



Mixture-Dependent Preference for Commitment

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Abstract

The literature on temptation and self-control is motivated by evidence of a preference for commitment. This literature has typically put forth models for preferences over menus of lotteries that satisfy the Independence axiom. Independence requires that the ranking of two menus is not affected if each is mixed (probabilistically) with a common third menu. In particular, the preference for commitment is invariant under Independence. We argue that intuitive behavior may require that the preference for commitment be affected by such mixing, and hence be *mixture-dependent*. To capture such behavior, we generalize Gul and Pesendorfer (2001) by replacing their Independence axiom with a suitably adapted version of the Mixture-Betweenness axiom of Chew (1989)-Dekel (1986). Axiomatizing the model involves a novel extension of the Mixture Space Theorem to preferences that satisfy Mixture-Betweenness.

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1 Introduction

1.1 Overview

A key motivation for the literature on temptation and self-control problems comes from evidence of a preference for commitment (Bryan et al. (2010), Gul and Pesendorfer (2007) and Laibson (1997)). For instance a dieter might strictly prefer to eat at a salad bar rather than at a restaurant that offers both salad (s) and burgers (b). Identifying a restaurant with the set of alternatives it offers, this agent therefore exhibits

$$\{s\} \succ \{s, b\}.$$

In a seminal paper, Gul and Pesendorfer (2001) (henceforth GP) provide an axiomatic model of temptation and self-control that characterizes a preference over menus (of lotteries) in a manner that permits such preference for commitment. A distinct feature of GP is that it maintains the Independence axiom (appropriately adapted to the domain). We observe that Independence implies that preference for commitment must be *mixture independent* in the following sense:

$$\{s\} \succ \{s, b\} \implies \{\alpha s + (1 - \alpha)i\} \succ \{\alpha s + (1 - \alpha)i, \alpha b + (1 - \alpha)i\},$$

where $\alpha s + (1 - \alpha)i$ and $\alpha b + (1 - \alpha)i$ are lotteries that yield s and b with probability α respectively and dish i with probability $(1 - \alpha)$.

The main motivation of this paper is the idea that preference for commitment may in fact be mixture-dependent.¹ To illustrate, consider a frugal vacationer who is planning a trip and needs to choose a hotel room. During her trip, she expects to use the room only to sleep. Hence, she strictly prefers a conventional room (c) to a fancy room (f). The vacationer can either choose to reserve the room in advance and commit to staying in the room she reserved, or choose the room once she arrives at the hotel. She believes that if she waits until she arrives, she will not feel tempted to choose f ; after all, she is very careful with her spending and will only be using the room to sleep. Hence,

$$\{c\} \sim \{c, f\}.$$

¹Although mixture-dependent preference for commitment has not been tested directly in a laboratory experiment, there exists evidence in the psychology literature that suggests it exists. In particular, Milkman (2012) findings suggests that temptation is sensitive to risk and thus, imply mixtures can affect the preference for commitment.

However, being a preferred member of the hotel, she receives a promotion: her name will be added to a raffle where the prize is a free night in the fancy room, which we will denote by f' , with full reimbursements if necessary. If she waits until she arrives at the hotel to choose a room, then she will face a choice between two lotteries: $\alpha c + (1 - \alpha)f'$ and $\alpha f + (1 - \alpha)f'$ where $(1 - \alpha)$ is the probability she wins the raffle. Since there is a possibility that she might be able to stay in the fancy room for free, she will be dreaming about it for the rest of the day. Hence, if she waits to choose the room until she arrives, by then the idea of having to stay in the conventional room if she loses the lottery will make her feel tempted to choose the fancy room. Thus, in order to avoid temptation and stick to her budget, she would rather book the conventional room in advance:

$$\{\alpha c + (1 - \alpha)f'\} \succ \{\alpha c + (1 - \alpha)f', \alpha f + (1 - \alpha)f'\}.$$

Building on GP, we provide a novel axiomatic model of temptation and self-control. Like GP, our model characterizes preference over menus of lotteries. The key difference is that our model can accommodate mixture-dependent preference for commitment; it does so by weakening Independence to a property we refer to as Mixture-Betweenness which adapts the Mixture-Betweenness axiom of Chew (1983) and Dekel (1986).

More specifically, let $\Delta(X)$ be the set of all lotteries with payoffs in X . GP consider a preference \succeq over the set of all menus of lotteries (subsets of $\Delta(X)$). The interpretation is that at an unmodeled second stage, a lottery is selected from the menu chosen ex-ante according to \succeq . GP axiomatize the following utility function for \succeq ,

$$V(x) = \max_{p \in x} \{u(p) + v(p) - \max_{q \in x} v(q)\},$$

for all menus x , where u and v are vNM utility functions over lotteries. For singleton menus, $V(\{p\}) = u(p)$ and thus u describes preference under commitment, which we interpret as describing the agent's normative view. The function v describes the agent's urges at the moment in which she will make a choice out of the menu. Without commitment, the agent is tempted to deviate from the choices prescribed by u and maximize v . Temptation can be resisted, but at the cost of self-control which is described by $\max_{q \in x} v(q) - v(p)$. Since u and v are vNM utility functions, they satisfy the standard Independence axiom. This is imposed only for reasons of analytical convenience.

The fact that in GP's model the normative and temptation utilities are linear precludes their model from accommodating mixture-dependent preference

for commitment. In fact this limitation exists also in subsequent generalizations of GP in the literature (Dekel et al. (2009), Chatterjee and Krishna (2009), Noor and Takeoka (2010, 2015), Stovall (2010) and Kopylov (2012)). In order to accommodate mixture-dependent preference for commitment we need to allow for non-linear u and v .

This motivates us to consider the Chew (1983)-Dekel (1986) model for preference over risk. Their model is motivated by the descriptive failure of the Independence axiom and generalizes vNM utility theory. In particular, it is an implicit utility model in which the utility γ of a lottery p is the unique solution of

$$\gamma = u(p, \gamma),$$

where $u(\cdot, \gamma)$ is a vNM utility function over lotteries for all γ . The main ingredient in the characterization of the model is the Mixture-Betweenness axiom which is a weakening of Independence that is compatible with behavior such as the Allais paradox.²

Our model combines GP and Chew-Dekel. The result is an implicit utility model in which the utility of a menu x is defined as the unique γ that solves

$$\gamma = \max_{p \in x} \{u(p, \gamma) + v(p, \gamma) - \max_{q \in x} v(q, \gamma)\},$$

where $u(\cdot, \gamma)$ and $v(\cdot, \gamma)$ are vNM utility functions over lotteries for all γ . It differs from GP by allowing the normative and temptation rankings to be non-linear and menu-dependent. In particular, both depend on the menu through the overall level of utility. The main reason to allow such systematic relation is that the strength of a temptation may be endogenous: it depends on what alternatives are available.³ Similarly, how normatively appealing an alternative is may also depend on what is available.

Notice that our model is inherently non-linear. Hence, standard tools cannot be used to axiomatize it because they rely on the Mixture Space Theorem (Herstein and Milnor (1953)) at a fundamental level. Thus, we are forced to take a different approach. In particular, we develop a novel extension of the Mixture Space Theorem to preferences that satisfy Mixture-Betweenness. Since the Mixture Space Theorem is central to decision theory, our extension is potentially useful for addressing issues in economics other than temptation. Hence, we view it as a separate contribution of the paper.

²A special case of Chew-Dekel is the well known “disappointment aversion” model of Gul (1991) which admits an explicit representation (Cerreia-Vioglio et al. (2020)).

³See Mihm and Ozbek (2018) for a discussion of why temptation preferences may be endogenous.

The paper proceeds as follows: The introduction concludes with a review of the relevant literature. Axioms and the implied representation of utility are described in Sections 2 and 3 respectively. Section 4 studies the special case of the model in which the normative utility is linear and relates it to a version of the Allais Paradox. Section 5 contains a discussion of mixture-dependent preference for commitment, axiomatic foundations for two special cases of the model and concludes with some observations of the behavior allowed by mixture-dependent preference for commitment in a consumption-savings context. Section 6 concludes with our version of the Mixture Space Theorem and a discussion of its potential applications. The proof of the new Mixture Space Theorem is provided in Appendix A. The remaining proofs are collected in Appendix B.

1.2 Literature Review

This paper contributes to the axiomatic literature on temptation and self-control (Gul and Pesendorfer (2001), Dekel et al. (2001), and for a survey of the subsequent literature see Lipman and Pesendorfer (2013)). The closest papers to ours are Noor and Takeoka (2010, 2015) and Liang et al. (2019). They also generalize GP by weakening Independence. Noor and Takeoka (2010) extends GP to a model with a convex self-control cost, a feature shared by the non-axiomatic model of Fudenberg and Levine (2006). Liang et al. (2019) enrich GP by endowing the agent with a stock of willpower that cannot be exceeded by the cost of self-control in any menu.⁴ In all of these models, the agent’s normative and temptation utilities satisfy the Independence axiom thereby ruling out mixture-dependent preference for commitment. Further, both models are motivated by behavior that does not rely on the fact that the object of choice are lotteries.

Dillenberger and Sadowski (2012) provide a model of shame for preferences over menus of monetary divisions between two agents, a dictator and a recipient. Their model adapts and extends GP, and can accommodate the following experimental finding: when subjects are presented with a menu of monetary divisions, they tend to behave altruistically whenever the recipient can observe the menu from which they are making the choice. However, at an ex-ante stage that is not observed by the recipient, some subjects are willing to give up part of their payoff in exchange for the removal of “fair” monetary

⁴A closely related paper, Masatlioglu et al. (2020) deals with menus of abstract alternatives rather than menus of lotteries. However, an early version included lotteries in the domain and imposed linearity on the temptation preferences.

divisions so that in the second stage they can choose “unfair” divisions without feeling any shame. Since the objects of choice in their set up are menus of deterministic alternatives, their analysis is silent about mixture-dependence.

Following Dillenberger and Sadowski (2012) and GP, Saito (2015) develops a model of impure shame and impure altruism (that is, shame and altruism driven by temptation) for preferences over menus of lotteries. His model satisfies the Independence axiom and thus, cannot accommodate mixture-dependent preference for commitment.

Dekel et al. (2009), Chatterjee and Krishna (2009), Stovall (2010) and Kopylov (2012) also generalize GP. Their models satisfy the Independence axiom. We believe that our Mixture Space Theorem and the arguments used in the proof of Theorem 3.1 can be used to generalize these models in the same way that we generalize GP.

Finally, outside the temptation literature but within the menus of lotteries literature, Ergin and Sarver (2010) derive a utility representation of costly contemplation. The model assumes that the agent chooses from a menu with imperfect knowledge of her preference over lotteries. In particular, the agent considers a set of possible preferences over lotteries where each of them satisfies the standard Independence axiom. Their key axiom, referred to as Aversion to Contingent Planning, is a weakening of our adaptation of the Mixture-Betweenness axiom.

2 Axioms

Let X be a finite set of cardinality n . A lottery is a probability measure over X . The set of all lotteries is denoted $\Delta(X)$ and \mathcal{X} denotes the set of all of its non-empty closed subsets. We endow \mathcal{X} with the topology generated by the Hausdorff metric.⁵ A menu is an element of \mathcal{X} . Generic menus will be denoted by x, y and z and generic lotteries will be denoted by p, q and r . Each lottery p can be identified with the singleton menu $\{p\} \in \mathcal{X}$. Thus, where it does not cause confusion we will abuse notation and write p instead of $\{p\}$ and $\Delta(X)$ instead of $\{\{p\} | p \in \Delta(X)\}$.

Our primitive is a preference \succeq over \mathcal{X} . We impose four axioms on \succeq of which the first three are from GP.

Weak Order \succeq is complete and transitive.

⁵Let d be any metric on $\Delta(X)$. For any $x, y \in \mathcal{X}$ and $p, q \in \Delta(X)$, define $d(p, y) \equiv \inf_{q \in y} d(p, q)$ and $d_h(x, y) = \max\{\sup_{p \in x} d(p, y), \sup_{q \in y} d(q, x)\}$. The topology generated by d_h is the Hausdorff metric topology.

Hausdorff Continuity $\{y|x \succeq y\}$ and $\{y|y \succeq x\}$ are closed for all $x \in \mathcal{X}$.

Set-Betweenness $x \succeq y$ implies $x \succeq x \cup y \succeq y$.

Set-Betweenness admits an interpretation in terms of temptation and self-control. To illustrate, consider the ranking $\{p\} \succ \{p, q\} \succ \{q\}$. The ranking $\{p\} \succ \{p, q\}$ is referred to as *preference for commitment*, it suggests that the agent expects to be tempted by q if she faces $\{p, q\}$. Thus, $\{p, q\} \succ \{q\}$ implies that the agent expects to be able to resist temptation if she faces $\{p, q\}$, but it will require costly self-control. Similarly, $\{p\} \succ \{p, q\} \sim \{q\}$ suggests that the agent expects to be overwhelmed by temptation if she faces $\{p, q\}$. Finally, the lack of preference for commitment in $\{p\} \sim \{p, q\} \succ \{q\}$ suggests that the agent does not expect to be tempted by q if she faces $\{p, q\}$.

Whenever $x \subset y$ and $x \succ y$ we say \succeq *has preference for commitment at y* . Under the temptation and self-control interpretation, preference for commitment at y reveals that there is some element in y that the agent expects to be tempted by and thus, would like to remove from the feasible set she will face in the second stage.

For any two menus x, y and $\alpha \in [0, 1]$, define the mixture $\alpha x + (1 - \alpha)y$ as the menu generated by the point-wise mixtures:

$$\alpha x + (1 - \alpha)y = \{r \in \Delta(X) | r = \alpha p + (1 - \alpha)q, p \in x, q \in y\}.$$

GP's fourth axiom formulates the standard vNM Independence axiom in the menus of lotteries setting.

Independence $x \succeq y$ implies $\alpha x + (1 - \alpha)z \succeq \alpha y + (1 - \alpha)z$ for all $\alpha \in [0, 1]$ and $z \in \mathcal{X}$.

GP, Dekel et al. (2001) and the literature that followed them adopt this axiom because of its normative appeal and the analytical convenience it offers. To understand their motivation consider an extension of the preference to the set of lotteries over \mathcal{X} , the interpretation being that randomization over menus is resolved before the second stage. Suppose that this extended preference satisfies the standard vNM Independence axiom: the preference between a lottery that yields with probability α a menu x and with probability $1 - \alpha$ a menu z (denoted by $\alpha \circ x + (1 - \alpha) \circ z$) and $\alpha \circ y + (1 - \alpha) \circ z$ is the same as the preference between x and y . If the agent is indifferent between uncertainty being resolved before the second stage or after the second stage, then she satisfies *Reduction*:

$$\alpha \circ x + (1 - \alpha) \circ y \sim \alpha x + (1 - \alpha)y \text{ for all } x, y \in \mathcal{X} \text{ and } \alpha \in [0, 1].$$

Observe that vNM Independence and Reduction imply Independence. However, we claim that temptation may lead to violations of vNM Independence. Recall the rankings in our example:

$$\begin{aligned} \{c\} &\sim \{c, f\} \\ \{\alpha c + (1 - \alpha)f'\} &\succ \{\alpha c + (1 - \alpha)f', \alpha f + (1 - \alpha)f'\}. \end{aligned} \tag{1}$$

The intuition behind these rankings implies that the agent would strictly prefer $\alpha \circ \{c\} + (1 - \alpha) \circ \{f'\}$ to $\alpha \circ \{c, f\} + (1 - \alpha) \circ \{f'\}$ because once there is a possibility she might be able to stay in the fancy room for free, she will prefer to stay in the fancy room rather than the conventional room. Hence, if the lottery $\alpha \circ \{c, f\} + (1 - \alpha) \circ \{f'\}$ yields $\{c, f\}$, she will feel tempted to choose f .

More generally, Independence implies that preference for commitment is mixture independent: if \succeq has preference for commitment at y , then \succeq also has preference for commitment at $\alpha y + (1 - \alpha)z$ for all $\alpha \in (0, 1]$ and $z \in \mathcal{X}$. This follows from the fact that under Independence,

$$\begin{aligned} x \succ y &\implies \alpha x + (1 - \alpha)z \succ \alpha y + (1 - \alpha)z \\ &\text{and} \\ x \subset y &\implies \alpha x + (1 - \alpha)z \subset \alpha y + (1 - \alpha)z. \end{aligned}$$

Hence, Independence needs to be weakened. We weaken it to Mixture-Betweenness. Intuitively, Mixture-Betweenness requires the indifference curves be linear but allows them to not be parallel. Thus, the ranking of two menus may change when each is mixed with a common third menu.

Mixture-Betweenness

$$\begin{aligned} x \succ y &\text{ implies } x \succ \alpha x + (1 - \alpha)y \succ y \text{ for all } \alpha \in (0, 1), \text{ and} \\ x \sim y &\text{ implies } x \sim \alpha x + (1 - \alpha)y \sim y \text{ for all } \alpha \in (0, 1). \end{aligned}$$

Mixture-Betweenness requires that if an agent prefers x to y , then the mixture between x and y has to be between these two menus in terms of preference. In particular, if an agent is indifferent between two menus, then any mixture of these two is equally good. Hence, it implies that the indifference sets are convex. Next, we consider what Mixture-Betweenness permits and what it rules out.

Mixture-dependent preference for commitment is permitted by Mixture-Betweenness. For instance, the rankings in (1) are consistent with Mixture-Betweenness because the latter is silent about behavior that involves more than two menus. However, it does rule out a specific class of mixture-dependent

preference for commitment. To illustrate, consider a dieter who prefers to go to a salad bar rather than a restaurant that offers salads (s) and burgers (b):

$$\{s\} \succ \{s, b\} \succeq \{b\}. \quad (2)$$

Suppose now that there is a small new restaurant that also offers salads and burgers, but its burgers are so popular that it sometimes runs out of them. Therefore, its menu offers salads and a lottery between burgers and salads. Mixture-Betweenness would then require that the dieter would also expect to feel temptation if she goes to the new restaurant because its menu is equal to a mixture between the salad bar and the restaurant. In particular, it is equal to

$$\{s, \alpha s + (1 - \alpha)b\},$$

where α is the probability that the restaurant runs out of burgers. Hence,

$$\{s\} \succ \{s, b\} \implies \{s\} \succ \{s, \alpha s + (1 - \alpha)b\}.$$

In general, Mixture-Betweenness requires that if $x \subset y$ and $x \succ y$, then $x \succ \alpha x + (1 - \alpha)y$. Thus, it implies that if the agent has preference for commitment at y , then the agent's preference for commitment is mixture independent whenever the mixture is with any subset of y that she would prefer to commit to.

