Measurable Ambiguity

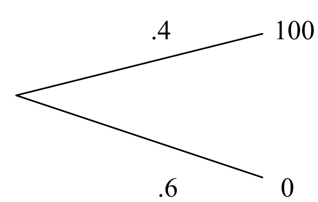
with Wolfgang Pesendorfer

August 2009

A Few Definitions

A Lottery is a (cumulative) probability distribution over monetary prizes.
It is a probabilistic description of the DMs uncertain situation.

$\mathcal{L}$ is the set of all lotteries.



A Few Definitions

A Lottery Preference is a utility function $V : \mathcal{L} \longrightarrow R$ over lotteries.

- In economics, often lotteries are the primitive.
- Empirical evidence does not come in the form of lotteries.
- The relevant probabilities are estimated.
- Assumptions are made about whether or not agents know (or agree on) these probabilities.

Definitions Continued

An Act is a nonprobabilistic description of the DMs uncertain situation.

Cloudy	Rainy	Snowy
100	-80	65

An act is less abstract than a lottery. It is more like real data.

Definitions Continued

Assessment is a the process of assigning subjective probabilities to events.

	.4	.35	.25
	Cloudy	Rainy	Snowy
f	100	-80	65
g	50	30	0
h	-20	-20	-20
…	…	…	…
…	…	…	…
…	…	…	…

Definitions Continued

Reduction enables the DM to interpret acts as lotteries.

	.4	.35	.25	
	Cloudy	Rainy	Snowy	Lotteries
f	100	-80	65	(.35, -80; .25, 65; .4, 100)
g	50	30	0	(.25, 0; .35, 30; .4, 50)
h	-20	-20	-20	(1, -20)
…	…	…	…	
…	…	…	…	
…	…	…	…	

Definitions Continued

A DM is Probabilistically Sophisticated if he evaluates acts f through

Assessment + Reduction + Lottery Preference

$$U(f) = V(G^f)$$

Add the phrase "as if" to the above as many times as you wish.

Three Approaches of Decision-Making under Uncertainty

The Pure Subjectivist (Bayesian) Approach

- All events can be assigned probabilities: Subjective Expected Utility Theory

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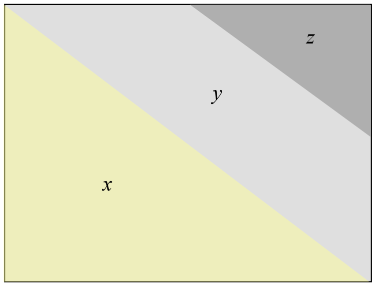
- Literature: Generalizations of EU Theory
- "Final" Model: Machina-Schmeidler (Probabilistic Sophistication)

Some Literature:

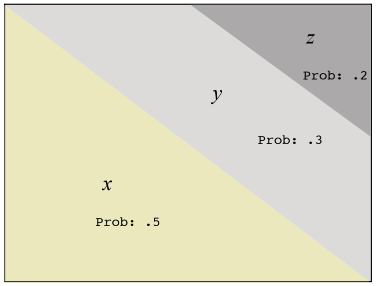
- Ramsey (1926)
- Savage (1954)
- All of the nonexpected utility literature
- Machina and Schmeidler (1992)

Fast Summary



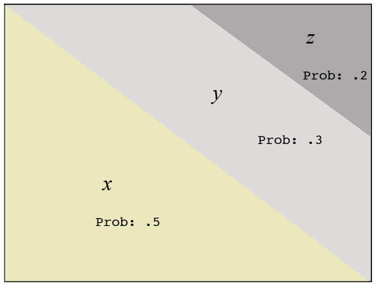
You win \$$x$, \$$y$ or \$$z$
Depending on where the dart lands

Subjective Probabilities



- The DM assesses probabilities to all events.

