tomorrow's income and prices m_{t+h}, p_{t+h} with today's prices p_t and today's preferences as represented by functions e_t and u_t. This artificial level of tomorrow's income is defined as

$$\hat{m}_{t+h} = e_t(u_t(m_{t+h}, p_{t+h}), p_t).$$

The idea is illustrated in Figure 2 in a situation where nominal income increases and the price of investment declines. The preference map corresponds to time t preferences as represented by $w_t(c, x)$. Point A is the observed situation at time t. Point B is the hypothetical choice using time t preferences when facing observed prices p_{t+h} and income m_{t+h}. Point C represents the choice that maintains such level of utility but with prices p_t. The index $\hat{m}_{t+h}/m_t$ compares two levels of income that correspond to the same price vector so it is filtering price changes. In the case depicted in Figure 2, the true quantity index is just reflecting the fact that the true output deflator is dropping with the price of investment, that is to say that income in real terms is growing more than nominal income m_{t+h}/m_t, as measured by using this particular numeraire. In regard of the definitions of u_t and e_t, it is straightforward to see that the true quantity index is independent of the choice of the numeraire provided that the price of equipment relative to consumption p_t remains unchanged.

The instantaneous Fisher-Shell index is defined as

$$g_t^{\rm FS} \doteq \frac{d}{dh}\left.\frac{\hat{m}_{t+h}}{m_t}\right|_{h=0} = \frac{1}{m_t}\left.\frac{d\hat{m}_{t+h}}{dh}\right|_{h=0},$$

that is, the instantaneous growth rate of the factor $\frac{\hat{m}_{t+h}}{m_t}$ when h gets arbitrarily small (for details see online Appendix). To compute this index note that

$$\left.\frac{d\hat{m}_{t+h}}{dh}\right|_{h=0} = e_{1,t}(u_t(m_t,p_t),p_t)\big(u_{1,t}(m_t,p_t)\dot{m}_t + u_{2,t}(m_t,p_t)\dot{p}_t\big)$$

where subscripts denote the partial derivatives with respect to the corresponding arguments.

To obtain an expression for all these derivatives let us go back to the dual and primal problems discussed above. Let μ be the Lagrange multiplier of the maximization problem in the definition of the indirect utility function, measuring the marginal contribution of income m to welfare w. We have, from the the primal problem

$$u_{1,t}(m_t,p_t) = \mu$$
$$u_{2,t}(m_t,p_t) = -\mu x_t,$$

and, since the expenditure function is the inverse of the indirect utility function,

$$e_{1,t}(u_t,p_t) = \frac{1}{\mu}.$$

As expected, the marginal contribution of income to welfare, $u_{1,t} = \mu$, is equal to the inverse of the marginal contribution of utility u to total expenditure, $e_{1,t} = 1/\mu$. Moreover, the negative marginal contribution of prices to welfare is $\partial u/\partial p = -\mu x$, since an increase in prices reduces income by x units. These properties are critical for the result below and they are directly related to the *money metric utility* nature of the Fisher-Shell index, which defines the hypothetical income $\hat{m}$ using the expenditure function to valuate changes in utility after controlling for changes in prices.

Using the three conditions above in the definition of the Fisher-Shell index, we conclude that

$$g_t^{\rm FS} = \frac{\dot{m}_t - x_t\dot{p}_t}{m_t} = \frac{\dot{m}_t}{m_t} - \frac{p_t x_t}{m_t}\frac{\dot{p}_t}{p_t}.$$

Notice that the marginal terms e_1, u_1 and u_2 in the definition of the Fisher-Shell index simplify as a direct consequence of the properties discussed in the paragraph above; all three are related to the marginal value of income μ. It is in this sense that money metric utility operates in the Fisher-Shell index. Since gains in welfare are measured as an equivalent variation, comparing the artificial level of income $\hat{m}_{t+h}$ with the nominal income m_t, and prices enter linearly in the budget constraint, gains in welfare are equal to the change in nominal income minus the contribution of prices to this change.

Finally, differentiate the definition of nominal income $m_t = c_t + p_t x_t$ with respect to time and define the equilibrium share of net investment to net income as $s_t \doteq p_t x_t / m_t$ to write

$$\frac{\dot{m}_t}{m_t} = (1 - s_t)\,\frac{\dot{c}_t}{c_t} + s_t \frac{\dot{x}_t}{x_t} + s_t \frac{\dot{p}_t}{p_t},$$

which implies that

$$g_t^{\mathrm{FS}} = (1 - s_t)\,\frac{\dot{c}_t}{c_t} + s_t \frac{\dot{x}_t}{x_t} \doteq g_t^{\mathrm{D}}$$

where g_t^{D} denotes the Divisia index. That is, we have shown that the Divisia index is a true quantity index in this framework, and as such it is a welfare measure. In regard of Figure 2, it is easy to see why the equivalent variation is equal to the Divisia index when the time increment (and therefore the change in prices) converges to zero. For the same reason, one can see that in discrete time and for price changes sufficiently small, the equivalence will hold approximately.[16]

The interpretation is straightforward. It is clear that g_t^{FS} is a measure of real growth since it is constructed as the growth rate of nominal income subtracting pure price changes, in this case the change of the relative price of investment p_t. The index only keeps changes in quantities. It is also clear that it is a true index because it is constructed from the representative household's preferences using standard theory.[17] The beauty of the result is that a national statistics office in this framework does not need to know people's preferences or production technology, neither the future consumption path, just current and past quantities and prices.

2.3 Household heterogeneity

In this section, we show that the reasoning above applies to a heterogeneous agents economy with different preferences, assets and income. Critical in the result is the fact that the utility representation of preferences emerging from the Bellman equation is

[16] Building on our work in continuous time, Duernecker et al. (2021) provide a formal proof in discrete time.

[17] This equivalence would come as no surprise to index number theorists. The Fisher ideal chain index is known to approximate in general some sort of true quantity index because both are bounded from above and below by the Laspeyres and Paasche indexes respectively. In continuous time, these indexes tend to each other as the time interval h tends to zero. Further, in general, the Divisia index coincides with the Fisher ideal chain index if the growth rates of consumption and investment are constant.

quasilinear, belonging to the Gorman family –see Gorman (1953, 1961). Let us develop a formal argument.

First, let us assume that there is a continuum of heterogeneous households of unit mass with household specific recursive preferences represented by the utility U_i generated by the differential equation

$$\frac{1}{dt}U_i({}_tC_i) = -f_i(c_{i,t}, U_i({}_tC_i)),$$

where ${}_tC_i$ represents the consumption path of household i and the household specific generating function f_i has the same properties as above. Second, at equilibrium capital is distributed across households according to φ_t, which maps any individual i at any time t into a quantity of capital $k_{i,t}$. Finally, assume that an equilibrium exists and is unique. Notice that this equilibrium will likely be different from the corresponding equilibrium with a representative household. The distribution of preferences and capital across individuals matters.

