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# Stone-Geary meets CES:

## An extended linear expenditure system

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**Abstract**: We reformulate the Stone-Geary utility function to incorporate non-unitary elasticities of substitution. We show that this extended linear expenditure system includes the standard Stone-Geary case but eliminates some of its restrictions. In particular, and most significantly, the proportions of expenditure over leftover income become price responsive under the extended demand system.

**Keywords**: Linear expenditure system, Constant elasticity of substitution, Calibrated demand functions.

**JEL codes**: C55, D11, D12

## 1. Introduction

The Stone-Geary utility function (Geary, 1950; Stone, 1954) is an extension of the Cobb-Douglas utility function that allows us to consider some minimum (or subsistence) requisite of consumption levels that need to be guaranteed before full consumption demand is determined. The vector of subsistence levels provides the coordinates for the shifting of the Cobb-Douglas function that yields the Stone-Geary extension. The nicest property of the Stone-Geary function is that its demand function yields a linear expenditure system (LES). For any good, total expenditure is the sum of a fixed level of expenditure allotted for the minimum consumption levels and a fixed

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proportion of the leftover income once the minimum consumptions are accounted for that indicates the variable part of consumption demand. In this text, we explore the extension of this idea to the CES utility function (Arrow et al, 1961). We shift the coordinates of the CES function using the vector of minimum consumptions and derive the demand and cost functions for this new extended system.

In Section 2, we recall the essential properties of the LES system and show a simple way to recalculate them from basic and well-known results. In Section 3 we present the properties of the CES shifted expenditure system whereas in Section 4 we discuss the calibration issues and procedures that would arise in the implementation of the extended demand system in numerical general equilibrium. Section 5 concludes.

## 2. The Cobb-Douglas Linear Expenditure System

For the *n*-good case the Stone-Geary utility function is defined by:

$$u(x_1, x_2, ..., x_n) = (x_1 - z_1)^{\alpha_1} \cdot (x_2 - z_2)^{\alpha_2} \cdot ... \cdot (x_n - z_n)^{\alpha_n} = \prod_{j=1}^{n} (x_j - z_j)^{\alpha_j} \qquad (1)$$

where $z_j \geq 0$ are minimum levels of consumption and $\alpha_j$ nonnegative weights that add up to 1. The utility maximization problem under the budget constraint imposed by income level $m$ and given minimum consumptions $z_j$ is:

$$Max \;\; u(x_1, x_2, ..., x_n) = \prod_{j=1}^{n} (x_j - z_j)^{\alpha_j}$$

$$st. \qquad \sum_{j=1}^{n} p_j x_j = m \text{ and } x_j \geq z_j$$

The solution of this problem is straightforward if we perform the change of variables:

$$y_i = x_i - z_i \text{ and } m' = \; m' = m - \sum_{j=1}^{n} p_j \cdot z_j \qquad (2)$$

We then solve the reformulated –and back to standard– Cobb-Douglas case to find:

$$y_i = \alpha_i \cdot m' / \, p_i$$

Undo the change of variable and we obtain:

$$x_j = z_j + \frac{\alpha_j}{p_j} \cdot (m - \sum_{i=1}^{n} p_i \cdot z_i) \tag{3}$$

It is well known, and straightforward to verify, that this solution yields a demand system that has the linear expenditure property. The expenditure on good *j* includes a fixed part –given by value at the current price of the minimal consumption of *j*– and a variable part that is $\alpha_j$ proportional to the leftover level of income after the consumer incurs in the expenditure of the minimal consumption levels:

$$p_j \cdot x_j = p_j \cdot z_j + \alpha_j \cdot (m - \sum_{i=1}^{n} p_i \cdot z_i) \tag{4}$$

Notice that with no minimum consumption levels, i.e. when all $z_j = 0$, the demand function in (3) reverts of course to the standard Cobb-Douglas demand function. Because of the linearity that we observe in expression (4), we commonly refer to this Stone-Geary demand system as the Linear Expenditure System (LES) as well.

