Preference Restrictions for Simple and Strategy-Proof Rules: Local and Weakly Single-Peaked Domains

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Abstract

We show that if a rule is strategy-proof, unanimous, anonymous and tops-only, then the preferences in its domain have to be local and weakly single-peaked, relative to a family of partial orders obtained from the rule by confronting at most three alternatives with distinct levels of support. Moreover, if this domain is enlarged by adding a non local and weakly single-peaked preference, then the rule becomes manipulable. We finally show that local and weak single-peakedness constitutes a weakening of known and well-studied restricted domains of preferences.

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1 Introduction

Agents, as members of a society, take collective decisions by choosing an alternative from a given set of social alternatives. It is then desirable that these choices are based on the preferences

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that agents have over the set of alternatives. However, since preferences are idiosyncratic and constitute agents' private information, they must be elicited. A rule is a systematic procedure that selects an alternative for each profile of preferences declared by the agents. Therefore, if the selected alternative depends, even partially, on the declared profile of preferences, a rule generates a decision problem for each agent: Given the true preference, what is the optimal preference that the agent should declare to the rule as its own?

A rule is strategy-proof if, for each agent, truth-telling is always optimal, regardless of the preferences declared by the other agents. Namely, at each profile of preferences and for each agent, to declare the (true) preference is a weakly dominant strategy in the game in normal form induced by the rule and the profile of preferences. The Gibbard-Satterthwaite theorem establishes the difficulties of designing strategy-proof and non-trivial rules, whenever agents' preferences are unrestricted: The class of all strategy-proof and unanimous rules is reduced to the unsatisfactory family of dictatorial rules.[1] Strategy-proofness is a strong requirement, but so is the assumption that agents' preferences are unrestricted. There is a wide and rich literature that studies the design of strategy-proof and non-trivial rules operating under domain restrictions.[2] Often these restrictions come from the fact that the set of alternatives has a particular structure (for example, it is a linearly ordered set) that suggests that only a subclass of preferences are plausible (for example, those that are single-peaked relative to the underlying linear order). And often, the class of all strategy-proof and simple (namely, unanimous, anonymous, and tops-only) rules are completely characterized and fully understood (for instance, the class of all anonymous generalized median voter schemes on the domain of single-peaked preferences).[3]

Much of this literature on restricted domains begins by identifying and proposing, somehow *ad hoc*, a particular domain restriction which, in turn, is only partially and heuristically justified. In some sense, the domain restriction is perceived as a guarantee (or as a sufficient condition) for the existence of simple and strategy-proof rules. Along the lines of a few and recent papers (that we will comment later on), here we ask the following question: Given a simple and strategy-proof rule on a domain of preferences, what is the fundamental property that the preferences in this domain must satisfy? And we ask this question in a fairly general setup: Without any assumption on the structure and cardinality of the set of alternatives, with no restriction on an *a priory* structure of the domain of preferences and with no limitation on whether the number of agents is even or odd. Nevertheless, we do also require (as all this literature does) that the

[1]A rule is unanimous if at those profiles where all agents agree that one alternative is the most-preferred one, the rule selects this alternative. A rule is dictatorial if there is an agent with the property that, at each profile of preferences, the rule selects the most-preferred alternative according to the preference of this agent.

[2]For earlier and influential papers on this literature see, for instance, Moulin (1980), Border and Jordan (1983), Barberà et al. (1991), Sprumont (1991) and Barberà et al. (1993).

[3]A rule is anonymous if the way it selects the alternative does not depend on the identity of the agents. A rule is tops-only if it only depends on the profile of the most-preferred alternatives (the profile of tops).

domain be rich in a specific way that we will make clear later on; otherwise, if the domain contained only few preferences, then strategy-proofness would lose all its bite, since small but arbitrary subsets of preferences could admit uninteresting simple and strategy-proof rules.

In our main result, given a simple and strategy-proof rule on a rich domain of preferences, we identify local and weak single-peakedness as the condition that all preferences in the domain have to satisfy. The main feature of this condition is that, in contraposition to single-peakedness, it does not require the existence of a unique partial order from which all local and weakly single-peaked preferences are defined. Instead, it requires that for each alternative x there is an underlying partial order relative to which two properties have to be satisfied by any local and weakly single-peaked preference with top on x. First, alternatives found moving away from x, in the increasing direction of the partial order relative to x, are progressively weakly less preferred, but no condition is imposed on the preference relation between any pair of alternatives found moving towards x. Second, for any alternative w that is not above x according to the partial order relative to x, the supremum between x and w (which in some sense is closer to x than to w) is weakly preferred to w (our richness condition imposed on the domain guarantees that this supremum does always exist). Our construction of this underlying partial order, relative to each alternative x, is involved and obtained by looking at the alternative chosen by the rule at profiles where (perhaps) x and two other alternatives receive distinct support of the agents. Hence, the identified domain is relative to the given simple and strategy-proof rule.

To reinforce the prominence of the domain identified in our main result, we show that the domain of local and weakly single-peaked preferences is maximal in the following sense: If a rule is simple and strategy-proof on a rich domain of preferences, then (i) this domain is a subset of the set of local and weakly single-peaked preferences relative to the rule, (ii) the rule remains strategy-proof on this larger domain, and (iii) if a non local and weakly single-peaked preference is added to the domain, then the rule is not any more strategy-proof on this enlarged domain.

Some earlier papers have asked questions similar to ours. Chatterji et al. (2013) assume from the very beginning that the number of agents is even, the set of alternatives is finite and the domain of preferences has a particular structure that induces a tree on the set of alternatives. They show that under these circumstances any rich domain that admits a simple and strategy-proof rule must be semi single-peaked. Here we show that the set of semi single-peaked preferences, relative to an induced two-agents rule and a threshold on the tree, coincides with the set of local and weakly single-peaked preferences induced by this rule. Chatterji and Massó (2018) also assume that the number of agents is even, but they consider finite as well as infinite sets of alternatives and, besides richness, no additional structure on the domain of preferences is imposed. They show that if a simple and strategy-proof rule operates on a rich domain of preferences, then the preferences in the domain have to satisfy a variant of single-peakedness (referred to as semilattice single-peakedness). They derive from an induced two-agents rule a partial order (that is a semilattice) from which the concept of semilattice single-peakedness can be defined.[4] Here we argue that, given a semilattice over a set of alternatives, the domain of local and weakly single-peaked preferences contains the domain of semilattice single-peaked preferences. We also show how other well-known restricted domains that admit simple and strategy-proof rules are local and weakly single-peaked. In particular, we argue that the domain of local and weakly single-peaked preferences contains all single-peaked preferences and all separable preferences (when the set of alternatives is the family of all subsets of a given set of objects, as in Barberà et al. (1991)). Chatterji et al. (2021) consider a setup where multiple private goods have to be assigned to a set of agents with entitlements, and their preferences may display satiation.[5] They show that if a rule is strategy-proof on a domain of preferences and if in addition it is tops-only, same-sided and individually rational with respect to the entitlements, then the preferences in the domain have to satisfy a variant of single-peakedness (referred to also as semilattice single-peakedness).

Our contribution to this literature is two-fold. First, our main result applies to any simple and strategy-proof rule, regardless of whether the set of agents is odd or even. And this is important because now it provides a complete answer to an interesting question that was only partially resolved. Second, the approaches in Chatterji et al. (2013) and Chatterji and Massó (2018) are based on identifying the domain restriction of a given simple and strategy-proof rule with an even number of agents by looking at the two-agents rule obtained by applying the original rule only to profiles where half of the agents declare one alternative as their top and the other half declare another alternative as their top. Our approach is more involved but delivers the result directly from any rule, independently of the number of agents. Moreover, we show that the approaches based on two-agents rules either implicitly omit more complex rules, for which our approach provides the desirable identification of the key feature of their domains, or else the identified domain is too large (*i.e.*, the property is too weak) because it still contains preferences that agents could use to manipulate the rule at some preference profiles.

The rest of the paper is organized as follows. Section 2 contains the preliminaries. Section 3 contains the definition of local and weak single-peakedness and the main result of the paper (Theorem 1) establishing that if a rule is simple and strategy-proof on a rich domain, then all preferences in the domain have to be local and weakly single-peaked. In Section 4 we introduce the notion of a maximal domain and show that the set of local and weakly single-peaked preferences is a maximal domain. This section also contains an example showing that the statement of Theorem 1 does not hold for non-rich domains of preferences and that the domain of separable preferences (in the context of Barberà et al., 1991) is not rich relative to a voting by quota but, nevertheless, it is contained in the domain of local and weakly single-peaked preferences relative to the voting by quota. Section 5 compares the domain of local and

[4] Bogomolnaia (1998) can be seen as an early formulation of this type of question in a model with finitely many alternatives and only two agents.

[5] This setting with several private goods can be seen as a multidimensional extension of the model with a single good considered by Sprumont (1991).

weakly single-peaked preferences with the domains of single-peaked preferences, semilattice single-peaked preferences, semi single-peaked preferences and separable preferences. Section 6 contains a discussion of our results comparing them to those obtained in Chatterji and Massó (2018) which require an even number of agents and are based on an induced two-agents rule. The Appendix at the end of the paper collects omitted proofs.

2 Preliminaries

Let $N = \{1, \ldots, n\}$ be a finite set of *agents*, with $n \geq 2$, and A be any set of *alternatives*. We do not assume any structure on the set of alternatives and, in particular, A can be finite or infinite. Each $i \in N$ has a *preference* $R_i \in \mathcal{D}$ over A, where $\mathcal{D}$ is a set of complete and transitive binary relations over A; namely, for all $x, y, z \in A$, either xR_iy or yR_ix, and xR_iy and yR_iz imply xR_iz. Given $x, y \in A$, xR_iy means that agent i considers alternative x to be at least as good as alternative y. We refer to the set $\mathcal{D}$ as a *domain* of preferences. Let P_i be the strict preference relation induced by $R_i \in \mathcal{D}$ where, for all $x, y \in A$, xP_iy if and only if xR_iy and yR_ix does not hold. We assume that the domain $\mathcal{D}$ satisfies two basic properties. First, for each $R_i \in \mathcal{D}$ there exists a unique alternative $t(R_i) \in A$, the *top* of R_i, such that $t(R_i)P_ix$ for all $x \in A \setminus \{t(R_i)\}$. Second, for each $x \in A$ there exists at least one $R_i \in \mathcal{D}$ such that $t(R_i) = x$. We say that a domain with these two properties is *basic*. Let $\mathcal{D}$ be a basic domain and let $x \in A$, we refer to $R_i^x \in \mathcal{D}$ as a generic preference with $t(R_i^x) = x$; often, when agent i could be any agent and no confusion can arise, we write R instead of R_i and R^x instead of R_i^x.

A *profile* (of preferences) $R = (R_1, \ldots, R_n) \in \mathcal{D}^N$ is an ordered list of preferences, one for each agent. Let $R \in \mathcal{D}^N$ be a profile. Denote by $t(R)$ the set of top-ranked alternatives at R; that is, $t(R) = \{x \in A \mid x = t(R_i) \text{ for some } i \in N\}$. To emphasize agent i's preference R_i in the profile R we often write it as (R_i, R_{-i}).

A *rule* on $\mathcal{D}$ is a mapping $f : \mathcal{D}^N \to A$ assigning to each profile $R \in \mathcal{D}^N$ an alternative $f(R) \in A$.

We now describe desirable properties that rules can satisfy, and in which we will be interested.

A rule $f : \mathcal{D}^N \to A$ is *strategy-proof* (on $\mathcal{D}$) if agents can never induce a strictly preferred alternative by misrepresenting their preferences; namely, for all $R \in D^N$, $i \in N$ and $R_i' \in \mathcal{D}$,

$$f(R_i, R_{-i})R_if(R_i', R_{-i}).$$

We say that agent i can *manipulate* f *at* R *via* R_i' if $f(R_i', R_{-i})P_if(R_i, R_{-i})$.

A rule $f : \mathcal{D}^N \to A$ is *unanimous* if for all $R \in D^N$ such that $t(R_i) = x$ for all $i \in N$, $f(R) = x$. Unanimity is a natural and weak form of efficiency: If all agents consider an alternative as being the most-preferred one, the rule should select it.

Anonymity requires that the rule treats all agents equally because the social outcome is selected without paying attention to the identities of the agents. To formally describe an anonymous rule on $\mathcal{D}$ define, for every profile $R \in \mathcal{D}^N$ and every one-to-one mapping $\sigma : N \to N$, the profile $R^\sigma = (R_{\sigma(1)}, \ldots, R_{\sigma(n)})$ as the $\sigma-$permutation of R, where for all $i \in N$, $R_{\sigma(i)}$ is the preference that agent $\sigma(i)$ had at profile R. The domain $\mathcal{D}^N$ is closed under permutations since it is the Cartesian product of the same set $\mathcal{D}$. A rule $f : \mathcal{D}^N \to A$ is *anonymous* if for every one-to-one mapping $\sigma : N \to N$, $f(R^\sigma) = f(R)$ for all $R \in \mathcal{D}^N$.