Subjective Probabilities



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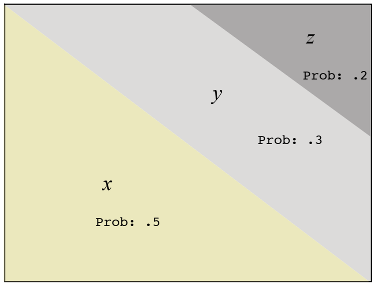

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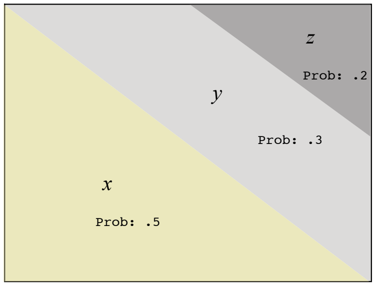

- The DM assesses probabilities to all events.
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The act becomes the lottery $(.5x; .3y; .2z)$
Yielding utility $.5u(x) + .3u(y) + .2u(z)$
Or more generally $U(.5x; .3y; .2z)$

Knightian Approach

Knightian uncertainty is risk that is immeasurable, not possible to calculate.
Wikipedia

"Uncertainty must be taken in a sense radically distinct from the familiar notion of Risk,.... The essential fact is that 'risk' means in some cases a quantity susceptible of measurement, while at other times it is something distinctly not of this character."
F. Knight

Some Literature:

- Knight (1921)
- Ellsberg (1961)
- Schmeidler (1989)
- Most of the ambiguity literature

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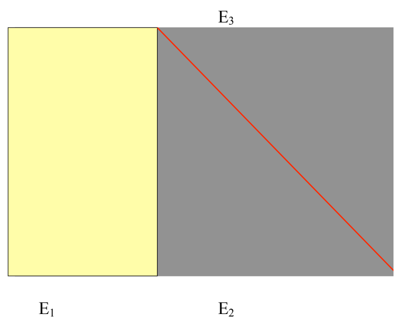
- "Final" Model: Uncertainty Averse Preferences Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2008)

Risky versus Ambiguous Events



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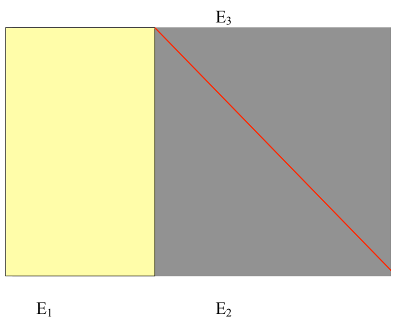

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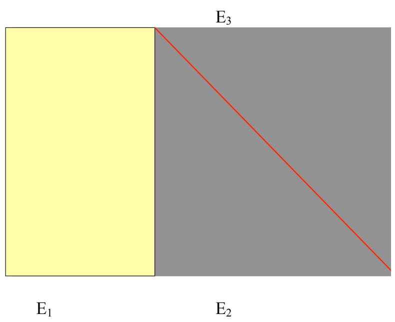

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DM prefers E_1 to E_2.

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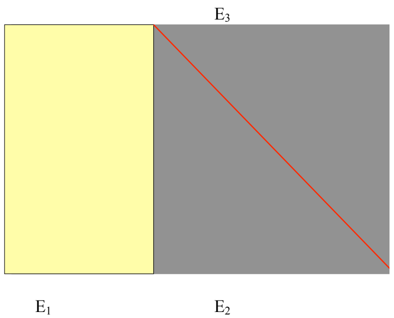

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DM prefers E_1 to E_2.

But also $E_2 \cup E_3$ to $E_1 \cup E_3$.

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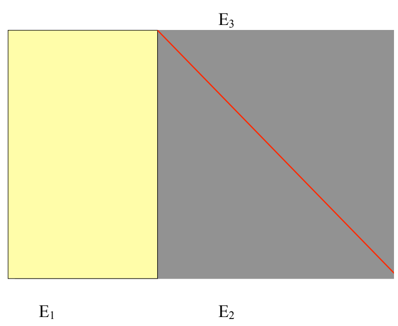

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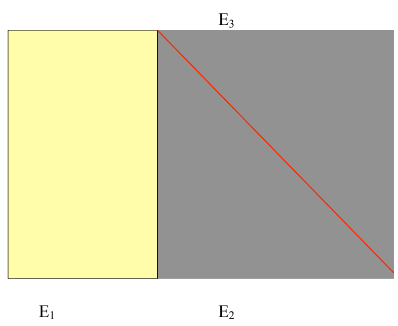