In the recursive competitive equilibrium representation of this economy, with exogenous state a_t and an equilibrium distribution of capital φ_t, the problem of a household i with capital $k_{i,t}$ can be written as

$$\begin{aligned} 0 = \quad \max \quad & f_i(c_i, v_i(k_{i,t}, a_t, \varphi_t)) + v_{i,1}(k_{i,t}, a_t, \varphi_t)x_i + \pi_{i,t} \\ \text{s.t.} \quad & c_i + p_t x_i = m_{i,t} \end{aligned}$$

where c_i and x_i are household's current consumption and net investment, respectively, p_t is the equilibrium price, and $m_{i,t}$ is the equilibrium net income of individual i. The term $\pi_{i,t}$ encompasses the differential terms of $v_i(k_{i,t}, a_t, \varphi_t)$ with respect to time that are exogenous to the problem of the consumer, i.e., those corresponding to a_t and φ_t.

As in Section 2.2, the optimization problem of household i is associated with the instantaneous utility function over consumption and net investment

$$w_{i,t}(c_i, x_i) \doteq f_{i,t}(c_i) + x_i,$$

where $f_{i,t}(c_i) \doteq f_i(c_i, v_i(k_{i,t}, a_t, \varphi_t))/v_{i,1}(k_{i,t}, a_t, \varphi_t)$. Notice that we have subtracted $\pi_{i,t}$ from the right hand side of the Bellman equation and then divided it by $v_{i,1}(k_{i,t}, a_t, \varphi_t)$. Since none of these two terms depend on c or x, such a transformation has no effect on the households program. The function $w_{i,t}(c_i, x_i)$ is maximized under the budget constraint $c_i + p_t x_i = m_{i,t}$. Since this utility representation is quasilinear, it belongs to the Gorman family. It is easy to show that the indirect utility and expenditure functions become

$$u_{i,t}(m_{i,t}, p_t) = A_{i,t}(p_t) + \frac{m_{i,t}}{p_t}$$

$$e_{i,t}(u_{i,t}, p_t) = p_t(u_{i,t} - A_{i,t}(p_t)),$$

where $A_{i,t}(p_t)$ is defined below. In fact, from the household problem, optimal consumption c_i solves

$$f'_{i,t}(c_i) = \frac{1}{p_t}.$$

By denoting the implicit solution for c_i as $c_{i,t}(p_t)$, it is then easy to show that

$$A_{i,t}(p_t) = f_{i,t}(c_{i,t}(p_t)) - \frac{c_{i,t}(p_t)}{p_t}.$$

Let us define the artificial level of household i's tomorrow income as in Section 2.2, i.e.,

$$\hat{m}_{i,t+h} = e_{i,t}(u_{i,t}(m_{i,t+h}, p_{t+h}), p_t) = p_t(A_{i,t}(p_{t+h}) - A_{i,t}(p_t)) + \frac{p_t}{p_{t+h}} m_{i,t+h},$$

which is linear on income due to the fact that preferences $w_{i,t}(c_i, x_i)$ are quasilinear. Like in Section 2.2, from the perspective of time t household i is better off at $t + h$ if $\hat{m}_{i,t+h} > m_{i,t}$.

Consistently with National Accounts, let us define aggregate income as

$$m_t = \int_i m_{i,t} di,$$

which also measures per capita income since population has been normalized to unity. Let us now define the tomorrow aggregate hypothetical income consistently with the definition of per capita income as

$$\tilde{m}_{t+h} = \int_i \hat{m}_{i,t+h} di.$$

Notice that if $\tilde{m}_{t+h} > m_t$, from the perspective of time t, equilibrium at $t + h$ Pareto dominates equilibrium at t since some agents can compensate others in a way that leaves all individuals at least as well as at time t but some of them better off.

Using the results above,

$$\tilde{m}_{t+h} = p_t\Big(\bar{A}_t(p_{t+h}) - \bar{A}_t(p_t)\Big) + \frac{p_t}{p_{t+h}} m_{t+h},$$

where

$$\bar{A}_t(p_t) = \int_i A_{i,t}(p_t) di.$$

As in Section 2.2, let us define the Fisher-Shell index for the economy with heterogeneous households as

$$\tilde{g}_t^{\mathrm{FS}} \doteq \frac{1}{m_t} \left.\frac{d\tilde{m}_{t+h}}{dh}\right|_{h=0}. \tag{3}$$

Operating on the definition of $\tilde{m}_{it+h}$ above

$$\left.\frac{d\tilde{m}_{t+h}}{dh}\right|_{h=0} = \dot{m}_t + \left(p_t \bar{A}'_t(p_t) - \frac{m_t}{p_t}\right)\dot{p}_t,$$

where

$$\bar{A}'_t(p_t) = \int_i A'_{i,t}(p_t)di = \int_i \left(f'_{i,t}c'_{i,t} - \frac{1}{p_t}c'_{i,t} + \frac{c_{i,t}}{p_t^2}\right)di = \frac{c_t}{p_t^2},$$

because $f'_{i,t} = 1/p_t$ and where $c_t = \int_i c_{it}di$ is consumption per capita. Then

$$\tilde{g}_t^{\rm FS} = \frac{\dot{m}_t}{m_t} - s_t\frac{\dot{p}_t}{p_t} = (1-s_t)\frac{\dot{c}_t}{c_t} + s_t\frac{\dot{x}_t}{x_t}, \tag{4}$$

where $s_t \doteq p_t x_t/m_t$ as before and $x_t = \int_i x_{it}di$ is net investment per capita. The Fisher-Shell index is, indeed, equal to the Divisia index, meaning that the growth rate in NIPA is a welfare measure irrespective of households being either homogeneous or heterogeneous. Of course, at equilibrium, consumption and investment in the heterogeneous agent economy may be growing at different rates than in the corresponding representative household model, and the saving rate may also be different. Consequently, even when the growth rate, as measured by the Divisia index is a welfare measure in both economies, these two economies may be growing at different rates.

The Fisher-Shell index defined above implicitly emerges from a utilitarian social welfare approach. To see this, let us apply a Fisher-Shell index to each household, a money metric representation of changes in household welfare given by the household specific income growth index

$$\tilde{g}_{i,t}^{\rm FS} = (1-s_{i,t})\frac{\dot{c}_{i,t}}{c_{i,t}} + s_{i,t}\frac{\dot{x}_{i,t}}{x_{i,t}}.$$

Which brings us to the result that households income should be deflated using household specific deflators including consumption and capital savings.[18] Aggregating the household specific Fisher-Shell indexes, weighting each household by the corresponding household income share $\pi_{i,t} = m_{i,t}/m_t$, we obtain again equation (4), i.e.,

$$\tilde{g}_t^{\rm FS} = \int_i \pi_{i,t}\, \tilde{g}_{i,t}^{\rm FS}\, di = (1-s_t)\frac{\dot{c}_t}{c_t} + s_t\frac{\dot{x}_t}{x_t},$$

where the growth rate of household i's income is the household specific Divisia index above. We have then showed that the aggregate Fisher-Shell index in (3) is consistent with a utilitarian social welfare approach where the weight of each individual is given

[18]For a discussion on the use of household specific CPI's to deflate household income see Prais (1959), Pollak (1980, 1998), and Ley (2005).

by her weight on total income. In other words, aggregate growth in the model economy is the welfare-based growth rate of income of all agents under an appropriate income transfer scheme.

