Optimal Monetary Policy with $r^* < 0$

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Abstract

We study the optimal monetary policy problem in a New Keynesian economy with a zero lower bound (ZLB) on the nominal interest rate, and in which the steady state natural rate (r^*) is negative. We show that the optimal policy aims to approach gradually a steady state with positive average inflation. Around that steady state, inflation and output fluctuate optimally in response to shocks to the natural rate. The central bank can implement that optimal outcome by means of an appropriate state-contingent rule, even though in equilibrium the nominal rate remains at zero most (or all) of the time. In order to establish that result, we derive sufficient conditions for local determinacy in a more general model with endogenous regime switches.

JEL Classification Numbers: E32, E52

Keywords: zero lower bound, New Keynesian model, decline in r*, equilibrium determinacy, regime switching models, secular stagnation

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1 Introduction

Over the past decade, a growing consensus has emerged among academic economists and policy-makers pointing to a substantial decline in the *natural* rate of interest in advanced economies.[1] Some of the likely sources of that decline, including a lower trend growth rate of productivity and demographic factors, suggest that the decline in the natural rate is likely to be highly persistent, or even permanent.[2]

A persistent decline in the average natural rate of interest –which, following convention, we henceforth refer to as r^*– has important implications for monetary policy, due to the presence of a zero lower bound (ZLB) on the nominal interest rate. Thus, and given the inflation target, a lower r^* will generally hamper the ability of monetary policy to stabilize the economy, bringing about more frequent episodes in which the ZLB becomes binding and the economy plunges into a recession with below-target inflation. Not surprisingly, the evidence of a decline in r^* has been a key motivation behind the monetary policy strategy reviews undertaken by many central banks in recent years.

On the research front, and as discussed in the literature review below, several authors have studied the problem of optimal monetary policy in the face of shocks that drive the natural rate of interest *temporarily* into negative territory. A common finding of those analyses is that the central bank finds it desirable to keep the short-term nominal rate at zero during those episodes, even for some time once the natural rate has returned to positive values. In all of those analyses, however, the natural rate tends to gravitate towards a positive mean, i.e. $r^* > 0$. By contrast, in the present paper we study the problem of optimal monetary policy under the ZLB constraint when the natural rate fluctuates around a mean that is *permanently negative*, i.e. $r^* < 0$.

As discussed below, that environment is of particular interest since the optimal policy implies *a binding ZLB constraint in the steady state*, a feature that is absent from conventional analyses that assume a positive steady state real rate. While the assumption of a negative r^* is at odds with the predictions of a standard macro framework with an infinite-lived representative consumer, it can be microfounded once the latter assumption is relaxed, e.g. by assuming an

[1] See, e.g. Brand and Mazelis (2019), Del Negro et al. (2019), Holston et al. (2017).

[2] See, e.g., Eggertsson et al. (2019).

overlapping generations structure. Furthermore, we believe the assumption of a negative r^* is more than a theoretical curiosum: recent estimates of the evolution of the natural rate in advanced economies display a downward trend that has already attained negative territory for some of them.[3] But even if one negates the current relevance of a negative r^*, that relevance can hardly be dismissed as a real possibility in a not too distant future, if the trends in some of the fundamental forces behind the recent decline in the natural rate were to persist or even strengthen further.

As much of the related literature, we cast our analysis of the optimal monetary policy problem in the context of an otherwise standard New Keynesian model subject to a ZLB constraint. A number of interesting results emerge from that exercise. Firstly, our findings show that the optimal policy aims at steering the economy *gradually* towards a steady state characterized by positive inflation, a positive output gap and a zero nominal rate. Thus, and even though the combination of a negative r^* and the ZLB constraint rules out the first-best outcome of a zero inflation steady state, the choice of a gradual transition (rather than an immediate jump to the new steady state) makes it possible for inflation to remain closer to zero –its efficient value– for a longer period, which is welfare improving.

Secondly, once the steady state is reached, inflation and output fluctuate optimally in response to shocks to the natural rate, even though the nominal rate remains at zero most of the time (*all* the time in our baseline calibration). We show that, behind the appearance of extreme passivity, the central bank can implement the optimal outcome by means of an appropriate state-contingent rule which calls for one-sided adjustments in the nominal rate in response to (off-equilibrium) deviations from the desired inflation and output paths. In order to establish that result, we derive and exploit a sufficient condition for local determinacy for a relatively general class of models with endogenous regime switches, a finding which we believe has some independent interest.

The rest of the paper is organized as follows. The remaining of the present section provides a brief review of the related literature. Section 2 formulates the optimal policy problem and derives the associated optimality conditions. Section 3 analyzes the economy's (deterministic) transitional dynamics under the optimal policy. Section 4 characterizes the fluctuations of infla-

[3]See, e.g., Brand and Mazelis (2019)

tion and output around the steady state, in response to natural rate shocks. Section 5 discusses the implementation of the optimal plan, deriving sufficient conditions on the coefficients of a proposed interest rate rule to support the optimal plan as a unique equilibrium. Section 6 concludes.

1.1 Related Literature

Our paper is related to a branch of the literature that studies the optimal design of monetary policy in the presence of a ZLB constraint on the nominal rate. Since Krugman (1998), a number of articles have studied optimal monetary policy with an occasionally binding zero lower bound (ZLB) on the nominal interest rate. Closest to us is the work by Eggertsson and Woodford (2003), Jung, Teranishi and Watanabe (2005), Adam and Billi (2006), and Nakov (2008), who analyze the problem of optimal policy under commitment in the basic New Keynesian model with a ZLB constraint. A different line of work has focused on the implications of the ZLB for the optimal choice of an inflation target, conditional on a given interest rate rule. Relevant papers include Coibion et al. (2012), Bernanke et al. (2019), and Andrade et al. (2020, 2021). In all the papers above, however, the natural interest rate becomes negative only temporarily, and the binding ZLB is a transitory phenomenon. In contrast, the analysis of the present paper assumes a negative steady state natural rate, and hence a long-lasting "secular stagnation" environment with a ZLB that is binding in steady state.

The present paper is also related to a rather different branch of the literature, one studying the conditions for equilibrium determinacy in regime-switching models. Applications of this literature have typically focused on regime switches driven by stochastic variations in the coefficients of a Taylor-type interest rate rule, which are often assumed to follow a finite-state Markov process. Prominent examples include Davig and Leeper (2007), Farmer et al. (2009) and Barthélemy and Marx (2019). The main difference in our approach is that we allow for *endogeneity* in the regime switches, i.e. the regime is a function of the state.[4] That endogeneity arises as a consequence of the particular nonlinearity embedded in the interest rate rule that implements the optimal allocation, which makes the effective coefficients of the corresponding

[4] Barthélemy and Marx (2017) also allow for endogeneity of the regime switches but only of a sort with continuous transition probabilities, which rules out the threshold switches that arise naturally in models with a ZLB constraint like ours.

linear model depend on the levels of inflation and output.[5]

2 The Optimal Monetary Policy Problem

The equilibrium conditions describing the economy's non-policy block are assumed to be given by

$$\pi_t = \beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t \tag{1}$$

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - r_t^n) \tag{2}$$

for $t = 0, 1, 2, ...$where π_t denotes inflation, y_t is the output gap, i_t is the short-term nominal rate and r_t^n is the natural rate of interest. Equation (1) is the familiar New Keynesian Phillips curve, which can be derived from the aggregation of firms' price setting decisions in an environment with price rigidities à la Calvo (1983). Equation (2) is the so-called dynamic IS equation, which results from combining an Euler equation for (log) aggregate consumption, a goods market clearing condition and an equation describing the evolution of output and the real interest rate under flexible prices.[6]

Variations in the natural rate of interest r_t^n are assumed to be described by

$$r_t^n = r^* + z_t \tag{3}$$

where $\{z_t\}$ follows an exogenous $AR(1)$ process with zero mean, autoregressive coefficient ρ_z and innovation variance σ_z^2. The unconditional mean of the natural rate is given by r^*, which coincides with the real interest rate in the deterministic steady state. Henceforth, we assume

$$r^* < 0 \tag{4}$$

In a companion appendix, we formally describe an environment where (1) and (2) obtain as equilibrium conditions, and where the steady state real interest rate may be negative. The proposed environment is a version of a New Keynesian model with overlapping generations à la

[5] One drawback of our approach, of limited consequence in our particular application, is that it only allows us to derive *sufficient* conditions for determinacy, i.e. we cannot establish necessity, in contrast with the papers mentioned above.

[6] See, e.g. Woodford (2003) or Galí (2015) for a derivation of (1) and (2) in a standard New Keynesian model. In a companion appendix, we show that similar equilibrium conditions obtain in an OLG version of the New Keynesian model that allows for a negative steady state real rate, as considered below.