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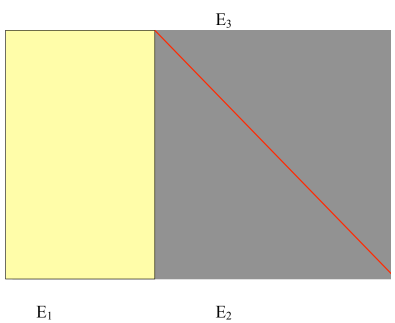

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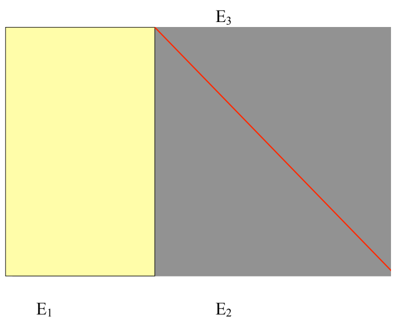

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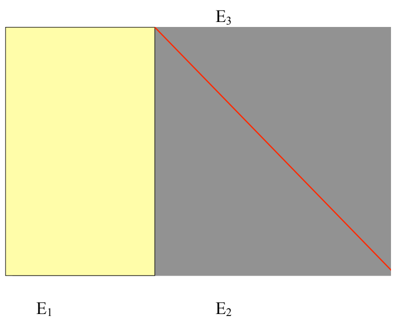

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If it existed would have to be $\frac{1}{3}$ but it doesn't.

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- “Final” Model: ?

Some Literature:

- Heath and Tversky (1991)
- Abdellaoui, Baillon, Placido and Wakker (2008)
- Chew and Sagi (2008)
- Ergin and Gul (2009)
- The Home Bias Literature
  Different risk aversion in different environments

A Two-Urn Example

Urn 1: n-colors, 1 ball for each color;

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- The DM is probabilistically sophisticated when choosing among risky prospects that depend on balls drawn from urn 1.

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- Since colors are interchangeable, we expect a decision maker to be probabilistically sophisticated when choosing among risky prospects that depend on balls drawn from urn 2.

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- Since colors are interchangeable, we expect a decision maker to be probabilistically sophisticated when choosing among risky prospects that depend on balls drawn from urn 2.

But the DM need not be indifferent between the Bet 1 and Bet 2.

Objectives:

- (1) Subjective Model of Choice under Uncertainty

  A (Simple) Representation for All Acts

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  A (Simple) Representation for All Acts

- (2) Multiple Sources and Environments

  Use the Framework to Address Experimental Evidence (Allais and Ellsberg)

- (3) Separate Uncertainty and Attitude to Uncertainty.

The Model

State space: Ω (cardinality of the continuum.)

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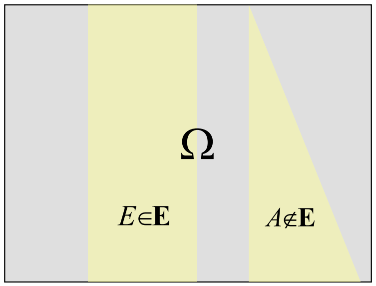
We axiomatize Expected Uncertain Utility (EUU).

An Example:



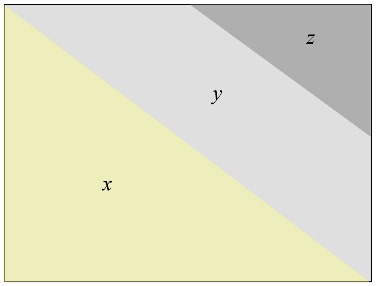
States: $\Omega = [0,1] \times [0,1]$

The Prior $(\mathcal{E}, \mu)$:

$\mathcal{E}$ is smallest σ-algebra that contains all full-height rectangles (like E) and all sets that have zero Lebesgue measure on the square.

$\mu([a,b] \times [0,1]) = b - a$ for $b \geq a$.