Two facts are critical for the main result in this section, i.e., that the Fisher-Shell index is equal to the Divisia index under heterogeneous households. First, as in the case of homogeneous households, nominal income is the metric used to measure households' utility. Gains in welfare are measured as gains in nominal income minus inflation. Second, the representation of preferences emerging from the Bellman is quasilinear, linear in current net investment. This property is not critical at all in the case of a representative household; in fact, in Section 2.2, we show that the Fisher-Shell index is equal to the Divisia index for a general function $w(c, x)$. Indeed, it is critical in this section, since we profit from the quasi linearity representation of preferences to show that aggregate utility gains, as measured by the Fisher-Shell index, are again equal to gains in nominal per capita income minus inflation.

3 Embodied technical progress

With the introduction in NIPA of constant quality price indexes for equipment investment, two related new facts emerged. First, the price of equipment investment permanently declines relative to the price of non-durable consumption, and second, equipment investment permanently grows faster than non-durable consumption implying that the investment to output ratio is permanently growing. To accommodate growth theory to these new facts, Greenwood et al. (1997), in their seminal paper, extend the Neoclassical growth model to a two-sector (consumption and investment) growth model with two sources of technical progress, consumption- and investment-specific technical change; the latter interpreted as embodied in capital goods.

This section describes a simple version of the two-sector AK model proposed by Rebelo (1991) and applies to it the Fisher-Shell index proposed in Section 2.2 to show that the BEA had good fundamental reasons to use a Fisher ideal chain index to measure output growth. As shown in Felbermayr and Licandro (2005), the two-sector AK model is the simplest endogenous growth model that replicates the observed permanent decline in the relative price of equipment and the permanent increase in the investment to output ratio.[19] We decided to use it instead of the original Greenwood et al. (1997) model, since the AK model has the advantage of jumping to the balanced growth path at the initial

[19]See also Acemoglu (2009, chapter 11.3), which follows Felbermayr and Licandro (2005) very closely.

time, which allows for an explicit solution of the value function. This will help illustrating the role of money metric utility and comparing it to consumption equivalent measures.

3.1 The two-sector AK model

The model in this section is based on Rebelo (1991), follows Felbermayr and Licandro (2005) closely, and entails all the characteristics that are relevant to the present discussion in the simplest possible framework. The stock of capital at each time t is k_t, from which a quantity $h_t \leq k_t$ is devoted to the production of the consumption good. Consumption goods technology is

$$c_t = h_t^\alpha,$$

where $\alpha \in (0, 1)$. The remaining stock $k_t - h_t \geq 0$ is employed in the production of new capital goods with a linear technology

$$\dot{k}_t = A(k_t - h_t),$$

where $A > 0$ is the marginal product of capital in the investment sector net of depreciation. There is a given initial stock of capital $k_0 > 0$. Again, we denote net investment $x_t = \dot{k}_t$.

The representative household has preferences over consumption paths represented by[20]

$$\int_0^\infty \frac{c_t^{1-\sigma}}{1-\sigma}\, \mathrm{e}^{-\rho t} dt, \tag{5}$$

that is, the additive case mentioned above, where $\rho > 0$ is the subjective discount rate and $\sigma \geq 0$ the inverse of the intertemporal elasticity of substitution.

In the absence of market failures, equilibrium allocations are solutions to the problem of a planner aiming at maximizing household's utility subject to the technological constraints. The Bellman equation associated with the planner's problem is

$$\begin{aligned} \rho v(k_t) = \max_{c,x} \quad & \frac{c^{1-\sigma}}{1-\sigma} + v'(k_t) x \\ \text{s.t.} \quad & x = A(k_t - c^{\frac{1}{\alpha}}). \end{aligned} \tag{6}$$

The value function $v(k_t)$ is (5) evaluated at equilibrium and represents the value of capital measured as the discounted flow of consumption utility. The return to assets, as measured

[20]This is a particular case of the general preferences in Section 2.1. Here the correspondence Γ is defined for every $k \geq 0$ as the set $\Gamma(k)$ of pairs $(c, \dot{k})$ such that there exists h with $0 \leq h \leq k$, $c \leq h^\alpha$, and $\dot{k} \leq A(k - h)$.

by the subjective discount rate ρ, is equal to the utility of current consumption plus the value of net investment, the later representing the utility of postposed consumption —the extra consumption that this additional capital will produce in the future.

As shown by Felbermayr and Licandro (2005), the equilibrium, time-invariant growth rate of capital is

$$\gamma = \frac{A-\rho}{1-\alpha(1-\sigma)}, \tag{7}$$

which we assume to be strictly positive.[21] From the feasibility constraints, it is clear that the growth rate of investment is also γ, and that $\alpha\gamma < \gamma$ is the growth rate of consumption. As shown in Felbermayr and Licandro (2005), competitive equilibrium allocations are balanced growth paths from $t \geq 0$.

Since $\alpha < 1$, as the stock of capital grows the investment sector becomes more productive with respect to the consumption goods sector. Differences in productivity causes the decline in investment prices relative to consumption goods prices. This difference in returns to scale can be interpreted, as put forward by Boucekkine et al. (2003), as a consequence of strong spillovers in the production of investment goods.[22] From the feasibility constraints, we can obtain the competitive equilibrium price of investment in terms of consumption units as the marginal rate of transformation:

$$p_t = -\frac{dc_t}{dx_t} = -\frac{dc_t}{dh_t}\frac{dh_t}{dx_t} = \frac{\alpha h_t^{\alpha-1}}{A}.$$

Since the stock of machines used in the consumption goods sector grows at the constant rate γ, the price of investment relative to consumption decreases at rate $(\alpha-1)\gamma < 0$.

The competitive equilibrium allocation displays the regularities observed in actual data. Investment grows faster than consumption since $\gamma > \alpha\gamma$. The relative price of investment decreases at rate $(\alpha-1)\gamma < 0$. The share of investment in income remains constant. To see this, take for example the consumption good as numeraire and define nominal income as in the general case as $m_t = c_t + p_t x_t$. From the equilibrium equations, one can show after some simple algebra that the investment share

$$s_t = \frac{p_t x_t}{m_t} = \frac{p_t x_t}{c_t + p_t x_t} = \frac{\alpha(A-\rho)}{\rho(1-\alpha)+\alpha\sigma A}$$

for all $t \geq 0$.

[21] Since $\sigma \geq 0 \geq 1-1/\alpha$, $\gamma > 0$ iff $A > \rho$.

[22] Cummins and Violante (2002) observe that their measure of investment-specific technical change occurs first in information technology and then accelerates in other industries. They conclude that information technology is a "general purpose" technology, an interpretation that matches well with the spillovers' interpretation. See also Boucekkine et al. (2005).

At this point it may be worth stressing that the choice of the consumption good as numeraire is inconsequential. The argument above follows equally if we choose to measure income in units of investment, $p_t^{-1}c_t + x_t$, or, for that matter, in any other arbitrary monetary unit provided that relative prices are respected.

3.2 Measuring real output growth

As in the general case of Section 2, in regard of the Bellman equation (2), the function

$$w_t(c, x) = \frac{c^{1-\sigma}}{1-\sigma} + v'(k_t)x$$

can be seen as representing preferences over current consumption and current net investment. Again, the constraint in the Bellman equation (2) can be replaced by the budget constraint $c + p_t x \leq m_t$ because the budget line is tangent to the production possibilities frontier locally at the optimum. In this example the utility representation $w_t(c, x)$ changes over time only because the marginal value of capital does.