Much less commonly derived is the cost function in the Stone-Geary utility case. Consider the cost minimization problem:

$$\begin{aligned} &Min \ \sum_{j=1}^{n} p_j x_j \\ &st. \quad u = \prod_{j=1}^{n} (x_j - z_j)^{\alpha_j} \text{ and } x_j \geq z_j \end{aligned}$$

We can also solve it quickly by performing the same change of variable as before:

$$\begin{aligned} &Min \ \sum_{j=1}^{n} p_j \cdot y_j = \sum_{j=1}^{n} p_j (x_j - z_j) \\ &st. \quad u = \prod_{j=1}^{n} y_j^{\alpha_j} \text{ and } y_j \geq 0 \end{aligned}$$

The optimal solution $y_j$ of this problem is independent of the constant expenditure term for minimal consumption levels $z_j$ in the objective function and we can omit it without loss of generality. Thus, we revert once again to the standard Cobb-Douglas cost minimization problem. The minimal cost for this Cobb-Douglas solution $y_j$ is:

$$\sum_{j=1}^{n} p_j \cdot y_j = e(p,u) = u \cdot \prod_{j=1}^{n} \alpha_j^{-\alpha_j} \cdot p_j^{\alpha_j}$$

Since the solution $y_j$ indicates the excess of demand $x_j$ over minimum consumption $z_j$, the function $e(p,u)$ is just the variable cost associated with the solution $y_j$. We undo the change of variable to obtain total minimum cost as:

$$c(p,u)=\sum_{j=1}^{n} p_j \cdot x_j=\sum_{j=1}^{n} p_j \cdot z_j+e(p,u)=\sum_{j=1}^{n} p_j z_j+u \cdot \prod_{j=1}^{n} \alpha_j^{-\alpha_j} \cdot p_j^{\alpha_j}$$

The total cost function $c(p, u)$ adds up the variable cost $e(p, u)$ and the fixed cost. Notice that in terms of production theory, we can reinterpret in a natural way this cost function as depicting a situation where a firm needs a minimum level of inputs before actually entering into production operations.

## 3. Extension to CES utility functions

The Constant Elasticity of Substitution (CES) (Arrow et al, 1961) utility function takes the form:

$$u(x_1,x_2,\ldots,x_n)=\left(\sum_{j=1}^{n} a_j \cdot x_j^{\theta}\right)^{\frac{1}{\theta}} \tag{5}$$

In this expression $\theta=(\sigma-1)/\sigma$ and $\sigma=1/(1-\theta)$ where $0 \leq \sigma < \infty$ is the elasticity of substitution, which takes non-negative values only, and thus $-\infty < \theta \leq 1$. Its lowest possible value is $\sigma = 0$, which corresponds to goods that are perfect complements. The CES utility function encompasses the Cobb-Douglas function when $\sigma = 1$. For the CES case, therefore, we may also consider the implications of the possible existence of minimum consumption levels $z_j \geq 0$. This would involve transforming the CES function in expression (5) to the shifted new utility function:

$$u(x_1,x_2,\ldots,x_n)=\left(\sum_{j=1}^{n} a_j \cdot (x_j-z_j)^{\theta}\right)^{\frac{1}{\theta}} \tag{6}$$

Again, with no minimum consumption levels expression (6) quickly reverts to expression (5), the standard CES function. The utility function in (6) is a displacement to coordinates ($z_1$, $z_2$, …, $z_n$) of the utility function in (5), which is centered at the origin. Hence, it inherits most of the properties (monotonicity, convexity, differentiability but not homotheticity) of the standard CES function.

### 3.1 The demand system

The quickest way to solve the utility maximization problem is to reformulate it from:

$$Max \;\; u(x_1, x_2, ..., x_n) = \left( \sum_{j=1}^{n} a_j \cdot (x_j - z_j)^{\theta} \right)^{\frac{1}{\theta}}$$

$$st. \qquad \sum_{j=1}^{n} p_j x_j = m \text{ and } x_j \geq z_j$$

to:

$$Max \;\; u(y_1, y_2, ..., y_n) = \left( \sum_{j=1}^{n} a_j \cdot y_j^{\theta} \right)^{\frac{1}{\theta}}$$

$$st. \qquad \sum_{j=1}^{n} p_j \cdot y_j = m' \text{ and } y_j \geq 0$$

where we have undertaken the same change of variables as in (2) above. The standard solution[2] to this CES problem is:

$$y_j = \frac{a_j^{\sigma} \cdot p_j^{-\sigma} \cdot m'}{\sum_{i=1}^{n} a_i^{\sigma} \cdot p_i^{1-\sigma}} = a_j^{\sigma} \cdot p_j^{-\sigma} \cdot \Delta_p^{\sigma-1} \cdot m' \tag{7}$$

with:

$$\Delta_p = \left( \sum_{i=1}^{n} a_i^{\sigma} \cdot p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