An Act, Its Envelope and Utility:



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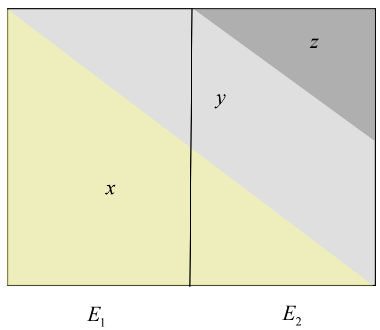

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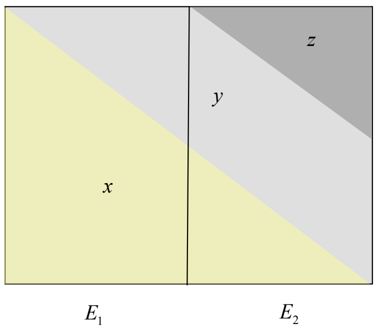

Suppose $x < y < z$

Envelope:

$\mathbf{f}_1 = x$

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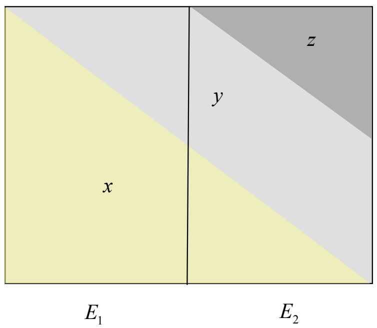

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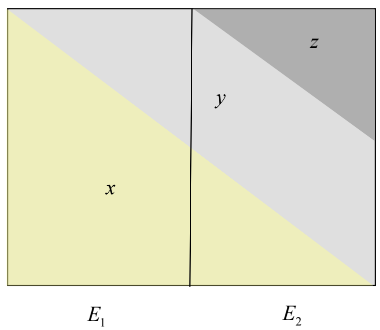

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$\mathbf{f}_1 = x$

$\mathbf{f}_2 = yE_1z$

$$U(f) = \mu(E_1)u(x, y) + \mu(E_2)u(x, z)$$

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Prior: is a $\sigma-$algebra $\mathcal{E}$ and a non-atomic (countably additive) probability measure μ.

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Definition: An envelope for $f \in \mathcal{F}$ is a function $\mathbf{f} : \Omega \to I$ such that

1. $\mathbf{f}$ is $\mathcal{E}$-measurable and $\mu(\{\mathbf{f}_1(\omega) \leq f(\omega) \leq \mathbf{f}_2(\omega)\}) = 1$
2. $\mathbf{g}$ satisfies (1) implies $\mu(\{\mathbf{g}_1(\omega) \leq \mathbf{f}_1(\omega) \leq \mathbf{f}_2(\omega) \leq \mathbf{g}_2(\omega)\}) = 1$.

Lemma 1: Let $(\mathcal{E}, \mu)$ be a prior and $f \in \mathcal{F}$. Then, f has an envelope.

Expected Uncertain Utility

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Definition: The preference $\succeq$ is an EUU if there is a prior $(\mathcal{E}, \mu)$ and an interval utility index u such that

$$U(f) = \int u(\mathbf{f}_1(\omega), \mathbf{f}_2(\omega)) d\mu$$

represents $\succeq$.

Expected Uncertain Utility

Given the prior μ we can define a bicumulative over prizes for every act f:

Bicumulative: Let $H_f(x, y) = \mu(\{\mathbf{f}_1 \leq x, \mathbf{f}_2 \leq y\})$.

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EUU:

$$U(f) = \int u(x, y) dH_f(x, y)$$

The bicumulative is analogous to cdf over prizes in the standard case.

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Under suitable assumptions,

A preference $\succeq$ on $\mathcal{F}$ has an EUU representation:

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or equivalently

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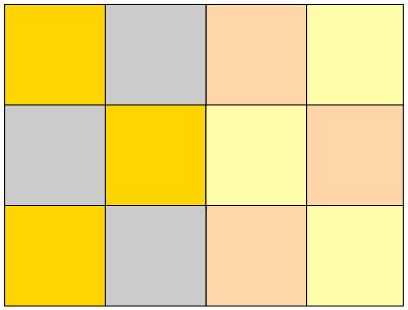
(2) Multiple Sources and Environments