A rule $f : \mathcal{D}^N \to A$ is *tops-only* if for all $R, R' \in \mathcal{D}^N$ such that $t(R_i) = t(R'_i)$ for all $i \in N$, $f(R) = f(R')$. Tops-onlyness constitutes a basic simplicity requirement. A tops-only rule $f : \mathcal{D}^N \to A$ can be written as $f : A^N \to A$. Accordingly, whenever f be tops-only we will use the notation $f(t(R_1), \ldots, t(R_n))$ interchangeably with $f(R_1, \ldots, R_n)$ and refer to $f : A^N \to A$, or to $f : A^n \to A$ if the rule is anonymous, as the *voting scheme* induced by the rule $f : \mathcal{D}^N \to A$. At a profile of tops $(t_1, \ldots, t_n) \in A^N$ we say that i *votes for* t_i (or *supports* t_i).

A voting scheme $f : A^N \to A$ is *strategy-proof on* $\mathcal{D}$ if for all $x = (x_1, \ldots, x_n) \in A^N$, $i \in N$ and $x'_i \in A$,

$$f(x_i, x_{-i}) R_i f(x'_i, x_{-i})$$

for all $R_i \in \mathcal{D}$ such that $t(R_i) = x_i$.

We will refer to a unanimous, anonymous and tops-only rule as a *simple* rule.

Our identification of the property that a preference domain must satisfy in order to admit a simple and strategy-proof rule will be based on different binary relations over A, obtained from the rule by confronting at most three alternatives with distinct levels of support. For this reason, we present below several notions and notations relative to a binary relation.

Fix a binary relation $\succeq$ over A. Given $x, y \in A$, we say that y dominates x according to $\succeq$ if $y \succeq x$. If $y \neq x$ we often write $y \succ x$ instead of $y \succeq x$. Suppose that $y \succeq x$, we define the interval $[x, y]_\succeq$ as the set

$$[x, y]_\succeq = \{z \in A \mid y \succeq z \text{ and } z \succeq x\}$$

and we refer to it as the $\succeq$-paths from x towards y. If $\succeq$ is reflexive, then $[x, x]_\succeq = \{x\}$ for all $x \in A$. By convention, for $y \not\succeq x$ we define $[x, y]_\succeq = \emptyset$.

Definition 1 *Let $\succeq$ be a binary relation over A. A domain $\mathcal{D}$ is **rich** on $(A, \succeq)$ if for all $x, y \in A$ with $[x, y]_\succeq \neq \emptyset$ and $z \notin [x, y]_\succeq$, there exist $R_i^x, R_i^y \in \mathcal{D}$ such that $y P_i^x z$ and $x P_i^y z$.*

Richness says that for any pair of alternatives x and y related by $\succeq$ and any alternative z not in a $\succeq$-path from x to y, a rich domain has to contain two preferences with the properties that for one of the preferences x is the top-ranked alternative and y is strictly preferred to z, and for the other preference y is the top-ranked alternative and x is strictly preferred to z. This notion of richness was proposed by Chatterji and Massó (2018). In Sections 4 and 5 we discuss the role of richness in our results and show that many (but not all) well-known domains are rich relative to the binary relation under which the domain restriction is founded.

Let $f : \mathcal{D}^N \to A$ be a simple rule. Of course, whether or not f is strategy-proof depends on its domain $\mathcal{D}$ but also on how its associated voting scheme $f : A^n \to A$ confers to agents the power to partially determine the selected alternative through their vote. Our objective is to extract from $f : A^n \to A$ the fundamental aspects to describe this power, which depends on the alternatives under consideration and shapes the basic features of $\mathcal{D}$ in order to assure that f is strategy-proof on it. For this purpose, we are going to define different binary relations that describe the treatment that the voting scheme $f : A^n \to A$ gives to alternatives, in relation to the power given to agents to impose one alternative, under certain voting configurations.

Let $f : \mathcal{D}^N \to A$ be a simple rule. Define the binary relation $\succeq_0^f$ *induced* by f over A as follows. For $y, z \in A$,

$$y \succeq_0^f z \iff f(y, \underbrace{z, \ldots, z}_{n-1 \text{ times}}) = y.$$

Namely, y dominates z according to $\succeq_0^f$ if f selects y with a single vote for y, even when z has the unanimous vote of the $n-1$ remaining agents; therefore, in the confrontation between y and z, the rule gives to y the strongest prominence. Since no confusion can arise, we will write $f(y, z, \ldots, z)$ instead of $f(y, \underbrace{z, \ldots, z}_{n-1 \text{ times}})$.

Let $A_0^f = \{y \in A \mid \text{there is } z \in A \text{ such that } z \succ_0^f y\}$ be the set of dominated alternatives according to $\succeq_0^f$. Given $x \in A$, define a *cover* of x, denoted by c^x, as follows. If $x \in A_0^f$, then c^x is an alternative in $A \setminus A_0^f$ such that $c^x \succ_0^f x$. If $x \in A \setminus A_0^f$, then $c^x = x$ (*i.e.*, x is the cover of itself). By definition, $c^x \notin A_0^f$ for all $x \in A$. When f is obvious from the context, we will often write $\succ_0$ and A_0 instead of $\succ_0^f$ and A_0^f.

In Lemma 1 we state some results concerning the binary relation $\succeq_0$ and the covers.

Lemma 1 *Let $f : \mathcal{D}^N \to A$ be a simple and strategy-proof rule and let $\succeq_0$ be the binary relation induced by f over A. If the domain D is rich on $(A_0, \succeq_0)$, then*

(i) $\succeq_0$ is a partial order,

(ii) if $x \in A_0$ and both c^x and $\bar{c}^x$ are a cover of x, then $c^x = \bar{c}^x$, and

(iii) for $x, y \in A$ such that $x \succ_0 y$, $c^y = c^x$.

Proof. See Appendix A.1. □

In general, the partial order $\succeq_0$ is not complete; by its definition, all pairs in $A \setminus A_0$ are unrelated. We will expand $\succeq_0$ by comparing y and z in $A \setminus A_0^f$ using x, with different levels of support, as a reference. If f chooses y at profiles where $t \in \{1, \ldots, n-2\}$ agents vote for x, one agent votes for y and the remaining $n-t-1$ agents vote for z, we will say that y $\succeq_t^{f,x}$-dominates z. And we will do that for each $x \in A$ and each $t \in \{1, \ldots, n-2\}$, so that the extension depends on x and t. Specifically, assume $n \geq 3$ and let $x \in A$ and $t \in \{1, \ldots, n-2\}$

be given. The binary relation $\succeq_t^{f,x}$ over $A \setminus A_0^f$ induced by f, x and t is defined by setting, for $y, z \in A \setminus A_0^f$,

$$y \succeq_t^{f,x} z \Longleftrightarrow f(\underbrace{x, \dots, x}_{t \text{ times}}, y, \underbrace{z, \dots \dots, z}_{n-t-1 \text{ times}}) = y.$$

Namely, y dominates z according to $\succeq_t^{f,x}$ if f selects y with a single vote for y, whenever x receives t votes and z has the unanimous vote of the $n-t-1$ remaining agents. The dominance of y over z is now mediated through the degree of support received by x.

We now define a property of a binary relation over A that will play a key role in the definition of local and weakly single-peaked preferences. Definition 2 below identifies conditions under which a given binary relation $\overline{\succeq}^{f,x}$ over A can be seen as the partially inverted extension of the binary relations induced by f and x. First, define $\overline{A}_0^f = A_0^f \cup \{c^y \in A \setminus A_0^f \mid y \in A_0^f\}$ as the set of dominated alternatives according to $\succeq_0^f$ together with their covers.

Definition 2 *Let $f : \mathcal{D}^N \to A$ be a simple rule and $x \in A$. The binary relation $\overline{\succeq}^{f,x}$ over A is the* ***partially inverted extension relative to x of $\succeq_0^f$ and $\succeq_1^{f,x}, \dots, \succeq_{n-2}^{f,x}$*** *(referred to as the* ***x-pie****) if the following conditions hold:*

(i) for $y, z \in \overline{A}_0^f$, $y \overline{\succeq}^{f,x} z$ if and only if $y \succeq_0^f z$.

(ii) for $y, z \in A \setminus A_0^f$, $y \overline{\succeq}^{f,x} z$ if and only if $z \succeq_t^{f,x} y$ for some $t \in \{1, \dots, n-2\}$,

(iii) if $y, z, w \in A$ are such that $y \overline{\succeq}^{f,x} z$ and $z \overline{\succeq}^{f,x} w$, then $y \overline{\succeq}^{f,x} w$, and

(iv) for $y \in A \setminus A_0^f$ and $z \in A_0^f$, $y \overline{\succeq}^{f,x} z$ implies $y \overline{\succeq}^{f,x} c^z$.

Namely, given a simple rule f and an alternative $x \in A$, a binary relation $\overline{\succeq}^{f,x}$ over A is the x-pie if the following conditions hold. When f is obvious from the context, we will often write $\overline{\succeq}^x$ instead of $\overline{\succeq}^{f,x}$. Condition (i) says that $\overline{\succeq}^x$ coincides with $\succeq_0$ over $\overline{A}_0$; $\succeq_0$ is the strongest form of domination and $\overline{\succeq}^x$ preserves it, independently of x. Condition (ii) says that for $y, z \in A \setminus A_0$, the x-pie between y and z is the inverted order according to some $\succeq_t^x$; the reason for this inversion is that it will allow us to describe more compactly the domain restriction that we are looking for. Condition (iii) says that the x-pie is transitive; this guarantees that $\overline{\succeq}^x$ is a partial order, and we will use it intensively. Lastly, condition (iv) says that for $y \in A \setminus A_0$ and $z \in A_0$, y is above z only if y is above the cover of z, both according the x-pie; this condition is essential to consistently define the binary relation $\overline{\succeq}^x$ over A considering at once its definition over $\overline{A}_0$, over $A \setminus \overline{A}_0$, and its transitivity. Notice that condition (iv) together with conditions (i) and (iii) imply that, for $y \in A \setminus A_0^f$ and $z \in A_0^f$, $y \overline{\succeq}^{f,x} z$ if and only if $y \overline{\succeq}^{f,x} c^z$. In Sections 4 and 5 we illustrate the definition of the x-pie for simple and strategy-proof rules on well-known domains of preferences; in particular, in Subsection 5.1 we do it for the case where f is a median voter scheme on the interval $A = [0, 1]$.

Remark 1 *Let $f : \mathcal{D}^N \to A$ be a simple rule. Then, for each $x \in A$, there is a unique x-pie. Moreover, by definition of $\overline{\succeq}^x$, if $y, z \in A$ are such that $y \in A_0$ and $y \overline{\succeq}^x z$, then $z \in A_0$.*

Lemma 5 in Appendix A.2 contains some results concerning the binary relations $\succeq_1^x, \ldots, \succeq_{n-2}^x$, and $\overline{\succeq}^x$ that we will use in the proofs of our results.

3 Local and weak single-peakedness: definition and main result

The notion of local and weak single-peakedness identified in our main result has the special feature, in contrast with the classical notion of single-peakedness, that there is no a unique underlying partial order from which the notion is defined. But instead, the underlying order is specific to the top alternative x of the preference to which the definition has to be applied. Namely, a preference R_i over A with top on x is local and weakly single-peaked relative to a partial order $\trianglerighteq^x$ over A if R_i satisfies two properties. First, alternatives found on the $\trianglerighteq^x$-paths starting at x (moving away from x but not going towards x) are progressively weakly less preferred. Second, any alternative w that does not lie on any $\trianglerighteq^x$-path starting at x, the supremum between x and w exists and it is weakly dispreferred to this supremum; observe that this supremum is an alternative somehow closer to x than w, and in this sense it can also be seen as a weakening of the notion of single-peakedness. Lemma 2 guarantees that, for the x-pie (*i.e.*, the partial order $\overline{\succeq}^x$ relative to x obtained from a given simple and strategy-proof rule f), this supremum coincides with the alternative chosen by f when one agent votes for x and all remaining $n-1$ agents vote for w. Our main result states that if a rich domain admits a simple and strategy-proof rule, then it is composed only of local and weakly single-peaked preferences, each preference R_i relative to the corresponding $t(R_i)$-pie. Since the content of Lemma 2 is required to define local and weak single-peakedness, we start by first stating it.