Mixture-Betweenness also imposes some structure on the agent's self-control. To illustrate, consider again the dieting agent. Suppose now that there are two restaurants that belong to the same chain and assume that one is smaller than the other. The smaller restaurant does not have a fixed menu. Rather, its menu is picked randomly by the chef (chef's pick). It can either be a salad or a burger. The larger restaurant not only offers the chef's pick, but in addition it has a fixed burger selection. Consistent with the preference for commitment she exhibited in (2), she prefers to go to the smaller restaurant. Hence,

$$\{\alpha s + (1 - \alpha)b\} \succ \{\alpha s + (1 - \alpha)b, b\}$$

where α is the probability that the chef's pick is a salad. If the dieter exerted costly self-control in $\{s, b\}$ (that is, if $\{s, b\} \succ \{b\}$ in (2)), then since the menu at the larger restaurant is a mixture between $\{s, b\}$ and $\{b\}$, Mixture-Betweenness implies that

$$\{\alpha s + (1 - \alpha)b, b\} \succ \{b\}.$$

Thus, if the dieter expects to resist temptation when facing $\{s, b\}$, then she must also expect to resist temptation when facing $\{\alpha s + (1 - \alpha)b, b\}$. Hence, whenever her preference for commitment is preserved in a mixture with the tempting alternative, so is her ability to resist temptation.

As a matter of fact, Mixture-Betweenness also requires the lack of self-control to be mixture independent for certain mixtures. For instance, suppose that the dieter thinks she will be overwhelmed by temptation if she goes to the old restaurant that offers salads and burgers (that is, if $\{s, b\} \sim \{b\}$ in (2)). Suppose now that she is also considering the large chain restaurant. Since the menu at the large chain restaurant is a mixture between $\{s, b\}$ and $\{b\}$, Mixture-Betweenness requires that

$$\{\alpha s + (1 - \alpha)b, b\} \sim \{b\}.$$

Thus, if the dieter expects to be overwhelmed by temptation when facing $\{s, b\}$, then she must also expect to be overwhelmed by temptation when facing $\{\alpha s + (1 - \alpha)b, b\}$. Hence, her lack of self-control is mixture independent when mixing with the overwhelming alternative.

We see therefore that the agent's self-control is mixture independent in a limited sense:

$$\text{If } \{s\} \succ \{s, b\}, \text{ then } \{s, b\} \succ \{b\} \text{ if and only if } \{\alpha s + (1 - \alpha)b, b\} \succ \{b\}.$$

This suggests that Mixture-Betweenness imposes a form of mixture independence of self control costs. In contrast, the model of convex self-control costs of Noor and Takeoka (2010) only allows for mixture dependent self-control and imposes mixture independence on the preference for commitment. To be precise, their model can accommodate behavior of the following type:

$$\begin{aligned} \{s\} &\succ \{s, b\} \sim \{b\} \\ \{\alpha s + (1 - \alpha)b, b\} &\succ \{b\}. \end{aligned} \tag{3}$$

The reason is that in their model, the marginal cost of self-control is increasing in the exertion of self-control. Hence, achieving “small” deviations from the tempting alternative is easier than large deviations. In terms of the rankings in (3), their model permits the dieter to believe that she will be able to choose $\alpha s + (1 - \alpha)b$ in the presence of b even if she expects to choose b from $\{s, b\}$ for a small enough α . Similarly, Mixture-Betweenness also rules out behavior related to concave self-control costs:

$$\begin{aligned} \{s\} &\sim \{s, b\} \succ \{b\} \\ \{s\} &\succ \{s, \alpha s + (1 - \alpha)b\}. \end{aligned}$$

3 Results

3.1 Representation Theorem

Say that \succeq is *non-trivial* if there exists $x, y \in \mathcal{X}$ such that $x \succ y$.

The central result of the paper is the following axiomatization of utility over menus.

Theorem 3.1. *A non-trivial preference \succeq satisfies Weak Order, Hausdorff Continuity, Set-Betweenness and Mixture-Betweenness if and only if there exist $u, v : \Delta(X) \times [0, 1] \rightarrow \mathbb{R}$ such that:*

1. $u(\cdot, \gamma)$ and $v(\cdot, \gamma)$ are vNM utility functions for all $\gamma \in [0, 1]$.
2. u is continuous in its second argument on the interval $(0, 1)$.
3. $u(\bar{p}, \gamma) = 1$ and $u(\underline{p}, \gamma) = 0$ for all $\gamma \in [0, 1]$ for some $\bar{p}, \underline{p} \in \Delta(X)$.
4. \succeq can be represented by a continuous utility function $V : \mathcal{X} \rightarrow [0, 1]$ where, for each $x \in \mathcal{X}$, $V(x)$ is the unique $\gamma \in [0, 1]$ that solves

$$\gamma = \max_{p \in x} \{u(p, \gamma) + v(p, \gamma) - \max_{q \in x} v(q, \gamma)\}.$$

One feature of our model that is not shared by any other model in the menus literature is that it does not reduce to expected utility over lotteries, that is, for menus that offer commitment: $x = \{p\}$ for some $p \in \Delta(X)$. Rather, it reduces to the Chew-Dekel model for preference under risk:

$$V(\{p\}) = u(p, V(\{p\})).$$

The fact that the normative preferences are Chew-Dekel is not necessarily shared by the temptation preferences. In particular, v may not define a utility over $\Delta(X)$ that has the Chew-Dekel form. To illustrate, consider the case in which u is independent of γ and $v(\cdot, \gamma) = -\gamma u(\cdot)$. Then, v does not define an implicit utility function over $\Delta(X)$. However, (u, v) do define an explicit utility function over \mathcal{X} :

$$V(x) = \frac{\max_{p \in x} u(p)}{1 + \max_{p \in x} u(p) - \min_{q \in x} u(q)}.$$

In this special case, the self-control cost is not additive, its multiplicative.

Say that (u, v) represents \succeq if it satisfies the conditions of Theorem 3.1. Note that it is not the case that any (u, v) that satisfies conditions 1-3 in Theorem 3.1 implicitly defines a utility function V as in condition 4. In the supplemental appendix (Section S.3) we provide a set of sufficient conditions for (u, v) which guarantees the existence of an implicit utility representation.

3.2 Uniqueness

To state the uniqueness properties of our model we require some additional terminology. Given any pair of functions $f, g : \Delta(X) \rightarrow \mathbb{R}$, we say that f is a *positive affine transformation* of g if there exist $a, b \in \mathbb{R}$ such that $a > 0$ and $f = ag + b$. Similarly, f is a *negative affine transformation* of g if there exist $a, b \in \mathbb{R}$ such that $a < 0$ and $f = ag + b$.

Theorem 3.2. *Let (u, v) and (u', v') be such that $u(\bar{p}, \gamma) = u'(\bar{p}, \gamma) = 1$ and $u(\underline{p}, \gamma) = u'(\underline{p}, \gamma) = 0$ for all $\gamma \in [0, 1]$ for some $\bar{p}, \underline{p} \in \Delta(X)$. Then both (u, v) and (u', v') represent \succeq if and only if for all $\gamma \in (0, 1)$, $u(\cdot, \gamma) = u'(\cdot, \gamma)$ and:*

1. *If $v(\cdot, \gamma)$ is a positive affine transformation of $u(\cdot, \gamma)$ or a constant, then $v'(\cdot, \gamma)$ is a positive affine transformation of $v(\cdot, \gamma)$ or a constant.*
2. *If $v(\cdot, \gamma) = -a_\gamma u(\cdot, \gamma) + b_\gamma$ for some $a_\gamma \geq 1$ and $b_\gamma \in \mathbb{R}$, then $v'(\cdot, \gamma) = a'_\gamma v(\cdot, \gamma) + b'_\gamma$ for some $a'_\gamma \geq \frac{1}{a_\gamma}$ and $b'_\gamma \in \mathbb{R}$.*
3. *If $v(\cdot, \gamma)$ is not a constant or a positive affine transformation of $u(\cdot, \gamma)$ and the condition in 2 does not hold, then $v'(\cdot, \gamma) = v(\cdot, \gamma) + b_\gamma$ for some $b_\gamma \in \mathbb{R}$.*

The uniqueness properties of u are completely characterized by the restriction of \succeq to $\Delta(X)$. In particular, for every $p \in \Delta(X)$, $V(p)$ is the unique $\gamma \in [0, 1]$ that solves

$$\gamma = u(p, \gamma),$$

where $u(\bar{p}, \gamma) = 1$ and $u(\underline{p}, \gamma) = 0$ for all $\gamma \in [0, 1]$. Dekel (1986) shows that such representations are unique.

Because u is completely characterized by the restriction of \succeq to menus that offer commitment, we interpret the utility function implicitly defined by u as the agent's commitment preferences.

To aid intuition for the uniqueness properties of v we describe why the conditions in Proposition 3.2 are sufficient. Assume (u, v) represents \succeq and

fix $\gamma \in (0, 1)$. If $v(\cdot, \gamma)$ is a constant or a positive affine transformation of $u(\cdot, \gamma)$, then for all $x \in \mathcal{X}$,

$$\arg \max_{p \in x} v(p, \gamma) = \arg \max_{p \in x} \{u(p, \gamma) + v(p, \gamma)\},$$

and

$$\max_{p \in x} \{u(p, \gamma) + v(p, \gamma) - \max_{p \in x} v(p, \gamma)\} = \max_{p \in x} u(p, \gamma).$$

Thus, replacing $v(\cdot, \gamma)$ with a constant or one of its positive affine transformations does not affect the representation. If $v(\cdot, \gamma) = -a_\gamma u(\cdot, \gamma) + b_\gamma$ for some $a_\gamma \geq 1$, then for all $x \in \mathcal{X}$,

$$\arg \max_{p \in x} \{v(p, \gamma)\} = \arg \min_{p \in x} \{u(p, \gamma)\}$$

and

$$\arg \min_{p \in x} \{u(p, \gamma)\} \subseteq \arg \max_{p \in x} \{u(p, \gamma) + v(p, \gamma)\}.$$

To see this, note that for any $a_\gamma \geq 1$ and $p, q \in \Delta(X)$ such that $u(p, \gamma) \geq u(q, \gamma)$,

$$\begin{aligned} u(p, \gamma) - u(q, \gamma) &\leq a_\gamma(u(p, \gamma) - u(q, \gamma)) \\ u(p, \gamma) - a_\gamma u(p, \gamma) + b_\gamma &\leq u(q, \gamma) - a_\gamma u(q, \gamma) + b_\gamma \\ u(p, \gamma) + v(p, \gamma) &\leq u(q, \gamma) + v(q, \gamma). \end{aligned}$$

Hence,

$$q \in \arg \min_{p \in x} \{u(p, \gamma)\} \implies q \in \arg \max_{p \in x} \{u(p, \gamma) + v(p, \gamma)\},$$

and

$$\max_{p \in x} \{u(p, \gamma) + v(p, \gamma) - \max_{q \in x} v(q, \gamma)\} = \min_{p \in x} \{u(p, \gamma)\},$$

for all $x \in \mathcal{X}$. Thus, if we replace $v(\cdot, \gamma)$ with $v(\cdot, \gamma) = a'_\gamma v(\cdot, \gamma) + b'_\gamma$ for some $a'_\gamma \geq \frac{1}{a_\gamma}$, then $v'(\cdot, \gamma)$ is also a negative affine transformation of $u(\cdot, \gamma)$ in which the coefficient multiplying $-u(\cdot, \gamma)$ is greater than or equal to one. Thus, the representation is not affected. Finally, note that if $v(\cdot, \gamma)$ is not a constant or a positive affine transformation of $u(\cdot, \gamma)$ and the condition in 2 does not hold, then for any $b_\gamma \in \mathbb{R}$,

$$\begin{aligned} \max_{p \in x} \{u(p, \gamma) + v(p, \gamma) + b_\gamma - \max_{q \in x} \{v(q, \gamma) + b_\gamma\}\} &= \max_{p \in x} \{u(p, \gamma) + v(p, \gamma) \\ &\quad - \max_{q \in x} v(q, \gamma)\}. \end{aligned}$$

Hence, replacing $v(\cdot, \gamma)$ with $v(\cdot, \gamma) + b_\gamma$ does not affect the representation.

3.3 Illustration

We conclude this section by illustrating our model in the context of our motivating example. Details concerning the calculations below can be found in the supplemental appendix (Section S.4). Recall the rankings in our motivating example:

$$\begin{aligned} \{c\} &\sim \{c, f\} \succ \{f\} \\ \{\alpha c + (1 - \alpha)f'\} &\succ \{\alpha c + (1 - \alpha)f', \alpha f + (1 - \alpha)f'\}. \end{aligned}$$

Consider the special case of our model in which the normative utility is linear and independent of the overall level of utility. Let

$$\begin{aligned} u(f') &= 1, & v(f', \gamma) &= 1 \\ u(c) &= \frac{1}{2}, & v(c, \gamma) &= \frac{1 - \gamma}{2} \\ u(f) &= 0, & v(f, \gamma) &= \frac{\gamma}{2} \end{aligned}$$

for all $\gamma \in [0, 1]$. For any lottery $p \in \Delta(\{c, f, f'\})$, we will abuse notation and write $u(p)$ and $v(p, \gamma)$ instead of $\sum_{a \in \{c, f, f'\}} p(a)u(a)$ and $\sum_{a \in \{c, f, f'\}} p(a)v(a, \gamma)$.

This special case of the model can accommodate the intuition behind the rankings in the example. The reason is that for $x = \{c, f\}$ and $y = \{\alpha c + (1 - \alpha)f', \alpha f + (1 - \alpha)f'\}$,

$$\begin{aligned} V(x) &= u(p) \\ V(y) &= u(p') + v(p', V(y)) - v(q, V(y)), \end{aligned}$$

where $p = c$, $p' = \alpha c + (1 - \alpha)f'$ and $q = \alpha f + (1 - \alpha)f'$. Hence, the model predicts that the agent does not expect to feel any temptation if she faces $\{c, f\}$ in the second stage. However, if she faces $\{\alpha c + (1 - \alpha)f', \alpha f + (1 - \alpha)f'\}$, then it suggests that she expects to choose $\alpha c + (1 - \alpha)f'$ and be tempted by $\alpha f + (1 - \alpha)f'$. Further,

$$\begin{aligned} V(\{f'\}) &> V(\{c\}) = V(\{c, f\}) > V(\{f\}) \\ V(\{\alpha c + (1 - \alpha)f'\}) &> V(\{\alpha c + (1 - \alpha)f', \alpha f + (1 - \alpha)f'\}), \end{aligned}$$

where

$$\begin{aligned} V(\{f'\}) &= 1, V(\{c\}) = \frac{1}{2}, V(\{c, f\}) = \frac{1}{2}, V(\{f\}) = 0 \\ V(\{\alpha c + (1 - \alpha)f'\}) &= 1 - \frac{\alpha}{2}, V(\{\alpha c + (1 - \alpha)f', \alpha f + (1 - \alpha)f'\}) = \frac{1}{1 + \alpha}. \end{aligned}$$

Hence, the model accommodates our motivating example.

4 Specialization

To the extent that a systematic study of temptation calls for us to attribute any non-standard behavior to temptation, it is natural to attribute Independence violations to temptation and self-control rather than to normative preferences. Accordingly, we study a specialization of our model that retains linearity of normative preference and attributes non-standard effects of randomization to temptation preferences. Consequently, we consider:

Commitment Independence

$$\{p\} \succeq \{q\} \text{ implies } \alpha\{p\} + (1 - \alpha)\{r\} \succeq \alpha\{q\} + (1 - \alpha)\{r\} \\ \text{for all } \alpha \in [0, 1] \text{ and } r \in \Delta(X).$$

Commitment Independence requires that the agent’s commitment preference satisfy the Independence axiom. Hence, a decision maker who views Independence over lotteries as an appealing normative property will satisfy it. Further, it does not restrict behavior over non-singleton menus and allows for violations of Independence.

Proposition 4.1. *Suppose \succeq satisfies the axioms of Theorem 3.1.*

Then \succeq also satisfies Commitment Independence if and only if it admits a representation as in Theorem 3.1 in which $u(\cdot, \gamma) = u(\cdot, \gamma')$ for all $\gamma, \gamma' \in [0, 1]$.

Apart from being appealing from a modeling perspective, it suggests a non-standard perspective on the Allais Paradox and is motivated by experimental evidence from the time and risk literature. In experiments, subjects often violate the standard vNM Independence axiom when choosing among lotteries that will be realized immediately. However, Weber and Chapman (2005) and Baucells and Heukamp (2010), show that when there is a “distance” between the time in which the choice among lotteries is made and the time in which the lotteries are realized, then they tend to satisfy vNM Independence.

Our model suggests a unified explanation for both facts: non-linear temptation preference drive the violations of Independence when choosing among lotteries that will be realized immediately; linear normative preferences drive the choices when the lotteries will be realized in the future. Another possible explanation of these facts is suggested by Noor and Takeoka (2010, 2015). They conjecture that self-control costs are the driving force behind these types of behavior.

5 Commitment and Mixtures

5.1 General Self-Control Models

In the introduction we claimed that GP cannot accommodate mixture-dependent preference for commitment because of the linearity of the normative and temptation preferences. Here we show that in fact, any temptation model that satisfies Set-Betweenness and, maintains linearity of the normative and temptation preferences cannot accommodate mixture-dependent preference for commitment.

Recall that \succeq has preference for commitment at y if there exists $x \subset y$ such that $x \succ y$. Noor and Takeoka (2010, 2015) show that any temptation model that satisfies Set-Betweenness and maintains linearity of the normative and temptation preference can be written in the following way:

$$V(x) = \max_{p \in x} \{u(p) - c(p, \max_{q \in x} v(q))\},$$

for all menus x , where u and v are vNM utility functions over lotteries and c satisfies the following three properties:

1. c is weakly increasing in its second argument and continuous in both arguments.
2. $c(p, v(q)) > 0$ implies $v(q) > v(p)$.
3. $u(p) > u(q)$ and $v(p) < v(q)$ implies $c(p, v(q)) > 0$.