We now undo the change of variable and obtain, from expression (7), that demand $x_j$ for good $j$ will be:

$$x_j = z_j + \frac{a_j^{\sigma} \cdot p_j^{-\sigma} \cdot (m - \sum_{i=1}^{n} p_i \cdot z_i)}{\sum_{i=1}^{n} a_i^{\sigma} \cdot p_i^{1-\sigma}} = z_j + a_j^{\sigma} \cdot p_j^{-\sigma} \cdot \Delta_p^{\sigma-1} \cdot (m - \sum_{i=1}^{n} p_i \cdot z_i) \tag{8}$$

The expenditure system becomes:

[2] See Varian (1992), Chapter 7, and Jehle and Reny (2011), Chapter 1.

$$p_j \cdot x_j = p_j \cdot z_j + s_j(p) \cdot \left( m - \sum_{i=1}^{n} p_i \cdot z_i \right) \tag{9}$$

with:

$$s_j(p) = \frac{a_j^{\sigma} \cdot p_j^{1-\sigma}}{\sum_{i=1}^{n} a_i^{\sigma} \cdot p_i^{1-\sigma}} = a_j^{\sigma} \cdot p_j^{1-\sigma} \cdot \Delta_p^{\sigma-1} \tag{10}$$

We first observe that the terms $s_j(p)$ are indeed proportions since clearly their sum over $j$ is 1. But unlike the LES case in (4), where the proportions of expenditure over leftover income are constant, here these proportions $s_j(p)$ are price and substitution elasticity dependent.

Notice also the following properties:

**Property 1**. When $\sigma = 1$, which represents the LES Cobb-Douglas case, the proportions become constant. Indeed, in this case:

$$s_j(p) = \frac{a_j^{\sigma} \cdot p_j^{1-\sigma}}{\sum_{i=1}^{n} a_i^{\sigma} p_i^{1-\sigma}} = \frac{a_j}{\sum_{i=1}^{n} a_i} = \alpha_j$$

and we recover the LES constancy property that follows from the usual Stone-Geary utility function.

**Property 2**. The own derivatives have opposite signs depending on the substitution elasticity:

$$\frac{\partial s_j(p)}{\partial p_i} > 0 \text{ for } 0 < \sigma < 1 \text{ and } \frac{\partial s_j(p)}{\partial p_i} < 0 \text{ for } \sigma > 1$$

If we take the derivative from expression (10) we obtain:

$$\frac{\partial s_j(p)}{\partial p_j} = \frac{(1-\sigma) \cdot a_j^{\sigma} \cdot \sum_{i \neq j} a_i^{\sigma} \cdot p_i^{1-\sigma}}{\left( \sum_{i=1}^{n} a_i^{\sigma} \cdot p_i^{1-\sigma} \right)^2}$$

We see that its sign depends only on whether $\sigma < 1$ or $\sigma > 1$. When goods tend to be complementary ($0 < \sigma < 1$) any increase in the price of good $j$ will require a larger fraction of the leftover income to be devoted to the good getting more expensive. The reason is that for complements consumptions tend to move in the same direction and the good getting relatively more expensive will be the most affected in terms of expenditure. The consumer needs to devote a greater proportion of the leftover income to purchase the good in question. We can see this more clearly in the limit case of perfect complements. In this extreme case, consumption proportions are constant and, even if the allotted income falls, the share of leftover income needed for the good whose price increase becomes larger. The opposite occurs when goods tend to be substitutes ($\sigma$ > 1).

**Property 3**. The cross derivatives have opposite signs depending on the substitution elasticity:

$$\frac{\partial s_j(p)}{\partial p_i} < 0 \text{ for } 0 < \sigma < 1 \text{ and } \frac{\partial s_j(p)}{\partial p_i} > 0 \text{ for } \sigma > 1$$

The same intuition as in Property 2 helps to explain why. If good *i* becomes more expensive, and goods are complements, demand for *j* will fall too but the share $s_j(p)$ becomes smaller for good *j* since it is getting relatively cheaper than good *i*.