Lemma 2 *Let $f : \mathcal{D}^N \to A$ be a simple and strategy-proof rule. For $x \in A$, let $\overline{\succeq}^x$ be the partially inverted extension over A relative to x, assume $\mathcal{D}$ is rich on $(A, \overline{\succeq}^x)$ and let $w \in A$ be such that $w \overline{\nsucceq}^x x$. Then, $\sup_{\overline{\succeq}^x}\{x, w\}$ exists. Moreover, $\sup_{\overline{\succeq}^x}\{x, w\} = f(x, w, \ldots, w)$.*

Proof. See Appendix A.3. □

Definition 3 *Let $x \in A$ and $\trianglerighteq^x$ be a partial order over A. The preference R^x is* ***local and weakly single-peaked over*** $(A, \trianglerighteq^x)$ *if, for all $y, z, w \in A$,*

(LWSP.1) *$z \trianglerighteq^x y \trianglerighteq^x x$ implies $y R^x z$, and*

(LWSP.2) *$w \ntrianglerighteq^x x$ implies $\sup_{\trianglerighteq^x}\{x, w\} R^x w$.*

Given $(A, \trianglerighteq^x)$, let $\mathcal{LWSP}(\trianglerighteq^x)$ be the set of all local and weakly single-peaked preferences over $(A, \trianglerighteq^x)$. Let $\{\trianglerighteq^r\}_{r \in A}$ be a family of partial orders over A, one for each alternative in A. We say that a domain is *local and weakly single-peaked over* $(A, \{\trianglerighteq^r\}_{r\in A})$, and denote it by $\mathcal{LWSP}(\{\trianglerighteq^r\}_{r\in A})$, if for each $x \in A$, each preference $R^x \in \mathcal{LWSP}(\{\trianglerighteq^r\}_{r\in A})$ is *local and weakly single-peaked* over $(A, \trianglerighteq^x)$. When the family of partial orders over A is the family of partially inverted extensions $\{\overline{\succeq}^{f,r}\}_{r\in A}$ for some simple rule $f : \mathcal{D}^N \to A$, we will often write $\mathcal{LWSP}(f)$ instead of $\mathcal{LWSP}(\{\overline{\succeq}^{f,r}\}_{r\in A})$ and refer to it as the set of local and weakly single-peaked preferences relative to f. Moreover, we say that $\mathcal{D}$ is rich relative to f if $\mathcal{D}$ is rich on $(A, \overline{\succeq}^{f,x})$ for each $x \in A$.

We now state the main result of the paper.

Theorem 1 *Let $f : \mathcal{D}^N \to A$ be a simple and strategy-proof rule and assume that $\mathcal{D}$ is rich relative to f. Then, $\mathcal{D} \subseteq \mathcal{LWSP}(f)$.*

Proof. See Appendix A.4. □

Although the proof of Theorem 1 is relegated to Appendix A.4, we now present a sketch of the proof for completeness. Assume $\mathcal{D}$ and $f : \mathcal{D}^N \to A$ fulfill the hypothesis of the theorem. Let $x \in A$ and $R^x \in \mathcal{D}$ be arbitrary. In order to show that (LWSP.2) holds, assume that $w \in A$ is such that $w \overline{\not\succeq}^x x$. We know, by Lemma 2, that $\sup_{\overline{\succeq}^x}\{x, w\} = f(x, w, \ldots, w)$. Therefore, by strategy-proofness and unanimity,

$$\sup_{\overline{\succeq}^x}\{x, w\} = f(x, w, \ldots, w) R^x f(w, \ldots, w) = w,$$

and (LWSP.2) follows. To show that (LWSP.1) holds, let $y, z \in A$ be such that $z \overline{\succeq}^x y \overline{\succeq}^x x$. Since x, y, z can be in either A_0 or $A \setminus A_0$, the proof proceeds by distinguishing among several cases, depending on whether x, y and z belong to A_0 or to $A \setminus A_0$, and in each of those cases establishing, with the help of Lemmata 1 and 5, that $y R^x z$ holds.

4 Maximal and rich domains

In this section we first show that the set of preferences that are local and weakly single-peaked relative to a given family of simple rules that are strategy-proof on a common, basic, and rich domain of preferences is maximal. Second, we show that the statement of Theorem 1 does not hold if the domain of the simple and strategy-proof rules is not rich. Third, for the setting where the set of alternatives A is the family of all subsets of a given set of objects (studied by Barberà et al., 1991) we exhibit a voting by quota f and an alternative x for which the set of separable preferences is not rich on $(A, \overline{\succeq}^{f,x})$. Hence, Theorem 1 cannot be applied. However, we will argue in Subsection 5.4 that the set of separable preferences is a subset (not necessarily strict) of the set of local and weakly single-peaked preferences relative to any voting by quota.

4.1 Maximality

To test the degree of arbitrariness of a domain restriction, and since the property of strategy-proofness on a domain is inherited by any of its subdomains, it is natural to ask to what extent it is possible to broaden the domain of a family of simple and strategy-proof rules while at the same time maintaining the properties of the rules on the larger domain. An important part of the literature on restricted domains has asked this question. Its answers have spawn a family of results, specific to each domain restriction. They say that the domain that satisfies the restriction and admits simple and strategy-proof rules is maximal in the sense that at least one of the rules becomes manipulable after adding a preference to the domain.[6] To state and prove that the domain of local and weakly single-peaked preferences has this property, we first present the notion of a maximal domain of preferences for a family of simple and strategy-proof rules on a common and basic domain.

Let $\mathcal{D}$ be a basic domain. Let $\mathcal{F} \subseteq \{f : \mathcal{D}^N \to A \mid f \text{ is simple and strategy-proof on } \mathcal{D}\}$ and, for each $f \in \mathcal{F}$, let $\widehat{f} : A^n \to A$ be the voting scheme induced by f. The domain $\mathcal{D}^\star$, with $\mathcal{D} \subseteq \mathcal{D}^\star$, is *maximal for* $\mathcal{F}$ if: (i) for each $f \in \mathcal{F}$, the voting scheme $\widehat{f}$ is strategy-proof on $\mathcal{D}^\star$ and (ii) for each $R_i \notin \mathcal{D}^\star$ with $|t(R_i)| = 1$, there is $f \in \mathcal{F}$ such that $\widehat{f}$ is not strategy-proof on $\mathcal{D}^\star \cup \{R_i\}$.

Proposition 1 states that, for any family of simple and strategy-proof rules, the domain of local and weakly single-peaked preferences relative to all rules in the family is maximal for the family. Corollary 1 states the consequence of Proposition 1 for the case when the family of simple and strategy-proof rules is a singleton.

Proposition 1 *Let $\mathcal{D}$ be a basic domain and let $\mathcal{F} \subseteq \{f : \mathcal{D}^N \to A \mid f$ is simple and strategy-proof on $\mathcal{D}\}$. If $\mathcal{D}$ is rich relative to f for each $f \in \mathcal{F}$, then $\bigcap_{f\in\mathcal{F}} \mathcal{LWSP}(f)$ is maximal for $\mathcal{F}$.*

Proof. See Appendix A.5. □

Corollary 1 *Let $\mathcal{D}$ be a basic domain and $f : \mathcal{D}^N \to A$ be a simple and strategy-proof rule on $\mathcal{D}$. If $\mathcal{D}$ is rich relative to f, then $\mathcal{LWSP}(f)$ is maximal for f.*

4.2 Example of a non-rich domain

We show here that the rich domain condition is indispensable for the statement of Theorem 1 to be true. The example below, taken from Chatterji and Massó (2018), exhibits a domain $\mathcal{D}$ and a simple and strategy-proof rule $f : \mathcal{D}^2 \to A$ such that $\mathcal{D}$ is not rich relative to f and $\mathcal{D}$

[6]Previous maximality results include, for example, Barberà et al. (1991), Serizawa (1995), Ching and Serizawa (1998), Barberà et al. (1999), Berga and Serizawa (2000), Berga (2002), Massó and Neme (2001, 2004), Hatsumi et al. (2014), and Achuthankutty and Roy (2018).

contains a non local and weakly single-peaked preference relative to f; namely, the conclusion of Theorem 1 may not hold if $\mathcal{D}$ is not rich relative to f.

Let $A = \{x, y, z, r, w\}$ be the set of alternatives and $\mathcal{D}$ the domain of five strict preferences:

P^x	P^y	P^z	P^r	P^w
x	y	z	r	w
z	z	w	w	x
r	r	r	z	y
w	w	x	y	z
y	x	y	x	r

Consider the strategy-proof and simple rule $f : \mathcal{D}^2 \longrightarrow A$ defined by the following table:

f	x	y	z	r	w
x	x	z	z	r	w
y	z	y	z	r	w
z	z	z	z	w	w
r	r	r	w	r	w
w	w	w	w	w	w

Remember that when there are only two agents, the partially inverted extension $\overline{\succeq}^x$ is equal to the binary relation $\succeq_0$ for each $x \in A$. This partially inverted extension is illustrated in Figure 1.[7]

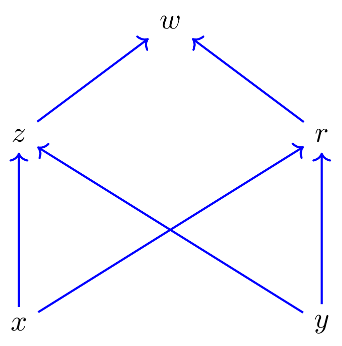

Figure 1: $\succeq_0$.

Notice that $\mathcal{D}$ is not rich relative to f since $r \notin [x, z]_{\succeq_0} \neq \emptyset$ and there does not exist $P^z \in \mathcal{D}$ such that $xP^z r$. Furthermore, $y \not\succeq_0 x$ and $\sup_{\succeq_0^f}\{x, y\}$ does not exist. Hence, $y \not\succeq_0 x$ and since

[7]In general, we represent a binary relation $\succ$ with arrows, where $x \rightarrow y$ means $y \succ x$ and all dominance relations that can be derived by transitivity are omitted.

$\sup_{\succeq_0^f}\{x,y\}P^x y$ does not hold, P^x does not satisfy (LWSP.2). Hence, $P^x \notin \mathcal{LWSP}(f)$ and so the conclusion of Theorem 1 may not hold if $\mathcal{D}$ is not rich relative to f.

4.3 The separable domain may not be rich relative to some voting by quota

Here, we first recall the notions of separable preferences and of voting by quota in the context of Barberà et al. (1991). Then, we exhibit a voting by quota for which the domain of separable preferences is not rich relative to it. Following Barberà et al. (1991), consider the social choice problem where a set N of agents has to choose a subset (possibly empty) of a given finite set of objects $\mathcal{K} = \{1, \ldots, K\}$. A prototypical example of this setting is when objects are candidates to become new members of the society, and they have to be elected by its current members. The set of alternatives A is the family $2^{\mathcal{K}}$ of all subsets of $\mathcal{K}$ which can be identified with the K -dimensional hypercube $\{0,1\}^{\mathcal{K}}$ by assigning to each subset $X \in A$ the vector $x \in \{0,1\}^{\mathcal{K}}$ where, for each $k \in \mathcal{K}$.

$$x_k = \begin{cases} 1 & \text{if } k \in X \\ 0 & \text{if } k \notin X. \end{cases}$$

An object is good (respectively, bad) if as a singleton set is strictly preferred (respectively, dispreferred) to the empty set. A preference P_i is separable if the division between good and bad objects guides the ordering between some specific pairs of subsets: Adding a good object to any set leads to a better set, while adding a bad object to any set leads to a worse set. Formally, a strict preference P_i over A is *separable* if for all $X \in A$ and $y \notin X$,

$$X \cup \{y\} P_i X \text{ if and only if } \{y\} P_i \{\emptyset\}.$$

Let $\mathcal{S}$ be the set of all strict separable preferences over A, and let $\mathcal{D}$ be a basic domain.

A rule $f : \mathcal{D}^N \to A$ is *voting by quota* if there exists a vector of quotas $q = (q_k)_{k\in\mathcal{K}}$ such that, for every $k \in \mathcal{K}$, $q_k \in \{1, \ldots, n\}$ and, for all $R \in \mathcal{D}^N$ and $k \in \mathcal{K}$,

$$k \in f(R) \text{ if and only if } |\{i \in N \mid k \in t(R_i)\}| \geq q_k.$$

Note that by definition, voting by quotas are very simple. They are tops-only and the selected subset of objects at each profile of preferences is obtained in a decomposable way, object-by-object: An object k is selected at a profile if and only if the number of agents that consider that this object is good is greater than or equal to the quota of k.

A corollary of the main result in Barberà et al. (1991) is the following characterization: A rule $f : \mathcal{S}^N \to A$ is simple and strategy-proof if and only if f is voting by quota.