Define the indirect utility $u_t(m_t, p_t)$ and the expenditure function $e_t(u_t, p_t)$ as in Section 2. Recall that the Fisher-Shell true quantity index compares income today m_t with the hypothetical level of income $\hat{m}_{t+h}$ that would be necessary to attain the level of utility associated with tomorrow's income and prices m_{t+h}, p_{t+h} with today's prices p_t and today's preferences as evaluated by e_t, u_t.

From the definition of g_t^{FS} in Section 2, we conclude that, for all $t \geq 0$,

$$g_t^{\text{FS}} = (1-s)\alpha\gamma + s\gamma = \frac{\alpha A(A-\rho)}{\rho(1-\alpha) + \alpha\sigma A}$$

and the interpretation is as in the general case.

3.3 On money metric utility

Money metric utility, implicit in the Fisher-Shell index, selects a particular representation of preferences that makes welfare to grow at the rate g^{FS}. This particular representation depends crucially on preferences and technology. The simple structure of this economy helps illustrate this point.

The two-sector AK model jumps to its balanced growth path at the initial time. A constant fraction of total capital will be permanently allocated to the production of consumption goods. Capital will permanently grow at the endogenous rate γ and consumption at rate $\alpha\gamma$. After substituting the optimal consumption path in (5), the value function reads

$$v(k_t) = Bk_t^{\alpha(1-\sigma)} \quad \text{with} \quad B = \frac{(A-\gamma)^{\alpha(1-\sigma)}}{(1-\sigma)\big(\rho - \alpha\gamma(1-\sigma)\big)} > 0. \tag{8}$$

Notice that both the exponent of k_t and B depend on preferences and technology.[23]

The argument is the following. The utility function in (5) is one among many representations of the same preference order—constant intertemporal elasticity of substitution preferences. The Fisher-Shell index choses another representation, the one that at equilibrium grows at rate g^{FS} and adopts nominal income at some reference time as its benchmark.

We build this particular representation of preferences in two steps. First, let us denote by $\hat{v}_t$ the equilibrium welfare of the representative agent at time t measured on an arbitrary unit. Let us then make two assumptions concerning $\hat{v}_t$, consistently with the main implicit assumptions of the Fisher-Shell index. First, let us assume that at the initial time, $t = 0$, $\hat{v}_0 = (c_0 + p_0 x_0)/\rho$. This is the money metric utility assumption that the return to assets is equal to nominal income at the reference time, here $t = 0$. Second, let us assume that $\hat{v}_t$ grows at the rate $g = g^{\text{FS}}$, meaning that $\dot{\hat{v}}_t = g\hat{v}_t$ and therefore

$$\hat{v}_t = \hat{v}_0 e^{gt}$$

for all $t \geq 0$. Consequently, if a utility representation of household preferences consistent with the Fisher-Shell index exists, it has to be that at equilibrium welfare is a potential function of k_t with exponent g/γ.

Let us now show that such a representation exists. We adopt the following alternative representation of the original preferences

$$\tilde{U}({}_tC) = \lambda U({}_tC)^\beta = \lambda \left(\int_t^\infty \frac{c_s^{1-\sigma}}{1-\sigma} e^{-\rho(s-t)} dt \right)^\beta$$

for some $\lambda, \beta > 0$. Since this new utility function represents the same preferences (5), the equilibrium path is the same. Hence, in equilibrium

$$\hat{v}(k_t) = \lambda v(k_t)^\beta = \hat{v}_0 e^{gt},$$

where

$$\lambda = \hat{v}_0 B^{-\beta} k_0^{-\alpha(1-\sigma)\beta}$$

[23]Notice that $B > 0$ requires the condition $\rho > \alpha A(1-\sigma)$ to hold. We impose this condition, since it is also needed for the discounted flow of consumption utility to be bounded at equilibrium. See Felbermayr and Licandro (2005).

and

$$\beta = \frac{g}{\alpha\gamma(1-\sigma)},$$

both depending on the parameters of preferences and technology, and λ additionally depending on both the initial capital stock and the initial nominal income. We have then shown that the growth rate as measured by the Divisia index is a welfare measure in the sense that it is equal to the growth rate of a particular representation of household preferences. The choice of this representation directly results from the key assumptions in money metric utility that welfare is measured in units of nominal income at some reference time.

Notice that at equilibrium the welfare of the representative household, $v(k_t)$ in the Bellman equation (6), measures the value of assets, represented here by the capital stock. Then, $\rho\, v(k_t)$ is the return to these assets as evaluated using the subjective discount rate ρ. From (6), at equilibrium the return to assets is equal to the utility of current consumption plus the value of current investment, the latter being assessed at the marginal value of capital $v'(k_t)$. Of course, welfare as measured by $v(k)$ is defined in an arbitrary unit: monotonic transformations of preferences will change the level of utility leaving the preference map intact; consequently, the growth rate of different representations will not be necessarily the same. To overcome this problem, as discussed in the introduction, consistently with money metric utility we adopt current income as a sensible norm to measure changes in welfare; we do it by using an equivalent variation measure. Since income as measured by National Accounts represents the return to the stock of assets, the Fisher-Shell quantity index and then the Divisia index are equivalent variation measures quantifying changes in the return to capital. Since the subjective discount rate in (6) is time independent, the Divisia index also measures changes in welfare.

SUMMARY: We can better understand now the main result in this paper that the growth rate of output is a welfare measure. When we compare $v(k_t)$ with $v(k_{t+h})$, we are comparing the welfare of the representative household at two different moments in time, as if households could costlessly decide entering the economy at any of these two moments. In a growing economy $v(k_{t+h}) > v(k_t)$, implying that a household will feel wealthier at $t+h$. $\hat{v}(k_t)$ is a money metric representation of the same preferences, giving a quantitative meaning to this intertemporal welfare comparison. Consequenlty, given initial conditions at any time t, omitting then the cost of moving from one moment in time to another, in the context of the two-sector AK model, households welfare gains are measured by the growth rate of output as measured by NIPA.

3.4 On consumption equivalent

When measuring welfare gains, macroeconomist usually apply consumption equivalent variations to deal with the previous referred problem of the representation of preferences.[24] In our context, the problem reduces to measure a hypothetical, proportional increase in the consumption path that makes an individual evaluating her welfare at time t indifferent between staying at t or being costless transferred to time $t + h$. Notice that consumption equivalent is a variation measure in terms of the entire consumption path instead of an equivalent variation in terms of current income.

Let us formulate the problem formally in the case of the two-sector AK model developed in this section. When c_t is evaluated at the equilibrium solution, the value function reads

$$v(k_t) = \int_0^\infty \frac{c_t^{1-\sigma}}{1-\sigma} e^{-\rho t} dt.$$

The hypothetical increase in the consumption path λ_h that makes an individual indifferent between staying at t or being costless transferred to $t + h$ must verify the condition

$$\lambda_h^{1-\sigma} v(k_t) = v(k_{t+h}).$$

The consumption equivalent variation λ_h directly depends on the length of the interval h. Substituting (8) into the previous condition and using the equilibrium condition $k_{t+h} = k_t e^{\gamma h}$, the consumption equivalent variation becomes

$$\lambda_h = \left(\frac{v(k_{t+h})}{v(k_t)} \right)^{\frac{1}{1-\sigma}} = e^{\alpha\gamma h}.$$

The growth rate of welfare consistent with consumption equivalent is then the derivate of λ_h with respect to h evaluated at $h = 0$, which in the case of the two-sector AK model reads

$$g^{\text{ce}} = \frac{1}{\lambda_h} \frac{d\lambda_h}{dh} \bigg|_{h=0} = \alpha\gamma.^{25}$$

In the two-sector AK economy, the consumption equivalent measure of the growth rate is equal to the growth rate of consumption. The result comes at no surprise, since the consumption path implicit in $v(k_{t+h})$ is the same as the consumption path in $v(k_t)$ multiplied by the factor $e^{\alpha\gamma h}$.