Intuitively, properties 1, 2 and 3 are the minimal properties c must possess in order to be interpretable as a self-control cost function. In particular, property 1 says that the higher the temptation, (weakly) higher the self-control is needed to resist it. Property 2 requires that if p is costly to choose in the presence of q , then it must be that q offers higher temptation utility. Property 3 provides a converse in that if q provides more temptation utility than p and there is conflict with the normative utility, then the cost of choosing p must be strictly positive.

Noor and Takeoka (2010) refer to this class of models as *general self-control* models, and identify them with the tuple (u, v, c) . As the next proposition shows, such models cannot accommodate mixture-dependent preference for commitment.

Proposition 5.1. *Let (u, v, c) be a general self-control model and \succeq the preference it represents. Then for all $x, y \in \mathcal{X}$:*

1. If \succeq has preference for commitment at x , then \succeq has preference for commitment at $\alpha x + (1 - \alpha)y$.
2. Suppose there is no preference for commitment at y . If \succeq does not have a preference for commitment at x , then \succeq does not have preference for commitment at $\alpha x + (1 - \alpha)y$.

To illustrate why mixture-dependent preference for commitment necessitates a violation of linearity of at least v , consider the following possible rankings involving binary menus:

- (*) $\{p\} \succ \{p, q\} \succeq \{q\}$ and $\{\alpha p + (1 - \alpha)r\} \sim \{\alpha p + (1 - \alpha)r, \alpha q + (1 - \alpha)r\}$
(**) $\{p\} \sim \{p, q\} \succ \{q\}$ and $\{\alpha p + (1 - \alpha)r\} \succ \{\alpha p + (1 - \alpha)r, \alpha q + (1 - \alpha)r\}$.

Let (u, v, c) be a general self-control model and \succeq the preference over menus it represents. Noor and Takeoka (2015) show that

$$\begin{aligned} \{p\} \succ \{p, q\} \succeq \{q\} &\implies v(q) > v(p) \\ \{p'\} \sim \{p', q'\} \succ \{q'\} &\implies v(p') \geq v(q'). \end{aligned}$$

Hence, the rankings in (*) imply

$$v(q) > v(p) \text{ and } v(\alpha p + (1 - \alpha)r) \geq v(\alpha q + (1 - \alpha)r).$$

Thus, by linearity of v , $v(q) > v(p)$ and $v(q) \leq v(p)$, an impossibility. Similarly, the rankings in (**) imply

$$v(p) \geq v(q) \text{ and } v(\alpha q + (1 - \alpha)r) \geq v(\alpha p + (1 - \alpha)r).$$

Hence, by linearity of v , $v(q) \geq v(p)$ and $v(q) < v(p)$, another impossibility.

5.2 Mixture Monotone Preference for Commitment

Here we specialize the model by focusing on the preference for commitment. In particular, we characterize two monotone patterns it can take under Commitment Independence and an additional assumption. In the supplementary appendix (Section S.5) we provide the corresponding results for the general model.

Mixture-Increasing Preference for Commitment

For all p and x such that \succeq has preference for commitment at $\alpha x + (1 - \alpha)\{p\}$ for some $\alpha \in [0, 1]$:

- I If $\{p\} \succ x$, then \succeq has preference for commitment at $\beta x + (1 - \beta)\{p\}$ for all $0 < \beta < \alpha$.
- II If $x \succ \{p\}$, then \succeq has preference for commitment at $\beta x + (1 - \beta)\{p\}$ for all $\beta > \alpha$.

Mixture-Increasing Preference for commitment restricts how preference for commitment can vary along mixtures. Indeed, Part I implies that if \succeq has preference for commitment at x , then \succeq has preference for commitment at any mixture between x and a superior menu that offers commitment $\{p\}$. On the other hand, Part II requires that if the agent has preference for commitment at a mixture between a menu that offers commitment $\{p\}$ and a superior menu x , then increasing the weight on the superior menu does not affect preference for commitment.

The special case of our model described in Section 3.3 satisfies this axiom (see the online appendix for a proof). Hence, it is compatible with our motivating example. Recall that in the example, the agent had preference for commitment at a mixture between $\{c, f\}$ and the superior menu $\{f'\}$ but not at $\{c, f\}$:

$$\begin{aligned} \{\alpha c + (1 - \alpha)f'\} &\succ \{\alpha c + (1 - \alpha)f', \alpha f + (1 - \alpha)f'\} \\ \{c\} &\sim \{c, f\}. \end{aligned}$$

Thus, the axiom only requires that the agent has preference for commitment at $\{\beta c + (1 - \beta)f', \beta f + (1 - \beta)f'\}$ for all $0 < \beta < \alpha$ which is consistent with the behavior prescribed by the model in Section 3.3.

To state the next result, we require some additional notation. Let p^*, p_* denote a fixed pair of lotteries such that $\{p^*\} \succeq x \succeq \{p_*\}$ for all $x \in \mathcal{X}$. Given our axioms such lotteries always exist.⁶

Theorem 5.1. *Assume \succeq satisfies the axioms of Proposition 4.1 and $\{p^*\} \sim \{p^*, p_*\}$. Then \succeq satisfies Mixture-Increasing Preference for Commitment if and only if there exists (u, v) that represents \succeq such that $0 < \gamma' < \gamma < 1$ implies that there exists $b_{\gamma, \gamma'} \in \mathbb{R}$ such that $v(\cdot, \gamma') + b_{\gamma, \gamma'}$ is a convex combination of u and $v(\cdot, \gamma)$.*

The assumption that $\{p^*\} \sim \{p^*, p_*\}$ implies that the agent does not expect to be tempted by p_* if she faces $\{p^*, p_*\}$ in the second stage. This assumption guarantees that for each $\gamma \in [0, 1]$, $v(\cdot, \gamma)$ is not a negative affine transformation

⁶Weak Order and Continuity imply that the ordering over singleton menus has a best and worst menu. Thus, by Set-Betweenness, all finite menus are between these two menus in terms of preference. Hence, by Continuity the same holds for any menu.

of u . Theorem 5.1 shows that Mixture-Increasing Preference for Commitment forces the following concrete relationship between u and v across different levels of utility: as utility decreases, u and $v(\cdot, \gamma)$ get “closer together”.

The following axiom characterizes the opposite case: as utility decreases, u and $v(\cdot, \gamma)$ move “further apart”.

Mixture-Decreasing Preference for Commitment

For all p and x such that \succeq has preference for commitment at $\alpha x + (1 - \alpha)\{p\}$ for some $\alpha \in [0, 1]$:

- I If $x \succ \{p\}$, then \succeq has preference for commitment at $\beta x + (1 - \beta)\{p\}$ for all $0 < \beta < \alpha$
- II If $\{p\} \succ x$, then \succeq has preference for commitment at $\beta x + (1 - \beta)\{p\}$ for all $\beta > \alpha$.

An agent modeled after this axiom would have mixture independent preference for commitment when mixing with inferior menus that offer commitment. Its interpretation is analogous to the interpretation of the previous axiom.

Part II of this axiom is inconsistent with the rankings in our motivating example. In particular, an agent that satisfies II that has preference for commitment at $\{\alpha c + (1 - \alpha)f', \alpha f + (1 - \alpha)f'\}$ would have preference for commitment at $\{c, f\}$. However, it can be motivated by similar examples. To illustrate, consider again the traveler but assume now that she wishes to stay at the fancy room but thinks that if she waits to choose the room until she arrives at the hotel, then she will be tempted to choose the conventional room to save money. Hence,

$$\{f\} \succ \{c, f\}.$$

Assume she receives the same promotion as in the motivating example: her name will be added to a raffle in which the prize is a free night in the fancy room. The possibility of staying for free in the fancy room makes her dream about it for the rest of the day. Hence, if she waits until she arrives to choose the room, by then she will not find the cheap room tempting. Hence,

$$\{\alpha f + (1 - \alpha)f'\} \sim \{\alpha c + (1 - \alpha)f', \alpha f + (1 - \alpha)f'\}.$$

This type of behavior is permitted by Mixture-Decreasing Preference for Commitment. The reason is that this axiom is silent about behavior that involves a mixture between a menu in which the agent has preference for commitment ($\{c, f\}$) and a superior menu that offers commitment ($\{f'\}$). However,

it is inconsistent with Part I of Mixture-Increasing Preference for Commitment. In particular, an agent that satisfies Part I of that axiom that has preference for commitment at $\{c, f\}$ would have preference for commitment at $\{\alpha c + (1 - \alpha)f', \alpha f + (1 - \alpha)f'\}$.

Theorem 5.2. *Assume \succeq satisfies the axioms of Proposition 4.1 and $\{p^*\} \sim \{p^*, p_*\}$. Then \succeq satisfies Mixture-Decreasing Preference for Commitment if and only if there exists (u, v) that represents \succeq such that $0 < \gamma' < \gamma < 1$ implies that there exists $b_{\gamma, \gamma'} \in \mathbb{R}$ such that $v(\cdot, \gamma) + b_{\gamma, \gamma'}$ is a convex combination of u and $v(\cdot, \gamma')$.*

5.3 Consumption-Savings

While our model is a representation for an ex ante preference over menus, it suggests that ex post choice is given by the choice correspondence defined by

$$c(x) = \arg \max_{p \in x} \{u(p, V(x)) + v(p, V(x))\}.$$

We conclude the discussion of mixture-dependent preference for commitment by providing some observations this choice correspondence in a consumption-savings context. In particular, we illustrate how our model can accommodate the behavior of an agent that is tempted to over-consume and under-save only when her income is stochastic. This prediction of the model seems particularly relevant to policymakers as it suggests that government programs that reduce income risk may have an unintended positive effect: they alleviate the agent's demand for commitment by helping them make normatively optimal choices.

Consider an agent that lives throughout 3 periods ($t = 0, 1, 2$). There is a single consumption good in period 1 and 2. We assume that the risk-free rate and other rates of return are exogenous, and that the agent cannot borrow against his second period income. Hence, it is WLOG to treat the risk-free rate as if it were zero. Finally, we assume that first period income is deterministic and equal to w_1 and that the second-period income y is allowed to be stochastic.

Consider the decision problem at $t = 1$ of an agent who exhibits mixture-dependent preference for commitment and has linear normative preferences:

$$\max_{c \leq w_1} \{\mathbb{E}[U_1(c) + U_2(w_1 - c + y) + V(c, w_1 - c + y, \gamma)]\}$$

where U_1 is the first period normative utility function, c the first period consumption, \mathbb{E} an expectation conditional on first-period information, U_2 the

second-period utility function, V is the temptation utility and γ is the unique solution of

$$\begin{aligned} \gamma = & \max_{c \leq w_1} \{ \mathbb{E}[U_1(c) + U_2(w_1 - c + y) + V(c, w_1 - c + y, \gamma)] \\ & - \max_{c' \leq w_1} V(c', w_1 - c + y, \gamma) \}. \end{aligned}$$

If V is independent of γ , then the model reduces to GP. Hence, temptation would affect the agent in the same way regardless of whether y is risky or not. To be precise, the agent would be tempted to over-consume (under-consume) and under-save (over-save) regardless of whether y is risky or not. This is no longer true under mixture-dependent preference for commitment. For instance, if V has the following form:

$$V(c_1, c_2, \gamma) = \begin{cases} U(c_1) + U(c_2) & \gamma \leq U(c_1) + U(c_2) \\ \frac{\gamma}{U(c_1) + U(c_2)} U(c_1) + U(c_2) & \gamma > U(c_1) + U(c_2) \end{cases}$$

the agent will only be tempted to over-consume *only* when her second period income is risky. Hence, she would only demand commitment when her second period income is stochastic.

6 Betweenness Mixture Space Theorem

In the introduction we discussed that one of the contributions of the paper is to provide a novel extension of the main result of Herstein and Milnor (1953), commonly known as the Mixture Space Theorem. Here we provide our extension.

A mixture space is a non-empty set \mathcal{M} which is endowed with an operation π ,

$$\begin{aligned} \pi : [0, 1] \times \mathcal{M} \times \mathcal{M} &\rightarrow \mathcal{M} \\ (a, x, y) &\rightarrow \pi_a(x, y), \end{aligned}$$

where π satisfies the following three properties:

- (A1) $\pi_1(x, y) = x$.
- (A2) $\pi_a(x, y) = \pi_{1-a}(y, x)$ for all $a \in [0, 1]$.
- (A3) $\pi_a(\pi_b(x, y), y) = \pi_{ab}(x, y)$ for all $a, b \in [0, 1]$.

Basically, a mixture space is an abstract version of a convex set.

Let \succeq be a binary relation over a mixture space (\mathcal{M}, π) . Consider the following axioms:

Weak Order \succeq is complete and transitive.

Continuity $x \succ y \succ z$ implies that there exist $a, b \in (0, 1)$ such that $\pi_a(x, z) \succ y \succ \pi_b(x, z)$.

Independence $x \succeq y$ implies $\pi_a(x, z) \succeq \pi_a(y, z)$ for all $z \in \mathcal{M}$ and $a \in [0, 1]$.

Theorem 6.1. (*Mixture Space Theorem*) *Let \succeq be a binary relation on a mixture space (\mathcal{M}, π) . Then the following two statements are equivalent:*

- a) \succeq satisfies Weak Order, Continuity and Independence.
- b) There exists a utility function $\Phi : \mathcal{M} \rightarrow \mathbb{R}$ that represents \succeq such that

$$\forall x, y \in \mathcal{M} \text{ and } a \in [0, 1], \quad \Phi(\pi_a(x, y)) = a\Phi(x) + (1 - a)\Phi(y). \quad (4)$$

Further, Φ is unique up to a positive affine transformation.

Say that a function $\Phi : \mathcal{M} \rightarrow \mathbb{R}$ is *mixture-linear* if it satisfies (4). The Mixture Space Theorem provides necessary and sufficient conditions for a preference to have a mixture-linear utility representation. Our theorem characterizes a representation where Φ instead conforms to an analogue of the Chew-Dekel utility representation. However, it is less general than the Mixture Space Theorem in the sense that it only applies to mixture spaces that satisfy the following additional property:

(A4) $\pi_a(\pi_b(x, y), z) = \pi_{ab}(x, \pi_{\frac{a(1-b)}{1-ab}}(y, z)) \quad \forall a, b \in [0, 1]$ such that $ab \neq 1$.

A4 requires that the order of the mixture does not matter. Hence, it is an associative property. Thus we refer to any mixture space that satisfies A4 as an associative mixture space. Not all mixture spaces are associative (see Mongin (2001) for an example). However, any space that is isomorphic to a convex subset of a linear space is an associative mixture space (examples include the frameworks employed in Anscombe and Aumann (1963), Dekel et al. (2001) and the probability simplex). Not all associative mixture spaces have this feature. In particular, Stone (1949) and Mongin (2001) show that the missing ingredient is the following property.

(A5) For any $a \in (0, 1)$ and $x \in \mathcal{M}$, $y \mapsto \pi_a(x, y)$ is injective.

Theorem 6.2. (Stone-Mongin) Let (\mathcal{M}, π) be an associative mixture space. Then the following two statements are equivalent:

- a) (\mathcal{M}, π) satisfies A5.
- b) (\mathcal{M}, π) is isomorphic to a convex subset of a linear space.

Thus, even though our result is not as general as the Mixture Space Theorem, it applies to settings that are more general than convex subsets of linear spaces.

In our version of the Mixture Space Theorem, we replace the Independence axiom with the following two axioms.

Mixture-Betweenness

- $x \succ y$ implies $x \succ \pi_a(x, y) \succ y$ for all $a \in (0, 1)$, and
- $x \sim y$ implies $x \sim \pi_a(x, y) \sim y$ for all $a \in (0, 1)$.

Strict Best and Worst There exist \bar{x}, \underline{x} such that $\bar{x} \succ x \succ \underline{x}$ for all $x \in \mathcal{M} \setminus \{\bar{x}, \underline{x}\}$.

To state our result we require some additional terminology. Say that a function $V : \mathcal{M} \rightarrow \mathbb{R}$ is *mixture-continuous* if for all $x, y \in \mathcal{M}$, $V(\pi_a(x, y))$ is continuous as a function of a .

Theorem 6.3. Let \succeq be a non-trivial binary relation on an associative mixture space (\mathcal{M}, π) . Then the following two statements are equivalent:

- a) \succeq satisfies Weak Order, Continuity, Mixture-Betweenness and Strict Best and Worst.
- b) There exists $\Phi : \mathcal{M} \times (0, 1) \rightarrow \mathbb{R}$ such that
 - 1.- Φ is continuous in its second argument on the interval $(0, 1)$.
 - 2.- Φ is mixture linear in its first argument for all $\gamma \in (0, 1)$.
 - 3.- $\Phi(\bar{x}, \gamma) = 1, \Phi(\underline{x}, \gamma) = 0$ for all $\gamma \in (0, 1)$.
 - 4.- $\Phi(x, \gamma) = \gamma$ has a unique solution for all $x \in \mathcal{M} \setminus \{\bar{x}, \underline{x}\}$.
 - 5.- \succeq can be represented by a mixture-continuous function $V : \mathcal{M} \rightarrow [0, 1]$ such that

$$V(x) = \begin{cases} 1 & x = \bar{x}, \\ \Phi(x, V(x)) & x \in \mathcal{M} \setminus \{\bar{x}, \underline{x}\}, \\ 0 & x = \underline{x}. \end{cases}$$

Further, $\Phi(\cdot, \gamma)$ is unique for all $\gamma \in (0, 1)$.

Mixture-Betweenness is obviously weaker than Independence. However, Strict Best and Worst is unrelated to Independence and limits the generality of our theorem. In the supplemental appendix (Section S.1) we show that it can be deleted if the associative mixture space is a compact and convex subset of a linear space and Continuity is replaced with the following axiom.

Strong Continuity The sets $\{y|y \succeq x\}$ and $\{y|x \succeq y\}$ are closed for all $x \in \mathcal{M}$.