Let $q = (q_k)_{k\in\mathcal{K}}$ be the vector of quotas associated to a voting by quota $f : \mathcal{D}^N \to A$. We start with a preliminary result, useful here and in Subsection 5.4, that identifies the set A_0^f

(induced by the binary relation $\succeq_0^f$) as the complementary family of subsets that contain all objects with quota 1 and none with quota n. Define

$$\begin{aligned} K_1 &= \{k \in \mathcal{K} \mid q_k = 1\}, \\ K_n &= \{k \in \mathcal{K} \mid q_k = n\}, \end{aligned}$$

and $\mathcal{X} = \{x \in \{0,1\}^{\mathcal{K}} \mid x_k = 1 \text{ if } k \in K_1 \text{ and } x_k = 0 \text{ if } k \in K_n\}$. Observe that each agent $i \in N$ is decisive for any object $k \in K_1$, since i can impose k by declaring that $k \in t(R_i)$, and i is a vetoer for any object $k \in K_n$, since i can assure that the chosen subset will not contain k by declaring that $k \notin t(R_i)$.

We show that $A_0 = A \setminus \mathcal{X}$ by verifying first that $A \setminus \mathcal{X} \subseteq A_0$. Assume $Y \notin \mathcal{X}$, and so $K_1 \cup K_n \neq \emptyset$.[8] To show that $Y \in A_0$, consider the subset $X \in 2^{\mathcal{K}}$ identified with the vector $x \in \{0,1\}^{\mathcal{K}}$ defined as follows:

$$x_k = \begin{cases} 1 & \text{if } k \in K_1 \\ 0 & \text{if } k \in K_n \\ y_k & \text{if } k \notin K_1 \cup K_n. \end{cases}$$

Note that $X \in \mathcal{X}$ and so $X \neq Y$. Then, for each $k \in \mathcal{K}$,

$$f(X, Y, \ldots, Y)_k = \begin{cases} 1 & \text{if } k \in K_1 \\ 0 & \text{if } k \in K_n \\ y_k & \text{if } k \notin K_1 \cup K_n, \end{cases}$$

which means that $f(X, Y, \ldots, Y) = X \succ_0 Y$. Hence, $Y \in A_0$.

We now verify that $A_0 \subseteq A \setminus \mathcal{X}$. To do so, assume $Y \in \mathcal{X}$ and to obtain a contradiction suppose that $Y \in A_0$. This means that there exists $X \in 2^{\mathcal{K}}$ such that

$$f(X, Y, \ldots, Y) = X \neq Y. \tag{1}$$

Then, for each $k \in \mathcal{K}$,

$$f(X, Y, \ldots, Y)_k = \begin{cases} 1 & \text{if } k \in K_1 \\ 0 & \text{if } k \in K_n \\ y_k & \text{if } k \notin K_1 \cup K_n, \end{cases}$$

where the first two lines follow because $Y \in \mathcal{X}$ and the third one because, for each $k \notin K_1 \cup K_n$, $2 \leq q_k \leq n-1$ and $q_k \leq x_k + (n-1)y_k$ if and only if $y_k = 1$. Hence, $f(X, Y, \ldots, Y) = Y$, a contradiction with (1).

Thus,

$$A_0 = A \setminus \mathcal{X}. \tag{2}$$

[8]If $K_1 \cup K_n = \emptyset$ then $\mathcal{X} = A$ and $A_0 = \emptyset$. Indeed, for $X, Y \in 2^{\mathcal{K}}$, $f(X, Y, \ldots, Y) = Y$ holds because for each $k \in \mathcal{K}$, $2 \leq q_k \leq n-1$ and so, $q_k \leq x_k + (n-1)y_k$ if and only $y_k = 1$. Hence, $X \not\succ_0 Y$ and $A_0 = \emptyset$.

To show that the set of separable preferences is not necessarily rich relative to a voting by quota, consider the case where $N = \{1,2,3\}$, $\mathcal{K} = \{x,y,z\}$ and the voting by quota $f : A^3 \to A$ is given by the vector of quotas $q = (q_x, q_y, q_z) = (1,2,2)$. Then, $K_1 = \{x\}$, $K_3 = \{\emptyset\}, \mathcal{X} = \{(1,0,0),(1,1,0),(1,0,1),(1,1,1)\}$ and, by (2), $A_0 = \{(0,0,0),(0,1,0),(0,0,1),(0,1,1)\}$. Figure 2 represents $\succeq_0$, where $(1,1,0) \succ_0 (0,1,0)$, $(1,1,1) \succ_0 (0,1,1)$, $(1,0,1) \succ_0 (0,0,1)$, and $(1,0,0) \succ_0 (0,0,0)$.

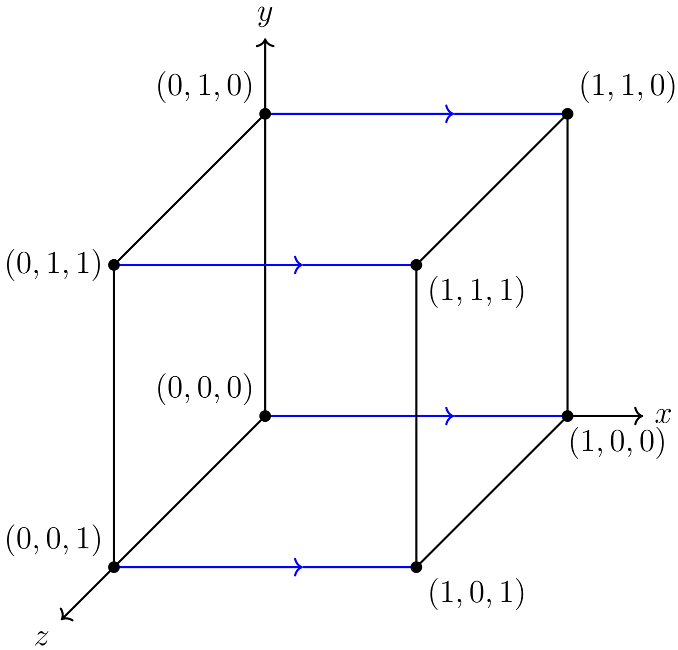

Figure 2: $\succeq_0$.

Fix $(1,1,1)$. To obtain $\succeq^{f,(1,1,1)}$ observe that

$$
\begin{aligned}
f((1,1,1),(1,1,1),(1,0,1)) &= (1,1,1), \text{ which means that } (1,1,1) \succ_1^{f,(1,1,1)} (1,0,1),\\
f((1,1,1),(1,1,1),(1,1,0)) &= (1,1,1), \text{ which means that } (1,1,1) \succ_1^{f,(1,1,1)} (1,1,0),\\
f((1,1,1),(1,0,1),(1,0,0)) &= (1,0,1), \text{ which means that } (1,0,1) \succ_1^{f,(1,1,1)} (1,0,0), \text{ and}\\
f((1,1,1),(1,1,0),(1,0,0)) &= (1,1,0), \text{ which means that } (1,1,0) \succ_1^{f,(1,1,1)} (1,0,0).
\end{aligned}
$$

We obtain $\succeq^{f,(1,1,1)}$ by inverting $\succ_1^{f,(1,1,1)}$ over $A \setminus A_0$ and maintaining the orders given by $\succeq_0$. Figure 3 represents the $(1,1,1)$-pie $\succeq^{f,(1,1,1)}$. Therefore, $(1,0,0) \succeq^{f,(1,1,1)} (0,1,1)$ and $(0,0,1) \notin [(0,1,1),(1,0,0)]_{\succeq^{f,(1,1,1)}}$. However, there does not exist a separable preference $P^{(0,1,1)}$ for which $(1,0,0)P^{(0,1,1)}(0,0,1)$. Hence, $\mathcal{LWSP}(f)$ is not rich relative to f.[9] Nevertheless, in Subsection

[9] It is useful to note that the domain of separable preferences is also not strongly path-connected, the general condition required in Chatterji et al. (2013)'s main result that identifies semi single-peakedness as the property of any strongly path-connected domain that admits a simple and strategy-proof rule with an even number of agents (Subsection 5.3 contains a brief description of this setting and result).

5.4 we will argue that any separable preference is local and weakly single-peaked relative to any voting by quota.

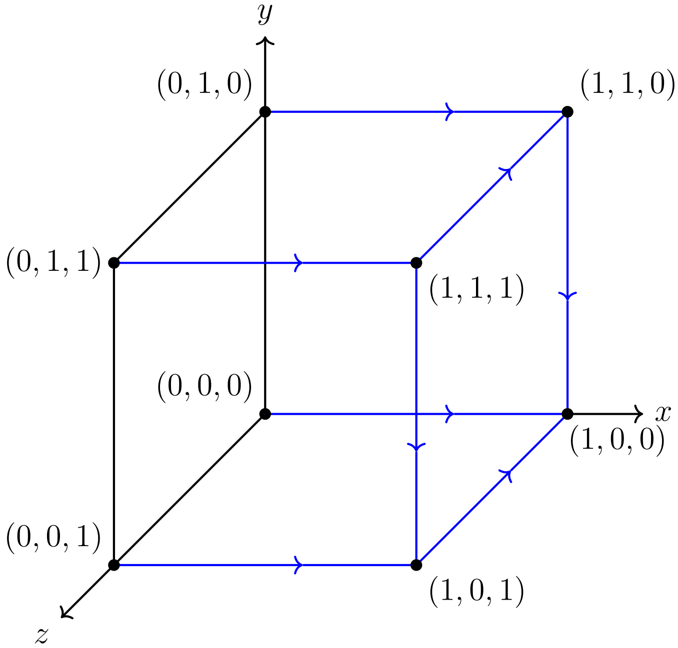

Figure 3: $\succsim^{f,(1,1,1)}$.

5 Examples

In this section we illustrate our results for single-peaked, semilattice single-peaked, semi single-peaked and separable domains of preferences. In each of these cases, we start with a structure on the set of alternatives A, we consider the class of all simple and strategy-proof rules on the domain of preferences that is meaningful according to the structure on A, and we argue that indeed the domain is contained in the set of all local and weakly single-peaked preferences relative to each rule in the class, identifying special cases (if any) for which this inclusion is not strict.

5.1 Single-peaked domains

We consider social choice problems where alternatives have objective traits that induce a "natural" linear order over A. For instance, a physical location of a public good (a library or a hospital) on a street, platforms offered by parties ordered on the left-right political spectrum, the temperature in a room or the minimum exempt in the income tax. It is important to note that the order over A must be unanimously perceived by all agents and, although it imposes meaningful restrictions on agents' preferences, they may still disagree on how to rank some pairs of alternatives. Black (1984) is the first to suggest that, according to the linear order over the set of alternatives, agents' preferences must be single-peaked. Moulin (1980) characterizes the class of all simple and strategy-proof rules on the domain of single-peaked preferences as the family of median voter schemes.

Following Moulin (1980), let the set of alternatives be the interval $A = [0, 1]$ of real numbers. A preference R_i is *single-peaked* over $[0, 1]$ if (i) there exists a unique alternative $t(R_i)$ such that $t(R_i)P_i x$ for all $x \in A \setminus \{t(R_i)\}$ and (ii) for all $x, y \in A$, if either $y < x < t(R_i)$ or $t(R_i) < x < y$ then xR_iy. Let $\mathcal{SP}$ be the domain of single-peaked preferences over $[0, 1]$. Given $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in [0, 1]^{n-1}$, define the *median voter* scheme associated to α, denoted by $f^\alpha : [0, 1]^N \to [0, 1]$, as follows. For each $(x_1, \ldots, x_n) \in [0, 1]^N$,

$$f^\alpha(x_1, \ldots, x_n) = median_{\leq}(\alpha_1, \ldots, \alpha_{n-1}, x_1, \ldots, x_n). \tag{3}$$

Note that $n - 1 + n = 2n - 1$ is an odd number and so the median is well-defined. The components of the vector α are called *fixed ballots*. Without loss of generality we can assume that $\alpha_1 \leq \cdots \leq \alpha_{n-1}$.

Consider the median voter scheme f^α associated to α, and let $y, z \in [0, 1]$ be two different alternatives. First, notice that

$$f^\alpha(y, z, \ldots, z) = y \text{ if and only if } z < y \leq \alpha_1 \text{ or } \alpha_{n-1} \leq y < z. \tag{4}$$

Therefore, $A_0 = [0, \alpha_1) \cup (\alpha_{n-1}, 1]$. Furthermore, if $x \in [0, \alpha_1)$, then $c^x = \alpha_1$; whereas if $x \in (\alpha_{n-1}, 1]$, then $c^x = \alpha_{n-1}$. By definition, $\overline{\succeq}^x$ coincides with $\succeq_0$ on $\overline{A}_0 = [0, \alpha_1] \cup [\alpha_{n-1}]$. This implies that, if $y, z \in \overline{A}_0$, we have $y \overline{\succeq}^x z$ if and only if $z \leq y \leq \alpha_1$ or $\alpha_{n-1} \leq y \leq z$. Figure 4 depicts $\succeq_0$.