[24] In the tradition of Lucas (1987), for example, Jones and Klenow (2016) apply a consumption equivalent measure to evaluate beyond GDP welfare gains.

[25] As expected, the consumption equivalent measure g^{ce} is invariant to any transformation of the underlying preferences. Proving it for $\hat{v}(k_t)$ is straightforward.

Summary: We must then conclude that, in the case of the two-sector AK model, the growth rate of consumption is a consumption equivalent measure of welfare gains that takes consumption instead of income as a norm. Unfortunately, apart from examples in artificial economies like this one, no statistical office is able to compute a consumption equivalent measure of welfare gains, since preferences, technology and the path of future consumption are unobservable.[26] The Divisia index, indeed, is a money metric equivalent variation measure in terms of current income, having the advantage of being independent of the particular form of preferences and technology, requiring much less information than the consumption equivalent measure to be computed.

4 Discussion

In the framework of two-sector dynamic general equilibrium models, Section 2 shows that the Divisia index is, in fact, a true quantity index. This is of substantive interest since the Fisher ideal chain index used in actual National Accounts approximates well the Divisia index, implying that in the framework of dynamic general equilibrium models the growth rate of output in NIPA is welfare-based. This section discusses the implications of this result. To make our main point clear, this section refers to additively separable preferences like in equation (5).

4.1 Investment matters

The following example makes it more clear why investment matters in the definition of output growth. Consider a world with embodied technical progress like the one in Greenwood and Yorukoglu (1997). Let the consumption path in this economy be depicted as in Figure 3. At time T there is an unexpected permanent technology shock to the investment sector: embodied technical progress accelerates. New machines, if produced and added to the capital stock, can make the productivity in the consumption goods sector grow faster indefinitely. In our example, hence, after observing the unexpected acceleration of investment specific technical change in T, the consumer finds optimal to initially reduce consumption in order to increase investment and, then, profit from technical progress. In this world, at time T, the drop in consumption reflects the

[26]The simplicity of the results relies on the fact that the two-sector AK economy is always at its balanced growth path. In the general case of a concave technology, like in Greenwood et al. (1997), if the economy is not at its balanced growth path, the calculation of the consumption equivalent growth rate is not straightforward and this results cannot be easily extended.

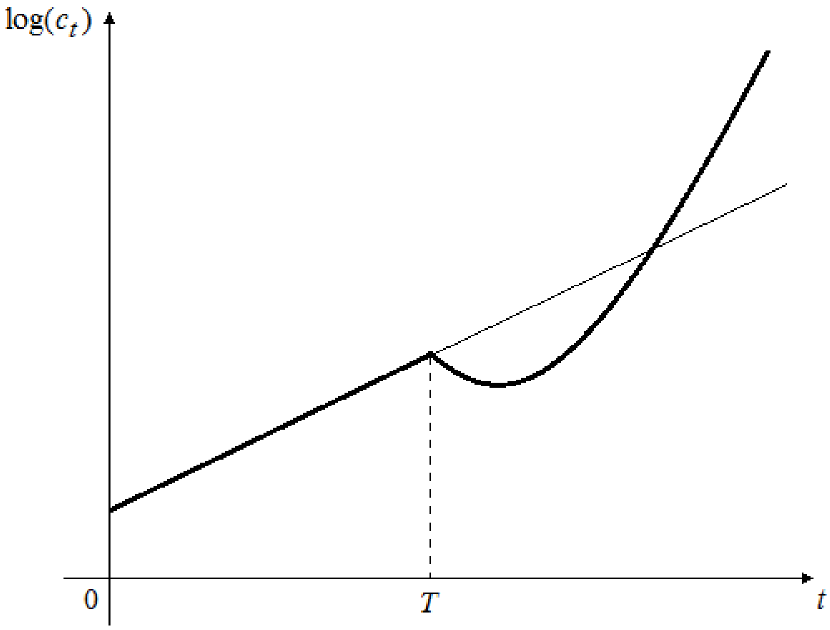

Figure 3: Investment matters

interest of the consumer in benefiting from faster growth thereon; if this move would have not increased her welfare, she would have chosen not to increase investment to remain in a lower growth path. Then, the consumption growth rate at time T does not measure welfare correctly. In fact, it has the opposite sign! However, the growth rate of output as measured by the Divisia index does, since it captures well the gains in welfare coming from the acceleration of technical progress and the associated optimal increase in investment. The key is that technical progress is assumed to be investment specific. Then, gains in productivity require new investments. The discussion above helps to illustrate why the growth rate of investment matters for output growth measurement. Faster growing investment today represents our best proxy for the preference for faster consumption growth tomorrow.

4.2 Net National Product

In connection with these considerations, the use of the Bellman equation makes it clear why production in National Accounts is measured as final demand. Since present and future consumption is all that matter for welfare, and net investment measures the value of the future consumption it will produce, a welfare measure of output growth has to weight the growth rate of both final demand components, consumption and net investment. This interpretation is consistent with Weitzman (1976)'s claim that "net national product is a proxy for the present discounted value of future consumption."[27] In fact, his equation (10) is in spirit equivalent to the Bellman equation (2), which rationalize our choice of taking current net income as the proper norm in the Fisher-Shell true quantity index. Nevertheless, it is important to note that Weitzman (1976) is not about output growth and its relation to welfare gains in the growth process, but about the level of output and its relation to the level of welfare. The main result of Weitzman's paper is that the level of nominal net national product is equal to the present value of current and future consumption (see again Hulten (2001, section 1.4.5)). The paper does not attempt to measure growth. In this sense, the non trivial question of the appropriate measurement of output growth has remained open until our days. The best result in this direction is in a subsequent paper by Asheim and Weitzman (2001). That paper builds a measure of the level of output and shows that output growth is a necessary and sufficient condition for welfare growth, but without providing any specific insight on how output growth should be measured. This papers gives a fundamental step ahead in this direction: by applying standard index number theory, we show that the precise way NIPA measures growth is welfare-based.

At this point it may be worth clarifying that, as pointed out by Weitzman (1976), it is not GDP but Net National Product (NNP) what matters for welfare.[28] Depreciated capital is a lost resource that does not contribute to welfare. If the depreciation rate is constant, however, net and gross investment grow at the same rate. Indeed, when investment grows faster than consumption, NNP grows slower than GDP since the share of net investment on net income is smaller than the corresponding share of gross investment.

4.3 Paradox of endowment vs production economies

It is important to note that a true quantity index of output growth is a welfare-based measure conditional on both preferences and technology. In other words, it does not reflect changes in welfare independently of the possibilities allowed by technology. The example below shows the interplay between technology and preferences in the definition of output growth emerging from index number theory applied to this family of problems.