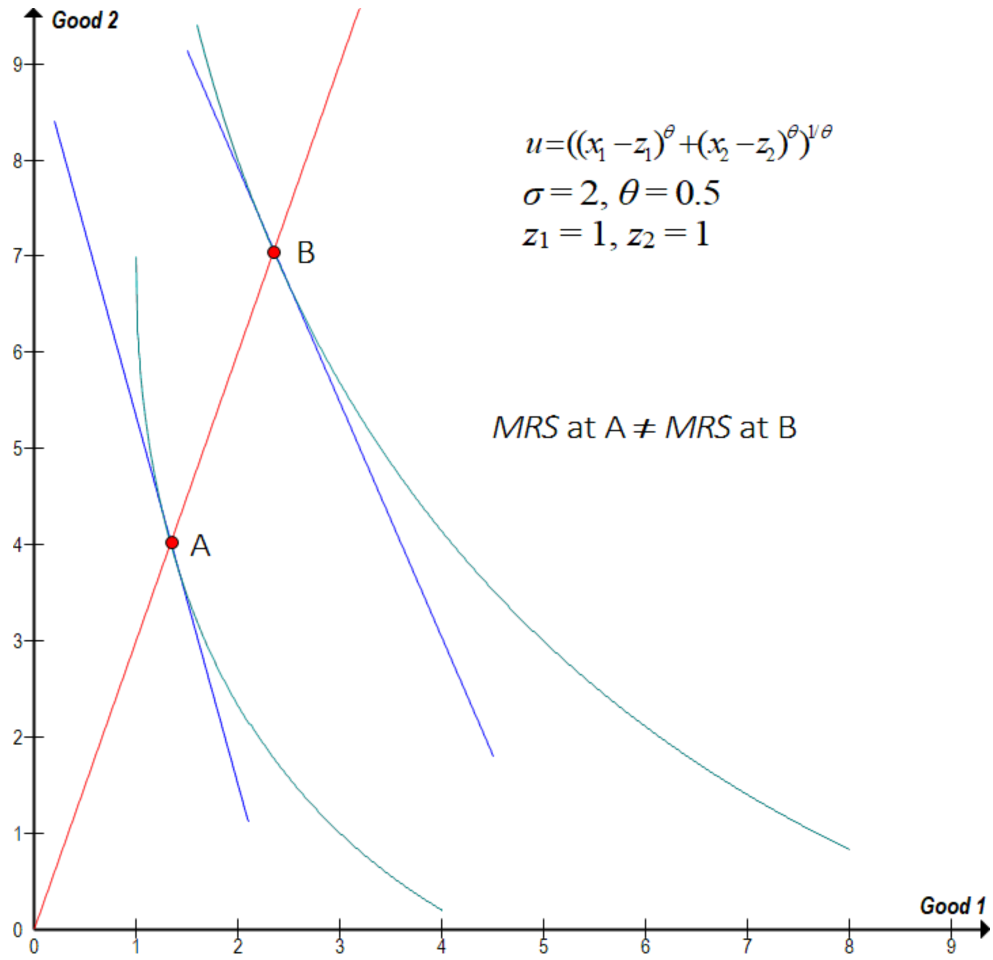
**Property 4**. The CES utility function with minimal consumptions is no longer homothetic. We calculate the marginal rate of substitution $MRS_{i,j}$ for the utility in (6) and obtain:

$$MRS_{i,j} = -\frac{a_i \cdot (x_i - z_i)^{\theta-1}}{a_j \cdot (x_j - z_j)^{\theta-1}}$$

Along the ray $x_j = \beta \cdot x_i$ with $\beta > 0$ it becomes:

$$MRS_{i,j} = -\frac{a_i \cdot (x_i - z_i)^{\theta-1}}{a_j \cdot (\beta \cdot x_i - z_j)^{\theta-1}}$$

whose value depends on the value of $x_i$ and thus the marginal rate of substitution is clearly not constant. One of the criticisms to the use (or abuse) of homothetic utilities in applied work is that the real world does not seem to be homothetic. The extended CES utility function with minimal consumption levels does not suffer from this problem since the *MRS* is not constant along a ray. Graph 1 illustrates the changing values of the marginal rate of substitution along a $\beta$ ray.