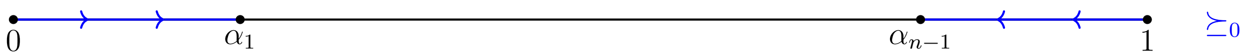

Figure 4: $\succeq_0$.

We show how f^α induces, for each $x \in A$, the x-pie $\overline{\succeq}^x$. Suppose that $n = 2$. Accordingly, $\alpha_1 = \alpha_{n-1}$, $A_0 = A \setminus \{\alpha_1\}$ and $\overline{A}_0 = [0, 1]$. Therefore, $y \succ_0 z$ holds if and only if either $z < y \leq \alpha_1$ or $\alpha_1 \leq y < z$. Hence, by (i) in the definition of the x-pie, $y \overline{\succ}^x z$ holds if and only if either $z < y \leq \alpha_1$ or $\alpha_1 \leq y < z$.

We now suppose that $n \geq 3$. There are two cases to consider:

Case 1: $\boldsymbol{x \notin [\alpha_1, \alpha_{n-1}]}$. First consider the case $x < \alpha_1$. We will show that

$$y \overline{\succ}^x z \text{ if and only if either (a) } 0 \leq z < y \leq \alpha_{n-1} \text{ or (b) } \alpha_{n-1} \leq y < z \leq 1. \tag{5}$$

($\Longleftarrow$) Assume (a) holds. We distinguish between two subcases.

(a.1) $\boldsymbol{y \le \alpha_1}$. Then, $y \succ_0 z$ and, by (i) in the definition of the x-pie, $y \overline{\succ}^x z$.

(a.2) $\boldsymbol{\alpha_1 < y}$. We further distinguish between two subcases.

- **(a.2.1) $\boldsymbol{\alpha_1 \le z < y}$.** Then, there exists $t \in \{1, \ldots, n-2\}$ such that $\alpha_{n-t-1} \le z \le \alpha_{n-t}$ and, by (3), $f^\alpha(\underbrace{x, \ldots, x}_{t \text{ times}}, z, \underbrace{y, \ldots \ldots, y}_{n-t-1 \text{ times}}) = z$ because the number of ballots at z or below is equal to $t + (n - t - 1) + 1 = n$; hence, $z \succ_t^x y$ and, by (ii) in the definition of the x-pie, $y \overline{\succ}^x z$.
- **(a.2.2) $\boldsymbol{0 \le z < \alpha_1 < y \le \alpha_{n-1}}$.** Then, by ($a$.2.1), $y \overline{\succ}^x \alpha_1$, and by ($a$.1) $\alpha_1 \overline{\succ}^x z$; by transitivity, $y \overline{\succ}^x z$.

Assume (b) holds. Since $y, z \in \overline{A}_0$, by (4), $y \succ_0 z$. By (i) in the definition of the x-pie, $y \overline{\succ}^x z$.

($\Longrightarrow$) Assume $y \overline{\succ}^x z$. If $y \in \overline{A}_0 = [0, \alpha_1] \cup [\alpha_{n-1}, 1]$, then $z \in A_0$ and $y \succ_0 z$. Therefore, $0 \le z < y \le \alpha_1$ or (b) $\alpha_{n-1} \le y < z \le 1$. We claim that if $y \in (\alpha_1, \alpha_{n-1})$, then $z \le \alpha_{n-1}$. Assume that $z > \alpha_{n-1}$. Thus, $z \in A_0$. Since $y \in A \backslash A_0$, $z \in A_0$, and $y \overline{\succ}^x z$ then, by condition (iv) in definition of the x-pie, $y \overline{\succ}^x c^z = \alpha_{n-1}$. Hence, $x < \alpha_1 < y < c^z$ and there is $t \in \{1, \ldots, n-2\}$ such that $c^z \succ_t^x y$. However,

$$f^\alpha(\underbrace{x, \ldots, x}_{t \text{ times}}, c^z, \underbrace{y, \ldots \ldots, y}_{n-t-1 \text{ times}}) = median_{\le}(\alpha_1, \ldots, \alpha_{n-1}, \underbrace{x, \ldots, x}_{t \text{ times}}, c^z, \underbrace{y, \ldots \ldots, y}_{n-t-1 \text{ times}}) \le y < c^z,$$

because the number of ballots at y or below is at least $t + 1 + n - t - 1 = n$, and this contradicts $c^z \succ_t^x y$. Therefore $z \le \alpha_{n-1}$. Now, assume that $z \in [y, \alpha_{n-1}]$. Hence, $z \in A \setminus A_0$ and there is $t \in \{1, \ldots, n-2\}$ such that and $z \succ_t^x y$. Hence, $x < \alpha_1 < y < z$. However,

$$f^\alpha(\underbrace{x, \ldots, x}_{t \text{ times}}, z, \underbrace{y, \ldots \ldots, y}_{n-t-1 \text{ times}}) = median_{\le}(\alpha_1, \ldots, \alpha_{n-1}, \underbrace{x, \ldots, x}_{t \text{ times}}, z, \underbrace{y, \ldots \ldots, y}_{n-t-1 \text{ times}}) \le y < z,$$

because the number of ballots at y or below is at least $t + 1 + n - t - 1 = n$, and this contradicts $z \succ_t^x y$. Therefore, $0 \le z < y < \alpha_{n-1}$, which is (a).

Now consider the case $x > \alpha_{n-1}$. Then, the x-pie $\overline{\succsim}^x$ can be constructed in a similar way. In fact, $y \overline{\succ}^x z$ if and only if either $0 \le z < y \le \alpha_1$ or $\alpha_1 \le y < z \le 1$.

The case for $n = 5$ is depicted in Figure 5.

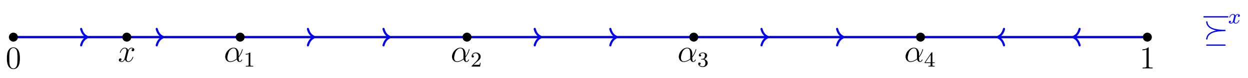

Figure 5: $\overline{\succsim}^x$ for the case $x \notin [\alpha_1, \alpha_{n-1}]$ with $n = 5$.

Case 2: $\boldsymbol{x \in [\alpha_1, \alpha_{n-1}]}$. We will show that $y\overline{\succ}^x z$ if and only if one of the following conditions holds: (a) $0 \leq z < y \leq \alpha_1$, (b) $\alpha_1 \leq y < z \leq x$, (c) $x \leq z < y \leq \alpha_{n-1}$, or (d) $\alpha_{n-1} \leq y < z \leq 1$.

($\Longleftarrow$) If either (a) or (d) holds, then $y \succ_0 z$ and, by (i) in the definition of the x-pie, $y\overline{\succ}^x z$. Suppose (b) holds. Then, there exists $t \in \{1, \ldots, n-2\}$ such that $\alpha_t < z \leq \alpha_{t+1}$ and, by (3), $f^\alpha(\underbrace{x, \ldots, x}_{t \text{ times}}, z, \underbrace{y, \ldots\ldots, y}_{n-t-1 \text{ times}}) = z$ because the number of ballots at z or below and at z or above is equal to $t + 1 + (n - t - 1) = n$; hence, $z \succ_t^x y$ and, by (ii) in the definition of the x-pie, $y\overline{\succ}^x z$. The analysis when (c) holds is similar to the case when (b) holds.

($\Longrightarrow$) Assume $y\overline{\succ}^x z$. If $y \in \overline{A}_0 = [0, \alpha_1] \cup [\alpha_{n-1}, 1]$, then $z \in A_0$ and $y \succ_0 z$. Therefore, $0 \leq z < y \leq \alpha_1$ or $\alpha_{n-1} \leq y < z \leq 1$, which are (a) or (d). First assume that $y \in (\alpha_1, x]$. We will show that $z \in (y, x]$. Assume that $z < \alpha_1$. Thus, $z \in A_0$. Since $y \in A \backslash A_0$, $z \in A_0$, and $y\overline{\succ}^x z$ then, by condition (iv) in definition of the x-pie, $y\overline{\succ}^x c^z = \alpha_1$. Hence, $z < c^z < y \leq x \leq \alpha_{n-1}$ and there is $t \in \{1, \ldots, n-2\}$ such that $c^z \succ_t^x y$. However,

$$f^\alpha(\underbrace{x, \ldots, x}_{t \text{ times}}, c^z, \underbrace{y, \ldots\ldots, y}_{n-t-1 \text{ times}}) = median_{\leq}(\alpha_1, \ldots, \alpha_{n-1}, \underbrace{x, \ldots, x}_{t \text{ times}}, c^z, \underbrace{y, \ldots\ldots, y}_{n-t-1 \text{ times}}) \geq y > c^z$$

because the number of ballots at y or above is at least $t+1+n-t-1 = n$, and this contradicts $c^z \succ_t^x y$. Now, assume that $z \in [\alpha_1, y)$. Hence, $z \in A \setminus A_0$ and there is $t \in \{1, \ldots, n-2\}$ such that $z \succ_t^x y$. Hence, $\alpha_1 \leq z < y \leq x \leq \alpha_{n-1}$. However,

$$f^\alpha(\underbrace{x, \ldots, x}_{t \text{ times}}, z, \underbrace{y, \ldots\ldots, y}_{n-t-1 \text{ times}}) = median_{\leq}(\alpha_1, \ldots, \alpha_{n-1}, \underbrace{x, \ldots, x}_{t \text{ times}}, z, \underbrace{y, \ldots\ldots, y}_{n-t-1 \text{ times}}) \geq y > z$$

because the number of ballots at y or above is at least $t+1+n-t-1 = n$, and this contradicts $z \succ_t^x y$. Therefore, $z \in (y, x]$, which is (b). In a similar way we can prove that if $y \in [x, \alpha_{n-1})$, then $z \in [x, y)$, which is (c).

The case for $n = 5$ is depicted in Figure 6.

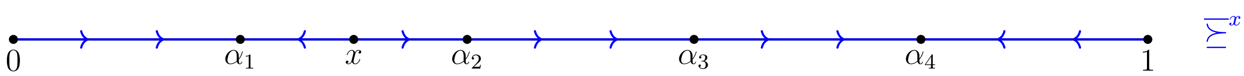

Figure 6: $\overline{\succeq}^x$ for the case $x \in [\alpha_1, \alpha_{n-1}]$ with $n = 5$.

Let $\mathcal{F} = \{f^\alpha \mid f^\alpha : \mathcal{SP}^N \to [0, 1]$ is a median voter scheme associated to $\alpha \in [0, 1]^{n-1}\}$ be the class of all simple and strategy-proof rules on $\mathcal{SP}$. Now we show that

$$\mathcal{SP} = \bigcap_{f^\alpha \in \mathcal{F}} \mathcal{LWSP}(f^\alpha);$$

namely, single-peakedness is the fundamental property of the domain of any simple and strategy-proof rule on a basic, common, and rich domain of preferences over $[0,1]$. We first check that, for each $f^\alpha \in \mathcal{F}$, the domain $\mathcal{SP}$ is rich relative to f^α.[10] Fix $f^\alpha \in \mathcal{F}$ and $w \in [0,1]$, and assume $z \notin [x,y]_{\overline{\succeq}^w} \neq \emptyset$. If $x = y$, $xP^y z$ and $yP^x z$ follow trivially for any $R^x, R^y \in \mathcal{SP}$. By the definition of the w-pie, either $x < y$ and $z \notin [x,y]$ or $y < x$ and $z \notin [y,x]$. But in each of the two cases there exist $R^x, R^y \in \mathcal{SP}$ such that $yP^x z$ and $xP^y z$. Hence, $\mathcal{SP}$ is rich on $([0,1], \overline{\succeq}^{f^\alpha,w})$ for all $w \in [0,1]$ and so $\mathcal{SP}$ is rich relative to f^α. By Theorem 1, $\mathcal{SP} \subset \mathcal{LWSP}(f^\alpha)$. Accordingly, $\mathcal{SP} \subseteq \bigcap_{f^\alpha \in \mathcal{F}} \mathcal{LWSP}(f^\alpha)$ holds. In order to see that $\bigcap_{f^\alpha \in \mathcal{F}} \mathcal{LWSP}(f^\alpha) \subseteq \mathcal{SP}$ holds as well, let $R^x \in \mathcal{LWSP}(f^\alpha)$ for each $f^\alpha \in \mathcal{F}$ and assume $R^x \notin \mathcal{SP}$. Then, there are $x, y, z \in [0,1]$ such that $z < y < x$ or $x < y < z$ and $zP^x y$. Without loss of generality, assume that $x < y < z$. Let $\widetilde{\alpha} \in [0,1]^{n-1}$ be such that $\widetilde{\alpha}_1 = \cdots = \widetilde{\alpha}_{n-1} = z$. By the definition of the x-pie, $z \overline{\succ}^{f^{\widetilde{\alpha}},x} y \overline{\succ}^{f^{\widetilde{\alpha}},x} x$ because $f^{\widetilde{\alpha}}(y, x, \ldots, x) = y$ and $f^{\widetilde{\alpha}}(z, y, \ldots, y) = z$. Then, by condition (LWSP.1) we have $yR^x z$, contradicting our hypothesis. Therefore, $\mathcal{SP} = \bigcap_{f^\alpha \in \mathcal{F}} \mathcal{LWSP}(f^\alpha)$.