The proof of Theorem 6.3 is similar *in spirit* to the proofs in Chew (1989), Dekel (1986) and Conlon (1995). However, they exploit properties of the probability simplex that have no evident analog in our framework. More specifically, Chew (1989) and Dekel (1986) use its geometric properties and Conlon (1995) uses its topological properties. Hence, to prove the theorem we were forced to develop novel arguments.

Due to its generality, Theorem 6.3 has several potential applications. In particular, it can be used to derive a natural counterpart of Dekel (1986) for the Anscombe-Aumann domain (Anscombe and Aumann (1963)). In this setting with uncertainty as opposed to risk, the result is a non-standard model of beliefs. More specifically, let Ω be a finite state space. Consider a preference \succeq over $\mathcal{F} = \{f : \Omega \rightarrow \Delta(X)\}$, set of all AA acts. In ongoing work, we use Theorem 6.3 as a stepping stone in the axiomatization of a representation for \succeq in which the utility of an AA act f is the unique γ that solves

$$\gamma = \int_{\Omega} u(f(\omega))\mu(d\omega, \gamma), \quad (5)$$

where u is a vNM utility function over lotteries and $\mu(\cdot, \gamma)$ is a probability measure over Ω for all γ .

In a separate project, we use our result to generalize Dekel et al. (2001) and therefore also Kreps (1979). We derive a counterpart of (5) in which the state space is subjective. In particular, we axiomatize a utility representation for preferences over menus of lotteries in which the utility of a menu x is the unique γ that solves

$$\gamma = \int_S \sup_{p \in x} u(p, s)\mu(ds, \gamma),$$

where S is a non-empty set (subjective state space), $u(\cdot, s)$ is a vNM utility function over lotteries for all $s \in S$ and $\mu(\cdot, \gamma)$ is a probability measure over S for all γ .

Appendix A Mixture Space Theorem

A.1 Sufficiency

Lemma A.1. *Let \succeq be a binary relation over a Mixture Space (\mathcal{M}, π) that satisfies Weak Order, Continuity and Mixture-Betweenness. Then:*

1. $x \succ y$ and $0 \leq a < b \leq 1$ implies $\pi_b(x, y) \succ \pi_a(x, y)$.

2. For all x, y and z the following sets are closed:

$$\{a | \pi_a(x, y) \succeq z\} \text{ and } \{a | z \succeq \pi_a(x, y)\}.$$

3. If $x \succ z \succ y$, then there exists a unique $a \in (0, 1)$ such that $z \sim \pi_a(x, y)$.

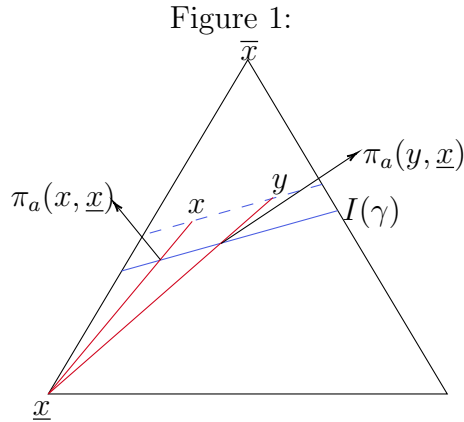
The proof of this lemma is provided in Appendix A.4. The argument for Part 2 is an adaptation to the one of Karni and Safra (2015).

Lemma A.1 shows that for each $x \in \mathcal{M}$ there exists a unique $\gamma(x) \in [0, 1]$ such that $x \sim \pi_{\gamma(x)}(\bar{x}, \underline{x})$. Define

$$V(x) \equiv \gamma(x).$$

Then, V is mixture-continuous and represents \succeq . Now we proceed with the construction of Φ .

For each $\gamma \in (0, 1)$, let x_γ denote $\pi_\gamma(\bar{x}, \underline{x})$ and $I(\gamma) = \{x | x \sim x_\gamma\}$. Our goal is to construct a mixture-linear $\Phi(\cdot, \gamma)$ such that it represents an artificial preference that has indifference curves that are “parallel” to $I(\gamma)$. Informally, x, y are in a “higher” artificial indifference curve parallel to $I(\gamma)$ if $\pi_a(x, \underline{x}) \sim x_\gamma$ and $\pi_a(y, \underline{x}) \sim x_\gamma$. Figure 1 illustrates this idea.



Similarly, x, y are in a “lower” artificial indifference curve parallel to $I(\gamma)$ if $\pi_a(x, \bar{x}) \sim x_\gamma$ and $\pi_a(y, \bar{x}) \sim x_\gamma$.

Consider the mapping $\lambda : \mathcal{M} \times (0, 1) \rightarrow [0, 1]$ given by

$$\lambda(x, \gamma) = \begin{cases} a|\pi_a(x, \underline{x}) \sim x_\gamma & V(x) > \gamma \\ 1 & V(x) = \gamma \\ b|\pi_b(x, \bar{x}) \sim x_\gamma & V(x) < \gamma. \end{cases}$$

By Lemma A.1, λ is well defined. λ can be used to define an artificial preference that has indifference curves parallel to $I(\gamma)$: if either $x, y \succ x_\gamma$ or $x_\gamma \succ x, y$, then x, y are in the same artificial indifference curve if and only if $\lambda(x, \gamma) = \lambda(y, \gamma)$. $\Phi(\cdot, \gamma)$ would represent such artificial preference. Hence, mixture linearity of $\Phi(\cdot, \gamma)$ and the definition of λ would imply that if $x \succ x_\gamma$,

$$\begin{aligned} \Phi(\pi_{\lambda(x, \gamma)}(x, \underline{x}), \gamma) &= \Phi(x_\gamma, \gamma) \\ \lambda(x, \gamma)\Phi(x, \gamma) &= \gamma \\ \frac{\gamma}{\lambda(x, \gamma)} &= \Phi(x, \gamma). \end{aligned}$$

Similarly, if $x_\gamma \succ x$, then

$$\Phi(x, \gamma) = 1 - \frac{1 - \gamma}{\lambda(x, \gamma)}.$$

Hence, $\Phi(\cdot, \gamma)$ must have the following form:

$$\Phi(x, \gamma) \equiv \begin{cases} \frac{\gamma}{\lambda(x, \gamma)} & V(x) > \gamma \\ \gamma & V(x) = \gamma \\ 1 - \frac{1 - \gamma}{\lambda(x, \gamma)} & V(x) < \gamma. \end{cases}$$

Notice that $\Phi(x, \gamma) = \gamma$ if and only if $V(x) = \gamma$. Further,

$$\begin{aligned} V(x), V(y) \geq \gamma &\implies \Phi(x, \gamma) \geq \Phi(y, \gamma) \iff \lambda(x, \gamma) \leq \lambda(y, \gamma) \\ V(x) \geq \gamma \geq V(y) &\implies \Phi(x, \gamma) \geq \Phi(y, \gamma) \\ V(x), V(y) \leq \gamma &\implies \Phi(x, \gamma) \geq \Phi(y, \gamma) \iff \lambda(x, \gamma) \geq \lambda(y, \gamma). \end{aligned}$$

We will show that $\Phi(\cdot, \gamma)$ is mixture linear and continuous in its second argument. In our proof we employ the following lemma that describes several properties of λ .

Lemma A.2. *If \succeq satisfies the axioms of Theorem 6.3, then λ satisfies the following properties:*

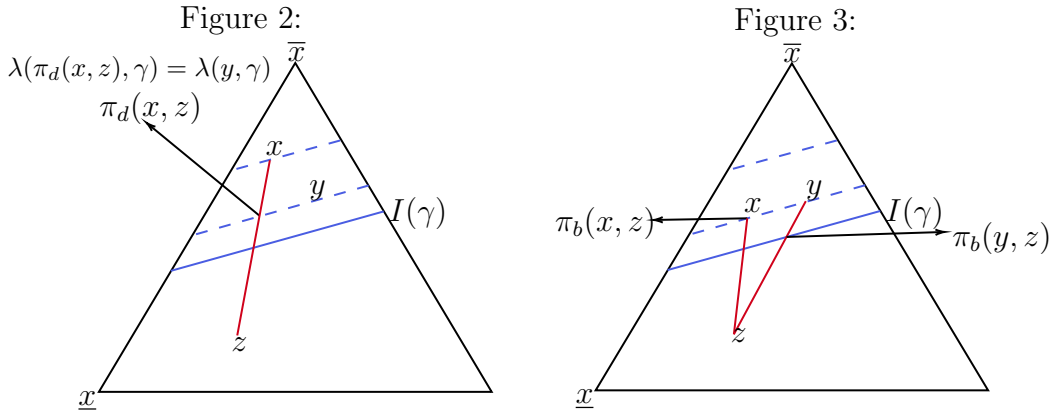
1. $V(x), V(y) \geq \gamma \implies \lambda(\pi_a(x, y), \gamma) = \frac{\lambda(x, \gamma)\lambda(y, \gamma)}{a\lambda(y, \gamma) + (1-a)\lambda(x, \gamma)}$ for all $a \in [0, 1]$.
2. $V(x), V(y) \leq \gamma \implies \lambda(\pi_a(x, y), \gamma) = \frac{\lambda(x, \gamma)\lambda(y, \gamma)}{a\lambda(y, \gamma) + (1-a)\lambda(x, \gamma)}$ for all $a \in [0, 1]$.
3. $V(x), V(y) > \gamma > V(z)$ and $\lambda(x, \gamma) < \lambda(y, \gamma)$
 $\implies \lambda(\pi_a(x, z), \gamma) = \lambda(y, \gamma)$ for a unique $a \in (0, 1)$.
4. $V(x), V(y) < \gamma < V(z)$ and $\lambda(x, \gamma) < \lambda(y, \gamma)$
 $\implies \lambda(\pi_a(x, z), \gamma) = \lambda(y, \gamma)$ for a unique $a \in (0, 1)$.
5. $V(x), V(y) > \gamma > V(z)$, $\lambda(x, \gamma) = \lambda(y, \gamma)$ and $\pi_a(x, z) \sim x_\gamma$
 $\implies \pi_a(y, z) \sim x_\gamma$.
6. $V(x), V(y) < \gamma < V(z)$, $\lambda(x, \gamma) = \lambda(y, \gamma)$ and $\pi_a(x, z) \sim x_\gamma$
 $\implies \pi_a(y, z) \sim x_\gamma$.

The proof is provided in Appendix A.4. Here we provide two figures that illustrate properties 3 (Figure 2) and 5 (Figure 3).

Notice that properties 1 and 2 imply that if x, y are both above or both below $I(\gamma)$ in terms of preferences, then

$$\lambda(\pi_a(x, y), \gamma) = (a\lambda(x, \gamma)^{-1} + (1-a)\lambda(y, \gamma)^{-1})^{-1}.$$

Hence, $\lambda(\pi_a(x, y), \gamma)$ is the harmonic weighted mean of $\lambda(x, \gamma)$ and $\lambda(y, \gamma)$.



$\Phi(\cdot, \gamma)$ is Mixture Linear

Fix $x, y \in \mathcal{X}, \gamma \in (0, 1)$ and $a \in (0, 1)$. Assume WLOG that $V(x) \geq V(y)$. Then, by Mixture-Betweenness, $V(x) \geq V(\pi_a(x, y)) \geq V(y)$.

There are four possible cases to consider:

(i) $V(x) \geq V(y) \geq \gamma$. (ii) $\gamma \geq V(x) \geq V(y)$.

(iii) $V(x) \geq V(\pi_a(x, y)) \geq \gamma \geq V(y)$. (iv) $V(x) \geq \gamma \geq V(\pi_a(x, y)) \geq V(y)$.

Since the proofs of (i) and (ii) are analogous, and the proofs of (iii) and (iv) are analogous we only consider (i) (Lemma A.3) and (iii) (Lemma A.4).

Lemma A.3. $V(x), V(y) \geq \gamma \implies \Phi(\pi_a(x, y), \gamma) = a\Phi(x, \gamma) + (1-a)\Phi(y, \gamma)$.

Proof. First note that if $V(x) = V(y) = \gamma$, then there is nothing to prove. Assume $V(x) > V(y) \geq \gamma$. Then, by Mixture-Betweenness, $V(\pi_a(x, y)) > \gamma$. Thus,

$$\begin{aligned} \Phi(\pi_a(x, y), \gamma) &= \frac{\gamma}{\lambda(\pi_a(x, y), \gamma)} \\ &= \frac{\gamma}{\frac{\lambda(x, \gamma)\lambda(y, \gamma)}{a\lambda(y, \gamma) + (1-a)\lambda(x, \gamma)}} \\ &= \frac{\gamma}{\lambda(x, \gamma)\lambda(y, \gamma)} (a\lambda(y, \gamma) + (1-a)\lambda(x, \gamma)) \\ &= a\frac{\gamma}{\lambda(x, \gamma)} + (1-a)\frac{\gamma}{\lambda(y, \gamma)} \\ &= a\Phi(x, \gamma) + (1-a)\Phi(y, \gamma). \end{aligned}$$

where the second equality follows from Part 1 of Lemma A.2. □

Lemma A.4. $V(x) > \gamma > V(y)$ and $V(\pi_a(x, y)) \geq \gamma \implies \Phi(\pi_a(x, y), \gamma) = a\Phi(x, \gamma) + (1-a)\Phi(y, \gamma)$.

Proof. Let $b \in (0, 1)$ be such that

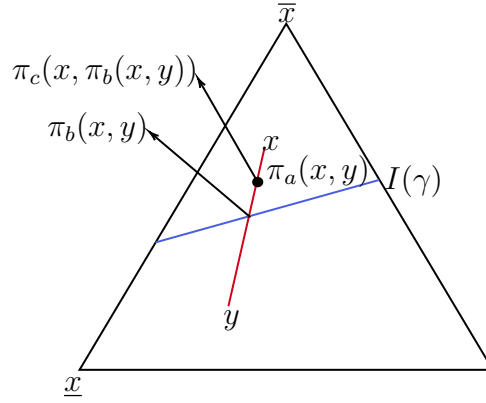
$$\pi_b(x, y) \sim x_\gamma.$$

The proof of this lemma is done in two steps.

Step 1: Calculate $\Phi(\pi_a(x, y), \gamma)$

Figure 4 illustrates the argument for this step.

Figure 4:



$\pi_a(x, y) \succeq x_\gamma$ implies $a \geq b$. Let $c = \frac{a-b}{1-b} \leq 1$. We claim that $\pi_a(x, y) = \pi_c(x, \pi_b(x, y))$. To prove this claim note that

$$\begin{aligned} \pi_c(x, \pi_b(x, y)) &= \pi_{1-c}(\pi_{1-b}(y, x), x) \\ &= \pi_{(1-c)(1-b)}(y, x) \\ &= \pi_{1-(1-c)(1-b)}(x, y). \end{aligned}$$

Further,

$$\begin{aligned} 1 - (1-c)(1-b) &= 1 - \left(1 - \frac{a-b}{1-b}\right)(1-b) \\ &= 1 - (1-b - (a-b)) \\ &= a. \end{aligned}$$

Hence, $\pi_a(x, y) = \pi_c(x, \pi_b(x, y))$. Since $V(x), V(\pi_b(x, y)) \geq \gamma$, by Part 1 of Lemma A.2,

$$\begin{aligned} \lambda(\pi_c(x, \pi_b(x, y)), \gamma) &= \frac{\lambda(x, \gamma)}{c + (1-c)\lambda(x, \gamma)} \\ &= \frac{\lambda(x, \gamma)}{\frac{a-b}{1-b} + \left(1 - \frac{a-b}{1-b}\right)\lambda(x, \gamma)} \\ &= (1-b) \frac{\lambda(x, \gamma)}{(a-b) + (1-a)\lambda(x, \gamma)}. \end{aligned}$$

Therefore,

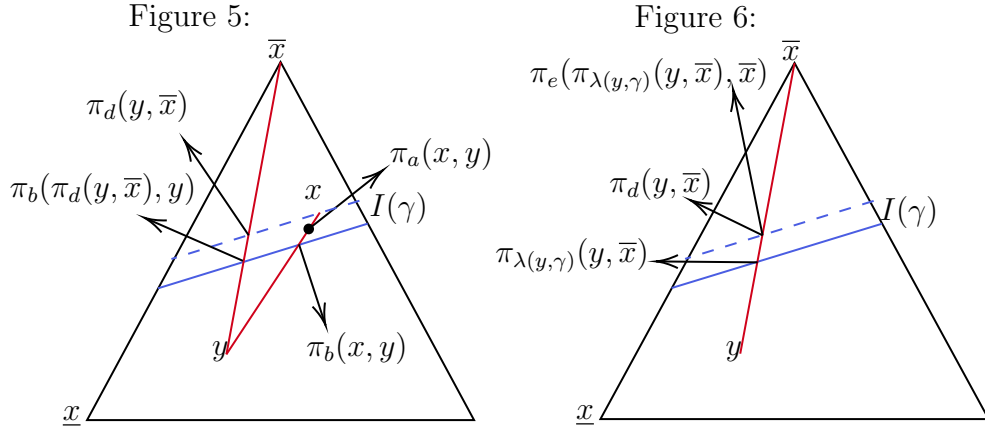
$$\begin{aligned}
\Phi(\pi_a(x, y), \gamma) &= \Phi(\pi_c(x, \pi_b(x, y)), \gamma) \\
&= \gamma \frac{1}{\lambda(\pi_c(x, \pi_b(x, y)), \gamma)} \\
&= \gamma \frac{1}{(1-b) \frac{\lambda(x, \gamma)}{(a-b) + (1-a)\lambda(x, \gamma)}} \\
&= \gamma \frac{a-b + (1-a)\lambda(x, \gamma)}{(1-b)\lambda(x, \gamma)}.
\end{aligned}$$

Step 2: Calculate $a\Phi(x, \gamma) + (1-a)\Phi(y, \gamma)$ and show it is equal to $\Phi(\pi_a(x, y), \gamma)$

To calculate $a\Phi(x, \gamma) + (1-a)\Phi(y, \gamma)$ we first need to derive the relation between $\lambda(y, \gamma)$, $\lambda(x, \gamma)$, γ and b . In particular, we need to show that

$$\lambda(y, \gamma) = \frac{(1-b)\lambda(x, \gamma)(\gamma-1)}{\lambda(x, \gamma)(\gamma-1) - b(\gamma - \lambda(x, \gamma))}. \quad (6)$$

To prove (6) we need to distinguish between two cases: (i) $\lambda(x, \gamma) > \lambda(\bar{x}, \gamma)$ and (ii) $\lambda(x, \gamma) \leq \lambda(\bar{x}, \gamma)$. The proof of both cases is analogous. Thus, we only consider (i).