Blanchard-Yaari, as developed in Galí (2021). In that environment the steady state real interest is not fully pinned down by the discount rate; instead it also depends on the extent to which income of any given cohort declines over time as a result of retirement or other shocks that make individuals leave employment permanently (e.g. skill obsolescence). That phenomenon tends to enhance savings, lowering the steady state real rate, which may take a negative value.

The monetary authority is assumed to choose at $t = 0$ a state-contingent sequence $\{y_t, \pi_t\}_{t=0}^{\infty}$ that minimizes the welfare loss function

$$\frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + \vartheta y_t^2\right)$$

subject to the sequence of constraints (1) and (2), as well the ZLB constraint

$$i_t \geq 0 \tag{5}$$

all for $t = 0, 1, 2, ..$[7]

Note that the ZLB constraint can be rewritten in terms of inflation and the output gap as:

$$r_t^n + \mathbb{E}_t\{\pi_{t+1}\} + \sigma(\mathbb{E}_t\{y_{t+1}\} - y_t) \geq 0 \tag{6}$$

for $t = 0, 1, 2, ..$

The (discounted) Lagrangian is given by:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{2}\left(\pi_t^2 + \vartheta y_t^2\right) - \xi_{1,t}(\pi_t - \kappa y_t - \beta\pi_{t+1}) - \xi_{2,t}[\pi_{t+1} + \sigma(y_{t+1} - y_t)]\right]$$

The associated optimality conditions are:

$$\pi_t = \xi_{1,t} - \xi_{1,t-1} + \beta^{-1}\xi_{2,t-1} \tag{7}$$

$$\vartheta y_t = -\kappa\xi_{1,t} - \sigma\xi_{2,t} + \sigma\beta^{-1}\xi_{2,t-1} \tag{8}$$

$$\xi_{2,t} \geq 0 \tag{9}$$

$$\xi_{2,t}\ \left[r_t^n + \mathbb{E}_t\{\pi_{t+1}\} + \sigma(\mathbb{E}_t\{y_{t+1}\} - y_t)\right] = 0 \tag{10}$$

[7] As discussed in the companion appendix, the previous loss function can be microfounded as the second order approximation to the expected welfare losses of individuals currently alive in a New Keynesian model with overlapping generations.

which should be interpreted as holding for each time period and each possible state. The previous conditions, combined with (1), (2), (3), (6) and initial conditions $\xi_{1,-1} = \xi_{2,-1} = 0$, describe the economy's equilibrium under the optimal policy.

In the next two sections, we characterize that equilibrium and provide simulations for a calibrated version of the model. First we study the transitional dynamics. Then we look at the economy's response to shocks in a neighborhood of the steady state.

3 Transitional Dynamics under the Optimal Monetary Policy

In the present section we focus on the transitional dynamics implied by the optimal policy. For simplicity, we focus on the deterministic case, with $r_t^n = r^* < 0$ for $t = 0, 1, 2, ...$

We start by characterizing the perfect foresight steady state under the optimal policy. In the (deterministic) steady state we must have $i = \pi + r \geq 0$ or, equivalently, $\pi \geq -r^* > 0$. In addition, it follows from (7)-(10) that under the optimal policy:

$$\pi = \beta^{-1}\xi_2 \geq 0$$

$$\vartheta y = -\kappa\xi_1 + \sigma(\beta^{-1} - 1)\xi_2$$

$$\xi_2 \geq 0 \ ; \ \xi_2(r^* + \pi) = 0$$

It is easy to check that the optimal policy requires that $i = 0$. To see this, note that if $i > 0$ then $\xi_2 = 0$ implying $\pi = 0$, which is inconsistent with a steady state. Thus the steady state under the optimal policy must satisfy:

$$\pi = -r^* > 0$$

$$y = \frac{1-\beta}{\kappa}\pi = -\frac{1-\beta}{\kappa}r^* > 0$$

$$\xi_2 = \beta\pi = -\beta r^* > 0$$

$$\begin{aligned} \xi_1 &= -\frac{\vartheta}{\kappa}y + \frac{\sigma(\beta^{-1} - 1)}{\kappa}\xi_2 \\ &= -\frac{(1-\beta)}{\kappa}\left(\sigma - \frac{\vartheta}{\kappa}\right)r^* \end{aligned}$$

Next we study the transitional dynamics, i.e. we characterize the equilibrium paths that satisfy

$$\widehat{\pi}_t = \beta\widehat{\pi}_{t+1} + \kappa\widehat{y}_t$$

$$\widehat{\pi}_t = \widehat{\xi}_{1,t} - \widehat{\xi}_{1,t-1} + \beta^{-1}\widehat{\xi}_{2,t-1}$$

$$\vartheta\widehat{y}_t = -\kappa\widehat{\xi}_{1,t} - \sigma\widehat{\xi}_{2,t} + \sigma\beta^{-1}\widehat{\xi}_{2,t-1}$$

$$\widehat{\xi}_{2,t} + \xi_2 \geq 0$$

$$\widehat{\pi}_{t+1} + \sigma(\widehat{y}_{t+1} - \widehat{y}_t) \geq 0$$

$$(\widehat{\xi}_{2,t} + \xi_2)\ [\widehat{\pi}_{t+1} + \sigma(\widehat{y}_{t+1} - \widehat{y}_t)] = 0$$

for $t = 0, 1, 2, ...$with initial conditions $\widehat{\xi}_{1,-1} = -\xi_1$ and $\widehat{\xi}_{2,-1} = -\xi_2$, and such that $\lim_{t\to\infty}\widehat{x}_t = 0$ for $\widehat{x}_t \in \{\widehat{\pi}_t, \widehat{y}_t, \widehat{\xi}_{1,t}, \widehat{\xi}_{2,t}\}$, where a "$\widehat{\ }$" symbol on a variable denotes deviations from the corresponding steady state value. In Appendix A we describe our approach to solving the above system of difference equations.

Figure 1 illustrates the transitional dynamics for a calibrated version of our economy. In particular, we assume $\sigma = 1$, $\beta = 0.99$, $\kappa = 0.1717$, $\vartheta = 0.0191$, which are values consistent with the baseline calibration in Galí (2015). In addition, we set $r = -0.0025$, implying an annualized steady state natural rate of minus 1 percent. Interest rates and the inflation rate are shown in annualized terms.

As shown in Figure 1, the transition to the steady state under the optimal policy is not immediate. Instead, the initial values of inflation and the output gap are significantly below their long run values of 1 and 0.058 percent, respectively, and adjust only gradually towards that steady state. In fact, inflation is negative for a few periods under our baseline calibration.[8] By choosing a path like the one depicted in Figure 1, the central bank succeeds in keeping inflation close to the first best temporarily, even though it is at the cost of a persistently negative output gap. Given the relative small weight of the latter under our baseline calibration ($\vartheta \simeq 0.02$), that choice turns out to be more desirable than jumping immediatly to the steady state (which would be perfectly feasible). The persistent low inflation and output gaps are consistent with the observed path for the real rate, which remains above its long run value r during the transition.

[8]The result of an optimal negative inflation in the short run is not general. In particular, it doesn't obtain when the weight on the output gap is raised sufficiently (e.g. when $\vartheta = 1$).

Most interestingly, the path for the real rate is entirely driven by expected inflation, since the nominal rate remains at the ZLB throughout the transition. Thus, the central bank manages to implement its nontrivial optimal plan while keeping the setting for its policy instrument unchanged. In section 4 below, we discuss how the central bank may implement the optimal outcome, given the multiplicity of equilibrium paths consistent with a constant nominal rate.

4 Aggregate Fluctuations under the Optimal Monetary Policy

In this section, we characterize the behavior of inflation and the output gap under the optimal policy in a neighborhood of the steady state, in response to shocks to the natural rate (i.e. fluctuations in z_t). The (local) equilibrium dynamics are described by the system of stochastic difference equations given by:

$$\widehat{\pi}_t = \beta\mathbb{E}_t\{\widehat{\pi}_{t+1}\} + \kappa\widehat{y}_t$$

$$\widehat{\pi}_t = \widehat{\xi}_{1,t} - \widehat{\xi}_{1,t-1} + \beta^{-1}\widehat{\xi}_{2,t-1}$$

$$\vartheta\widehat{y}_t = -\kappa\widehat{\xi}_{1,t} - \widehat{\xi}_{2,t} + \beta^{-1}\widehat{\xi}_{2,t-1}$$

$$\widehat{\xi}_{2,t} + \xi_2 \geq 0$$

$$\sigma(\mathbb{E}_t\{\widehat{y}_{t+1}\} - \widehat{y}_t) + \mathbb{E}_t\{\widehat{\pi}_{t+1}\} + z_t \geq 0$$

$$[\widehat{\xi}_{2,t} + \xi_2][\sigma(\mathbb{E}_t\{\widehat{y}_{t+1}\} - \widehat{y}_t) + \mathbb{E}_t\{\widehat{\pi}_{t+1}\} + z_t] = 0$$

for $t = 0, 1, 2, ...$with initial conditions now given by $\widehat{\xi}_{1,-1} = 0$ and $\widehat{\xi}_{2,-1} = 0$. Appendix B describes our approach to determining the solution to the system above.