Graph 1: Non-homothetic CES shifted utility function.

### 3.2 The cost function

If we substitute the solution in (7) into the CES utility function $u(y)$ we derive the indirect utility function. With a little bit of tedious algebra we can show that it adopts the form:

$$v(p,m') = m' \cdot \Delta_p^{-1}$$

We swiftly find the cost function using duality:

$$e(p,u) = e(p,v(p,m')) = m' = v(p,m') \cdot \Delta_p = u \cdot \Delta_p$$

Recall, however, that $e(p, u)$ measures just the cost of the excess consumption over subsistence levels. We can derive the total cost function once again undoing the change of variable. For the $y_j$ minimal cost solution we have:

$$e(p,u) = \sum_{j=1}^{n} p_j \cdot y_j = \sum_{j=1}^{n} p_j \cdot (x_j - z_j)$$

Therefore total cost is:

$$c(p,u) = \sum_{j=1}^{n} p_j \cdot x_j = \sum_{j=1}^{n} p_j \cdot z_j + e(p,u) = \sum_{j=1}^{n} p_j \cdot z_j + u \cdot \Delta_p \qquad (11)$$

Total cost contains a fixed part and a variable part. In the absence of minimum consumption levels, the cost function reverts to the standard CES cost function. Even though the variable cost function satisfies the homotheticity property, i.e. $e(p,u) = u \cdot e(p,1),$ the total cost function does not.

## 4. Calibration issues in empirical applications

The common use of the Stone-Geary utility function in the implementation of numerical general equilibrium models is clearly a way to improve the usual but more restrictive Cobb-Douglas assumption. Similarly, the shifted CES utility system improves upon the LES approach by eliminating two of its restrictions. One is the unitary substitution elasticity implicit in the Cobb-Douglas function, the other is the fixed proportions of excess consumption over minimum levels that we observe in the LES system, which turn out to be unrealistically independent of the evolution of the price vector *p*. The shifted CES function does not suffer these problems. The question is how to implement the extended CES utility in numerical general equilibrium models. The calibration goal is to have an empirical specification of expression (6) that has the property that when we use it to maximize utility, the solution endogenously yields the very same observed empirical data registered in some database, such as in input-output (I-O) or Social Accounting Matrix (SAM) databases.

A look at (6) shows that we need values for the $n$ share coefficients $a_j$, the elasticity of substitution $\sigma$ (which gives us $\theta$) and the $n$ minimum consumption levels $z_j$. We usually borrow the elasticity of substitution from available econometric estimates. This leaves 2$n$ parameters to be determined, or "calibrated": the $n$ $a_j$ coefficients and the $n$ minimal consumptions $z_j$. If we use I-O or SAM data, however, we only have $n$ observations available, which are the currency values (Euros, Dollars, etc) of consumption for the $n$ goods registered in the I-O database.

The first calibration trick is to assume that all prices reflected in the database are unitary. This entails a redefinition of the units in such a way that one physical unit has the worth of one currency unit. With this redefinition, all observed data in the database are now both value as well as physical units (for the appropriate unit, of course). If the price vector for goods is implicitly made to be equal to the unit vector **1**, the proportions in expression (10) become for all $j$:

$$s_j(\mathbf{1}) = \frac{a_j^{\sigma} \cdot 1_j^{1-\sigma}}{\sum_{i=1}^{n} a_i^{\sigma} \cdot 1_i^{1-\sigma}} = \frac{a_j^{\sigma}}{\sum_{i=1}^{n} a_i^{\sigma}}$$

The second step in the calibration uses the fact that, from expression (9), the income elasticity of demand $\eta_j$ is easily seen to be:

$$\eta_j = \frac{\partial x_j}{\partial m} \cdot \frac{m}{x_j} = s_j(p) \cdot \frac{m}{p_j \cdot x_j} = s_j(p) \cdot \alpha_j(p)^{-1}$$