Assume $\lambda(x, \gamma) > \lambda(\bar{x}, \gamma)$. By Part 3 of Lemma A.2 there exists a unique $d \in (0, 1)$ such that $\lambda(\pi_d(y, \bar{x}), \gamma) = \lambda(x, \gamma)$. First we show that $\lambda(y, \gamma) = 1 - b(1-d)$. Figure 5 illustrates the argument we use to prove this.

By Part 5 of Lemma A.2, $\pi_b(\pi_d(y, \bar{x}), y) \sim x_\gamma$. Further,

$$\begin{aligned}\pi_b(\pi_d(y, \bar{x}), y) &= \pi_b(\pi_{1-d}(\bar{x}, y), y) \\ &= \pi_{b(1-d)}(\bar{x}, y) \\ &= \pi_{1-b(1-d)}(y, \bar{x}).\end{aligned}$$

Hence $\pi_{\lambda(y, \gamma)}(y, \bar{x}) = \pi_b(\pi_d(y, \bar{x}), y)$ and $\lambda(y, \gamma) = 1 - b(1 - d)$.

Next, we need to replace d in $\lambda(y, \gamma) = 1 - b(1 - d)$ with $e\lambda(y, \gamma)$ where $e = \frac{\gamma - \lambda(x, \gamma)}{\lambda(x, \gamma)(\gamma - 1)}$. Figure 6 illustrates how we do this.

First we show that $e \in (0, 1)$ and $\pi_d(y, \bar{x}) = \pi_e(\pi_{\lambda(y, \gamma)}(y, \bar{x}), \bar{x})$. To prove this, first note that $\gamma = \lambda(\bar{x}, \gamma)$ so $\lambda(x, \gamma) > \lambda(\bar{x}, \gamma)$ and $\gamma \in (0, 1)$ imply $e > 0$. Further, $\lambda(x, \gamma) \in (0, 1)$. Thus,

$$\begin{aligned}\gamma &> \lambda(x, \gamma)\gamma \\ \gamma - \lambda(x, \gamma) &> \lambda(x, \gamma)\gamma - \lambda(x, \gamma) \\ \frac{\gamma - \lambda(x, \gamma)}{\lambda(x, \gamma)(\gamma - 1)} &< 1.\end{aligned}$$

Next, note that

$$\pi_e(\pi_{\lambda(y, \gamma)}(y, \bar{x}), \bar{x}) = \pi_{e\lambda(y, \gamma)}(y, \bar{x}).$$

Thus, by Part 1 of Lemma A.2,

$$\lambda(\pi_e(\pi_{\lambda(y, \gamma)}(y, \bar{x}), \bar{x}), \gamma) = \frac{\lambda(\bar{x})}{e\lambda(\bar{x}, \gamma) + (1 - e)}.$$

Notice that if $e = \frac{\gamma - \lambda(x, \gamma)}{\lambda(x, \gamma)(\gamma - 1)}$, then

$$\begin{aligned}\pi_e(\pi_{\lambda(y, \gamma)}(y, \bar{x}), \bar{x}) &= \frac{\lambda(\bar{x}, \gamma)}{\left(\frac{\gamma - \lambda(x, \gamma)}{\lambda(x, \gamma)(\gamma - 1)}\right)\lambda(\bar{x}, \gamma) + \frac{\gamma(\lambda(x, \gamma) - 1)}{\lambda(x, \gamma)(\gamma - 1)}} \\ &= \frac{\gamma\lambda(x, \gamma)(\gamma - 1)}{\gamma(\gamma - \lambda(x, \gamma)) + \gamma(\lambda(x, \gamma) - 1)} \\ &= \frac{\lambda(x, \gamma)(\gamma - 1)}{(\gamma - \lambda(x, \gamma)) + (\lambda(x, \gamma) - 1)} \\ &= \lambda(x, \gamma).\end{aligned}$$

Hence by the uniqueness in Part 3 of Lemma A.2, $d = e\lambda(y, \gamma)$. Thus,

$$\begin{aligned}\lambda(y, \gamma) &= 1 - b(1 - d) \\ &= 1 - b(1 - e\lambda(y, \gamma)) \\ &= \frac{1 - b}{1 - be} \\ &= \frac{(1 - b)\lambda(x, \gamma)(\gamma - 1)}{\lambda(x, \gamma)(\gamma - 1) - b(\gamma - \lambda(x, \gamma))}.\end{aligned}$$

Finally, we finish the proof of this step by using (6) to show $a\Phi(x, \gamma) + (1 - a)\Phi(y, \gamma) = \Phi(\pi_a(x, y))$.

$$\begin{aligned}a\Phi(x, \gamma) + (1 - a)\Phi(y, \gamma) &= a\frac{\gamma}{\lambda(x, \gamma)} + (1 - a)\left(1 - \frac{1 - \gamma}{\lambda(y, \gamma)}\right) \\ &= a\frac{\gamma}{\lambda(x, \gamma)} \\ &\quad + (1 - a)\left(1 - (1 - \gamma)\frac{\lambda(x, \gamma)(\gamma - 1) - b(\gamma - \lambda(x, \gamma))}{(1 - b)\lambda(x, \gamma)(\gamma - 1)}\right) \\ &= a\frac{\gamma}{\lambda(x, \gamma)} + (1 - a)\left(1 - (1 - \gamma)\frac{b(\gamma - \lambda(x, \gamma)) - \lambda(x, \gamma)(\gamma - 1)}{(1 - b)\lambda(x, \gamma)(\gamma - 1)}\right) \\ &= a\frac{\gamma}{\lambda(x, \gamma)} \\ &\quad + (1 - a)\left(\frac{(1 - b)\lambda(x, \gamma) - (b(\gamma - \lambda(x, \gamma)) - \lambda(x, \gamma)(\gamma - 1))}{(1 - b)\lambda(x, \gamma)}\right) \\ &= a\frac{\gamma}{\lambda(x, \gamma)} + \frac{1 - a}{(1 - b)\lambda(x, \gamma)}(\lambda(x, \gamma)\gamma - b\gamma) \\ &= \frac{\gamma}{\lambda(x)(1 - b)}(a - ab + \lambda(x, \gamma) - b - a\lambda(x, \gamma) + ab) \\ &= \gamma\frac{(a - b) + \lambda(x, \gamma)(1 - a)}{\lambda(x, \gamma)(1 - b)} \\ &= \Phi(\pi_a(x, y), \gamma).\end{aligned}$$

□

Φ is continuous in its second argument

To show that Φ is continuous in its second argument it is enough to show that λ is continuous in its second argument.

Lemma A.5. *λ is continuous in its second argument.*

Proof. Fix $x \in \mathcal{M}$, $\gamma \in (0, 1)$ and γ_n such that $\gamma_n \rightarrow \gamma$. There are three possible cases: (i) $V(x) > \gamma$, (ii) $V(x) < \gamma$ and (iii) $V(x) = \gamma$. The proof of the three cases is analogous. Hence, we only consider (ii).

Assume $V(x) < \gamma$. By standard arguments it is without loss to assume $\gamma_n > V(x)$ for all n . Let $\lambda_n = \lambda(x, \gamma_n)$ and assume towards a contradiction that $\lambda_n \not\rightarrow \lambda(x, \gamma)$. Then, there exists a neighborhood $B(\lambda(x, \gamma))$ such that $\lambda_n \notin B(\lambda(x, \gamma))$. Let λ_m denote the corresponding subsequence. Since λ_m is a subsequence in $[0, 1]$, there exists a convergent subsequence λ_l with limit $\lambda^* \neq \lambda(x, \gamma)$. Then, either $\pi_{\lambda^*}(x, \underline{x}) \succ \pi_{\lambda(x, \gamma)}(x, \underline{x})$ or $\pi_{\lambda(x, \gamma)}(x, \underline{x}) \succ \pi_{\lambda^*}(x, \underline{x})$. The proof that either case leads to a contradiction is analogous. Hence, we only consider the case in which $\pi_{\lambda^*}(x, \underline{x}) \succ \pi_{\lambda(x, \gamma)}(x, \underline{x})$.

By Continuity there exists z such that $\pi_{\lambda^*}(x, \underline{x}) \succ z \succ \pi_{\lambda(x, \gamma)}(x, \underline{x})$. Further, by Lemma A.1, there exists N such that $l > N$ implies $\pi_{\lambda_l}(x, \underline{x}) \succ z$. Since $\pi_{\lambda_l}(x, \underline{x}) \sim \pi_{\gamma_l}(\bar{x}, \underline{x})$ for all l and $\pi_{\lambda(x, \gamma)}(x, \underline{x}) \sim x_\gamma$, then by Lemma A.1, $\pi_{\gamma_l}(\bar{x}, \underline{x}) \succ z$ for all $l > N$ and $\gamma_l \rightarrow \gamma$ imply that $\pi_\gamma(\bar{x}, \underline{x}) \succeq z$, a contradiction. \square

A.2 Necessity

Say that Φ represents \succeq if it satisfies the conditions of Theorem 6.3. The proof of necessity of Weak Order, Continuity and Strict Best and Worst is routine. Hence, we only show that the representation satisfies Mixture-Betweenness.

Preliminaries

Lemma A.6. *Assume Φ represents \succeq . Then for any x such that $V(x) \in (0, 1)$, the following two properties hold:*

1. $\Phi(x, \gamma) < \gamma$ implies $V(x) < \gamma$.
2. $\Phi(x, \gamma) > \gamma$ implies $V(x) > \gamma$.

Proof. Assume towards a contradiction that there exists γ and x such that $V(x) \geq \gamma$ and $\Phi(x, \gamma) < \gamma$. If $V(x) = \gamma$, then by the unique solution property $\Phi(x, \gamma) = \gamma$, a contradiction. Thus, $V(x) > \gamma$. Since $\Phi(x, \gamma) < \gamma < 1$, there exists $a \in (0, 1)$ such that $a\Phi(x, \gamma) + (1 - a) = \gamma$. Hence, $V(\pi_a(x, \bar{x})) = \gamma < V(x) < 1$. Since V is mixture-continuous, there exists $b \in (0, 1)$ such that $V(\pi_b(x, \bar{x})) = V(x)$. Hence, $V(x) = b\Phi(x, V(x)) + (1 - b) = bV(x) + (1 - b)$, a contradiction.

Property 2 Follows from an analogous argument, the only difference is that one needs to use \underline{x} instead of \bar{x} to derive a contradiction. \square

Next, we complete the proof. Showing that $V(x) = V(y)$ implies $V(x) = V(\pi_a(x, y)) = V(y)$ for all $a \in (0, 1)$ is straight forward. Assume $V(x) > V(y)$ and let $\gamma = V(\pi_a(x, y))$. Then

$$\gamma = a\Phi(x, \gamma) + (1 - a)\Phi(y, \gamma).$$

Notice that if $\gamma = V(x)$ or $\gamma = V(y)$, then the unique solution property implies a contradiction of $V(x) > V(y)$. Further, by Lemma A.6,

$$\begin{aligned} \gamma > V(x) &\implies \gamma > \Phi(x, \gamma), \Phi(y, \gamma) \\ \gamma < V(y) &\implies \gamma < \Phi(x, \gamma), \Phi(y, \gamma). \end{aligned}$$

In both cases $\gamma = a\Phi(x, \gamma) + (1 - a)\Phi(y, \gamma)$ does not have a solution. Hence, $\gamma \in (V(y), V(x))$.

A.3 Uniqueness

Preliminaries

The proof will employ the following elementary lemma which states that if two preferences satisfy Weak Order, Continuity, Independence, share an indifference curve and rank two elements in the same way, one above the indifference curve and one below, then they are equal.

Lemma A.7. *Let \succeq and \succeq' be two binary relations on a Mixture Space (\mathcal{M}, π) that satisfy Weak Order, Continuity and Independence. Assume*

1. $\{x|x \sim \pi_a(\bar{x}, \underline{x})\} = \{x|x \sim' \pi_a(\bar{x}, \underline{x})\}$ for some $a \in (0, 1)$.
2. $\bar{x} \succ \underline{x}$ and $\bar{x} \succ' \underline{x}$.

Then $\succeq = \succeq'$.

Proof. Assume towards a contradiction that there exist x, y such that $x \succ y$ and $y \succeq' x$. Since \succeq is complete, then there are two possible cases: $y \succeq \pi_a(\bar{x}, \underline{x})$ and $\pi_a(\bar{x}, \underline{x}) \succeq y$. The proof that either case leads to a contradiction is analogous. Hence, we only consider the case in which $y \succeq \pi_a(\bar{x}, \underline{x})$.

Assume $y \succeq \pi_a(\bar{x}, \underline{x})$. Then there exists b such that $\pi_b(y, \underline{x}) \sim \pi_a(\bar{x}, \underline{x})$. Hence, $\pi_b(y, \underline{x}) \sim' \pi_a(\bar{x}, \underline{x})$. Since $x \succ y$ and $y \succeq' x$, by Independence,

$$\pi_b(x, \underline{x}) \succ \pi_b(y, \underline{x}) \sim \pi_a(\bar{x}, \underline{x}) \text{ and } \pi_a(\bar{x}, \underline{x}) \sim \pi_b(y, \underline{x}) \succeq' \pi_b(x, \underline{x}).$$

Thus, there exists $c < b$ such that $\pi_c(x, \underline{x}) \sim \pi_a(\bar{x}, \underline{x})$ and $\pi_a(\bar{x}, \underline{x}) \succ' \pi_c(x, \underline{x})$ a contradiction. \square

Next, we complete the proof. First we show that if Φ and Φ' represent \succeq , then $V = V'$. First note that

$$V(\pi_\gamma(\bar{x}, \underline{x})) = \gamma \text{ and } V'(\pi_\gamma(\bar{x}, \underline{x})) = \gamma$$

for all $\gamma \in (0, 1)$. Hence, $V(x)$ and $V'(x)$ are the unique γ such that $x \sim \pi_\gamma(\bar{x}, \underline{x})$. Thus, $V = V'$.

To see that $\Phi = \Phi'$, let \succeq_γ and $\succeq_{\gamma'}$ be the preference represented by $\Phi(\cdot, \gamma)$ and $\Phi'(\cdot, \gamma)$ respectively. Then,

$$\{x|x \sim \pi_\gamma(\bar{x}, \underline{x})\} = \{x|x \sim' \pi_\gamma(\bar{x}, \underline{x})\}.$$

Thus, by Lemma A.7, $\succeq_\gamma = \succeq_{\gamma'}$. By the Mixture Space Theorem, $\Phi(\cdot, \gamma)$ is the unique mixture-linear representation of \succeq_γ in which $\Phi(\bar{x}, \gamma) = 1$ and $\Phi(\underline{x}, \gamma) = 0$. Hence, $\Phi = \Phi'$.

A.4 Proofs of Lemma A.1 and A.2

Preliminaries

Lemma A.8. *Assume (\mathcal{M}, π) is an Associative Mixture Space. Then, for all $x, y, z \in \mathcal{M}$ and $a, b, c \in [0, 1]$ such that $ca + (1 - c)b \neq 0$,*

$$\pi_c(\pi_a(x, y), \pi_b(z, y)) = \pi_{ca+(1-c)b}\left(\pi_{\frac{ca}{ca+(1-c)b}}(x, z), y\right).$$

Proof. Fix $x, y, z \in \mathcal{M}$ and $a, b, c \in [0, 1]$ such that $ca + (1 - c)b \neq 0$. Then,

$$\begin{aligned} \pi_c(\pi_a(x, y), \pi_b(z, y)) &= \pi_{ca}\left(x, \pi_{\frac{c(1-a)}{1-ac}}(y, \pi_b(z, y))\right) \\ &= \pi_{ca}\left(x, \pi_{\frac{1-c}{1-ac}}(\pi_b(z, y), y)\right) \\ &= \pi_{ca}\left(x, \pi_{\frac{b(1-c)}{1-ac}}(z, y)\right) \\ &= \pi_{ca+(1-c)b}\left(\pi_{\frac{ca}{ca+(1-c)b}}(x, z), y\right) \end{aligned}$$

Where the first inequality follows from (A4), the second from (A2), the third from (A3) and the fourth from (A4). \square

Proof of Lemma A.1

Part 1: If $a = 0$ or $b = 1$, then Mixture-Betweenness implies $\pi_b(x, y) \succ \pi_a(x, y)$. Suppose $0 < a < b < 1$. Then, there exists $c \in (0, 1)$ such that $cb = a$. Further, by Mixture-Betweenness, $\pi_b(x, y) \succ y$ and $\pi_b(x, y) \succ \pi_c(\pi_b(x, y), y) =$

$\pi_a(x, y)$.

Part 2: Fix $x, y, z \in \mathcal{M}$ and let $A = \{a | \pi_a(x, y) \succeq z\}$. WLOG assume $x \succeq y$. If $A = \emptyset$ or a singleton, then there is nothing to prove. Assume $A \neq \emptyset$ and $|A| > 1$. Define $a^* = \inf A$. If $a^* \in A$, then by Part 1 $A = [a^*, 1]$. Hence, A is closed. Assume towards a contradiction that $a^* \notin A$. Thus, $z \succ \pi_{a^*}(x, y)$. Since $|A| > 1$, then $x \succ z \succ \pi_{a^*}(x, y)$. Hence, by Continuity, there exists b such that

$$\begin{aligned} z \succ \pi_b(x, \pi_{a^*}(x, y)) &= \pi_{1-b}(\pi_{(1-a^*)}(y, x), x) \\ &= \pi_{(1-b)(1-a^*)}(y, x) = \pi_{1-(1-b)(1-a^*)}(x, y). \end{aligned}$$

Note that $1-(1-b)(1-a^*) = a^* + b(1-a^*) > a^*$. Hence, $z \succ \pi_{1-(1-b)(1-a^*)}(x, y)$ and $1-(1-b)(1-a^*) > a^*$, a contradiction of the fact that a^* is the infimum.