Figure 2 displays the equilibrium path for inflation and the output gap under the optimal policy, given a sequence of realized values of the shock $\{z_t\}$, drawn from an $AR(1)$ process with $\rho_z = 0.5$ and $\sigma_z = 0.0025$. The remaining parameters are kept at their baseline settings. The top-left box of the Figure displays the simulated path of the natural rate (in black) and the actual real rate (in blue). Note that the latter is much smoother than the former, which reflects the central bank's inability to match one-for-one fluctuations in the natural rate, due to the ZLB constraint. As a result, the central bank can't prevent some fluctuations in inflation and the output gap around the steady state, as illustrated in the two bottom plots. Furthermore the nominal rate remains at the ZLB throughout the simulation, as shown on the top-right plot. Thus, the central bank manages to steer the economy along the optimal path without changing the settings for its policy instrument, and keeping it instead constant at its steady state level. The reason why it does not lower the nominal rate in the face of negative natural rate shocks is clear: the ZLB prevents it from doing so. Perhaps less obvious is why it keeps the nominal rate at zero even when the natural rate lies above its steady state value. Intuitively, the anticipation that the central bank will keep the interest rate lower than the natural rate when the latter is high helps stabilize inflation and the output gap when the natural rate is low (and can thus not be matched due to the ZLB). That policy, which relies on the forward looking nature of aggregate demand and inflation, can thus be viewed as a form of forward guidance. In the simulation shown in Figure 2, the contemporaneous stabilizing gains from raising the nominal rate above zero, in order to bring it closer to the natural rate when the latter is high do not compensate the gains in earlier periods with a low natural rate from the anticipation of a constant zero nominal rate in the future. As a result, the nominal rate remains at the ZLB throughout the simulation.

The previous property is not general, however. In particular, the central bank may find it desirable to deviate from the constant zero nominal rate policy in response to an increase in the natural rate of interest that is sufficiently large and which may thus induce very high inflation if not counteracted at least partly by an increase in the nominal rate. This is illustrated in Figure 3, which shows a simulation of equilibrium fluctuations in a calibrated economy identical to that underlying the simulations of Figure 2 except for a higher shock volatility, with $\sigma_z = 0.0075$. Thus, in the simulation shown in Figure 3 there are three episodes in which the central bank optimally chooses to raise the nominal rate above zero, even if only briefly. Roughly speaking, those episodes can be seen to take place when two conditions are met simultaneously: (i) the natural interest rate is unusually high, and (ii) this has not been preceded by a recent episode with an unusually low natural rate, for in the latter case it would be desirable to keep the nominal rate "low for longer" for the reasons discussed above. Note, however, that the nominal rate remains unchanged at the ZLB for much of the simulation.

How the central bank manages to steer the economy as required by the solution to its optimal policy problem while keeping the nominal rate unchanged most or all of the time is the subject of the next section.

5 Implementing the Optimal Monetary Policy when the ZLB Constraint is Nearly-Always Binding

Let (i_t^*, y_t^*, π_t^*) denote the central bank's optimal plan, i.e. the solution to the policy problem analyzed in the previous sections. Consider next deviations from the optimal plan satisfying the equilibrium conditions (1), (2) and (5). Formally, and letting $\widetilde{\pi}_t \equiv \pi_t - \pi_t^*$, $\widetilde{y}_t \equiv y_t - y_t^*$ and $\widetilde{i}_t \equiv i_t - i_t^*$, we have

$$\widetilde{\pi}_t = \beta \mathbb{E}_t\{\widetilde{\pi}_{t+1}\} + \kappa \widetilde{y}_t \tag{11}$$

$$\widetilde{y}_t = \mathbb{E}_t\{\widetilde{y}_{t+1}\} - \frac{1}{\sigma}(\widetilde{i}_t - \mathbb{E}_t\{\widetilde{\pi}_{t+1}\}) \tag{12}$$

as well as the ZLB constraint

$$\widetilde{i}_t \geq -i_t^* \tag{13}$$

for all t.

We complement the previous equations with the following interest rate rule

$$\widetilde{i}_t = \phi_{\pi,t} \left|\widetilde{\pi}_t\right| + \phi_{y,t} \left|\widetilde{y}_t\right| \tag{14}$$

where $(\phi_{\pi,t}, \phi_{y,t}) \geq 0$. According to the rule, the central bank commits to deviating from the nominal rate path prescribed by the optimal plan whenever inflation and/or the output gap deviate from their corresponding optimal paths. The fact that the adjustment of the nominal rate is proportional to the absolute value of those deviations guarantees that $i_t \geq i_t^* \geq 0$, thus meeting the ZLB constraint (13) at all times, even on any off-equilibrium path.

Note that $\widetilde{\pi}_t = \widetilde{y}_t = \widetilde{i}_t = 0$ for all t is always a solution to the system (11)-(14). Our objective is to study the conditions (if any) on (ϕ_π, ϕ_y) that guarantee that the previous solution is (locally) unique or, equivalently, that the optimal plan is effectively implemented.

We tackle this problem in two stages. First we specify the time-varying interest rate rule in a way that allows us to reformulate our model of the deviations from the optimal plan as a regime switching model. In particular we assume that (14) takes the form of a piecewise linear rule, given by

$$\widetilde{i}_t = \begin{cases} \phi_\pi^{(1)}\widetilde{\pi}_t + \phi_y^{(1)}\widetilde{y}_t & \text{if } \widetilde{\pi}_t \geq 0 \text{ and } \widetilde{y}_t \geq 0 \\ -\phi_\pi^{(2)}\widetilde{\pi}_t - \phi_y^{(2)}\widetilde{y}_t & \text{if } \widetilde{\pi}_t < 0 \text{ and } \widetilde{y}_t < 0 \\ \phi_\pi^{(3)}\widetilde{\pi}_t - \phi_y^{(3)}\widetilde{y}_t & \text{if } \widetilde{\pi}_t \geq 0 \text{ and } \widetilde{y}_t < 0 \\ -\phi_\pi^{(4)}\widetilde{\pi}_t + \phi_y^{(4)}\widetilde{y}_t & \text{if } \widetilde{\pi}_t < 0 \text{ and } \widetilde{y}_t \geq 0 \end{cases} \tag{15}$$

where $(\phi_\pi^{(i)}, \phi_y^{(i)}) \geq 0$ for $i \in \{1, 2, 3, 4\}$. Thus, we allow for coefficients in the interest rate rule that depend on the sign configuration of the deviations $(\widetilde{\pi}_t, \widetilde{y}_t)$. The resulting system, consisting of (11), (12) and (15), can be viewed as a regime switching model, with *endogenous* regime switches.

In a second stage, to which we turn next, we apply a novel result that allows us to establish sufficient conditions for the (local) uniqueness of the solution of our endogenous regime switching model. Given its potential interest beyond the problem at hand, we first state our result for a more general setting before we apply it to the model above.

5.1 A Sufficient Condition for Equilibrium Determinacy of a (Possibly Endogenous) Regime Switching Model

Consider a regime switching model whose equilibrium is described by a system of difference equations of the form:

$$\mathbf{x}_t = \mathbf{A}_t \mathbb{E}_t\{\mathbf{x}_{t+1}\} \tag{16}$$

where $\mathbf{x}_t$ is an $(n \times 1)$ vector of non-predetermined variables and A_t is an $(n \times n)$ matrix. We assume $\mathbf{A}_t \in \mathcal{A}$ where $\mathcal{A} \equiv \{\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, ..., \mathbf{A}^{(Q)}\}$ is a finite set of $(n \times n)$ nonsingular matrices. The evolution of $\mathbf{A}_t$ over time is left unspecified. It may evolve exogenously, e.g. according to a Markov process. Alternatively $\mathbf{A}_t$ may vary endogenously, i.e. as a function of current or lagged values of $\mathbf{x}_t$.

It is clear that $\mathbf{x}_t = 0$ for all t is a solution to (16). Our goal is to establish *sufficient* conditions on $\mathcal{A}$ that guarantee that $\mathbf{x}_t = 0$ all t is the only bounded solution to (16). We take this to be the case if $\lim_{T\to+\infty} \mathbb{E}_t\{\|\mathbf{x}_{t+T}\|\} > M\|\mathbf{x}_t\|$ for any $M > 0$ and $\mathbf{x}_t \neq 0$, and where $\|\cdot\|$ is the usual L^2 norm.