Here $\alpha_j(p) = p_j \cdot x_j / m$ is the share of expenditure on good $j$ over total income $m$. We can calculate these shares from the given database where all prices are redefined to be 1. Hence:

$$s_j(\mathbf{1}) = \eta_j \cdot \alpha_j(\mathbf{1})$$

Provided estimates of the income elasticity of demand for all $n$ goods are available, we can calibrate the proportions $s_j(\mathbf{1})$ using these elasticity estimates and the shares obtained from the data. For a given substitution elasticity value $\sigma$, we can now solve the linear system:

$$s_j(\mathbf{1}) = \frac{a_j^\sigma}{\sum_{i=1}^{n} a_i^\sigma} \qquad j = 1,2,\ldots,n$$

and obtain the coefficients $a_j$. This system, however, has one redundant equation. Indeed, if $a_j$ is a solution so is $\mu \cdot a_j$ for any positive scalar $\mu$. We can therefore add an extra independent equation that makes, for example, the sum of all the $a_j$ coefficients to be 1 and then proceed to solve.

The final step is to determine the subsistence levels $z_j$. For this the same procedure as in the standard LES case will work. Define the coefficient:

$$-\phi = \frac{m}{m - \sum_{j=1}^{n} p_j \cdot z_j} = \frac{m}{m - \Sigma}$$

with $0 \le \Sigma \le m$. When $\Sigma = 0$ no income would be devoted to purchase minimum consumptions and then $\phi = -1$ whereas when $\Sigma \to m$ then $\phi \to -\infty$ an in this case no income would be available for the variable part of consumption demand. The coefficient $\phi$ is known as the Frisch parameter (Frisch, 1959) and measures the flexibility in the distribution of income between the fixed and the variable parts, with maximum flexibility when $\phi = -1$ and null flexibility when $\phi \to -\infty$. We substitute in expression (9) to obtain:

$$p_j \cdot x_j = p_j \cdot z_j + s_j(p) \cdot \left( m - \sum_{i=1}^{n} p_i \cdot z_i \right) = p_j \cdot z_j + s_j(p) \cdot \frac{m}{-\phi}$$

Solve for $z_j$ recalling that all prices are set to $\mathbf{1}$ in the initial data:

$$z_j = x_j - s_j(\mathbf{1}) \cdot \frac{m}{-\phi}$$

In this expression we know the values of $x_j$ and $m$ (from the data) and $s_j(\mathbf{1})$ (from step 2 above) so if we borrow $\phi$ from the econometrics literature we have all we need to calibrate the minimum consumptions $z_j$.

As an illustration of the calibration procedure, we borrow income elasticities calculated for the two-digit 12 ECOICOP sectors for Spain for the Spanish economy from Garcia-Villar (2018). The fact that these estimated income elasticities are not unitary challenges the typical use of homothetic utility functions in numerical general equilibrium since, most commonly, the selection of preferences give rise to unitary income elasticity[3]. This justifies departing from homothetic functions and endorses the use of LES or the here proposed CES extended Stone-Geary utility functions. We also use the reported value of the Frisch parameter ($\phi = -2$) from Deaton and Muellbauer (1980) along with two sensible small deviations around it. The central elasticity of substitution value corresponds to the widely used Cobb-Douglas case ($\sigma = 1$) but, again, we introduce deviations from this unitary elasticity value to appraise the sensitivity of the results. Finally, we use expenditure data for the same classification of goods taken from the ECOICOP data published by the National Institute of Statistics for 2017[4]. Table 1 shows the calibration of the utility function coefficients whereas Table 2 shows the calibrated minimum levels of consumption.

Table 1: Utility coefficients for alternative substitution elasticity values and $\phi = -2$