5.2 Semilattice single-peaked domains

We now consider social choice problems where the set of alternatives is endowed with a partial order $\succeq$ that is a semilattice.[11] For instance, A could be a tree or a river with its tributaries where the flow of the water is represented by the increasing order of $\succeq$.

Let $(A, \succeq)$ be a semilattice and let $x \in A$ be an alternative. According to Chatterji and Massó (2018), a preference R^x is *semilattice single-peaked* over $(A, \succeq)$ if R^x is local and weakly single-peaked over $(A, \succeq)$.[12] Let $\mathcal{SSP}(\succeq)$ be the set of all semilattice single-peaked preferences over $(A, \succeq)$. Then, preferences in $\mathcal{SSP}(\succeq)$ in the river allegory of the semilattice correspond to the situation where agents can move from where they are located (their top locations) only downstream, but not upstream. Thus, any pair of non-top locations that are not connected by the flow of the water can be ordered in any way.

Bonifacio and Massó (2021) characterize the class of all simple and strategy-proof rules on $\mathcal{SSP}(\succeq)$. This class consists of the supremum rule and the generalized quota-supremum rules whose definitions, according to Bonifacio and Massó (2021), are as follows.

The *supremum* rule, denoted as $\sup_{\succeq} : \mathcal{SSP}(\succeq)^N \to A$, is defined by setting, for each profile $R = (R_1, \ldots, R_n) \in \mathcal{SSP}(\succeq)^N$,

$$\sup_{\succeq}(R_1, \ldots, R_n) = \sup_{\succeq}\{t(R_1), \ldots, t(R_n)\}.$$

Given $R \in \mathcal{SSP}(\succeq)^N$ and $x \in A$, let $N(R, x) = \{i \in N \mid t(R_i) = x\}$ be the set of agents whose

[10] To verify that $\mathcal{SP}$ is rich on $([0,1], \leq)$ is immediate.

[11] Let $\succeq$ be a partial order over A. Then, $(A, \succeq)$ is a *semilattice* if $\sup_{\succeq}\{x, y\}$ exists for each pair $x, y \in A$.

[12] We have decided to not use the term semilattice in the name of the domain identified here because partially inverted extensions are not necessarily semilattices; for instance, the x-pie $\overline{\succeq}^x$ in Figure 6 is not a semilattice because $\sup_{\overline{\succeq}^x}\{y, z\}$ does not exist for any pair of alternatives y and z such that $y < \alpha_1$ and $z > \alpha_4$.

top at R is x. Assume $(A, \succeq)$ has a supremum, denoted as $\alpha \equiv \sup_{\succeq} A$, and let

$$A^{\star}(\succeq) = \{x \in A \mid \text{ for each } y \in A \setminus \{\alpha\},\, x \not\succeq y \text{ and } y \not\succeq x\}$$

be the set of alternatives that, according to $\succeq$, are not related to any other alternative but α. Observe that $A^{\star}(\succeq)$ may be empty and $\alpha \notin A^{\star}(\succeq)$. A quota system $q = \{q^x\}_{x \in A^{\star}(\succeq)}$ assigns to each $x \in A^{\star}(\succeq)$ an integer $1 \leq q^x \leq n$, which we also refer to as a quota, satisfying the following two properties.

(QS.1) There is $x \in A^{\star}(\succeq)$ such that $1 \leq q^x < n$.

(QS.2) For any two distinct alternatives $x, y \in A^{\star}(\succeq)$, $q^x + q^y > n$.

Let $(A, \succeq)$ be a semilattice such that $\sup_{\succeq} A$ exists. The rule $f : \mathcal{SSP}(\succeq)^N \to A$ is a *generalized quota-supremum* rule if there exists a quota system $q = \{q^x\}_{x \in A^{\star}(\succeq)}$ such that, for every $R \in \mathcal{SSP}(\succeq)^N$,

$$f(R) = \begin{cases} x & \text{if } x \in A^{\star}(\succeq) \text{ and } |N(R, x)| \geq q^x \\ \sup_{\succeq}(R) & \text{otherwise.} \end{cases}$$

Let $(A, \succeq)$ be a semilattice, and let $f : \mathcal{SSP}(\succeq)^N \to A$ be either the $\sup_{\succeq}$ rule or a generalized quota-supremum rule. In Appendix A.6 we establish that $\mathcal{SSP}(\succeq) \subsetneq \mathcal{LWSP}(f)$ and show that the inclusion is strict only if f is a generalized quota-supremum rule with the property that $q^x = 1$ for an alternative $x \in A^{\star}(\succeq)$.

5.3 Semi single-peaked domains

Chatterji et al. (2013) started this literature that aims to identify the structure of a domain of preferences if it has to admit a simple and strategy-proof rule. The novelty of their approach is that the structure on the set of alternatives underlying their notion of semi single-peakedness is uncovered from $\mathcal{D}$, precisely because it admits such a rule. Roughly, their setting is as follows. Let $\mathcal{D}$ be a domain of strict preferences over a finite set of alternatives A. Alternatives $x, y \in A$ are strongly connected (denoted by $x \leftrightarrow y$) if there exist $P^x, P^y \in \mathcal{D}$ such that x and y are respectively ranked first and second according to P^x, the reverse is true according to P^y, and P^x and P^y coincide over $A \setminus \{x, y\}$. They postulate the basic richness assumption that $\mathcal{D}$ is strongly path-connected (every pair of alternatives has a path of strong connections). They first show that if $\mathcal{D}$ is strongly path-connected and it admits a simple and strategy-proof rule, then the graph G induced on A by $\mathcal{D}$ through $\leftrightarrow$ is a tree. Consequently, for every pair of alternatives (nodes of G) $x, y \in A$, there is a unique path p linking them, denoted by $\langle x, y\rangle$. They show first that if a domain $\mathcal{D}$ is strongly path-connected, n is even and $\mathcal{D}$ admits a simple and strategy-proof rule, then $\mathcal{D}$ is a set of semi single-peaked preferences. Second, if $\mathcal{D}$ is a semi single-peaked domain, then there exists a simple and strategy-proof rule $f : \mathcal{D}^n \to A$ for any integer n.

The notion of semi single-peakedness relies on the tree G induced on A by $\mathcal{D}$ through $\leftrightarrow$, and according to Chatterji and Massó (2018) it is as follows. Given alternatives $x, y, z \in A$, let $\pi(z, \langle x, y\rangle)$ denote the projection of z on the path $\langle x, y\rangle$ that is defined as the unique alternative $w \in A$ such that $\langle x, z\rangle \cap \langle y, z\rangle = \langle w, z\rangle$. A path p is *maximal* if it cannot be extended by adding more edges at either one of the two ends. Fix a particular alternative α on the tree A (call it the threshold), and use it to specify a threshold on every maximal path p, denoted by $\lambda(p)$, as $\lambda(p) = \pi(\alpha, p)$. Thus, for every maximal path p, if it contains the alternative α, set $\lambda(p) = \alpha$; otherwise, the threshold $\lambda(p)$ is the unique alternative that lies on every path from an alternative on the path p to alternative α. Chatterji et al. (2013) say that a preference R is *semi single-peaked* with respect to a tree G and a threshold $\alpha \in A$ if, for each maximal path p and all $x, y \in p$, the following two conditions hold.

(i) $x, y \in \langle t(R), \lambda(p)\rangle$ and $x \in \langle t(R), y\rangle$ imply xRy, and

(ii) $\lambda(p) \in \langle t(R), x\rangle$ implies $\lambda(p)Rx$.[13]

Chatterji et al. (2013) obtain the necessity of such property by looking at the two-agent voting scheme $\overleftrightarrow{f}^{\alpha} : A^2 \to A$, where for all $x, y \in A$,

$$\overleftrightarrow{f}^{\alpha}(x, y) = \pi(\alpha, \langle x, y\rangle).$$[14]

Chatterji et al. (2013) show that the rule $\overleftrightarrow{f}^{\alpha}$ is simple and strategy-proof on the domain of semi single-peaked preferences with respect to the tree G and the threshold α. According to Chatterji and Massó (2018), define the binary relation $\succsim^{\alpha}$ over A by setting, for all $x, y \in A$,

$$x \succsim^{\alpha} y \text{ if and only if } x = \pi(\alpha, \langle x, y\rangle).$$

They argue that (i) the domain consisting of all semi single-peaked preferences is rich on $(A, \succsim^{\alpha})$, (ii) for all $x, y \in A$, $\overleftrightarrow{f}^{\alpha}(x, y) = \sup_{\succsim^{\alpha}}\{x, y\}$, (iii) $(A, \succsim^{\alpha})$ is a semilattice, and (iv) the set of strict semilattice single-peaked preferences over $(A, \succsim^{\alpha})$ coincides with the set of semi single-peaked preferences with respect to the tree G and the threshold α. Thus, since $\overleftrightarrow{f}^{\alpha}$ is a supremum rule for the semilattice $\succsim^{\alpha}$, the analysis presented in Section 5.2 implies that, for strict preferences, $\mathcal{LWSP}(\overleftrightarrow{f}^{\alpha}) = \mathcal{SSP}(\succsim^{\alpha})$. Consequently, for strict preferences, the set of local and weakly single-peaked preferences relative to $\overleftrightarrow{f}^{\alpha}$ coincides with the set of semi single-peaked preferences with respect to the tree G and the threshold α.

[13] Conditions (i) and (ii) geometrically resemble conditions (LWSP.1) and (LWSP.2), respectively.

[14] Observe that from any simple rule $g : \mathcal{D}^n \to A$ with n even, one can define the two-agent voting scheme $f : A^2 \to A$ by setting, for each $x, y \in A$, $f(x, y) = g(\underbrace{x, \ldots, x}_{\frac{n}{2}\text{ times}}, \underbrace{y, \ldots, y}_{\frac{n}{2}\text{ times}})$. This construction makes evident the need to assume that the number of agents n is even.

5.4 Separable domains

We come back to the case of separable preferences already considered in Subsection 4.3 to argue that the set of separable preferences is contained in the domain of local and weakly single-peaked preferences relative to any voting by quota f, despite this domain is not necessarily rich relative to f.

The following geometric representation of separability will be useful. Let P be a separable preference over $2^{\mathcal{K}}$ and, by iterating the definition of separability, the top alternative $t(P)$ is the subset of all good objects (*i.e.*, $t(P) = \{k \in \mathcal{K} \mid \{k\}P\{\emptyset\}\}$). Define the bottom alternative according to P, denoted by $b(P)$, as the alternative for which $YPb(P)$ for all $Y \in A \backslash \{b(P)\}$. Then, the bottom alternative is the subset of all bad objects (*i.e.*, $b(P) = \{k \in \mathcal{K} \mid \{\emptyset\}P\{k\}\}$). Hence, for all $k \in \mathcal{K}$,

$$b(P)_k = \begin{cases} 0 & \text{if } t(P)_k = 1 \\ 1 & \text{if } t(P)_k = 0. \end{cases}$$

Figure 7 represents, for the case where $\mathcal{K} = \{x, y, z\}$, the separable preference $P^{(1,1,1)}$, where an arrow from X to Y, differing from only one object (*i.e.*, X and Y are two contiguous vertices in $2^{\mathcal{K}}$), means now that XPY. In general, separability of P means that along each path starting at $t(P)$ and finishing at $b(P)$ the preference is decreasing, and all other pairs of subsets can be freely ordered by P.