Let us define the induced matrix norm $\left\|\mathbf{A}^{(q)}\right\| \equiv \max_{\mathbf{x}} \left\|\mathbf{A}^{(q)}\mathbf{x}\right\|$ subject to $\|\mathbf{x}\| = 1$. In addition, define $\alpha \equiv \max\{\left\|\mathbf{A}^{(1)}\right\|, \left\|\mathbf{A}^{(2)}\right\|, ... \left\|\mathbf{A}^{(Q)}\right\|\}$. Note that nonsingularity of $\mathbf{A}^{(q)}$ for

$q = 1, 2, ...Q$ implies $\alpha > 0$.

Theorem *[sufficient condition for determinacy]***:** If $\alpha < 1$, then $\mathbf{x}_t = 0$ for all t is the only bounded solution to (16)

Proof: See Appendix C

Remark: note that $\left\|\mathbf{A}^{(q)}\right\| < 1$ implies that all the eigenvalues of $\mathbf{A}^{(q)}$ lie within the unit circle (though the converse is not true). See Appendix D for a proof. Hence our sufficient condition $\alpha < 1$ also implies that $\mathbf{x}_t = 0$ is the unique bounded solution to the single regime model $\mathbf{x}_t = \mathbf{A}^{(q)}\mathbb{E}_t\{\mathbf{x}_{t+1}\}$, for $q = 1, 2, ..., Q$.

5.2 Application to the Problem of Optimal Policy Implementation

Next, we apply the result of the previous subsection to the problem of implementation of the optimal monetary policy analyzed above. Recall that the dynamics of feasible deviations from the optimal policy allocation are described by (11), (12) and (15), with the latter effectively defining four regimes. Plugging (15) into (12) to eliminate $\widetilde{i}_t$, and after some straightforward substitutions, we can represent the dynamics for $\mathbf{x}_t \equiv [\widetilde{y}_t, \widetilde{\pi}_t]'$ as in (16), with

$$\mathbf{A}^{(1)} \equiv \frac{1}{\sigma + \phi_y^{(1)} + \kappa\phi_\pi^{(1)}} \begin{bmatrix} \sigma & 1 - \beta\phi_\pi^{(1)} \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y^{(1)}) \end{bmatrix}$$

$$\mathbf{A}^{(2)} \equiv \frac{1}{\sigma - \phi_y^{(2)} - \kappa\phi_\pi^{(2)}} \begin{bmatrix} \sigma & 1 + \beta\phi_\pi^{(2)} \\ \sigma\kappa & \kappa + \beta(\sigma - \phi_y^{(2)}) \end{bmatrix}$$

$$\mathbf{A}^{(3)} \equiv \frac{1}{\sigma - \phi_y^{(3)} + \kappa\phi_\pi^{(3)}} \begin{bmatrix} \sigma & 1 - \beta\phi_\pi^{(3)} \\ \sigma\kappa & \kappa + \beta(\sigma - \phi_y^{(3)}) \end{bmatrix}$$

$$\mathbf{A}^{(4)} \equiv \frac{1}{\sigma + \phi_y^{(4)} - \kappa\phi_\pi^{(4)}} \begin{bmatrix} \sigma & 1 + \beta\phi_\pi^{(4)} \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y^{(4)}) \end{bmatrix}$$

corresponding to the four regimes defined above (i.e., $Q = 4$).

The blue (dark) areas in Figures 4a-4d display the configurations of $(\phi_\pi^{(q)}, \phi_y^{(q)})$ values for which $\left\|\mathbf{A}^{(q)}\right\| < 1$, for $q \in \{1, 2, 3, 4\}$. Thus, to the extent that the central bank adopts rule (15) with state-contingent coefficients that fall within those regions, no deviations from the desired allocation will be consistent with a (bounded) equilibrium, and hence the rule will indeed implement the desired allocation (y_t^*, π_t^*), while satisfying the ZLB constraint.

As discussed above, the norm condition $\left\|\mathbf{A}^{(q)}\right\| < 1$ is stronger than the usual eigenvalue condition for determinacy in a model with no regime switches. The grey (light) areas in Figure 4 display the configurations of inflation and output coefficients that meet the eigenvalue criterion but not the norm one. The previous result is consistent with the finding in Barthélemy and Marx (2019), in the context of a New Keynesian model with exogenous switches in the interest rate rule coefficients, showing that indeterminacy may emerge even if each of the regimes adheres to the Taylor principle (i.e. it satisfies the eigenvalue condition for uniqueness in the corresponding single regime economy).

Note also that there is a nonempty intersection for the determinacy regions shown in Figures 4.a-4.d, which corresponds to that of regime $q = 4$.[9] Thus, any configuration of inflation and output gap coefficients within this region will make it possible to support the optimal allocation as a unique equilibrium by means of a version of rule (15) with *constant* coefficients, i.e. $(\phi_\pi^{(q)}, \phi_y^{(q)}) = (\phi_\pi, \phi_y)$ for $q = 1, 2, 3, 4$.

Finally, a word about some of the rule's implications. The rule instructs the central bank to keep the interest rate at the level i_t^*, consistent with the optimal policy, and to deviate from it only if inflation and/or output deviate from their optimal values, π_t^* and y_t^*. If the rule coefficients satisfy the sufficient condition for a unique equilibrium (as assumed in our simulations), those deviations never materialize ex-post. While the previous feature can be shown to be common to any interest rule that implements a given feasible allocation, a specific characteristic of our rule is that all its implied off-equilibrium deviations are *positive*, i.e. they involve raising the nominal interest rate above i_t^*. That property guarantees that that the ZLB constrained is never violated, not even on off-equilibrium paths, given that $i_t^* \geq 0$ for all t (with $i_t^* = 0$ most of the time in our simulations). Needless to say, some of the off-equilibrium interest rate movements called for by the rule may be perceived ex-post as being suboptimal (e.g. raising the interest rate if inflation falls below its desired level), but this sort of time inconsistency is inherent to optimal policies under commitment even in the absence of the ZLB constraint, their benefits arising from the (desirable) effects of their anticipation (as it is the case here).

[9] The previous property applies to our baseline calibration, it is not necessarily general.

6 Concluding Remarks

An eventual permanent decline in the natural rate of interest to negative levels raises the prospect that central bankers may feel compelled to keep the policy rate at zero for an indefinite period. The analysis in the present paper shows that monetary policy can keep influencing macro outcomes in that environment, in the face of continuous shocks that may impinge on the economy.

More specifically, we have studied the optimal monetary policy problem in a New Keynesian economy with a zero lower bound (ZLB) on the nominal interest rate, and in which the natural rate of interest has a negative mean, i.e. $r^* < 0$. We have shown that the optimal policy in that environment aims to approach *gradually* a steady state with positive average inflation. A gradualist approach minimizes welfare losses by keeping inflation close to zero for longer.

Around that steady state, inflation and the output gap have been shown to fluctuate in response to shocks to the natural rate, since the central bank is unable to fully stabilize those variables at their (first-best) zero value due to the ZLB constraint. Under the optimal policy, persistent fluctuations in the output gap and inflation coexist with a nominal rate that remains at its ZLB most (or all) of the time.

Finally we have shown that the central bank can implement the optimal policy as a (locally) unique equilibrium by means of an appropriate state-contingent rule. In order to establish that result, we derive a sufficient condition for local determinacy in a more general model with endogenous regime switches, a finding that may be of interest beyond the problem studied in the present paper.