[27] Weitzman's argument is developed in an optimal growth model with linear utility and the proof is based on the assumption that current income remains constant over time. In his own words, he gets "the right answer, although for the wrong reason." To be precise, Weitzman's claim should be restated as "net national product is *the return to capital,* a proxy for the present discounted value of future consumption." The text in italics is ours.

[28] Since the model economy in Section 2 is closed, GDP and GNP are equal, as well as NDP and NNP.

Consider the following example that clarifies further the meaning of a welfare measure in this context. For the two-sector AK model in Section 3, take any configuration of parameters such that, for example, the growth rates of consumption and investment at equilibrium are 2% and 6%, respectively, and the investment share is 20%. We are implicitly assuming α equal to 1/3. The Divisia index tells us that this economy will be growing at 2.8%. Alternatively, consider an *endowment* economy with exactly the same preferences and the same equilibrium consumption flow. In this economy, consumption is mana from haven. Indeed, an individual would be indifferent between living in the AK or in the endowment economy, since she will get the same consumption path, that she will evaluate using the same preferences. In the endowment economy, indeed, index number theory will associate income with current consumption; the Divisia index will then measure output grow as consumption growth; 2% in our example.

Why is it the case that two economies where people have identical preferences and face exactly the same consumption path do not grow at the same rate? The reason is that a true quantity index takes current income as a norm and current income is defined differently; at any time, both economies share the same consumption utility, but investment goods are produced only in the production economy. These seemingly paradoxical example illustrate well the intimate relation between preferences (what we would prefer to do) and technology (what we can do) when measuring output growth. Indeed, in this particular example, both measures of output growth are welfare-based and consistent with NIPA methodology. The example makes also clear the implications of measuring production as final demand: since there is no investment in the endowment economy, output growth becomes identical to consumption growth.

This intimate relation between what we would prefer to do and what we can do implies that the measurement of welfare and productivity cannot be two different exercises. For instance, some authors use the GDP deflator to obtain real production and the deflator of private consumption to obtain real income as if they where two different concepts —see for example Ribarsky et al. (2016). In the abstract economies above this amounts to identify output growth with the Divisia index g_t^D and income growth with the growth rate of consumption $g_{c,t} = \dot{c}_t/c_t$. Our results show that the only relevant deflator is that of output.

4.4 Growth accounting

To end this discussion, let us review the implications for growth accounting. In terms of model representations of actual economies, the introduction of more than one sector with different growth rates raises the practical and conceptual issue of how output growth has to be measured. The choice of the appropriate output growth rate affects every quantitative exercise based on the measurement of growth. This is the case in the literature on growth accounting under embodied technical change, the so-called Solow-Jorgenson controversy.[29] To measure the contribution of investment specific technical change to growth, Hulten (1992) measures growth (his equation (7)) following Jorgenson (1966). He suggests a raw addition of consumption and investment units, calling the outcome quality-adjusted output. Using our notation, this strategy amounts to $c_t + x_t$. Greenwood et al. (1997) note that, in their setting, adding consumption and effective investment turns the economy into a standard Solow (1960) growth model with no embodied technical change. Greenwood et al. (1997) correctly state that any aggregation requires the different quantities to be expressed in a common unit and they adopt the consumption good as their standard. For this purpose, investment has to be multiplied by its relative price, in our notation their choice of output level would be $y_t = c_t + p_t x_t$.[30] What the present paper shed light on is that the issue is not the units used to measure real output *levels* but the choice of the right index of real output *growth*. In this sense, we follow Licandro et al. (2002) and conclude that the "true" contribution of ETC to output growth, reflecting welfare changes, has to be measured using NIPA methodology as in Cummins and Violante (2002).

4.5 A word of caution

We have to be careful in the way we interpret the output growth rate in this framework. Since raising the growth performance of an economy is costly, it is well-known in endogenous growth theory that there exists an optimal growth rate.[31] In the case of the two-sector AK model above, the optimal growth rate of capital is γ –see equation (7). Let us then assume, for example, that the two-sector AK model is at equilibrium growing at its optimal growth rate, but at time $t = 0$ an uninformed government decides to introduce some incentives to promote growth, for example by subsidizing capital production and then distorting the private return to capital. At the time of the reform, $t = 0$, the economy starts growing faster at the cost of a reduction in welfare. From this time

[29] See Hercowitz (1998) for a review of the Solow-Jorgenson controversy.

[30] In their setting, this choice looks somewhat natural because the investment sector uses as input the consumption good. In their notation $y_t = c_t + p_t x_t$ is total output in the non-durable sector, even if only c_t is consumed and the remaining production $p_t x_t$ is allocated to the investment sector.

[31] See for example Section 4 in Aghion and Howitt (1992).

on ahead, the growth rate of output in the distorted economy, like in Section 3.3, will measure welfare gains, which will be larger than in the efficient economy. Unfortunatelly, the initial welfare losses will not be captured by National Accounts, since changes in the value of assets are in general not registered.

Let us formalize the previous statement by following the same steps as in Section 3.3. The value function of the distorted economy reads, for $t \geq 0$,

$$v_d(k_t) = B_d \; k_{d,t}^{\alpha(1-\sigma)} \quad \text{with} \quad B_d = \frac{(A-\gamma_d)^{\alpha(1-\sigma)}}{(1-\sigma)\big(\rho - \alpha\gamma_d(1-\sigma)\big)}.$$

where

$$\gamma_d = \frac{\tau A - \rho}{1-\alpha(1-\sigma)} \quad \text{and} \quad k_{d,t} = k_0 e^{\gamma_d t}.$$

The distortion introduced by the subsidy is represented by the wedge $\tau > 1$. It is easy to see that $|B_d| < |B|$ and decreasing in $\tau > 1$, meaning that at $t = 0$ the policy generates welfare losses, which are larger the larger the distortion is. Paradoxically, welfare in the distorted economy is growing faster; reflecting the fact that there exists a finite time $t_d > 0$ from which $v_d(k_{d,t})$ becomes larger than $v(k_t)$.

Summary: The fact that in this framework the growth rate of output as measured by NIPA is welfare-based, measuring gains in welfare, does not imply that any policy that increases the growth rate is welfare improving. For an equilibrium path, the growth rate of output measures gains in welfare. A policy or a shock modifying the equilibrium path may have, in addition to changes in the growth rate, an initial change in welfare that is not generally measured by National Accounts. This relates to the debate on the inclusion of capital gains in National Accounts –see Fagereng et al. (2019).

5 Conclusions and extensions

This paper studies the welfare properties of the growth rate of real output. Its main contribution is to show that real output growth as assessed by NIPA is a money metric measure of welfare gains in the precise sense of equivalent variation. More precisely, it shows that a Fisher-Shell true quantity index is equal to the Divisia index when applied to a continuous time two-sector dynamic general equilibrium economy with heterogeneous households, general recursive preferences and general technology transforming production factors (capital and labor) into consumption and investment. It turns out that the type of chain indexes used by National Accounts to compute real output growth is well approximated by the Divisia index.

This result is illustrated in the framework of the two-sector AK model, which replicates the well-know stylized facts that investment permanently grows faster than consumption and that the relative price of investment permanently declines. More important, changes in the growth rate of investment induced by changes in embodied technical progress turn out to be a relevant part of welfare gains along an equilibrium path. Investment then matters for the growth rate of real output be a welfare measure.