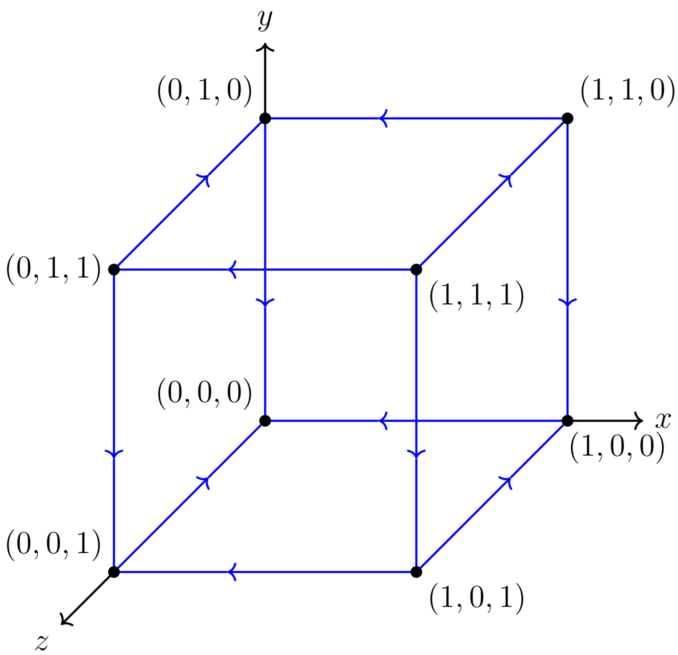

Figure 7: Path representation of $P^{(1,1,1)}$.

We first exhibit a voting by quota f with the property that the domain of local and weakly single-preferences relative to f coincides with the set of separable preferences.

Let $N = \{1, 2, 3\}$, $\mathcal{K} = \{x, y, z\}$ and consider the voting by quota $f : A^3 \to A$ where $q_x = q_y = q_z = 2$. By (2), $A_0^f = \emptyset$. It is easy to see that Figure 7 also represents the $(1, 1, 1)$-pie $\succeq^{f,(1,1,1)}$. Hence, $\mathcal{LWSP}(\succeq^{f,(1,1,1)})$ coincides with the set of separable preferences with top in $(1, 1, 1)$. It can be verified that this is always the case; namely, $\mathcal{LWSP}(\succeq^{f,X}) = \mathcal{S}$ holds for any voting by quota f with the property that $K_1 \cup K_n = \emptyset$ and any alternative $X \in 2^{\mathcal{K}}$.

We now come back to the voting by quota f defined by $q = (q_x, q_y, q_z) = (1, 2, 2)$ already presented in Subsection 4.3, where Figure 3 represents $\succeq^{f,(1,1,1)}$. Consider a preference $P^{(1,1,1)}$. To be local and weakly single-peaked, $P^{(1,1,1)}$ has to satisfy two families of conditions, those coming from its definition. First, only the two paths $(1, 0, 0) \succ^{f,(1,1,1)} (1, 0, 1) \succ^{f,(1,1,1)} (1, 1, 1)$ and $(1, 0, 0) \succ^{f,(1,1,1)} (1, 1, 0) \succ^{f,(1,1,1)} (1, 1, 1)$ satisfy the hypothesis of (LWSP.1). Therefore, local and weak single-peakedness requires that $(1, 0, 1)P^{(1,1,1)}(1, 0, 0)$ and $(1, 1, 0)P^{(1,1,1)}(1, 0, 0)$ must hold, but this is the case for any strict separable preference with top on $(1, 1, 1)$. Second, we list the four cases satisfying the hypothesis of (LWSP.2), and their corresponding restriction for $P^{(1,1,1)}$.

$$
\begin{array}{lll}
(0,0,1) \not\succeq^{f,(1,1,1)} (1,1,1) & \text{and} & \sup_{\succeq^{f,(1,1,1)}}\{(0,0,1),(1,1,1)\} = (1,0,1)P^{(1,1,1)}(0,0,1) \\
(0,1,1) \not\succeq^{f,(1,1,1)} (1,1,1) & \text{and} & \sup_{\succeq^{f,(1,1,1)}}\{(0,1,1),(1,1,1)\} = (1,1,1)P^{(1,1,1)}(0,1,1) \\
(0,1,0) \not\succeq^{f,(1,1,1)} (1,1,1) & \text{and} & \sup_{\succeq^{f,(1,1,1)}}\{(0,1,0),(1,1,1)\} = (1,1,0)P^{(1,1,1)}(0,1,0) \\
(0,0,0) \not\succeq^{f,(1,1,1)} (1,1,1) & \text{and} & \sup_{\succeq^{f,(1,1,1)}}\{(0,0,0),(1,1,1)\} = (1,0,0)P^{(1,1,1)}(0,0,0).
\end{array}
$$

But any strict separable preference with top on $(1, 1, 1)$ satisfies the four conditions. Hence, the set of separable preferences with top on $(1, 1, 1)$ is a strict subset of $\mathcal{LWSP}(\succeq^{f,(1,1,1)})$ and so $\mathcal{S} \subsetneq \mathcal{LWSP}(f)$. It can be verified that this is the case for any voting by quota f with the property that $K_1 \cup K_n \neq \emptyset$ and any alternative $X \in 2^{\mathcal{K}}$.

Let $\mathcal{V}b\mathcal{Q} = \{f^q \mid f^q : \mathcal{S}^N \to 2^{\mathcal{K}}$ is a voting by quota associated to $q = (q_k)_{k \in \mathcal{K}}\}$ be the class of all simple and strategy-proof rules on $\mathcal{S}$. It is easy to verify that

$$
\mathcal{S} = \bigcap_{f^q \in \mathcal{V}b\mathcal{Q}} \mathcal{LWSP}(f^q);
$$

namely, separability is the fundamental property of the domain of any simple and strategy-proof rule on a common and basic domain of preferences over $2^{\mathcal{K}}$.

6 Discussion

Our paper contributes to the literature that aims to identify the crucial property that a domain of preferences must satisfy if it has to admit a simple and strategy-proof rule. Our setting does not require any assumption on the structure of the set of alternatives. Moreover, and this is one of our most significant contributions, we do not require that the number of agents be even, a somehow awkward assumption in Chatterji et al. (2013) and Chatterji and Massó (2018). As we have already mentioned, their approaches require an even number of agents since they obtain the restrictions on $\mathcal{D}$ because of the existence of a simple and strategy-proof rule $g : \mathcal{D}^N \to A$, not from g itself but rather from an induced voting scheme $f : A^2 \to A$, defined by looking at the alternative chosen by g *only* at profiles of tops where the set of agents is divided into exactly two halves: those that support one alternative and those that support the other one. Namely, given a voting scheme $g : A^n \to A$, define $f : A^2 \to A$ by setting, for each $(x, y) \in [0, 1]^2$,

$$f(x, y) = g(\underbrace{x, \ldots, x}_{\frac{n}{2}\text{ times}}, \underbrace{y, \ldots, y}_{\frac{n}{2}\text{ times}}).^{15}$$

Of course, this construction doe not only leave aside the obvious case where n is odd (covered with our approach) but it also has an important consequence: The implications on $\mathcal{D}$ obtained from f are weaker than the implications obtained from the original g. That is, some preferences are admissible in $\mathcal{D}$ as the domain of f, but they are not as the domain of g.

To see that this two-agent approach does not fully squeeze the domain implications of admitting a simple and strategy-proof rule when $n > 2$ is even, consider now the case where $A = [0, 1]$, $n = 4$ and the median voter scheme $g : [0, 1]^4 \to [0, 1]$ associated to three fixed ballots $0 \leq \alpha_1 < \alpha_2 < \alpha_3 \leq 1$, defined on an arbitrary basic domain of preferences $\mathcal{D}$, as presented in Subsection 5.1. The approach in Chatterji and Massó (2018) consists of defining a voting scheme $f : [0, 1]^2 \to [0, 1]$ by setting, for every $(x, y) \in [0, 1]^2$,

$$f(x, y) = g(x, x, y, y).$$

Then, they define the binary relation $\succeq^f$ over $[0, 1]$ as follows. For every $x, y \in [0, 1]$,

$$x \succeq^f y \text{ if and only if } f(x, y) = x.$$

Observe that the binary relation $\succeq^f$ coincides with $\succeq^f_0$ and so with $\overline{\succeq}^{f,x}$ for each $x \in A$. Chatterji and Massó (2018) show that $\mathcal{D}$ is a set of semilattice single-peaked preferences over $([0, 1], \succeq^f)$,[16] provided that g is strategy-proof and $\mathcal{D}$ is rich on $(A, \succeq^f)$. In contrast, we obtain the domain of local and weakly single-peaked preferences $\mathcal{LWSP}(g)$ using the original g. But instead, the binary relation to which conditions (LWSP.1) and (LWSP.2) in Definition 3 are required to hold for a preference R^x is specific to the top-ranked alternative x, and relative to the partial order $\overline{\succeq}^{g,x}$, which is different from $\succeq^f$. In this case $\mathcal{SSP}(\succeq^f) = \mathcal{LWSP}(f)$, but this domain is too large for g to be strategy-proof on it. To see this, consider x, y and z such that $\alpha_2 < x < y < z < \alpha_3$ and a preference $R^x \in \mathcal{SSP}(\succeq^f)$ such that $zP^x y$, which is possible because $\alpha_2 = \sup_{\succeq^f}[0, 1]$ and $x \succ^f y \succ^f z$. Hence, because $z \overline{\succ}^{g,x} y \overline{\succ}^{g,x} x$, $R^x \notin \mathcal{LWSP}([0, 1], \overline{\succeq}^{g,x})$. To see that g is not strategy-proof on $\mathcal{SSP}(\succeq^f)$, consider the profile of tops $t = (x, y, z, z)$. Then, $g(t) = y$ and agent 1 can manipulate g at t via any R^z_1 (by declaring z) since $g(z, y, z, z) = zP^x_1 y = g(x, y, z, z)$.

[15]It is easy to see that f inherits the properties of simplicity and strategy-proofness from g.

[16]Namely, each preference in $\mathcal{D}$ has to satisfy conditions (LWSP.1) and (LWSP.2) relative to $\succeq^f$ in our Definition 3 of local and weak single-peakedness.

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A Appendix

A.1 Properties of $\succeq_0$ and the covers

Before proving Lemma 1, we present two useful lemmata.

Lemma 3 *Let $x \in A_0$ and assume c^x and $\bar{c}^x$ are two different covers of x. Then,*

$$f(c^x, \bar{c}^x, \ldots, \bar{c}^x) = \bar{c}^x \text{ and } f(\bar{c}^x, c^x, \ldots, c^x) = c^x.$$

Proof. Let $f(c^x, \bar{c}^x, \ldots, \bar{c}^x) = z \neq \bar{c}^x$. By strategy-proofness, $f(z, \bar{c}^x, \ldots, \bar{c}^x) = z$ implying that $z \succ_0 \bar{c}^x$, a contradiction with $\bar{c}^x \in A \setminus A_0$. Similarly, $f(\bar{c}^x, c^x, \ldots, c^x) = c^x$. □

Lemma 4 *Let $x \in A_0$ and assume c^x and $\bar{c}^x$ are two different covers of x. Then, $f(R) = \bar{c}^x$ for each $R \in \mathcal{D}^N$ such that $t(R) = \{x, c^x, \bar{c}^x\}$ and $|N(R, c^x)| = 1$.*

Proof. Note that, as $|t(R)| \geq 3$, $n \geq 3$. Let $k = |N(R, x)|$. The proof is by induction on k.

Basis step: We prove that $f(x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = \bar{c}^x$ through two claims.

Claim 1: $\boldsymbol{f(x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) \in \{x, c^x, \bar{c}^x\}}$. Assume otherwise; *i.e.*, $f(x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = y \notin \{x, c^x, \bar{c}^x\}$. We show that $y \not\succ_0 x$. Assume $y \succ_0 x$. Since $\bar{c}^x$ covers x, $\bar{c}^x \in A \setminus A_0$ and hence $y \not\succ_0 \bar{c}^x$. Thus, $\bar{c}^x \notin [x, y]_{\succeq 0}$. By richness, there is $R_2^x \in \mathcal{D}$ such that $yP_2^x\bar{c}^x$. The hypothesis $f(x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = y$ and strategy-proofness imply $f(x, y, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = y$. In addition, $\bar{c}^x \succ_0 x$ implies $f(\bar{c}^x, x, \ldots, x) = \bar{c}^x$ and, by a repeated use of strategy-proofness, we obtain $f(x, x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = \bar{c}^x$. Therefore,

$$f(x, y, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = yP_2^x\bar{c}^x = f(x, x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}),$$

and agent 2 manipulates f, contradicting strategy-proofness. Thus, $y \not\succ_0 x$ and, accordingly, $y \notin [x, \bar{c}^x]_{\succeq 0}$. By richness, there is $R_1^x \in \mathcal{D}$ such that $\bar{c}^x P_1^x y$. Using Lemma 3,

$$f(\bar{c}^x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = \bar{c}^x P_1^x y = f(x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}),$$

and agent 1 manipulates f, contradicting strategy-proofness.