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APPENDIX A: Solving for the transitional dynamics under the optimal policy

We seek to determine the solution to the system

$$\widehat{\pi}_t = \beta\widehat{\pi}_{t+1} + \kappa\widehat{y}_t$$

$$\widehat{y}_t \leq \widehat{y}_{t+1} + \sigma^{-1}\widehat{\pi}_{t+1}$$

$$\widehat{\pi}_t = \widehat{\xi}_{1,t} - \widehat{\xi}_{1,t-1} + \beta^{-1}\widehat{\xi}_{2,t-1}$$

$$\vartheta\widehat{y}_t = -\kappa\widehat{\xi}_{1,t} - \sigma\widehat{\xi}_{2,t} + \sigma\beta^{-1}\widehat{\xi}_{2,t-1}$$

$$\widehat{\xi}_{2,t} \geq \beta r^*$$

$$(\widehat{\xi}_{2,t} - \beta r)\ \ [\widehat{\pi}_{t+1} + \sigma(\widehat{y}_{t+1} - \widehat{y}_t)] = 0$$

for $t = 0, 1, 2, ...$and such that $\lim_{t\to\infty}\widehat{\mathbf{x}}_t = 0$ for $\widehat{\mathbf{x}}_t \in \{\widehat{\pi}_t, \widehat{y}_t, \widehat{\xi}_{1,t}, \widehat{\xi}_{2,t}\}$,

We conjecture that $i_t > 0$ for $t < t^*$ and $i_t = 0$ for $t \geq t^*$, for some $t^* \geq 0$ to be determined. In that case, for $t = t^*$, $t^* + 1, ...$we have

$$\widehat{\pi}_t = \beta\widehat{\pi}_{t+1} + \kappa\widehat{y}_t$$

$$\widehat{y}_t = \widehat{y}_{t+1} + \sigma^{-1}\widehat{\pi}_{t+1}$$

$$\widehat{\pi}_t = \widehat{\xi}_{1,t} - \widehat{\xi}_{1,t-1} + \beta^{-1}\widehat{\xi}_{2,t-1}$$

$$\vartheta\widehat{y}_t = -\kappa\widehat{\xi}_{1,t} - \sigma\widehat{\xi}_{2,t} + \sigma\beta^{-1}\widehat{\xi}_{2,t-1}$$

with $\widehat{\xi}_{2,t^*-1} = -\xi_2 = \beta r^*$ and an initial condition for $\widehat{\xi}_{1,t^*-1}$ to be determined below.

Next we derive the canonical representation. Note that:

$$\begin{aligned}
\widehat{\pi}_t &= \beta\widehat{\pi}_{t+1} + \kappa\widehat{y}_t \\
&= (\beta + \sigma^{-1}\kappa)\widehat{\pi}_{t+1} + \kappa\widehat{y}_{t+1}
\end{aligned}$$

$$\begin{aligned}
\widehat{\xi}_{1,t-1} &= \widehat{\xi}_{1,t} - \widehat{\pi}_t + \beta^{-1}\widehat{\xi}_{2,t-1} \\
&= \widehat{\xi}_{1,t} - (\beta + \sigma^{-1}\kappa)\widehat{\pi}_{t+1} - \kappa\widehat{y}_{t+1} + \sigma^{-1}\vartheta\widehat{y}_t + \sigma^{-1}\kappa\widehat{\xi}_{1,t} + \widehat{\xi}_{2,t} \\
&= (1 + \sigma^{-1}\kappa)\widehat{\xi}_{1,t} + (\sigma^{-2}\vartheta - \sigma^{-1}\kappa - \beta)\widehat{\pi}_{t+1} + (\sigma^{-1}\vartheta - \kappa)\widehat{y}_{t+1} + \widehat{\xi}_{2,t}
\end{aligned}$$

$$\begin{aligned}
\widehat{\xi}_{2,t-1} &= \sigma^{-1}\beta\vartheta\widehat{y}_t + \sigma^{-1}\beta\kappa\widehat{\xi}_{1,t} + \beta\widehat{\xi}_{2,t} \\
&= \beta\widehat{\xi}_{2,t} + \sigma^{-1}\beta\kappa\widehat{\xi}_{1,t} + \sigma^{-1}\beta\vartheta\widehat{y}_{t+1} + \sigma^{-2}\beta\vartheta\widehat{\pi}_{t+1}
\end{aligned}$$

More compactly,

$$\begin{bmatrix} \widehat{\pi}_t \\ \widehat{y}_t \\ \widehat{\xi}_{1,t-1} \\ \widehat{\xi}_{2,t-1} \end{bmatrix} = \begin{bmatrix} \beta + \sigma^{-1}\kappa & \kappa & 0 & 0 \\ \sigma^{-1} & 1 & 0 & 0 \\ \sigma^{-2}\vartheta - \sigma^{-1}\kappa - \beta & \sigma^{-1}\vartheta - \kappa & 1 + \sigma^{-1}\kappa & 1 \\ \sigma^{-2}\beta\vartheta & \sigma^{-1}\beta\vartheta & \sigma^{-1}\beta\kappa & \beta \end{bmatrix} \begin{bmatrix} \widehat{\pi}_{t+1} \\ \widehat{y}_{t+1} \\ \widehat{\xi}_{1,t} \\ \widehat{\xi}_{2,t} \end{bmatrix}$$

or, letting $\widehat{\mathbf{x}}_t \equiv [\widehat{\pi}_t, \widehat{y}_t]'$ and $\widehat{\boldsymbol{\xi}}_t \equiv [\widehat{\xi}_{1,t}, \widehat{\xi}_{2,t}]'$

$$\begin{bmatrix} \widehat{\mathbf{x}}_t \\ \widehat{\boldsymbol{\xi}}_{t-1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & 0 \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{x}}_{t+1} \\ \widehat{\boldsymbol{\xi}}_t \end{bmatrix}$$

The eigenvalues of $\mathbf{A}$ correspond to those of $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$. Each has two real eigenvalues, one inside and one outside the unit circle. Thus the solution is (locally) unique and has the following state-space representation:

$$\widehat{\mathbf{x}}_t = C\widehat{\boldsymbol{\xi}}_{t-1} \tag{17}$$

$$\widehat{\boldsymbol{\xi}}_t = B\widehat{\boldsymbol{\xi}}_{t-1} \tag{18}$$

for $t = t^*, t^* + 1, \ldots$with $\widehat{\boldsymbol{\xi}}_{t^*-1} = [\widehat{\xi}_{1,t^*-1}, \beta r^*]'$.

For $t = 0, 1, \ldots, t^* - 1$ we have $i_t > 0$ and $\widehat{\xi}_{2,t} = \beta r$. The equilibrium dynamics are given by

$$\widehat{\pi}_t = \beta\widehat{\pi}_{t+1} + \kappa\widehat{y}_t$$

$$\vartheta\widehat{y}_t = -\kappa\widehat{\xi}_{1,t} + \sigma(1-\beta)r \tag{19}$$

$$\widehat{\pi}_t = \widehat{\xi}_{1,t} - \widehat{\xi}_{1,t-1} + r^* \tag{20}$$

for $t = 0, 1, \ldots, t^* - 1$ with initial condition $\widehat{\xi}_{1,-1} = -\xi_1 = \frac{(1-\beta)}{\kappa}\left(\sigma - \frac{\vartheta}{\kappa}\right) r$ and terminal condition $\widehat{\pi}_{t^*} = c_{11}\widehat{\xi}_{1,t^*-1} - c_{12}\xi_2 = c_{11}\widehat{\xi}_{1,t^*-1} + c_{12}\beta r^*$

Combining the previous equations to eliminate output:

$$\begin{aligned} \widehat{\pi}_t &= \beta\widehat{\pi}_{t+1} + \kappa\widehat{y}_t \\ &= \beta\widehat{\pi}_{t+1} - \frac{\kappa^2}{\vartheta}\widehat{\xi}_{1,t} + \frac{\kappa}{\vartheta}\sigma(1-\beta)r^* \end{aligned}$$

For $t = 1, \ldots, t^* - 2$ we have

$$\widehat{\xi}_{1,t} - \widehat{\xi}_{1,t-1} = \beta\widehat{\xi}_{1,t+1} - \beta\widehat{\xi}_{1,t} - \frac{\kappa^2}{\vartheta}\widehat{\xi}_{1,t} + \left(\frac{\kappa\sigma}{\vartheta} - 1\right)(1-\beta)r^*$$

or, rearranging terms

$$\widehat{\xi}_{1,t} = \gamma\widehat{\xi}_{1,t-1} + \beta\gamma\widehat{\xi}_{1,t+1} + \delta r^*$$

where $\gamma \equiv \frac{\vartheta}{(1+\beta)\vartheta+\kappa^2}$ and $\delta \equiv \gamma\left(\frac{\kappa\sigma}{\vartheta}-1\right)(1-\beta)>0$.

For $t=0$,

$$\widehat{\xi}_{1,0}=\beta\gamma\widehat{\xi}_{1,1}+(\delta r^*-\gamma\xi_1)$$

For $t=t^*-1$,

$$\widehat{\xi}_{1,t^*-1}-\widehat{\xi}_{1,t^*-2}=\beta[b_{11}\widehat{\xi}_{1,t^*-1}+b_{12}\beta r^*]-\beta\widehat{\xi}_{1,t^*-1}-\frac{\kappa^2}{\vartheta}\widehat{\xi}_{1,t^*-1}+\left(\frac{\kappa\sigma}{\vartheta}-1\right)(1-\beta)r^*$$

or, rearranging terms,

$$[1-\beta b_{11}+\beta+\frac{\kappa^2}{\vartheta}]\widehat{\xi}_{1,t^*-1}=\widehat{\xi}_{1,t^*-2}+\left[b_{12}\beta^2+\left(\frac{\kappa\sigma}{\vartheta}-1\right)(1-\beta)\right]r^*$$

or, rearranging terms,

$$\widehat{\xi}_{1,t^*-1}=\frac{1}{1-\beta b_{11}+\beta+\frac{\kappa^2}{\vartheta}}\widehat{\xi}_{1,t^*-2}+\frac{b_{12}\beta^2+\left(\frac{\kappa\sigma}{\vartheta}-1\right)(1-\beta)}{1-\beta b_{11}+\beta+\frac{\kappa^2}{\vartheta}}r^*$$

Thus, we have a system S with t^* unknowns ($\widehat{\xi}_{1,0},\widehat{\xi}_{1,1},...,\widehat{\xi}_{1,t^*-1}$) and equal number of equations.