In general, this paper can be seen as a recall to macroecnomics that index number theory has an important role to play for the understanding of the welfare properties of GDP growth. In particular, this approach may be of great relevance for the use on index number theory to rationalize the Penn World Tables methodology –see Neary (2004), Van Veelen and Van der Weide (2008), and Jones and Klenow (2016). When comparing welfare across countries, we face the critical issue that preferences may change from one country to another. The Fisher-Shell index is an appropriate true quantity index in this case, since it was designed to make welfare comparisons between agents with different preferences. Evaluating it in a dynamic framework by the mean of the Bellman equation seems to be the appropriate approach.

We are now in the position of answering the following fundamental question: What do we mean in this context when we claim that *the growth rate of output as measured by NIPA is a money metric representation of welfare gains in the precise sense of equivalent variation?*

1. *The growth rate in NIPA measures gains in welfare.* In a dynamic general equilibrium framework, social wealth is the discounted flow of consumption utility that an optimal use of current assets will generate. The value function associated to the social planner problem, when evaluated at equilibrium, measures then the value of these assets. Output is nothing else that the return to assets. The Bellman equation is a representation of preferences relating the value of assets to their return. In this sense, the growth rate of output as measured by the Fisher-Shell index is a measure of the growth rate of the return to assets. In the context of additively separable preferences and constant subjective discount rate, the growth rates of output and wealth are equal, the growth rate of output in NIPA measuring welfare gains.

2. *The growth rate in NIPA measures gains in welfare between two different moments in time disregarding the cost of moving from one to the other.* What does a Fisher-Shell index measure in this context? In a dynamic general equilibrium model, household welfare is the discounted flow of consumption utility some initial stock

of assets leads to. The cost endured to accumulate this initial stock of assets is disregarded. In the same line, the Fisher-Shell index compares household welfare at two different moments in time, disregarding the transitional cost of accumulating assets between these two periods. In other terms, the growth rate in NIPA measures how much our welfare increases today, irrespective of the cost we have payed to be here now.

3. *The growth rate in NIPA is a measure of welfare gains, but it is not the only one.* Money metric in this context means that we adopt, as National Accounts do, current income as our norm. The Divisia index then measures welfare gains in current income units, by implicitly choosing a particular representation of preferences consistent with this metric. Consumption equivalent is an alternative measure of welfare gains that adopts the consumption good as its metric, but it is more difficult to measure in practice.

4. *The growth rate in NIPA is a measure of welfare gains conditional on technology, reflecting then changes in production.* The Fisher-Shell index in this context is a measure of welfare gains that depends on both preferences and technology. Economies with the same preferences and consumption path, but different production structures may grow at different rates if they entail a different production and income path.

5. *Net investment also matters when measuring welfare gains.* Investment also matters when measuring welfare gains, since investment growth entails the impact on overall welfare of the induced changes in future consumption. But, it is net investment that matters for welfare.

6. *Growing faster is not necessarily Pareto improving.* The fact that in this framework the growth rate of output as measured by NIPA reflects gains in welfare, does not imply that any policy increasing the growth rate is welfare improving. For an equilibrium path, the growth rate of output measures gains in welfare. A policy or shock modifying the equilibrium path may have, in addition to changes in the growth rate, initial welfare gains or losses usually unmeasured by National Accounts.

Let us finally comment on some dimensions in which this approach could be extended. Broaden it to economies with multiple durable and non-durable goods seems straightforward, as well as to study the welfare properties of the growth rate in open economies. The approach could also be applied to many forms of non-optimal equilibria.

Notice that, in this case, the production possibility frontier will not be tangent to an indifference curve at equilibrium, and hence the generalization will not be straightforward. However, if the representative household is price taker in all markets, irrespective of the fact that prices are distorted, at equilibrium the budget constraint will be tangent to an indifference curve. Under theses circumstances, index number theory could be applied to compare different points in the equilibrium path in a similar way we do in Section 2. In particular, for a stationary economy moving from a distorted to a non distorted equilibrium, the Divisia index will measure the welfare gains period by period.

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A Quantity indexes in continuous time

A.1 Growth factors

In continuous time, let us define a growth factor Γ_{t+h}^{t} as the gross rate of growth of an arbitrary variable between a base time t and a current time $t+h$, $h \geq 0$. When $h \leq 0$, Γ_{t+h}^{t} measures the gross rate of growth between the base time $t+h$ and the current time t. In the jargon of National Accounts, Γ_{t+h}^{t} is referred as a volume index. Let us then define the instantaneous growth rate of the underline variable at time $t+h$ when the base time is t as

$$g_{t+h}^{t} = \frac{d\Gamma_{t+h}^{t}}{dh}\frac{1}{\Gamma_{t+h}^{t}}. \tag{9}$$

Notice that in continuous time, $h \geq 0$, the derivate of a growth factor at any time $t+h$ is equal to the growth rate of the variable itself at $t+h$. Let z_t be a continuous-time variable and $\Gamma_{t+h}^{t} = z_{t+h}/z_t$ the growth factor. Apply (9) to get

$$\frac{d\Gamma_{t+h}^{t}}{dh}\frac{1}{\Gamma_{t+h}^{t}} = \frac{\dot{z}_{t+h}}{z_{t+h}}.$$

This way of defining the instantaneous growth rate may look odd but it may be useful in those cases in which we have an index like Γ_{t+h}^{t} but no explicit variable giving rise to it like z_t in this example.

Using the notation introduced in Section 2, the starting point in index number theory is some nominal aggregate income $c_t + p_t x_t$. Remind that we have adopted consumption as the numeraire so that its price is normalized to one while the price of investment in consumption units is p_t. Laspeyres quantity indexes use time t (the base time) prices as weights based on the following growth factor

$$\mathcal{L}_{t+h}^{t} = \frac{c_{t+h} + p_t x_{t+h}}{c_t + p_t x_t}.$$

It does allow to compute the growth rate of output by putting all nominal values at base time prices. Notice that in this framework the *real* unit in which quantities are measured is nominal income $c_t + p_t x_t$ at the base time. Paasche indexes take current prices as weights by defining the growth factor as

$$\mathcal{P}_{t-h}^{t} = \frac{c_t + p_t x_t}{c_{t-h} + p_t x_{t-h}},$$

$h \geq 0$. Real output growth is measured at current t prices.

The Fisher ideal growth factor with time base t and current time $t+h$, $h \geq 0$, is defined as

$$\mathcal{F}_{t+h}^{t} = \left(\mathcal{L}_{t+h}^{t}\mathcal{P}_{t}^{t+h}\right)^{\frac{1}{2}}. \tag{10}$$

The definition in equation (9) is also applied in Section 2.2 to the Fisher-Shell quantity index since we have a well-defined factor $\hat{m}_{t+h}/m_t$. Notice that in the definition of g_t^{FS}, we use the property that $\lim_{h\to 0}\hat{m}_{t+h} = m_t$.