The proof that $\{a | z \succeq \pi_a(x, y)\}$ is closed is analogous, the only difference is that instead of using the infimum to derive a contradiction, one needs to use the supremum.

Part 3: Follows from Part 2.

Proof of Lemma A.2

The proofs of Parts 2, 4 and 6 are analogous to the proofs of Parts 1, 3 and 5 respectively. Hence, we only prove Parts 1, 3 and 5.

Part 1: Fix x, y such that $V(x), V(y) \geq \gamma$ and $a \in (0, 1)$. Then, by Mixture-Betweenness, $V(\pi_a(x, y)) \geq \gamma$. Further,

$$\pi_{\lambda(x, \gamma)}(x, \underline{x}) \sim x_\gamma \sim \pi_{\lambda(y, \gamma)}(y, \underline{x}).$$

Hence, by Mixture-Betweenness, $\pi_c(\pi_{\lambda(x, \gamma)}(x, \underline{x}), \pi_{\lambda(y, \gamma)}(y, \underline{x})) \sim x_\gamma$ for all $c \in (0, 1)$. Notice that if for some $c \in (0, 1)$

$$\begin{aligned} \pi_c(\pi_{\lambda(x, \gamma)}(x, \underline{x}), \pi_{\lambda(y, \gamma)}(y, \underline{x})) &= \pi_{\frac{\lambda(x, \gamma)\lambda(y, \gamma)}{a\lambda(y, \gamma) + (1-a)\lambda(x, \gamma)}}(\pi_a(x, y), \underline{x}) \\ \implies \lambda(\pi_a(x, y), \gamma) &= \frac{\lambda(x, \gamma)\lambda(y, \gamma)}{a\lambda(y, \gamma) + (1-a)\lambda(x, \gamma)}. \end{aligned}$$

Thus, it is enough to show that the first equality holds.

By Lemma A.8, for all $c \in [0, 1]$

$$\pi_c(\pi_{\lambda(x, \gamma)}(x, \underline{x}), \pi_{\lambda(y, \gamma)}(y, \underline{x})) = \pi_{c\lambda(x, \gamma) + (1-c)\lambda(y, \gamma)}\left(\pi_{\frac{c\lambda(x, \gamma)}{c\lambda(x, \gamma) + (1-c)\lambda(y, \gamma)}}(x, y), \underline{x}\right).$$

In particular, if $c = \frac{a\lambda(y,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}$, then

$$\begin{aligned} c\lambda(x,\gamma) + (1-c)\lambda(y,\gamma) &= \frac{a\lambda(y,\gamma)\lambda(x,\gamma)}{a\lambda(y,\gamma) + (1-a)\lambda(x,\gamma)} + \frac{(1-a)\lambda(x,\gamma)\lambda(y,\gamma)}{a\lambda(y,\gamma) + (1-a)\lambda(x,\gamma)} \\ &= \frac{\lambda(x,\gamma)\lambda(y,\gamma)}{a\lambda(y,\gamma) + (1-a)\lambda(x,\gamma)} \end{aligned}$$

and

$$\begin{aligned} \frac{c\lambda(x,\gamma)}{c\lambda(x,\gamma) + (1-c)\lambda(y,\gamma)} &= \frac{\frac{a\lambda(y,\gamma)\lambda(x,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}}{\frac{a\lambda(y,\gamma)\lambda(x,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)} + \frac{(1-a)\lambda(x,\gamma)\lambda(y,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}} \\ &= a. \end{aligned}$$

Hence, $\pi_c(\pi_{\lambda(x,\gamma)}(x, \underline{x}), \pi_{\lambda(y,\gamma)}(y, \underline{x})) = \pi_{\frac{\lambda(x,\gamma)\lambda(y,\gamma)}{a\lambda(y,\gamma)+(1-a)\lambda(x,\gamma)}}(\pi_a(x, y), \underline{x})$.

Part 3: Fix $x, y, z \in \mathcal{X}$ such that $V(x), V(y) > \gamma > V(z)$ and $\lambda(x, \gamma) < \lambda(y, \gamma)$. Let $b \in (0, 1)$ be such that

$$\pi_b(x, z) \sim x_\gamma.$$

Then, by Part 1,

$$\lambda(\pi_a(x, \pi_b(x, z))) = \frac{\lambda(x, \gamma)}{a + (1-a)\lambda(x, \gamma)}.$$

Define $\phi(a) = \frac{\lambda(x, \gamma)}{a + (1-a)\lambda(x, \gamma)}$. Then, $\phi(0) = 1$, $\phi(1) = \lambda(x, \gamma)$ and $\phi'(a) < 0$ for all $a \in (0, 1)$. Hence, there exists a unique $a \in (0, 1)$ such that $\phi(a) = \lambda(y, \gamma)$.

Part 5: Let a_1 and a_2 be such that

$$x_\gamma \sim \pi_{a_1}(x, z) \text{ and } x_\gamma \sim \pi_{a_2}(y, z).$$

Assume towards a contradiction that $a_2 > a_1$ (the case in which $a_1 > a_2$ is analogous). Then, $\pi_{a_2}(x, z) \succ x_\gamma$ and $\lambda(\pi_{a_2}(x, z), \gamma) < 1$. Since $\lambda(x, \gamma) = \lambda(y, \gamma)$, then by Part 1, for all $c \in (0, 1)$

$$\lambda(\pi_c(x, \pi_{a_2}(y, z)), \gamma) = \lambda(\pi_c(y, \pi_{a_1}(x, z)), \gamma).$$

However,

$$\begin{aligned} \pi_c(x, \pi_{a_2}(y, z)) &= \pi_{1-c}(\pi_{a_2}(y, z), x) \\ &= \pi_{(1-c)a_2}(y, \pi_{\frac{(1-c)(1-a_2)}{1-(1-c)a_2}}(z, x)) \\ &= \pi_{(1-c)a_2}(y, \pi_{\frac{c}{1-(1-c)a_2}}(x, z)). \end{aligned}$$

Thus, for $c = \frac{a_2}{1+a_2}$,

$$\pi_c(x, \pi_{a_2}(y, z)) = \pi_{\frac{a_2}{1+a_2}}(y, \pi_{a_2}(x, z)) \text{ and } \pi_c(y, \pi_{a_1}(x, z)) = \pi_{\frac{a_2}{1+a_2}}(y, \pi_{a_1}(x, z)).$$

Therefore, $\lambda(\pi_{\frac{a_2}{1+a_2}}(y, \pi_{a_2}(x, z)), \gamma) = \lambda(\pi_{\frac{a_2}{1+a_2}}(y, \pi_{a_1}(x, z)), \gamma)$ implies

$$\frac{\lambda(y, \gamma)\lambda(\pi_{a_2}(x, z))}{c + (1-c)\lambda(y, \gamma)} = \frac{\lambda(y, \gamma)}{c + (1-c)\lambda(y, \gamma)}$$

$$\lambda(\pi_{a_2}(x, z)) = 1,$$

a contradiction.

Appendix B Proof of Theorem 3.1

B.1 Sufficiency

$\Delta(X)$ compact implies \mathcal{X} is compact (Aliprantis and Border (2006), Theorem 3.71(3)). Let $\mathcal{K}(\Delta(X))$ denote the set of all closed and convex menus of lotteries. By the Blaschke Selection Theorem (Schneider (2014) Theorem 1.8.5), $\mathcal{K}(\Delta(X))$ is compact.

Lemma B.1. $\mathcal{K}(\Delta(X))$ is an associative mixture space.

Proof. By Lemmas S.2 and S.3 in Dekel et al. (2007), there exists a mixture preserving bijection from $\mathcal{K}(\Delta(X))$ to a convex subset of a linear space. Hence, by Theorem 6.2, $\mathcal{K}(\Delta(X))$ is an Associative Mixture Space. \square

Let $\bar{p}, \underline{p} \in \Delta(X)$ denote a fixed pair of lotteries such that $\{\bar{p}\} \succeq x \succeq \{\underline{p}\}$ for all $x \in \mathcal{X}$. Given our axioms, such lotteries always exist (see footnote 4). For each $\gamma \in (0, 1)$, let $\{p_\gamma\}$ denote $\gamma\{\bar{p}\} + (1-\gamma)\{\underline{p}\}$.

Define V and Φ as in the proof of Theorem 6.3 using $\{\bar{p}\}$ and $\{\underline{p}\}$ as the best and worst menus. Then, $\Phi(\cdot, \gamma)$ is mixture linear for all $\gamma \in (0, 1)$ and, $\Phi(\{\bar{p}\}, \gamma) = 1$ and $\Phi(\{\underline{p}\}, \gamma) = 0$ for all $\gamma \in [0, 1]$. Further, for all x such that $\{\bar{p}\} \succ x \succ \{\underline{p}\}$, $V(x) = \gamma$ is the unique solution of

$$\gamma = \Phi(x, \gamma).$$

An identical argument to the one in the proof of Theorem S.1.1 in the supplemental material shows that $\Phi(\cdot, \gamma)$ and V are continuous for all $\gamma \in (0, 1)$. In what follows we will use $\Phi(\cdot, \gamma)$ to construct the representation in Theorem 3.1 for the case in which $\gamma \in (0, 1)$. Afterwards we construct the representation

for the case in which $\gamma = 1$ and $\gamma = 0$.

Step 1: Extend V and $\Phi(\cdot, \gamma)$ to \mathcal{X} .

For each menu $x \in \mathcal{X}$, let $ch(x)$ denote its convex hull. Extend V by letting $V(x) = V(ch(x))$ for all $x \in \mathcal{X} \setminus \mathcal{K}(\Delta(X))$. We claim that V represents \succeq . To prove this, it is enough to show that our axioms imply $x \sim ch(x)$ for all $x \in \mathcal{X}$.

Lemma B.2. *Let \succeq be a binary relation over \mathcal{X} that satisfies Weak Order, Continuity and Mixture-Betweenness. Then $x \sim ch(x)$ for all $x \in \mathcal{X}$.*

Proof. Let $K = |X|$ and assume, by way of contradiction, that there exists x such that $x \not\sim ch(x)$. Then, by Mixture-Betweenness, $\alpha x + (1-\alpha)ch(x) \not\sim ch(x)$ for all $\alpha \in (0, 1)$. This is a contradiction. In particular, Lemma S.6 in the supplemental appendix of Dekel et al. (2007) shows that $\alpha x + (1-\alpha)ch(x) = ch(x)$ for all $\alpha \in [0, \frac{1}{K}]$. \square

Extend Φ by letting $\Phi(x, \gamma) = \Phi(ch(x), \gamma)$ for all $x \in \mathcal{X} \setminus \mathcal{K}(\Delta(X))$. Then,

$$\begin{aligned} \Phi(\alpha x + (1-\alpha)y, \gamma) &= \Phi(ch(\alpha x + (1-\alpha)y), \gamma) \\ &= \Phi(\alpha ch(x) + (1-\alpha)ch(y), \gamma) \\ &= \alpha \Phi(ch(x), \gamma) + (1-\alpha)\Phi(ch(y), \gamma) \\ &= \alpha \Phi(x, \gamma) + (1-\alpha)\Phi(y, \gamma) \end{aligned}$$

for all $x, y \in \mathcal{X}$ and $\alpha \in [0, 1]$. Thus, the extension of Φ is also mixture linear in its first argument and continuous in its second.

Finally, continuity of $\Phi(\cdot, \gamma)$ and V on \mathcal{X} follows from the fact that for all $x, y \in \mathcal{X}$, $d_h(ch(x), ch(y)) \leq d_h(x, y)$.

Step 2: Show that $\Phi(\cdot, \gamma)$ satisfies Set-Betweenness.

Fix $x, y \in \mathcal{X}$ and $\gamma \in (0, 1)$. Assume WLOG that $\Phi(x, \gamma) \geq \Phi(y, \gamma)$. There are three possible cases: (i) $\Phi(x, \gamma) \geq \Phi(y, \gamma) \geq \gamma$. (ii) $\Phi(x, \gamma) \geq \gamma \geq \Phi(y, \gamma)$. (iii) $\gamma \geq \Phi(x, \gamma) \geq \Phi(y, \gamma)$.

Since the proofs of (i) and (iii) are analogous, we only consider (i) and (ii).

(i) $\Phi(x, \gamma) \geq \Phi(y, \gamma) \geq \gamma$.

By Lemma A.6, $V(x), V(y) \geq \gamma$. Thus, by Set-Betweenness, $V(x \cup y) \geq \gamma$. By construction, $\Phi(x, \gamma) \geq \Phi(y, \gamma)$ if and only if $\lambda(x, \gamma) \leq \lambda(y, \gamma)$ so

$$\lambda(x, \gamma)x + (1 - \lambda(x, \gamma))\{\underline{p}\} \sim \{p_\gamma\} \succeq \lambda(x, \gamma)y + (1 - \lambda(x, \gamma))\{\underline{p}\}.$$

Hence, by Set-Betweenness,

$$\begin{aligned} & \lambda(x, \gamma)x + (1 - \lambda(x, \gamma))\{\underline{p}\} \\ & \succeq \lambda(x, \gamma)x + (1 - \lambda(x, \gamma))\{\underline{p}\} \cup \lambda(x, \gamma)y + (1 - \lambda(x, \gamma))\{\underline{p}\} \\ & \succeq \lambda(x, \gamma)y + (1 - \lambda(x, \gamma))\{\underline{p}\}. \end{aligned}$$

However,

$$\begin{aligned} & \lambda(x, \gamma)x + (1 - \lambda(x, \gamma))\{\underline{p}\} \cup \lambda(x, \gamma)y + (1 - \lambda(x, \gamma))\{\underline{p}\} \\ & = \lambda(x, \gamma)(x \cup y) + (1 - \lambda(x, \gamma))\{\underline{p}\}. \end{aligned}$$

Thus,

$$\{p_\gamma\} \succeq \lambda(x, \gamma)(x \cup y) + (1 - \lambda(x, \gamma))\{\underline{p}\}.$$

Hence, $\lambda(x \cup y, \gamma) \geq \lambda(x, \gamma)$ so $\Phi(x, \gamma) \geq \Phi(x \cup y, \gamma)$.

A similar argument shows that $\Phi(x \cup y, \gamma) \geq \Phi(y, \gamma)$, the only difference is that one needs to use y ($\lambda(y, \gamma)$) instead of x ($\lambda(x, \gamma)$).

(ii) $\Phi(x, \gamma) \geq \gamma \geq \Phi(y, \gamma)$.

By Lemma A.6, $V(x) \geq \gamma \geq V(y)$. Hence, by Set-Betweenness, $V(x) \geq V(x \cup y) \geq V(y)$. There are two cases: $V(x \cup y) \geq \gamma$ or $\gamma \geq V(x \cup y)$.

If $V(x \cup y) \geq \gamma$, then, by Lemma A.6, $\Phi(x \cup y, \gamma) \geq \Phi(y, \gamma)$. Hence, we only need to show $\Phi(x, \gamma) \geq \Phi(x \cup y, \gamma)$. By Mixture-Betweenness,

$$\lambda(x, \gamma)x + (1 - \lambda(x, \gamma))\{\underline{p}\} \sim \{p_\gamma\} \succeq \lambda(x, \gamma)y + (1 - \lambda(x, \gamma))\{\underline{p}\}.$$

Thus, by Set-Betweenness,

$$\begin{aligned} & \lambda(x, \gamma)x + (1 - \lambda(x, \gamma))\{\underline{p}\} \\ & \succeq \lambda(x, \gamma)x + (1 - \lambda(x, \gamma))\{\underline{p}\} \cup \lambda(x, \gamma)y + (1 - \lambda(x, \gamma))\{\underline{p}\} \\ & \succeq \lambda(x, \gamma)y + (1 - \lambda(x, \gamma))\{\underline{p}\}. \end{aligned}$$

However,

$$\begin{aligned} & \lambda(x, \gamma)x + (1 - \lambda(x, \gamma))\{\underline{p}\} \cup \lambda(x, \gamma)y + (1 - \lambda(x, \gamma))\{\underline{p}\} \\ & = \lambda(x, \gamma)x \cup y + (1 - \lambda(x, \gamma))\{\underline{p}\}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \lambda(x, \gamma)x + (1 - \lambda(x, \gamma))\{\underline{p}\} \\ & \succeq \lambda(x, \gamma)x \cup y + (1 - \lambda(x, \gamma))\{\underline{p}\} \\ & \succeq \lambda(x, \gamma)y + (1 - \lambda(x, \gamma))\{\underline{p}\}. \end{aligned}$$

Hence, $\lambda(x \cup y, \gamma) \geq \lambda(x, \gamma)$ so $\Phi(x, \gamma) \geq \Phi(x \cup y, \gamma)$.

A similar argument establishes that if $\gamma \geq V(x \cup y)$, then $\Phi(x, \gamma) \geq \Phi(x \cup y, \gamma) \geq \Phi(y, \gamma)$. The only difference is that if $\gamma \geq V(x \cup y)$, then, by Lemma A.6, $\Phi(x, \gamma) \geq \Phi(x \cup y, \gamma)$. Therefore, one only needs to show that $\Phi(x \cup y, \gamma) \geq \Phi(y, \gamma)$. An analogous argument to the previous one in which $\lambda(x, \gamma)$ is replaced by $\lambda(y, \gamma)$ proves that $\Phi(x \cup y, \gamma) \geq \Phi(y, \gamma)$.

Step 3: Show that there exists $u(\cdot, \gamma)$ and $v(\cdot, \gamma)$ such that for all $\gamma \in (0, 1)$ and $x \in \mathcal{X}$,

$$\Phi(x, \gamma) = \max_{p \in x} \{u(p, \gamma) + v(p, \gamma) - \max_{q \in x} v(q, \gamma)\}.$$

Restrict \succeq to $\Delta(X)$, then by Proposition 1 in Dekel (1986) there exists a unique $u : \Delta(X) \times (0, 1) \rightarrow \mathbb{R}$ such that $u(\cdot, \gamma)$ is a vNM utility function for all $\gamma \in (0, 1)$, u is continuous in its second argument on the open interval $(0, 1)$, $u(\bar{p}, \gamma) = 1$, $u(\underline{p}, \gamma) = 0$ and $V(\{p\})$ is the unique $\gamma \in [0, 1]$ that solves

$$\gamma = u(p, \gamma).$$

Hence, $\Phi(\{p\}, \gamma) = u(p, \gamma)$ for all $p \in \Delta(X)$ and $\gamma \in (0, 1)$.