Algorithm:

(1) Conjecture $t^*=0$. Solve (18) with $\widehat{\boldsymbol{\xi}}_{-1}=[-\xi_1,-\xi_2]'$. If $\widehat{\xi}_{2,t}>-\xi_2$ for $t=0,1,2,...$ then conjecture is validated. Otherwise move on to (2)

(2) Increase t^* by one period. Solve (18) with $\widehat{\boldsymbol{\xi}}_{t^*-1}=[\widehat{\xi}_{1,t^*-1},-\xi_2]'$ where $\widehat{\xi}_{1,t^*-1}$ is obtained by solving system S. If $\widehat{\xi}_{2,t}>-\xi_2$ for $t=t^*,t^*+1,...$ then conjecture is validated and move on to (3). Otherwise back to (2).

(3) Solve for $\widehat{\pi}_t,\widehat{y}_t$ using (17) for for $t=t^*,t^*+1,...$ and (19) and (20) for $t=0,1,...,t^*-1$.

APPENDIX B: Solving for the local equilibrium dynamics under the optimal policy

We use the numerical algorithm for solving rational expectations models as implemented in the CompEcon toolkit of Miranda and Fackler (2002). In particular, we solve for the optimal policy x as a function of the state s, when equilibrium is governed by a system of the form

$$f[s_t,x_t,E_t h(s_{t+1},x_{t+1})]=\xi_t$$

where s follows the state transition function

$$s_{t+1} = g(s_t, x_t, \varepsilon_{t+1})$$

and x_t and ξ_t in our case satisfy the following Kuhn-Tucker condition

$$i_t \geqslant 0, \quad \xi_{2t} \geqslant 0, \quad i_t > 0 \Rightarrow \xi_{2t} = 0.$$

The solution is obtained with the collocation method, which consists of approximating the expectation functions by linear combinations of known basis functions, θ_j. The corresponding coefficients, c_j, are determined by requiring the approximating function to satisfy the equilibrium equations *exactly* at n collocation nodes:

$$h[s, x(s)] \approx \sum_{j=1}^{n} c_j \theta_j(s)$$

For a given value of the coefficient vector c, the equilibrium policies x_i are computed at the n collocation nodes s_i by solving a standard root-finding problem. The coefficient vector c is updated solving the n-dimensional linear system

$$\sum_{j=1}^{n} c_j \theta_j (s_i) = h (s_i, x_i)$$

The previous iterative procedure is repeated until the distance between successive values of c becomes sufficiently small. To approximate the expectation functions, we discretize the innovation to r_t^n using a K-node Gaussian quadrature scheme:

$$Eh[s, x(s)] \approx \sum_{k=1}^{K} \sum_{j=1}^{n} \omega_k c_j \theta_j [g (s_i, x, \varepsilon_k)]$$

where ε_k and ω_k are Gaussian quadrature nodes and weights chosen so that the discrete distribution approximates the continuous univariate normal distribution $N(0, \sigma^2)$. We use linear splines on a uniform grid of 200 points for values of the natural rate of interest between -10 percent and $+10$ percent, so that each point on the grid corresponds to 10 basis points.

APPENDIX C: Proof of Theorem [sufficiency conditions for determinacy]

From (16) we have $\mathbf{x}_{t+T-1} = \mathbf{A}_{t+T-1}\mathbb{E}_{t+T-1}\{\mathbf{x}_{t+T}\}$ from which it follows that

$$\|\mathbf{x}_{t+T-1}\| = \|\mathbf{A}_{t+T-1}\mathbb{E}_{t+T-1}\{\mathbf{x}_{t+T}\}\| \leq \alpha \|\mathbb{E}_{t+T-1}\{\mathbf{x}_{t+T}\}\|$$

Accordingly, and given that $\alpha > 0$, can write

$$\begin{aligned}
\|\mathbb{E}_{t+T-1}\{\mathbf{x}_{t+T}\}\| &\geq \frac{1}{\alpha}\,\|\mathbf{x}_{t+T-1}\| \\
&= \frac{1}{\alpha}\,[\mathbb{E}_{t+T-2}\{\|\mathbf{x}_{t+T-1}\|\} + \xi_{t+T-1}] \\
&\geq \frac{1}{\alpha}\,[\|\mathbb{E}_{t+T-2}\{\mathbf{x}_{t+T-1}\}\| + \xi_{t+T-1}]
\end{aligned}$$

where $\xi_{t+k} \equiv \|\mathbf{x}_{t+k}\| - \mathbb{E}_{t+k-1}\{\|\mathbf{x}_{t+k}\|\}$. To simplify the notation, define $y_{t+k} \equiv \|\mathbb{E}_{t+k}\{\mathbf{x}_{t+k+1}\}\|$. Thus we have

$$y_{t+T-1} \geq \frac{1}{\alpha}\; y_{t+T-2} + \frac{1}{\alpha}\xi_{t+T-1}$$

which can be iterated recursively to yield

$$y_{t+T-1} \geq \frac{1}{\alpha^{T-1}}\; y_t + \frac{1}{\alpha}\xi_{t+T-1} + \frac{1}{\alpha^2}\xi_{t+T-2} + ... + \frac{1}{\alpha^{T-1}}\xi_{t+1}$$

Taking expectations conditional on information available in period t on both sides, and using the law of iterated expectations (which implies $\mathbb{E}_t\{\xi_{t+k}\} = \mathbb{E}_t\{\mathbb{E}_{t+k-1}\{\xi_{t+k}\}\} = 0$ for $k = 1, 2, 3, ..$) we can write

$$\mathbb{E}_t\{y_{t+T-1}\} \geq \frac{1}{\alpha^{T-1}}\; y_t$$

or, equivalently,

$$\begin{aligned}
\mathbb{E}_t\{\|\mathbb{E}_{t+T-1}\{\mathbf{x}_{t+T}\}\|\} &\geq \frac{1}{\alpha^{T-1}}\;\|\mathbb{E}_t\{\mathbf{x}_{t+1}\}\| \\
&= \frac{1}{\alpha^{T-1}}\;\left\|\mathbf{A}_t^{-1}\mathbf{x}_t\right\|
\end{aligned}$$

which in turn allows us to write

$$\begin{aligned}
\mathbb{E}_t\{\|\mathbf{x}_{t+T}\|\} &= \mathbb{E}_t\{\mathbb{E}_{t+T-1}\{\|\mathbf{x}_{t+T}\|\}\} \\
&\geq \mathbb{E}_t\{\|\mathbb{E}_{t+T-1}\{\mathbf{x}_{t+T}\}\|\} \\
&\geq \frac{1}{\alpha^{T-1}}\;\left\|\mathbf{A}_t^{-1}\mathbf{x}_t\right\|
\end{aligned}$$

Define $\delta_q \equiv \min_{\mathbf{x}} \left\|(\mathbf{A}^{(q)})^{-1}\mathbf{x}\right\|$ subject to $\|\mathbf{x}\| = 1$. Note that nonsingularity of $\mathbf{A}^{(q)}$ implies $\delta_q > 0$. Let $\delta \equiv \min\{\delta_1, \delta_2, .., \delta_Q\}$. Thus, it follows that

$$\mathbb{E}_t\{\|\mathbf{x}_{t+T}\|\} \geq \frac{\delta}{\alpha^{T-1}}\;\;\|\mathbf{x}_t\|$$