A.2 Fixed-base quantity indexes in continuous time

Traditional measures of real growth stem from fixed-base quantity indexes. The most common among them are the Laspeyres and Paasche indexes referred in the online Appendix A1. From the online Appendix A1, the Laspeyres factor of change between t and $t+h$ is

$$\mathcal{L}_{t+h}^{t} = \frac{c_{t+h} + p_t x_{t+h}}{c_t + p_t x_t},$$

for $h \geq 0$, where t represents the base time and $t+h$ the current time. In continuous time, the Laspeyres index $g_{t+h}^{\mathcal{L}\,t}$ is the instantaneous growth rate of factor $\mathcal{L}_{t+h}^{t}$ as a function of h—see equation (9). That is,

$$g_{t+h}^{\mathcal{L}\,t} = \frac{d\mathcal{L}_{t+h}^{t}}{dh}\frac{1}{\mathcal{L}_{t+h}^{t}} = \frac{\dot{c}_{t+h} + p_t \dot{x}_{t+h}}{c_{t+h} + p_t x_{t+h}},$$

which measures the instantaneous real growth rate at $t+h$ for the given base time t. The Laspeyres index is popular because it is conceptually simple.

However, if the relative price of investment permanently declines and substitution makes real investment permanently grow faster than real consumption, as observed in the data, the Laspeyres index tends to give too much weight to investment as we depart from the base time t. In order to illustrate it, let us assume the economy is at a balanced growth path with constant investment and consumption shares, s and $1-s$ respectively, $s \in (0,1)$, the relative price of investment goods p_t declining at a constant rate, and investment and consumption growing at the constant rates g_x and g_c, respectively, $g_x > g_c > 0$.

Note, indeed, that the Laspeyres fixed-base index reads

$$g_{t+h}^{\mathcal{L}\,t} = \frac{c_{t+h}}{c_{t+h} + p_t x_{t+h}} g_c + \frac{p_t x_{t+h}}{c_{t+h} + p_t x_{t+h}} g_x. \tag{11}$$

Since x_{t+h} grows relative to c_{t+h}, it is easy to see that along a balanced growth path the weight of consumption in the Laspeyres fixed-base index decreases and the weight of investment increases with h. This effect is known in the index numbers literature as the *substitution bias.* Fast growing items when weighted using past (relatively high) prices are overweighted, overstating the real growth rate of output. The effect is larger the farther we are from the base time, converging to the growth rate of investment as h goes to infinity.

The Paasche index uses current prices as a base, instead of past prices, and hence tends to understate real growth as we go back in time. The Passche factor is

$$\mathcal{P}_{t-h}^{t} = \frac{c_t + p_t x_t}{c_{t-h} + p_t x_{t-h}}$$

for all $h \geq 0$ and the growth rate

$$g_{t-h}^{\mathcal{P}\,t} = \frac{d\mathcal{P}_{t-h}^{t}}{dh}\frac{1}{\mathcal{P}_{t-h}^{t}} = \frac{c_{t-h}}{c_{t-h} + p_t x_{t-h}} g_c + \frac{p_t x_{t-h}}{c_{t-h} + p_t x_{t-h}} g_x, \tag{12}$$

under the assumption that the growth rates of both consumption and investment are constant. As h grows, so $t-h$ decreases, the weight of consumption increases because x_{t-h}/c_{t-h} decreases, converging to the growth rate of consumption as h goes to infinity.

For the arguments developed above, both Laspeyres and Paasche fixed-base indexes yield poor measures of real growth when output components grow at different rates because of changing relative prices. The farther we are from the base time, the more the Laspeyres index overstates growth, and the more the Paasche index underesates it.[32]

Indeed, it is easy to see that in continuous time both Laspeyres and Paasche quantity indexes are equal to the Divisia index when evaluated at t:

$$\left.\frac{d\mathcal{L}_{t+h}^{t}}{dh}\frac{1}{\mathcal{L}_{t+h}^{t}}\right|_{h=0} = \left.\frac{d\mathcal{P}_{t-h}^{t}}{dh}\frac{1}{\mathcal{P}_{t-h}^{t}}\right|_{h=0} = (1-s_t)g_{ct} + s_t g_{xt},$$

where $s_t = \frac{p_t x_t}{c_t + p_t x_t}$ is the investment share, $g_{ct} = \frac{\dot{c}_t}{c_t}$ the growth rate of consumption and $g_{xt} = \frac{\dot{x}_t}{x_t}$ the growth rate of investment.[33] Given that in continuous time, both Laspeyres and Paasche quantity indexes are equal to the Divisia index at t, it is easy to show that the Fisher ideal index is equal too. It is trivial to see that this property also applies to the Fisher ideal index.

A.3 Chained-type quantity indexes in continuous time

In this appendix, we use our simple framework to review the BEA methodology.[34] The introduction by the BEA of quality corrections in equipment prices in the mid-eighties

[32]Updating regularly the base is not a solution because it would imply a permanent revision of past growth performance. It posses the additional problem of multiple real growth measures for each period, each of them affected differently for the substitution bias depending on the associated base period.

[33]In discrete time, the weights of consumption and investment growth rates in the Laspeyres and Paasche indexes are different from current income shares.

[34]Young (1992) is a non-technical presentation of the methodological changes introduced in NIPA. Whelan (2002, 2003) provides a more detailed guide into the new methods in use at BEA to measure real growth. For economic index number theory see Diewert (1993), Triplett (1992) and Fisher and Shell (1998).

revealed a persistent declining pattern in the price of equipment relative to the price of non-durable consumption goods. Since then, real investment appears to be growing much faster than real non-durable consumption. In this new scenario, fixed-base quantity indexes face the severe substitution bias problem explained in the online Appendix A2 above. For this reason, the BEA moved to a chain-type index based on a Fisher ideal index computed for contiguous periods.[35] Let us first define a Fisher ideal index to them define a Fisher ideal chain index both in continuous time.

Let us now define a Fisher ideal chain (factor) index for the time interval $(0, T)$, where $t = 0$ represents now the *reference time* (in contraposition to the *base time*). The key assumption of chain indexes is that the base time moves with t, by taking t as the base time when computing the growth rate at time t. From the online Appendix A.2, for any time $t \in (0, T)$, the instantaneous growth rate of the Fisher ideal index is

$$g_t^{\mathcal{F}} = \frac{d\mathcal{F}_{t+h}^t}{dh} \frac{1}{\mathcal{F}_{t+h}^t}\bigg|_{h=0} = (1 - s_t)g_{ct} + s_t g_{xt}.$$

Even if there is a trend in relative prices, inducing the substitution of one good for another, the chain-type index allows weights to change continuously to avoid the emergence of any substitution bias.

Let us assume that s_t, g_{ct} and g_{xt} are continuous function of t, then the Fisher ideal index $g_t^{\mathcal{F}}$ is continuous too. A Fisher ideal chain (factor) index $\mathcal{C}_t^{\mathcal{F}}$ is defined by the differential equation

$$\dot{\mathcal{C}}_t^{\mathcal{F}} = g_t^{\mathcal{F}} \mathcal{C}_t^{\mathcal{F}},$$

$\mathcal{C}_0^{\mathcal{F}} = 1$, which solution is

$$\mathcal{C}_t^{\mathcal{F}} = e^{\int_0^t g_s^{\mathcal{F}} ds}.$$

A chain factor index for a time interval $t \in (0, T)$ is build in two stages. First, at any time $t \in (0, T)$ a growth rate is computed using t as the base time. Second, the time t growth rates computed at the first stage are chain in order to build growth factors in an interval of time $t \in (0, T)$. Notice that fixed-base factor indexes are equal to one at the base time. In the case of chain indexes base times are changing. For this reason, the time at which the factor index is set equal to one is now called the reference time.

[35]Diewert (1993) provides a clear explanation of the index suggested by Fisher (1922).