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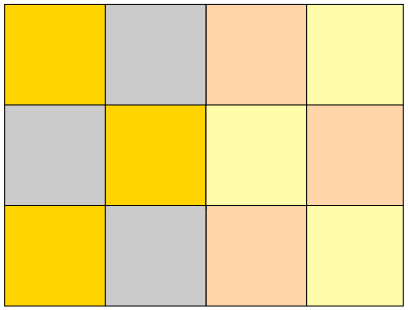

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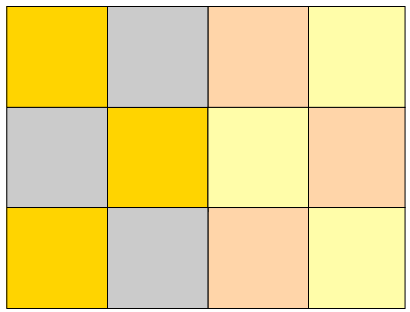

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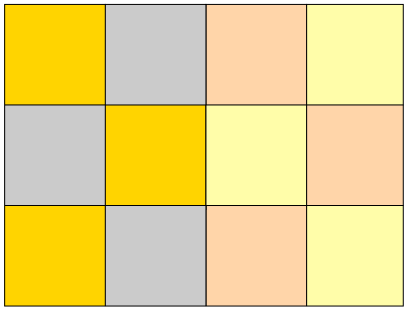

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Each urn (or collection of events: rows, columns and colors) is a a source and the collection of all bets (acts) that depend on a particular source is an environment.

The DM can be more risk averse when betting on columns than when betting on colors.

Sources and Environments

- Let $\mathcal{C}$ be a collection of sets (a λ-system).
- $\mathcal{F}_{\mathcal{C}} = \{f \in \mathcal{F} : f \text{ is } \mathcal{C} - \text{measurable}\}$

For example, let

$$\mathcal{C}_1 = \{\text{G(ray), O(range), Y(ellow), P(each)}\}$$

$$\mathcal{C} = \{\text{all events that depend only on color, G, Y, G} \cup \text{Y etc.}\}$$

$$\mathcal{F}_{\mathcal{C}} = \{\text{all acts that depend only on color}\}$$

- Suppose each color has the same probability and each column K_i has the same probability $(1/4)$.
- Consider the two bets: $100Y0$ and $100K_10$.
- Suppose the DM utility function satisfies

$$U(40) = U(100Y0) > U(100K_10) = U(35)$$

Hence, the DM prefers betting on color to betting on column. Equivalently the DM is more risk averse when betting on columns than when betting on colors.

- If $(\mathcal{C}, \pi)$ is a probability measure (Assessment), then each $f \in \mathcal{F}_{\mathcal{C}}$ can be assigned a cdf (Lottery) G^f (Reduction).
- The Assessment) makes $f \in \mathcal{F}_{\mathcal{C}}$ into a source and $\mathcal{F}_{\mathcal{C}}$ into an environment.
- Then, the DM has a lottery preference V so that he assigns utility $V(G^f)$ to each f.

Whether or not $\mathcal{F}_{\mathcal{C}}$ is an environment is subjective as is the lottery preference V on $\mathcal{F}_{\mathcal{C}}$.

Sources, Environments and EUU

- So far, the definitions of Source and Environment don't require EUU preferences.
- How many sources does a typical EUU preference have?
- What kind of lottery preferences does an EUU preference have in these environments?
- How do these environments enable EUU theory to address experimental and empirical evidence (Allais, Ellsberg, Home Bias)?

(2) Multiple Sources and Environments in EUU Theory

Definition: u is strongly symmetric if it has the form

$$u(x, y) = (v(x) + v(y))/2$$

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- EUU's with the same prior have (essentially) the same environments.
- This is the sense in which (3) Separation is achieved.
- We call $\mathcal{F}_\mathcal{C}$ a Regular Environment for $(\mathcal{E}, \mu)$ if it is an environment for some $(\mathcal{E}, \mu, u)$ with u not strongly symmetric.

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- address Allais-type evidence.