By Lemmas 2, 4 and 5 in GP, there exists a vNM function $v(\cdot, \gamma)$ such that

$$\Phi(x, \gamma) = \max_{p \in x} \{u(p, \gamma) + v(p, \gamma) - \max_{q \in x} v(q, \gamma)\}$$

for all $x \in \mathcal{X}$.

Step 4: Construct $v(\cdot, 1)$ and $v(\cdot, 0)$.

The construction of $v(\cdot, 0)$ and $v(\cdot, 1)$ are analogous. Thus, we only show the latter and specify how to adapt the argument for the former. Since \succeq satisfies Set-Betweenness, we will restrict our attention to binary menus. Our construction of $v(\cdot, 1)$ is similar to the construction in the proof of Theorem 2 of Noor and Takeoka (2015).

Let $\mathcal{P} = \{p | \{p\} \succeq \{q\} \text{ for all } q\}$. There are two possible cases: (i) $\{p\} \sim \{p, q\}$ for all $p \in \mathcal{P}, q \in \Delta(X)$ and (ii) there exist $p \in \mathcal{P}$ and $q \in \Delta(X)$ such that $\{p\} \succ \{p, q\}$.

(i) $\{p\} \sim \{p, q\}$ for all $p \in \mathcal{P}, q \in \Delta(X)$

Let $v(\cdot, 1) = 0$. Then, $\max_{p \in \{p, q\}} u(p, 1) = 1$ if and only if $p \in \mathcal{P}$ or $q \in \mathcal{P}$.

Hence,

$$\max_{p' \in \{p, q\}} u(p', 1) = 1 \text{ if and only if } V(\{p, q\}) = 1.$$

- (ii) There exist $p \in \mathcal{P}$ and $q \in \Delta(X)$ such that $\{p\} \succ \{p, q\}$
 Define \succeq^T over $\Delta(X)$ as follows

$$\begin{aligned} p \succeq^T q &\text{ if and only if } \{p\} \sim \{p, q\} \succ \{q\} \text{ and } p \in \mathcal{P} \\ q \succ^T p &\text{ if and only if } \{p\} \succ \{p, q\} \text{ and } p \in \mathcal{P}. \end{aligned}$$

We will show that there exists a vNM utility function $v(\cdot, 1)$ such that $p \succeq^T q$ implies $v(p, 1) \geq v(q, 1)$ and $q \succ^T p$ implies $v(q, 1) > v(p, 1)$. We claim that if such function exists, then

$$1 = \max_{p' \in \{p, q\}} \{u(p', 1) + v(p', 1) - \max_{q' \in \{p, q\}} v(q', 1)\} \text{ if and only if } V(\{p, q\}) = 1.$$

To prove this, let

$$\Phi(x, 1) = \max_{p \in x} \{u(p, 1) + v(p, 1) - \max_{q \in x} v(q, 1)\}.$$

First, assume $V(\{p, q\}) = 1$. If $p, q \in \mathcal{P}$, then $u(p, 1) = u(q, 1) = 1$. Then $\Phi(\{p, q\}, 1) = 1$ because $\Phi(\{p\}, 1) = \Phi(\{q\}, 1) = 1$ and $\Phi(\cdot, 1)$ satisfies Set-Betweenness. If $p \in \mathcal{P}$ and $q \notin \mathcal{P}$, then $\{p, q\} \succ \{q\}$. Hence, by construction, $v(p, 1) \geq v(q, 1)$. Thus, $\Phi(\{p, q\}, 1) = u(p, 1) = 1$. Finally, note that if $p, q \notin \mathcal{P}$, then, by Set-Betweenness, $V(\{p, q\}) < 1$.

Next, assume $1 = \Phi(\{p, q\}, 1)$. Since $v(p', 1) - \max_{q' \in \{p, q\}} v(q', 1) \leq 0$ and $u(p, 1) \leq 1$, then $1 = \Phi(\{p, q\}, 1)$ implies that $p \in \mathcal{P}$ or $q \in \mathcal{P}$. If $p, q \in \mathcal{P}$, then $V(\{p\}) = V(\{q\}) = 1$ so by Set-Betweenness $V(\{p, q\}) = 1$. Assume towards a contradiction that $p \in \mathcal{P}, q \notin \mathcal{P}$ and $V(\{p, q\}) < 1$. Then, $q \succ^T p$ so $v(q, 1) > v(p, 1)$. Further, $p \in \mathcal{P}$ implies $u(p, 1) = 1$. Hence, $\Phi(x, 1) < 1$ a contradiction.

Finally, we will show that such vNM utility function exists.

Let $\bar{0}$ denote the zero vector and $\mathcal{T} = cl(\{\sum_{i=1}^n \lambda_i(p_i - q_i) | n \in \mathbb{N}, \lambda_i > 0 \text{ and } p_i \succ q_i \text{ or } p_i \succeq^T q_i \text{ for } i = 1, \dots, n\})$. By Lemma 2 in Fishburn (1975), there exists $v(\cdot, 1)$ such that $p \succ^T q$ implies $v(p, 1) > v(q, 1)$ and $p \succeq^T q$ implies $v(p, 1) \geq v(q, 1)$ if $\bar{0} \notin \mathcal{T}$. We prove this in three steps.

Step 4.1: Show that if $\{p_t\} \sim \{p_t, q_t\}$ for $t = 1, \dots, n$ and $p_t \in \mathcal{P}$ for all t , then $\{\sum_t \lambda_t p_t\} \sim \{\sum_t \lambda_t p_t, \sum_t \lambda_t q_t\}$ for all $\lambda \in \Delta(\{1, \dots, n\})$.

The proof is by induction. Fix $p_1, p_2 \in \mathcal{P}$ such that $\{p_1\} \sim \{p_1, q_1\}$ and $\{p_2\} \sim \{p_2, q_2\}$. By Mixture-Betweenness, $\{\lambda p_1 + (1 - \lambda)p_2\} \succeq x$ for all $x \in \mathcal{X}$. Assume towards a contradiction that $\{\lambda p_1 + (1 - \lambda)p_2\} \succ \{\lambda p_1 + (1 - \lambda)p_2, \lambda q_1 + (1 - \lambda)q_2\}$.

By Mixture-Betweenness,

$$\begin{aligned} \{p_2, q_2\} &\succ \{\lambda q_1 + (1 - \lambda)p_2, \lambda q_1 + (1 - \lambda)q_2\} \\ \{p_1, q_1\} &\succ \{\lambda p_1 + (1 - \lambda)q_2, \lambda q_1 + (1 - \lambda)q_2\}. \end{aligned}$$

Thus, by Set-Betweenness,

$$\{p_2, q_2\} \succ \{\lambda q_1 + (1 - \lambda)p_2, \lambda q_1 + (1 - \lambda)q_2, \lambda p_1 + (1 - \lambda)q_2\}.$$

Assume WLOG that

$$\begin{aligned} &\{\lambda q_1 + (1 - \lambda)p_2, \lambda q_1 + (1 - \lambda)q_2, \lambda p_1 + (1 - \lambda)q_2\} \\ &\succeq \{\lambda p_1 + (1 - \lambda)p_2, \lambda q_1 + (1 - \lambda)q_2\}. \end{aligned}$$

Thus, By Set-Betweenness,

$$\{p_2, q_2\} \succ \{\lambda q_1 + (1 - \lambda)p_2, \lambda q_1 + (1 - \lambda)q_2, \lambda p_1 + (1 - \lambda)q_2, \lambda p_1 + (1 - \lambda)p_2\}.$$

However,

$$\begin{aligned} &\{\lambda q_1 + (1 - \lambda)p_2, \lambda q_1 + (1 - \lambda)q_2, \lambda p_1 + (1 - \lambda)q_2, \lambda p_1 + (1 - \lambda)p_2\} \\ &= \lambda\{p_1, q_1\} + (1 - \lambda)\{p_2, q_2\}, \end{aligned}$$

a contradiction.

Induction step: Suppose the result is true for n . We will now show it holds for $n + 1$.

Fix $p_t \in \mathcal{P}$ such that $p_t \sim \{p_t, q_t\}$ for all $t = 1, \dots, n + 1$. By the induction hypothesis $\{\sum_{t=1}^n \frac{\lambda_t}{\sum_{t=1}^n \lambda_t} p_t\} \sim \{\sum_{t=1}^n \frac{\lambda_t}{\sum_{t=1}^n \lambda_t} p_t, \sum_{t=1}^n \frac{\lambda_t}{\sum_{t=1}^n \lambda_t} q_t\}$. Hence, by the base case, $\{\sum_{t=1}^{n+1} \lambda_t p_t\} \sim \{\sum_{t=1}^{n+1} \lambda_t p_t, \sum_{t=1}^{n+1} \lambda_t q_t\}$.

Step 4.2: Show that if $\{p_t\} \succ \{p_t, q_t\}$ $t = 1, \dots, n$ and $p_t \in \mathcal{P}$ for all t , then $\{\sum_{t=1}^n \lambda_t p_t\} \succ \{\sum_{t=1}^n \lambda_t p_t, \sum_{t=1}^n \lambda_t q_t\}$ for all $\lambda \in \Delta(\{1, \dots, n\})$.

The proof is by induction. Fix $p_1, p_2 \in \mathcal{P}$ and $\lambda \in [0, 1]$ such that $\{p_1\} \succ \{p_1, q_1\}$ and $\{p_2\} \succ \{p_2, q_2\}$. By Mixture-Betweenness, $\{\lambda p_1 + (1 - \lambda)p_2\} \succeq \{\lambda p_1 + (1 - \lambda)p_2, \lambda q_1 + (1 - \lambda)q_2\}$. Assume towards a contradiction that $\{\lambda p_1 + (1 - \lambda)p_2\} \sim \{\lambda p_1 + (1 - \lambda)p_2, \lambda q_1 + (1 - \lambda)q_2\}$. First note that if

$$\{\lambda p_1 + (1 - \lambda)p_2, q_2\} \sim \{p_2\},$$

then by the previous step,

$$\{\lambda p_1 + (1 - \lambda)p_2, \lambda p_1 + (1 - \lambda)q_2\} \sim \{p_2\}.$$

However

$$\{\lambda p_1 + (1 - \lambda)p_2, \lambda p_1 + (1 - \lambda)q_2\} = \lambda\{p_1\} + (1 - \lambda)\{p_2, q_2\} \prec \{p_2\},$$

a contradiction. Similarly, if

$$\{\lambda p_1 + (1 - \lambda)p_2, q_1\} \sim \{p_1\},$$

by the previous step,

$$\{\lambda p_1 + (1 - \lambda)p_2, \lambda q_1 + (1 - \lambda)p_2\} \sim \{p_1\}.$$

However,

$$\{\lambda p_1 + (1 - \lambda)p_2, \lambda q_1 + (1 - \lambda)p_2\} = \lambda\{p_1, q_1\} + (1 - \lambda)\{p_2\} \prec \{p_1\},$$

a contradiction. Hence,

$$\begin{aligned} \{\lambda p_1 + (1 - \lambda)p_2\} \succ \{\lambda p_1 + (1 - \lambda)p_2, q_1\}, \{\lambda p_1 + (1 - \lambda)p_2\} \succ \{\lambda p_1 + (1 - \lambda)p_2, q_2\} \\ \text{and } \{\lambda p_1 + (1 - \lambda)p_2\} \sim \{\lambda p_1 + (1 - \lambda)p_2, \lambda q_1 + (1 - \lambda)q_2\}. \end{aligned}$$

Let $p = \lambda p_1 + (1 - \lambda)p_2$. By continuity, there exists $\alpha > \lambda > \beta$ such that

$$\{p\} \succ \{p, \alpha q_1 + (1 - \alpha)q_2\} \sim \{p, \beta q_1 + (1 - \beta)q_2\}.$$

Let ν be such that

$$\nu(\alpha q_1 + (1 - \alpha)q_2) + (1 - \nu)(\beta q_1 + (1 - \beta)q_2) = \lambda q_1 + (1 - \lambda)q_2.$$

Then, by Mixture-Betweenness,

$$\begin{aligned} \{p, \alpha q_1 + (1 - \alpha)q_2\} &\sim \nu\{p, \alpha q_1 + (1 - \alpha)q_2\} + (1 - \nu)\{p, \beta q_1 + (1 - \beta)q_2\} \\ &\sim \{p, \beta q_1 + (1 - \beta)q_2\}. \end{aligned}$$

However,

$$\begin{aligned} &\nu\{p, \alpha q_1 + (1 - \alpha)q_2\} + (1 - \nu)\{p, \beta q_1 + (1 - \beta)q_2\} \\ &= \{p, \lambda q_1 + (1 - \lambda)q_2, \nu p + (1 - \nu)(\beta q_1 + (1 - \beta)q_2), \nu(\alpha q_1 + (1 - \alpha)q_2) + (1 - \nu)p\} \\ &= \{p, \lambda q_1 + (1 - \lambda)q_2\} \cup [\nu\{p\} + (1 - \nu)\{p, \beta q_1 + (1 - \beta)q_2\}] \\ &\quad \cup [(1 - \nu)\{p\} + \nu\{p, \alpha q_1 + (1 - \alpha)q_2\}]. \end{aligned}$$

Notice that

$$\begin{aligned} & \{p, \lambda q_1 + (1 - \lambda)q_2\} \succ \{p, \alpha q_1 + (1 - \alpha)q_2\} \\ & \nu\{p\} + (1 - \nu)\{p, \beta q_1 + (1 - \beta)q_2\} \succ \{p, \beta q_1 + (1 - \beta)q_2\} \\ & (1 - \nu)\{p\} + \nu\{p, \alpha q_1 + (1 - \alpha)q_2\} \succ \{p, \alpha q_1 + (1 - \alpha)q_2\} \end{aligned}$$

So by Set-Betweenness,

$$\begin{aligned} \nu\{p, \alpha q_1 + (1 - \alpha)q_2\} + (1 - \nu)\{p, \beta q_1 + (1 - \beta)q_2\} & \succ \{p, \alpha q_1 + (1 - \alpha)q_2\} \\ & \sim \{p, \beta q_1 + (1 - \beta)q_2\}, \end{aligned}$$

a contradiction of Mixture-Betweenness.

Induction step: Suppose the result is true for n . We will show it is true for $n + 1$.

Fix $p_t \in \mathcal{P}$ such that $\{p_t\} \succ \{p_t, q_t\}$ for $t = 1, \dots, n + 1$. By the induction hypothesis $\{\sum_{t=1}^n \frac{\lambda_t}{\sum_{t=1}^n \lambda_t} p_t\} \succ \{\sum_{t=1}^n \frac{\lambda_t}{\sum_{t=1}^n \lambda_t} p_t, \sum_{t=1}^n \frac{\lambda_t}{\sum_{t=1}^n \lambda_t} q_t\}$. Hence, by the base case, $\{\sum_t \lambda_t p_t\} \succ \{\sum_t \lambda_t p_t, \sum_t \lambda_t q_t\}$.

Step 4.3: show that $0 \notin \mathcal{T}$.

We only show that $0 \notin \text{int}(\mathcal{T})$ because Continuity implies that if $0 \notin \text{int}(\mathcal{T})$, then $0 \notin \mathcal{T}$.

Fix $w \in \mathcal{T}$, then

$$w = \sum_{t=1}^n \lambda_t p_t + \sum_{t=n+1}^N \lambda_t q'_t - \sum_{t=1}^n \lambda_t q_t - \sum_{t=n+1}^N \lambda_t p'_t$$

where $p_t \in \mathcal{P}$, $\{p_t\} \sim \{p_t, q_t\}$ for $t = 1, \dots, n$ and $p'_t \in \mathcal{P}$, $\{p'_t\} \succ \{p'_t, q'_t\}$ for $t = n + 1, \dots, N$, and $\lambda_t > 0$ for all t . Let

$$\begin{aligned} p &= \sum_t \frac{\lambda_t}{\sum_{t=1}^N \lambda_t} p_t \text{ and } p' = \sum_{t=n+1}^N \frac{\lambda_t}{\sum_{t=1}^N \lambda_t} p'_t, \\ q &= \sum_{t=1}^n \frac{\lambda_t}{\sum_{t=1}^N \lambda_t} q_t \text{ and } q' = \sum_{t=n+1}^N \frac{\lambda_t}{\sum_{t=1}^N \lambda_t} q'_t. \end{aligned}$$

Assume $w = \bar{0}$. Let $\lambda^* = \sum_{t=1}^n \frac{\lambda_t}{\sum_{t=1}^N \lambda_t}$. Then, $\lambda^* p + (1 - \lambda^*) q' = \lambda^* q + (1 - \lambda^*) p'$. Note that by steps 4.1 and 4.2, $\{p\} \sim \{p, q\}$ and $\{p'\} \succ \{p', q'\}$. Hence, by Mixture-Betweenness,

$$\begin{aligned} \{\lambda^* p + (1 - \lambda^*) p'\} & \succ \lambda^* \{p\} + (1 - \lambda^*) \{p', q'\} \\ & = \{\lambda^* p + (1 - \lambda^*) p', \lambda^* p + (1 - \lambda^*) q'\} \\ & = \{\lambda^* p + (1 - \lambda^*) p', \lambda^* q + (1 - \lambda^*) p'\} \\ & = \lambda^* \{p, q\} + (1 - \lambda^*) \{p'\} \sim \{\lambda^* p + (1 - \lambda^*) p'\}, \end{aligned}$$

a contradiction.

To conclude the proof we outline the construction of $v(\cdot, 0)$. Let $\mathcal{Q} = \{q \in \Delta(X) \mid \{p\} \succeq \{q\} \text{ for all } p \in \Delta(X)\}$. There are two possible cases: (i) $\{p, q\} \sim \{q\}$ for all $p \in \Delta(X)$ and $q \in \mathcal{Q}$, and (ii) there exist $p \in \Delta(X)$ and $q \in \mathcal{Q}$ such that $\{p, q\} \succ \{q\}$.