Thus, $\alpha < 1$ implies

$$\lim_{T \to +\infty} \mathbb{E}_t\{\|\mathbf{x}_{t+T}\|\} > M \|\mathbf{x}_t\|$$

for any $M > 0$ and $\mathbf{x}_t \neq 0$. *QED.*

APPENDIX D

Let $\mathbf{A}$ be a nonsingular matrix with $\|\mathbf{A}\| < 1$. Thus, $0 < \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x} < 1$ for all $\mathbf{x}$ such that $\|\mathbf{x}\| = 1$. Let $\mathbf{Q}$ be the matrix of (orthonormal) eigenvectors of $\mathbf{A}'\mathbf{A}$ and let Υ be the corresponding (diagonal) matrix with (real) eigenvalues on its diagonal. Thus, $\mathbf{A}'\mathbf{A}\mathbf{Q} = \mathbf{Q}\Upsilon$ with $\mathbf{Q}'\mathbf{Q} = \mathbf{I}$. Hence $\mathbf{Q}'\mathbf{A}'\mathbf{A}\mathbf{Q} = \Upsilon$, with all diagonal elements of Υ between zero and one. Thus we can write $\mathbf{A}'\mathbf{A} = \mathbf{Q}\Upsilon\mathbf{Q}'$ or, equivalently, $\mathbf{A}'\mathbf{Q}\mathbf{Q}'\mathbf{A} = (\mathbf{Q}\Upsilon^{\frac{1}{2}})(\boldsymbol{\Upsilon}^{\frac{1}{2}}\mathbf{Q}')$ implying $\mathbf{A}'\mathbf{Q} = \mathbf{Q}\Upsilon^{\frac{1}{2}}$. Thus the eigenvalues of $\mathbf{A}'$ (and, hence, of $\mathbf{A}$, since both share the same characteristic polynomial) are given by the diagonal elements of $\Upsilon^{\frac{1}{2}}$ and are thus real and between zero and one. This is precisely the condition for determinacy of a single regime model.

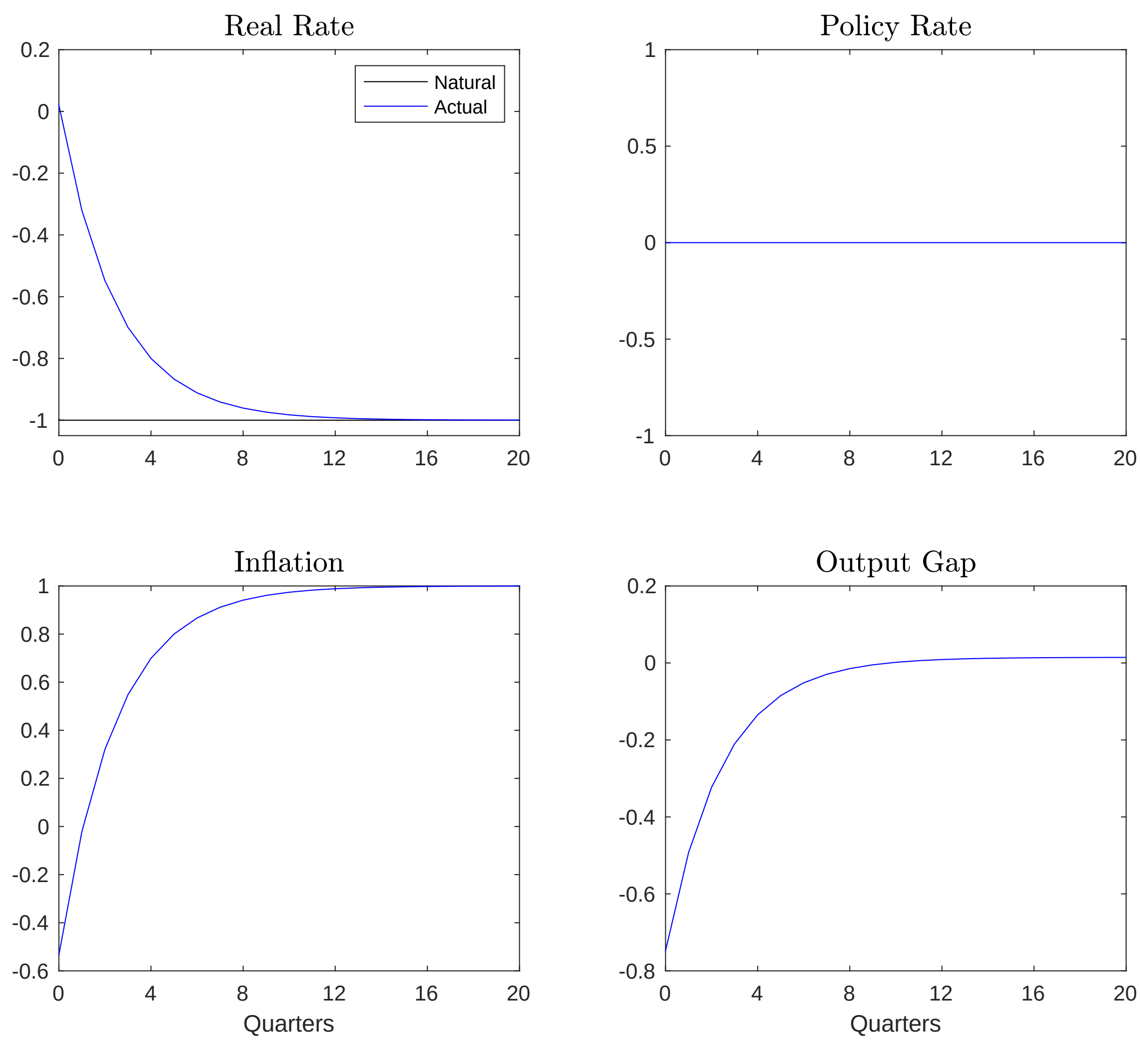

Figure 1: Transitional dynamics under the optimal monetary policy. Percent deviations from steady state in annualized terms.

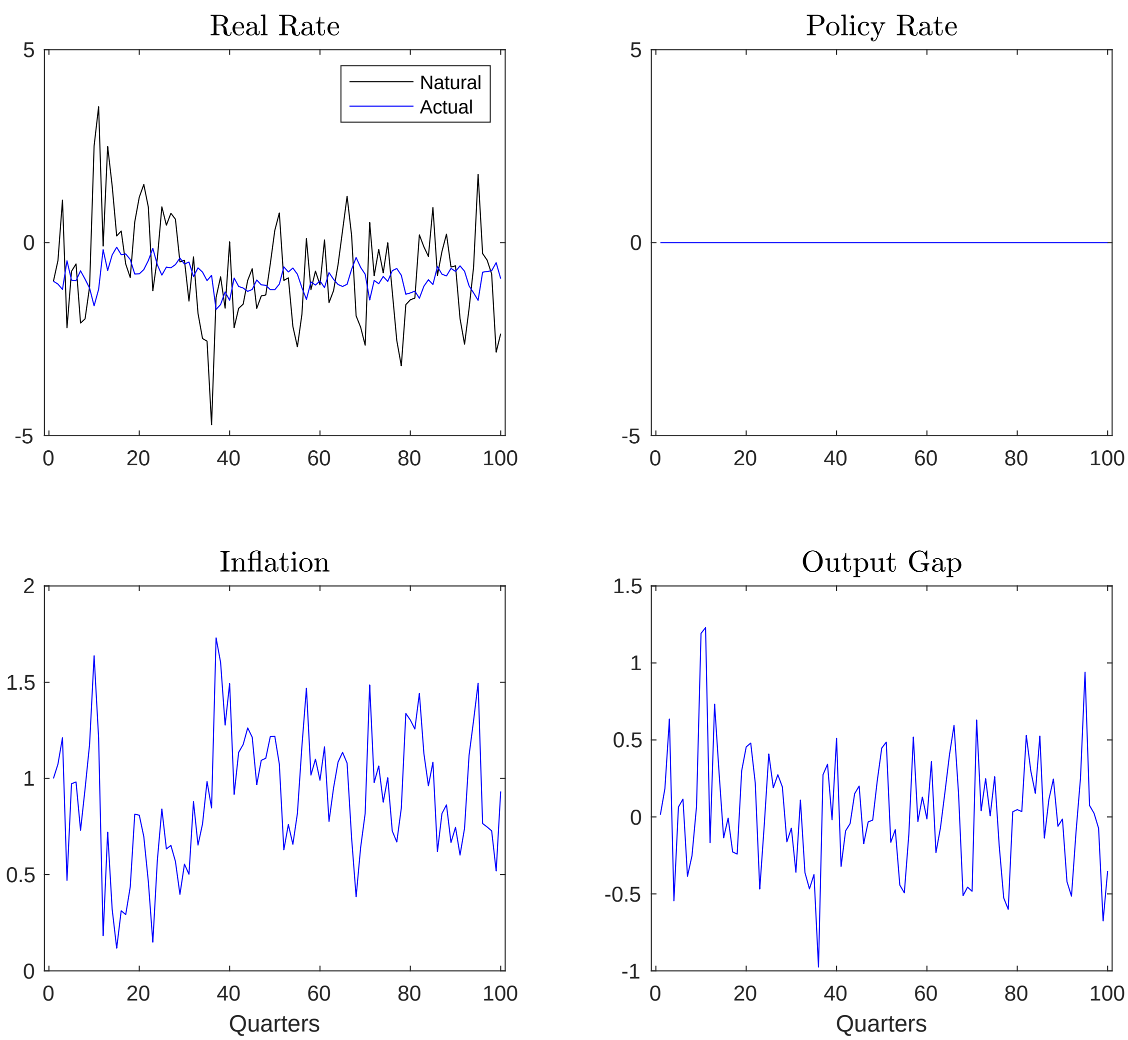

Figure 2: Aggregate fluctuations under the optimal monetary policy with baseline calibration. Percent deviations from steady state in annualized terms.