| **Goods** | **Coefficients** | $\sigma$ = 0.75 | $\sigma$ = 1 | $\sigma$ = 1.25 |
|---|---|---|---|---|
| 1.-Food and non-alcoholic beverages | $a_1$ | 0.087 | 0.093 | 0.095 |
| 2.-Alcoholic beverages and tobacco | $a_2$ | 0.006 | 0.013 | 0.019 |
| 3.-Clothing | $a_3$ | 0.058 | 0.069 | 0.075 |
| 4.-Housing | $a_4$ | 0.165 | 0.151 | 0.140 |
| 5.-Household articles | $a_5$ | 0.051 | 0.062 | 0.069 |
| 6.-Health | $a_6$ | 0.032 | 0.044 | 0.052 |
| 7.-Transportation | $a_7$ | 0.259 | 0.211 | 0.183 |
| 8.-Communication services | $a_8$ | 0.009 | 0.016 | 0.024 |
| 9.-Recreational services | $a_9$ | 0.070 | 0.079 | 0.084 |
| 10.-Education | $a_{10}$ | 0.013 | 0.022 | 0.030 |
| 11.-Hotels and restaurants | $a_{11}$ | 0.175 | 0.157 | 0.145 |
| 12.-Other services | $a_{12}$ | 0.074 | 0.083 | 0.086 |

Source: Our computations

[3] The econometrics literature provide ample evidence for non-unitary income elasticities. See Lecocq & Robin (2006), Christensen (2014) and García-Enriquez & Echevarría (2016).
[4] https://www.ine.es/jaxiT3/Tabla.htm?t=24765&L=0

Table 2: Minimum consumptions for alternative values of the Frisch parameter $\phi$

| Goods | **Expenditure data 2017** | **Minima** | $\phi$ = -1.75 | $\phi$ = -2 | $\phi$ = -2.25 |
|---|---|---|---|---|---|
| 1.-Food and non-alcoholic beverages | 76.042 | $z_1$ | 47.268 | 50.864 | 53.662 |
| 2.-Alcoholic beverages and tobacco | 9.927 | $z_2$ | 6.054 | 6.538 | 6.914 |
| 3.-Clothing | 28.043 | $z_3$ | 6.781 | 9.439 | 11.506 |
| 4.-Housing | 162.431 | $z_4$ | 115.939 | 121.750 | 126.270 |
| 5.-Household articles | 24.762 | $z_5$ | 5.542 | 7.944 | 9.813 |
| 6.-Health | 18.149 | $z_6$ | 4.502 | 6.208 | 7.535 |
| 7.-Transportation | 67.890 | $z_7$ | 2.819 | 10.953 | 17.279 |
| 8.-Communication services | 17.209 | $z_8$ | 12.188 | 12.816 | 13.304 |
| 9.-Recreational services | 30.770 | $z_9$ | 6.289 | 9.349 | 11.729 |
| 10.-Education | 7.668 | $z_{10}$ | 0.855 | 1.707 | 2.369 |
| 11.-Hotels and restaurants | 55.588 | $z_{11}$ | 7.008 | 13.081 | 17.804 |
| 12.-Other services | 41.864 | $z_{12}$ | 16.332 | 19.524 | 22.006 |

Source: National Institute of Statistics, García-Villar (2018) for the income elasticities, and our computations. Expenditure data in millions of current Euros.

## 5. Concluding remarks

The Stone-Geary linear expenditure system correctly captures some rigidity properties of consumption demand, namely, the likely existence of minimum or subsistence levels of consumption for some goods. Actual consumption goes over these limits but cannot go under. Excess consumption over these minimal levels is apportioned using fixed share coefficients only when some level of leftover income is available. This real-world property, however, turns out to be price insensitive as long as the shifted utility function is restricted to be of the Cobb-Douglas variety. When we contemplate wider substitution possibilities, as is the case with CES displaced utility functions, the shares of excess consumption become price sensitive, capturing a more realistically empirical property. This extension to non-unitary substitution elasticities yields expenditures systems that are both linear in the mathematical structure but variable in the values of the excess consumption shares. The fact that CES displaced utilities gives rise to price responsive coefficients, hence improving the reaction capacities of consumers before changing prices, may endow the modeling of numerical general equilibrium with a more consistent demand platform which in turn may provide more reliable, and more real-world grounded, welfare assessment of policies. Additionally, the LES or the proposed CES extended function, being non-homothetic, both reflect the empirical nature of income elasticities, since plenty of econometrics evidence suggest their values are not unitary.

Notice also that our emphasis here on utility theory can quickly be reinterpreted in terms of production theory. In this case, the technology would presume a minimum size of primary factors, labor and capital, and in a very natural way the derived cost function would separate variable and fixed production costs in contrast with the usual practice of numerical general equilibrium, where the introduction of fixed costs follows ad-hoc rules with little theoretical backing.

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