(i) $\{p, q\} \sim \{q\}$ for all $p \in \Delta(X)$ and $q \in \mathcal{Q}$

Let $v(\cdot, 0) = -u(\cdot, 0)$. Then,

$$\max_{p' \in \{p, q\}} \{u(p', 0) + v(p', 0) - \max_{q' \in \{p, q\}} v(q', 0)\} = \min_{p' \in \{p, q\}} u(p', 0).$$

Further,

$$\min_{p' \in \{p, q\}} u(p', 0) = 0 \iff p \in \mathcal{Q} \text{ or } q \in \mathcal{Q} \iff V(\{p, q\}) = 0.$$

(ii) There exist $p \in \Delta(X)$ and $q \in \mathcal{Q}$ such that $\{p, q\} \succ \{q\}$

Define \succeq_T over $\Delta(X)$ as follows:

$q \succeq_T p$ if and only if $\{p\} \succ \{p, q\} \sim \{q\}$ and $q \in \mathcal{Q}$

$p \succ_T q$ if and only if $\{p, q\} \succ \{q\}$ and $q \in \mathcal{Q}$.

An identical argument to the one in the construction of $v(\cdot, 1)$ shows there exists v' such that $p \succeq_T q$ implies $v'(p) \geq v'(q)$, $p \succ_T q$ implies $v'(p) > v'(q)$. Let $v(\cdot, 0) = -u(\cdot, 0) + v'$, we will now show that

$$0 = \max_{p' \in \{p, q\}} \{u(p, 0) + v(p, 0) - \max_{q' \in \{p, q\}} v(q, 0)\} \text{ if and only if } V(\{p, q\}) = 0. \quad (7)$$

To prove this, let

$$\Phi(x, 0) = \max_{p \in x} \{u(p, 0) + v(p, 0) - \max_{q \in x} v(q, 0)\}.$$

First, assume $V(\{p, q\}) = 0$. If $p, q \in \mathcal{Q}$, then $u(p, 0) = u(q, 0) = 0$. Thus, by Set-Betweenness, $\Phi(\{p, q\}, 0) = 0$. If $q \in \mathcal{Q}$ and $p \notin \mathcal{P}$, then, by construction, $-u(q, 0) > -u(p, 0)$ and $v(q, 0) \geq v(p, 0)$. Thus, $\Phi(\{p, q\})$ is equal to

$$v'(q) - v'(q) + u(q, 0) = u(q, 0) = 0.$$

Hence, $\Phi(\{p, q\}, 0) = 0$. Finally, note that if $p, q \notin \mathcal{Q}$, then $V(\{p, q\}) > 0$.

Next, assume towards a contradiction that $\Phi(\{p, q\}, 0) = 0$ and $V(\{p, q\}) > 0$. If $p, q \notin \mathcal{Q}$, then $u(p, 0) > 0$ and $u(q, 0) > 0$. Hence, by Set-Betweenness,

$\Phi(\{p, q\}, 0) > 0$. Therefore, either $p \in \mathcal{Q}$ or $q \in \mathcal{Q}$. If $p, q \in \mathcal{Q}$, then $V(\{p\}) = V(\{q\}) = 0$. Hence, by Set-Betweenness, $V(\{p, q\}) = 0$ a contradiction. If $q \in \mathcal{Q}$ and $p \notin \mathcal{Q}$, then $V(\{p, q\}) > 0$ implies $v'(p) > v'(q)$. Hence, $\Phi(\{p, q\}, 0)$ is equal to either $v'(p) - v'(q)$ or $u(p, 0)$. Either way, $\Phi(\{p, q\}, 0) \neq 0$, a contradiction.

B.2 Necessity

The proof of necessity of Weak Order and Hausdorff Continuity is routine. Further, the proof of necessity of Mixture-Betweenness is identical to the proof given in Appendix A. Hence, we only show that the representation satisfies Set-Betweenness.

Assume (u, v) represent \succeq . Fix x, y such that $V(x) \geq V(y)$,

$$p_z \in \arg \max_{p \in z} \{u(p, V(z)) + v(p, V(z))\} \text{ and } q_z \in \arg \max_{q \in z} v(q, V(z))$$

for $z \in \{x, y, x \cup y\}$. Then, by the unique solution property, $V(z) = V(\{p_z, q_z\})$ for all $z \in \{x, y, x \cup y\}$.

There are 4 possible cases: (i) $p_{x \cup y}, q_{x \cup y} \in x$. (ii) $p_{x \cup y}, q_{x \cup y} \in y$. (iii) $p_{x \cup y} \in x \setminus y, q_{x \cup y} \in y \setminus x$. (iv) $p_{x \cup y} \in y \setminus x, q_{x \cup y} \in x \setminus y$.

If $p_{x \cup y}, q_{x \cup y} \in x$, then $V(x) = V(x \cup y)$ and if $p_{x \cup y}, q_{x \cup y} \in y$, then $V(x \cup y) = V(y)$. Hence we only consider cases (iii) and (iv).

(iii) $p_{x \cup y} \in x \setminus y, q_{x \cup y} \in y \setminus x$.

By definition of $p_{x \cup y}$ and $q_{x \cup y}$,

$$\begin{aligned} V(x) &\geq \max_{p \in \{p_{x \cup y}, q_x\}} \{u(p, V(x)) + v(p, V(x))\} - \max_{q \in \{p_{x \cup y}, q_x\}} v(q, V(x)) \\ V(x \cup y) &\leq \max_{p \in \{p_{x \cup y}, q_x\}} \{u(p, V(x \cup y)) + v(p, V(x \cup y))\} - \max_{q \in \{p_{x \cup y}, q_x\}} v(q, V(x \cup y)) \\ V(x \cup y) &\geq \max_{p \in \{p_y, q_{x \cup y}\}} \{u(p, V(x \cup y)) + v(p, V(x \cup y))\} - \max_{q \in \{p_y, q_{x \cup y}\}} v(q, V(x \cup y)) \\ V(y) &\leq \max_{p \in \{p_y, q_{x \cup y}\}} \{u(p, V(y)) + v(p, V(y))\} - \max_{q \in \{p_y, q_{x \cup y}\}} \{v(q, V(y))\}. \end{aligned}$$

Therefore, by Lemma A.6, $V(x) \geq V(\{p_{x \cup y}, q_x\}) \geq V(x \cup y) \geq V(\{p_y, q_{x \cup y}\}) \geq V(y)$ so $V(x) \geq V(x \cup y) \geq V(y)$.

(iv) $p_{x \cup y} \in y \setminus x, q_{x \cup y} \in x \setminus y$.

In this case, by definition of $p_{x \cup y}$ and $q_{x \cup y}$,

$$\begin{aligned} V(y) &\geq \max_{p \in \{p_{x \cup y}, q_y\}} \{u(p, V(y)) + v(p, V(y))\} - \max_{q \in \{p_{x \cup y}, q_y\}} v(q, V(y)) \\ V(x \cup y) &\leq \max_{p \in \{p_{x \cup y}, q_y\}} \{u(p, V(x \cup y)) + v(p, V(x \cup y))\} - \max_{q \in \{p_{x \cup y}, q_y\}} v(q, V(x \cup y)) \\ V(x \cup y) &\geq \max_{p \in \{p_x, q_{x \cup y}\}} \{u(p, V(x \cup y)) + v(p, V(x \cup y))\} - \max_{q \in \{p_x, q_{x \cup y}\}} v(q, V(x \cup y)) \\ V(x) &\leq \max_{p \in \{p_x, q_{x \cup y}\}} \{u(p, V(x)) + v(p, V(x))\} - \max_{q \in \{p_x, q_{x \cup y}\}} v(q, V(x)) \end{aligned}$$

Hence, Lemma A.6 implies that $V(y) \geq V(\{p_{x \cup y}, q_y\}) \geq V(x \cup y) \geq V(\{p_x, q_{x \cup y}\}) \geq V(x) \geq V(y)$ so $V(x) = V(x \cup y) = V(y)$.

B.3 Uniqueness

Sufficiency is given in the text. Here we prove necessity.

Assume (u, v) and (u', v') represent \succeq . Fix $\gamma \in (0, 1)$ and let $\Phi(\cdot, \gamma)$ and $\Phi(\cdot, \gamma')$ denote the GP functionals induced by $(u(\cdot, \gamma), v(\cdot, \gamma))$ and $(u(\cdot, \gamma'), v(\cdot, \gamma'))$ respectively.

By an identical argument to the one in the uniqueness part of Theorem 6.3, $\Phi(x, \gamma) = \Phi'(x, \gamma)$ for all $x \in \mathcal{K}(\Delta(X))$ and $\gamma \in (0, 1)$. In particular, $\Phi(\{p\}, \gamma) = \Phi'(\{p\}, \gamma)$ for all $p \in \Delta(X)$ and $\gamma \in (0, 1)$. Hence, $u = u'$. Fix $\gamma \in (0, 1)$. If $v(\cdot, \gamma)$ is a constant or a positive affine transformation of $u(\cdot, \gamma)$. Then,

$$\Phi'(x, \gamma) = \max_{p \in x} \{u(p, \gamma)\}$$

for all $x \in X$. Thus, by GP (p. 1414), either $v'(\cdot, \gamma)$ is a constant or a positive affine transformation of $u(\cdot, \gamma)$.

If $v(\cdot, \gamma)$ is not a constant or a positive affine transformation of $u(\cdot, \gamma)$. Then, by GP (p.1414), there are two cases: either there exist p, q such that $\Phi(\{p\}, \gamma) > \Phi(\{p, q\}, \gamma) > \Phi(\{q\}, \gamma)$ or $\Phi(\{p\}, \gamma) > \Phi(\{p, q\}, \gamma) = \Phi(\{q\}, \gamma)$ for all p, q such that $\Phi(\{p\}, \gamma) > \Phi(\{q\}, \gamma)$. If there exist p, q such that $\Phi(\{p\}, \gamma) > \Phi(\{p, q\}, \gamma) > \Phi(\{q\}, \gamma)$, then since $u(\cdot, \gamma)$ is unique, by GP's Theorem 4, $v'(\cdot, \gamma) = v(\cdot, \gamma) + b_\gamma$. If $\Phi(\{p\}, \gamma) > \Phi(\{p, q\}, \gamma) = \Phi(\{q\}, \gamma)$ for all p, q such that $\Phi(\{p\}, \gamma) > \Phi(\{q\}, \gamma)$, then by GP (p.1414), $v(\cdot, \gamma)$ and $v'(\cdot, \gamma)$ are negative affine transformation of $u(\cdot, \gamma)$. Hence, we only need to rule out the case in which $v(\cdot, \gamma) = -a_\gamma u(\cdot, \gamma) + b_\gamma$ and $a_\gamma \in (0, 1)$. To see that this is impossible let p_γ be such that $\gamma = u(p_\gamma, \gamma)$. Then, $u(\bar{p}, \gamma) = 1 > u(p_\gamma, \gamma)$. Hence, $\Phi(\{p_\gamma, \bar{p}\}, \gamma) = \gamma$. However, if $a_\gamma < 1$, then

$$u(\bar{p}, \gamma) - a_\gamma u(\bar{p}, \gamma) = 1 - a_\gamma > u(p_\gamma, \gamma) - a_\gamma u(p_\gamma, \gamma) = \gamma - a_\gamma \gamma.$$

Hence,

$$\Phi(\{p_\gamma, \bar{p}\}, \gamma) = 1 - a_\gamma + a_\gamma \gamma \neq \gamma.$$

Therefore, $a_\gamma \geq 1$.

B.4 Proof of Proposition 4.1

Necessity is straightforward. Here we prove sufficiency. Recall that for any p , $V(p)$ is the unique scalar γ such that

$$\{p\} \sim \gamma\{\bar{p}\} + (1 - \gamma)\{\underline{p}\}.$$

Fix $\gamma \in (0, 1)$ and let \succeq_γ be the preference over lotteries represented by $u(\cdot, \gamma)$. Note that $\{\{p\}|\{p\} \sim_\gamma \gamma\{\bar{p}\} + (1 - \gamma)\{\underline{p}\}\} = \{\{p\}|\{p\} \sim \gamma\{\bar{p}\} + (1 - \gamma)\{\underline{p}\}\}$. Further, $u(\bar{p}, \gamma) = 1 > u(\underline{p}, \gamma) = 0$. Hence, by Lemma A.7, for the restriction of \succeq to singleton menus, $\succeq_\gamma = \succeq$ for all $\gamma \in (0, 1)$. By the vNM Expected Utility Theorem, there exists a unique expected utility function u^* such that u^* represents the restriction of \succeq to singleton menus, $u^*(\bar{p}) = 1$ and $u^*(\underline{p}) = 0$. Hence $u^* = u(\cdot, \gamma)$ for all $\gamma \in (0, 1)$. Finally, by continuity, $u(\cdot, 1) = u(\cdot, 0) = u^*$.

B.5 Proof of Proposition 5.1

Let (u, v, c) represent \succeq . Then, by Theorem 1 in Noor and Takeoka (2010), \succeq satisfies Commitment Independence and Set-Betweenness. In what follows we will use the following property of c (Noor and Takeoka (2010 p.134)):

4. $c(p, v(p)) = 0$ for all $p \in \Delta(X)$. For any menu z , let q_z denote an arbitrary element of $\arg \max_{p \in z} v(p)$.

Part 1

Assume towards a contradiction that \succeq has preference for commitment at x but not at $\alpha x + (1 - \alpha)y$. Then, there exist $p_x \in \arg \max_{p \in x} u(p)$ and $p_y \in \arg \max_{p \in y} u(p)$ such that

$$c(\alpha p_x + (1 - \alpha)p_y, \alpha v(q_x) + (1 - \alpha)v(q_y)) = 0.$$

Further, since \succeq has preference for commitment at x , $c(p_x, v(q_x)) > 0$ and $u(p_x) > u(q_x)$. Hence, by Property 2, $v(q_x) > v(p_x)$. Therefore,

$$u(\alpha p_x + (1 - \alpha)p_y) > u(\alpha q_x + (1 - \alpha)q_y)$$

and

$$v(\alpha q_x + (1 - \alpha)q_y) > v(\alpha p_x + (1 - \alpha)p_y).$$

Thus, by Property 3, $c(\alpha p_x + (1 - \alpha)p_y, \alpha v(q_x) + (1 - \alpha)v(q_y)) > 0$ a contradiction.

Part 2

Assume towards a contradiction that \succeq does not have preference for commitment at x and y but has preference for commitment at $\alpha x + (1 - \alpha)y$. Then, there exist $p_x \in \arg \max_{p \in x} u(p)$ and $p_y \in \arg \max_{p \in y} u(p)$ such that $c(p_x, v(q_x)) = 0$ and $c(p_y, v(q_y)) = 0$. Since \succeq has preference for commitment at $\alpha x + (1 - \alpha)y$, then $c(\alpha p_x + (1 - \alpha)p_y, \alpha v(q_x) + (1 - \alpha)v(q_y)) > 0$. Hence, by Property 2, $\alpha v(q_x) + (1 - \alpha)v(q_y) > \alpha v(p_x) + (1 - \alpha)v(p_y)$. Thus, either $v(q_x) > v(p_x)$ or $v(q_y) > v(p_y)$. Since the proof that $v(q_y) > v(p_y)$ leads to a contradiction is analogous to the proof that $v(q_x) > v(p_x)$ leads to a contradiction, we only consider the case in which $v(q_x) > v(p_x)$.

By Property 2, $v(q_x) > v(p_x)$ implies $u(q_x) = u(p_x)$. Further, by Property 4, $\alpha u(p_x) + (1 - \alpha)u(p_y) > \alpha u(q_x) + (1 - \alpha)u(q_y)$. Hence, $u(p_y) > u(q_y)$. Therefore, it must be the case that $c(\alpha q_x + (1 - \alpha)p_y, \alpha v(q_x) + (1 - \alpha)v(p_y)) > 0$. Thus, by Property 2,

$$\alpha v(q_x) + (1 - \alpha)v(p_y) > v(\alpha q_x + (1 - \alpha)p_y)$$

a contradiction.

B.6 Mixture Monotone Preference for Commitment without Commitment Independence

The proof of Theorems 5.1 and 5.2 are analogous. Thus, we only show the proof of the latter. Further, the proof of necessity is identical to the one of Theorem S.5.2 in the online appendix. Hence, we only show sufficiency of the axioms.

Sufficiency

Let (u, v') be the representation of \succeq constructed using $\{p^*\}$ and $\{p_*\}$ as the best and worst menus. Then, $\bar{p} = p^*$ and $\underline{p} = p_*$. By the uniqueness properties of (u, v') there exists v such that if $v'(\cdot, \gamma)$ is a constant or a positive affine transformation of $u(\cdot)$, then $v(\cdot, \gamma) = u(\cdot)$ and (u, v) represents \succeq . Let \succeq_γ be the preference over menus represented by the GP functional induced by $(u(\cdot), v(\cdot, \gamma))$.

An identical argument to the one in the proof of Theorem S.5.2 in the online appendix establishes that $v(\cdot, \gamma)$ is not a negative affine transformation

of $u(\cdot)$ and that if \succeq_γ has preference for commitment at y and $1 > \gamma > \gamma' > 0$, then $\succeq_{\gamma'}$ also has preference for commitment at y .

Fix $\gamma \in (0, 1)$. If \succeq_γ does not have preference for commitment, then $u = v(\cdot, \gamma)$. Therefore, $v(\cdot, \gamma)$ is a convex combination of u and $v(\cdot, \gamma')$. If \succeq_γ has preference for commitment at some menu, then by Theorems 4 and 8 in GP, there exists a joint positive affine transformation of $(u(\cdot), v(\cdot, \gamma))$ such that $v(\cdot, \gamma)$ is a convex combination of $u(\cdot)$ and $v(\cdot, \gamma')$.⁷⁸

To conclude, notice that if the coefficient multiplying $u(\cdot)$ of the joint affine transformation of $(u, v(\cdot, \gamma))$ is not equal to 1, then $v(\cdot, \gamma)$ is an affine transformation of u , an impossibility.

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⁷Let $f, g, f', g : \Delta(X) \rightarrow \mathbb{R}$. Say that (f, g) are a joint positive affine transformation of (f', g') if there exists $a \in \mathbb{R}_{++}, b_f, b_g \in \mathbb{R}$ such that $f = af' + b_f$ and $g = ag' + b_g$.

⁸Here we are citing the version of Theorem 8 in Gul and Pesendorfer (2001) stated in Gul and Pesendorfer (2004).

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