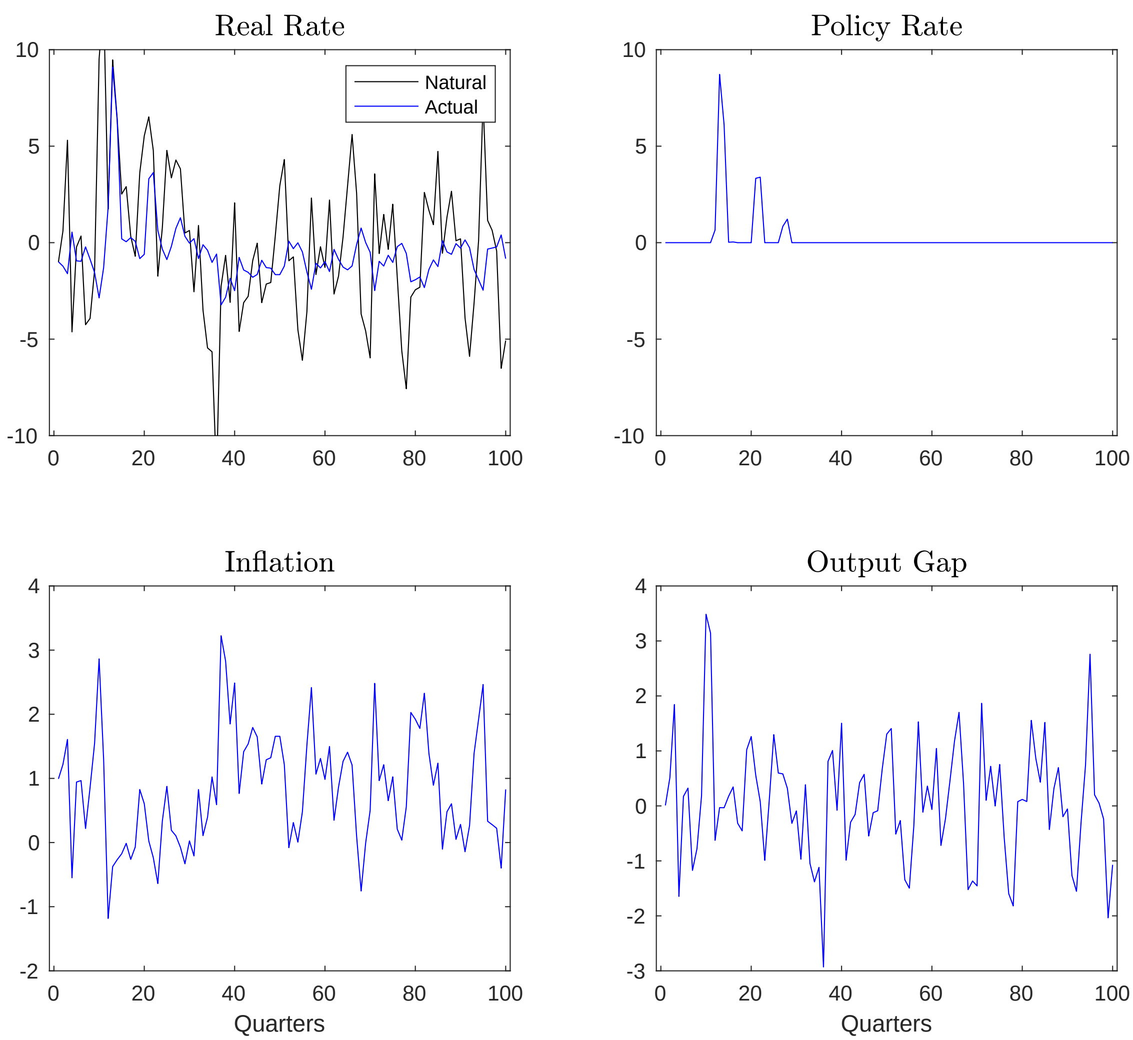

Figure 3: Aggregate fluctuations under the optimal monetary policy with higher shock volatility. Percent deviations from steady state in annualized terms.

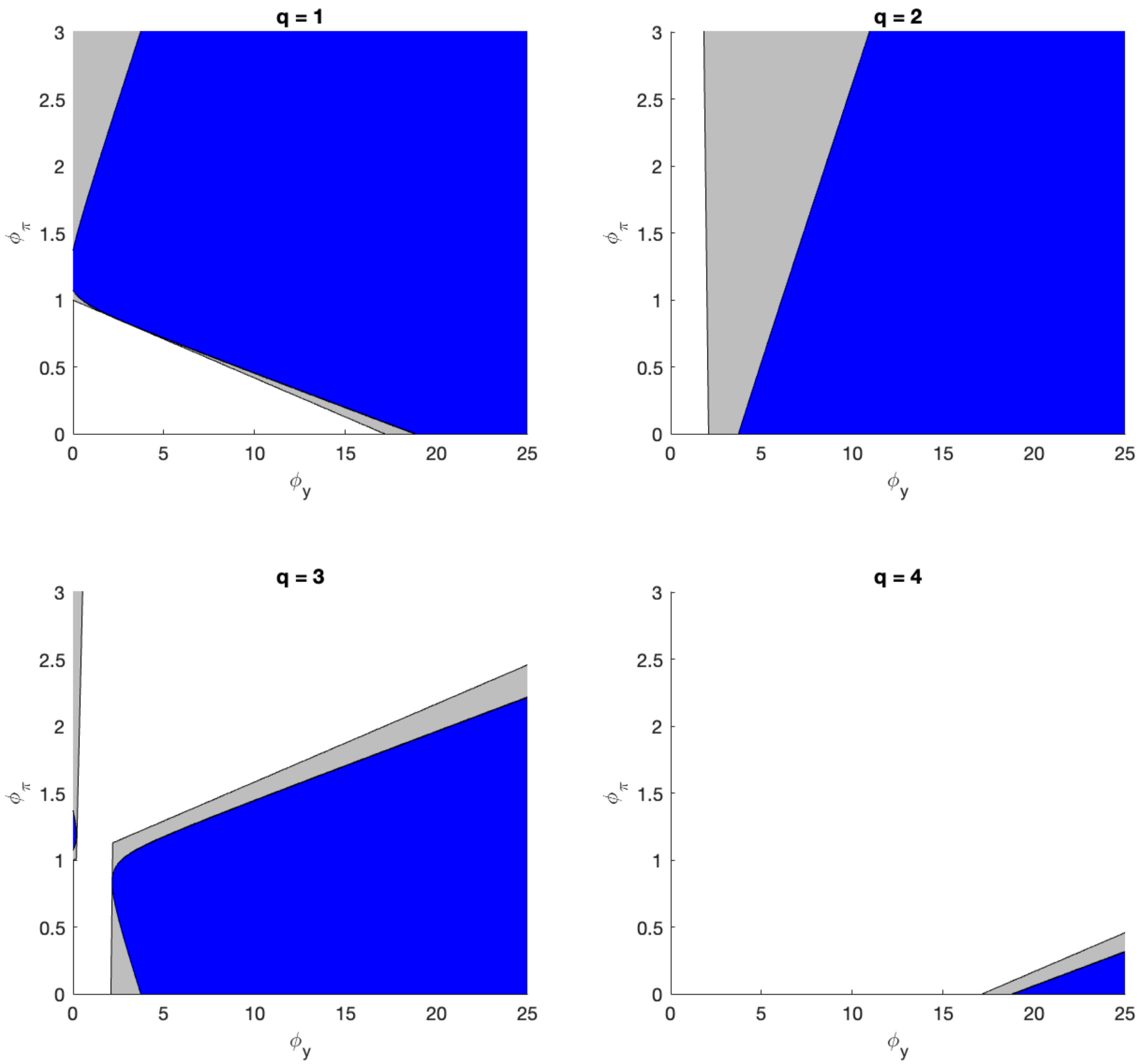

Figure 4: Implementation of the optimal monetary policy with state-contingent interest rate rule. Blue (dark) areas show values of the rule coefficients consistent with the norm condition for determinacy, while grey (light) areas show the values that meet only the standard eigenvalue condition.