Multiple Environments: Some Properties

- Every EUU has every source.
- The prior alone determines if $\mathcal{F}_{\mathcal{C}}$ is an environment for $(\mathcal{E}, \mu, u)$.
- Risk attitude depends u.
- One environment for the the EUU $(\mathcal{E}, \mu, u)$ is $\mathcal{F}_{\mathcal{E}}$, the Ideal environment.
- Every EUU is an expected utility maximizer in its ideal environment.

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- In other environments, the EUU $(\mathcal{E}, \mu, u)$ is a nonexpected utility maximizer

Regular Environments and Lottery Preferences for EUU

Proposition 2: For any interval utility u, there exists a sequence of lottery preferences V_n^u and for any regular environment $\mathcal{F}_{\mathcal{C}}$ of $(\mathcal{E}, \mu)$, there exists a sequence $a_n \geq 0$, $\sum a_n = 1$ such that

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$$U(f) = V(G^f) = \sum_n a_n V_n^u(G^f)$$

represents $(\mathcal{E}, \mu, u)$. Furthermore, for any such sequence a_n and $(\mathcal{E}, \mu)$, there exists a regular environment $\mathcal{F}_{\mathcal{C}}$ such that

$$U(f) = \sum_n a_n V_n(G^f)$$

for all $f \in \mathcal{F}_{\mathcal{C}}$.

Allais and Uncertainty Aversion

Allais Paradox:

$$V(100) > V\big(150, 4/5; 0, 1/5\big)$$

but

$$V\big(100, 2/5; 0, 3/5\big) < V\big(150, 1/2; 0, 1/2\big)$$

Allais Reversals

Definition: A lottery preference V is prone to Allais-reversals if there is an environment γ so that we can find

- a lottery F
- prizes x, y where x is weakly worse than all other prizes in the support of F
- $\alpha \in (0, 1)$

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- $V(y) > V(F)$
- $V(\alpha y + (1 - \alpha)x) < V(\alpha F + (1 - \alpha)x)$.

Rank Dependent EU

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PTF's that have an inverted S-shape are (a) consistent with Allais reversals and (b) have some supporting experimental evidence (Starmer (2000)).

Polynomial Utility and Special Cases

Recall: A sequence $a_n \geq 0$ such that $\sum_n a_n = 1$ characterizes a regular environment and in each environment $\{a_n\}$ the EUU with interval utility u has lottery preference

$$U(f) = \sum_n a_n V_n^u(G^f)$$

We call the sequence a_n the uncertainty measure of the corresponding environment.

- $V_1^u(G) = \int u(x,x)dG(x)$. Hence V_1^u is an EU preference. The environment $a_1 = 1$ is an EU environment.

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- More generally, whenever $u(x,y) = \alpha v(x) + (1-\alpha)v(y)$ for some v, every environment is an RDEU environment. This RDEU has the desired inverted S-shape whenever $\{a_n\}$ is sufficiently uncertain.

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- More generally, whenever $u(x,y) = \alpha v(x) + (1-\alpha)v(y)$ for some v, every environment is an RDEU environment. This RDEU has the desired inverted S-shape whenever $\{a_n\}$ is sufficiently uncertain.
- V_2^u is the quadratic utility of Machina (1982), Chew, Epstein and Segal (1991) (with utility index u). Hence, $a_2 = 1$ is the quadratic utility environment.

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- u is maximally pessimistic if there exist some v such that $u(x, y) = v(x)$ for all x, y.

Proposition 3: Let $(\mathcal{E}, \mu, u)$ be an EUU. Then, the following conditions are equivalent

(1) The EUU $(\mathcal{E}, \mu, u)$ is strongly uncertainty averse;

(2) u is maximally pessimistic and concave.

Uncertainty of Environments

Definition: The environment $\mathcal{F}_{\mathcal{A}}$ is more uncertain than the environment $\mathcal{F}_{\mathcal{B}}$ if every strongly uncertainty averse EUU prefers $f \in \mathcal{F}_{\mathcal{A}}$ to $g \in \mathcal{F}_{\mathcal{B}}$ whenever f and g yield the same lottery.

Proposition 4: $\mathcal{F}_{\mathcal{B}}$ more uncertain than $\mathcal{F}_{\mathcal{A}}$ if and only if

$$\sum_n b_n t^n \leq \sum_n a_n t^n$$

for all $t \in [0,1]$, where $\{a_n\}$ and $\{b_n\}$ are the uncertainty measures of $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{B}}$ respectively.

- We write $\mathcal{F}_{\mathcal{B}} \succeq_{mu} \mathcal{F}_{\mathcal{A}}$ (or equivalently $\{b_n\} \succeq_{mu} \{a_n\}$) to mean "$\mathcal{F}_{\mathcal{B}}$ is more uncertain than $\mathcal{F}_{\mathcal{A}}$."
- $b_{n+1} = 1$ and $a_n = 1$ implies $\{b_n\} \succeq_{mu} \{a_n\}$.
- Not all environments can be ranked. For example, $a_2 = 1$ and $a_1 = a_4 = 1/2$ cannot be ranked.

Risk Loving under Extreme Uncertainty:

The EUU is risk loving under extreme uncertainty if, for sufficiently uncertain environments, there are lotteries that the DM prefers to their expected value.

Definition u displays risk loving under extreme uncertainty if there exists an environment $\mathcal{F}_{\mathcal{A}}$ and a lottery F such that $\mathcal{F}_{\mathcal{B}} \succeq_{mu} \mathcal{F}_{\mathcal{A}}$ implies $U(f) > U(z)$ whenever $f \in \mathcal{F}_{\mathcal{B}}$, $G^f = F$ and z is the mean of F.

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Ellsberg One Urn Example

3 balls, red, blue or green. 1 ball is red. Intuitively, $\{r\}$ and $\{b, g\}$ have unambiguous probability 1/3 and 2/3. But, $\{g\}$ and $\{r, b\}$ are ambiguous.

What would it mean for a model (an EUU model) to *explain* or *rationalize* the Ellsberg One-Urn Example?

- N is a nonempty finite set; $\mathcal{N}$ is the set of subsets of N.
- P be the set of all probabilities on $\mathcal{N}$ and $\iota \in P$.
- $\mathcal{M} \subset \mathcal{N}$ is a collection of sets (a λ-system).

The collection $(N, \mathcal{M}, \iota)$ is an urn experiment if for all $K \in \mathcal{N} \backslash \mathcal{M}$, there exist $p \in P$ such that $p(M) = \iota(M)$ for all $M \in \mathcal{M}$ and $p(K) \neq \iota(K)$.

Given any prior $(\mathcal{E}, \mu)$, a collection of subsets $\mathcal{C}_o$ of Ω is unambiguous if there exists a source $\mathcal{A}$ such that $\mathcal{C}_o \subset \mathcal{A}$. The event $A \subset \Omega$ is ambiguous wrt $\mathcal{C}_o$ if there exists no source $\mathcal{B}$ such that $\mathcal{C}_o \cup \{A\} \subset \mathcal{B}$.

Ellsberg One Urn Example is an Urn Example

$\mathcal{M} = \{\{r\}, \{b, g\}\}$
ι is any probability such that $\iota\{r\} = 1/3$ and $\iota\{b, g\} = 2/3$

Zhang's (1997) 4 color urn

2 balls: red, blue, green, or yellow.
1 balls is red or blue
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Intuitively unambiguous events are $\{r, b\}, \{g, y\}, \{r, g\}, \{b, y\}$ and each has $\iota = 1/2$.

Rationalizing Urn Experiments

The prior $(\mathcal{E}, \mu)$ rationalizes the urn experiment (N, μ, ι) if there exists an onto mapping $T : \Omega \rightarrow N$ such that $\mathcal{C}_o := \{T^{-1}(M) \mid M \in \mathcal{M}\}$ is unambiguous and every $T^{-1}(L)$ for $L \in \mathcal{N} \backslash \mathcal{M}$ is ambiguous wrt $\mathcal{C}_o$.

Proposition 6: Every prior rationalizes every urn experiment.

Conclusion

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- separation of uncertainty attitude and uncertainty perception.

EUU has:

- significant overlap with many existing models (Choquet EU, Maxmin EU, α-Maxmin EU.)
- few behavioral restrictions; more of a framework than a "theory."