Claim 2: $\boldsymbol{f(x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = \bar{c}^x}$. By Claim 1, $f(x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) \in \{x, c^x, \bar{c}^x\}$. Assume first that $f(x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = x$. As $\bar{c}^x \succ_0 x$, by a repeated use of strategy-proofness, we obtain $f(x, x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = \bar{c}^x$. Therefore,

$$f(x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = xP_2^x\bar{c}^x = f(x, x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}),$$

contradicting strategy-proofness. Assume now that $f(x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = c^x$. By definition of c^x and $\bar{c}^x$, $c^x \notin [x, \bar{c}^x]_{\succeq 0}$. Therefore, by richness, there is R_1^x such that $\bar{c}^x P_1^x c^x$. By Lemma 3, $f(\bar{c}^x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = \bar{c}^x$. Hence,

$$f(\bar{c}^x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = \bar{c}^x P_1^x c^x = f(x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}),$$

contradicting strategy-proofness. Thus, $f(x, c^x, \underbrace{\bar{c}^x, \ldots, \bar{c}^x}_{n-2 \text{ times}}) = \bar{c}^x$.

Claims 1 and 2 prove the basis step of the induction.

Inductive step: Assume now that

$$f(\underbrace{x, \ldots, x}_{k \text{ times}}, c^x, \underbrace{\bar{c}^x, \ldots\ldots, \bar{c}^x}_{n-k-1 \text{ times}}) = \bar{c}^x. \tag{6}$$

We want to show that

$$f(\underbrace{x,\dots,x}_{k+1 \text{ times}},c^x,\underbrace{\bar{c}^x,\dots\dots,\bar{c}^x}_{n-k-2 \text{ times}}) = \bar{c}^x. \tag{7}$$

We will prove (7) through several claims.

Claim 3: $\boldsymbol{f(\underbrace{x,\dots,x}_{k+1 \text{ times}},c^x,\underbrace{\bar{c}^x,\dots\dots,\bar{c}^x}_{n-k-2 \text{ times}}) \in \{x,c^x,\bar{c}^x\}}$. Assume the claim is not true. Then,

$$f(\underbrace{x,\dots,x}_{k+1 \text{ times}},c^x,\underbrace{\bar{c}^x,\dots\dots,\bar{c}^x}_{n-k-2 \text{ times}}) = y \notin \{x,c^x,\bar{c}^x\}. \tag{8}$$

First, we show that $\bar{c}^x \not\succ_0 y$. If not, we have $f(\bar{c}^x,y,\dots,y) = \bar{c}^x$. Using anonymity once and strategy-proofness repeatedly, (8) becomes

$$f(\bar{c}^x,x,\underbrace{y,\dots,y}_{n-2 \text{ times}}) = y.$$

Therefore, for any $R_2^y \in \mathcal{D}$, we have

$$f(\bar{c}^x,x,\underbrace{y,\dots,y}_{n-2 \text{ times}}) = yP_2^y\bar{c}^x = f(\bar{c}^x,\underbrace{y,\dots,y}_{n-1 \text{ times}}),$$

contradicting strategy-proofness. Thus, $\bar{c}^x \not\succ_0 y$. As $\bar{c}^x \not\succ_0 y$, it follows that $y \notin [x,\bar{c}^x]_{\succeq_0}$. Then, by richness, there is $R_{k+1}^x \in \mathcal{D}$ such that $\bar{c}^x P_{k+1}^x y$. Therefore, by anonymity, (6) and (8),

$$f(\underbrace{x,\dots,x}_{k \text{ times}},\bar{c}^x,c^x,\underbrace{\bar{c}^x,\dots,\dots,\bar{c}^x}_{n-k-2 \text{ times}}) = \bar{c}^x P_{k+1}^x y = f(\underbrace{x,\dots,x}_{k+1 \text{ times}},c^x,\underbrace{\bar{c}^x,\dots\dots,\bar{c}^x}_{n-k-2 \text{ times}}),$$

contradicting strategy-proofness.

Claim 4: $\boldsymbol{f(\underbrace{x,\dots,x}_{k+1 \text{ times}},c^x,\underbrace{\bar{c}^x,\dots\dots,\bar{c}^x}_{n-k-2 \text{ times}}) \neq x}$. Assume otherwise. Then, as $\bar{c}^x \succ_0 x$ and repeated use of strategy-proofness imply $f(\underbrace{x,\dots,x}_{k+2 \text{ times}},\underbrace{\bar{c}^x,\dots\dots,\bar{c}^x}_{n-k-2 \text{ times}}) = \bar{c}^x$, we have, for any $R_{k+2}^x \in \mathcal{D}$, that

$$f(\underbrace{x,\dots,x}_{k+1 \text{ times}},c^x,\underbrace{\bar{c}^x,\dots\dots,\bar{c}^x}_{n-k-2 \text{ times}}) = xP_{k+2}^x\bar{c}^x = f(\underbrace{x,\dots,x}_{k+2 \text{ times}},\underbrace{\bar{c}^x,\dots\dots,\bar{c}^x}_{n-k-2 \text{ times}}),$$

contradicting strategy-proofness.

Claim 5: $\boldsymbol{f(\underbrace{x,\dots,x}_{k+1 \text{ times}},c^x,\underbrace{\bar{c}^x,\dots\dots,\bar{c}^x}_{n-k-2 \text{ times}}) \neq c^x}$. Assume otherwise. By anonymity and the inductive hypothesis (6),

$$f(\underbrace{x,\dots,x}_{k \text{ times}},\bar{c}^x,c^x,\underbrace{\bar{c}^x,\dots\dots,\bar{c}^x}_{n-k-2 \text{ times}}) = \bar{c}^x. \tag{9}$$

By Lemma 3, $c^x \notin [x, \bar{c}^x]_{\succeq_0}$ and so by richness there is $R^x_{k+1} \in \mathcal{D}$ such that $\bar{c}^x P^x_{k+1} c^x$. Therefore, by (9),

$$f(\underbrace{x, \ldots, x}_{k \text{ times}}, \bar{c}^x, c^x, \underbrace{\bar{c}^x, \ldots\ldots, \bar{c}^x}_{n-k-2 \text{ times}}) = \bar{c}^x P^x_{k+1} c^x = f(\underbrace{x, \ldots, x}_{k+1 \text{ times}}, c^x, \underbrace{\bar{c}^x, \ldots\ldots, \bar{c}^x}_{n-k-2 \text{ times}}),$$

contradicting strategy-proofness.

Notice that Claims 3, 4 and 5 imply (7). This completes the proof of Lemma 4. □

We are now ready to prove Lemma 1. We first remember its statement for completeness.

Lemma 1 *Let $f : \mathcal{D}^N \to A$ be a simple and strategy-proof rule and let $\succeq_0$ be the binary relation induced by f over A. If the domain D is rich on $(A_0, \succeq_0)$, then*

(i) *$\succeq_0$ is a partial order,*

(ii) *if $x \in A_0$ and both c^x and $\bar{c}^x$ are a cover of x, then $c^x = \bar{c}^x$, and*

(iii) *for $x, y \in A$ such that $x \succ_0 y$, $c^y = c^x$.*

Proof. Assume that the hypothesis of Lemma 1 hold.

Item (i): We have to show that $\succeq_0$ is reflexive, antisymmetric and transitive.

Reflexivity: By unanimity of f, $\succeq_0$ is reflexive.

Antisymmetry: Assume $x \succeq_0 y$ and $y \succeq_0 x$. Hence, $f(x, y, \ldots, y) = x$ and $f(x, \ldots, x, y) = y$. By strategy-proofness, $f(x, y, \ldots, y) = x$ implies $f(x, \ldots, x, y) = x$. Therefore $x = y$, and $\succeq_0$ is antisymmetric.

Transitivity: Assume $x \succeq_0 y$ and $y \succeq_0 z$, so that $f(x, y, \ldots, y) = x$ and $f(y, z, \ldots, z) = y$. We want to show that $f(x, z, \ldots, z) = x$. Assume otherwise; *i.e.*, $x \not\succeq_0 z$, and so $x \neq z$, $x \neq y$ and $y \neq z$. Hence, $z \notin [y, x]_{\succeq_0}$. Two cases are possible:

1. $\boldsymbol{f(x, z, \ldots, z) \equiv w \notin \{x, y\}}$**.** Since $f(y, z, \ldots, z) = y$ holds and f is strategy-proof,

$$wR^x y \text{ for all } R^x \in \mathcal{D}; \tag{10}$$

otherwise, agent 1 manipulates f at $(x, z, \ldots, z)$ via any R^y_1. If $w \notin [y, x]_{\succeq_0}$, by richness, there exists $R^x \in \mathcal{D}$ such that $yP^x w$, a contradiction with (10). Hence, $w \in [y, x]_{\succeq_0}$, which means that $x \succeq_0 w \succeq_0 y$; namely, $f(w, y, \ldots, y) = w$ and $f(x, w, \ldots, w) = x$. By strategy-proofness and $f(x, z, \ldots, z) = w$, we have $f(x, w, \ldots, w) = w$, which together with $f(x, w, \ldots, w) = x$ imply that $w = x$, a contradiction with the hypothesis that $w \notin \{x, y\}$.

2. $\boldsymbol{f(x, z, \ldots, z) \in \{x, y\}}$**.** Assume $f(x, z, \ldots, z) = y$. Since $x \neq y$, by strategy-proofness we have that $f(x, y, \ldots, y) = y$, a contradiction with $x \succeq_0 y$. Hence, $f(x, z, \ldots, z) = x$. Then, $x \succeq_0 z$, and $\succeq_0$ is transitive.

Item (ii): Let $x \in A$ and let c^x and $\bar{c}^x$ be two different covers of x. If $x \in A \setminus A_0$, by definition of cover, $c^x = x = \bar{c}^x$ and therefore $c^x = \bar{c}^x$. If $x \in A_0$, consider any profile $R \in \mathcal{D}^N$ such that $t(R) = \{x, c^x, \bar{c}^x\}$, $|N(R, c^x)| = 1$ and $|N(R, \bar{c}^x)| = 1$. Since $t(R) = \{x, c^x, \bar{c}^x\}$ and $|N(R, c^x)| = 1$, by Lemma 4, $f(R) = \bar{c}^x$. Also, since $t(R) = \{x, c^x, \bar{c}^x\}$ and $|N(R, \bar{c}^x)| = 1$, by Lemma 4, $f(R) = c^x$. This implies $c^x = \bar{c}^x$, contradicting our hypothesis. Therefore, $c^x = \bar{c}^x$.

Item (iii): Assume $x \succ_0 y$. As $c^x \succeq_0 x$, by transitivity $c^x \succ_0 y$. As $c^x \in A \setminus A_0$, $c^x = c^y$.

This completes the proof of Lemma 1. $\square$

A.2 Properties of $\succeq_1^x, \ldots, \succeq_{n-2}^x$ and $\overline{\succeq}^x$

For $x \in A$ and $t \in \{1, \ldots, n-2\}$, define recursively

$$A_t^{f,x} = \left\{ y \in (A \setminus A_0^f) \setminus \left(\cup_{k=1}^{t-1} A_k^{f,x} \right) \mid \text{there is } z \in A \text{ such that } z \succ_t^{f,x} y \right\}$$

as the set of dominated alternatives in $(A \setminus A_0^f) \setminus \left(\cup_{k=1}^{t-1} A_k^{f,x} \right)$ according to $\succeq_t^{f,x}$ where, by convention, $\cup_{k=1}^{t-1} A_k^{f,x} = \emptyset$ for $t = 1$, and if $n = 2$, $A_t^{f,x} = \emptyset$ for all x and t. When, f is obvious from the context, we will also often write $\succeq_t^x$ and A_t^x instead of $\succeq_t^{f,x}$ and $A_t^{f,x}$.

Lemma 5 *Let $f : \mathcal{D}^N \to A$ be a simple and strategy-proof rule. For $x \in A$, let $\overline{\succeq}^x$ be the x-pie over A relative to x, and assume the domain $\mathcal{D}$ is rich on $(A, \overline{\succeq}^x)$. Then,*

(i) $\succeq_t^x$ is reflexive and antisymmetric for all $t \in \{1, \ldots, n-2\}$,

(ii) $c^x \succeq_{n-2}^x y$ for all $y \in A \setminus A_0$,

(iii) $A \setminus A_0 = \{c^x\} \cup A_1^x \cup \cdots \cup A_{n-2}^x$ with $\{c^x\} \cap A_t^x = \emptyset$ and $A_t^x \cap A_{t'}^x = \emptyset$ for all $t, t' \in \{1, \ldots, n-2\}$ and $t' \neq t$,

(iv) if $y, z \in A \setminus A_0$ are such that $y \succ_t^x z$ for some $t \in \{1, \ldots, n-2\}$, then $y \in (\cup_{k=t}^{n-2} A_k^x) \cup \{c^x\}$ and $z \in (\cup_{k=1}^{t} A_k^x) \cup \{c^x\}$, and

(v) $\overline{\succeq}^x$ is a partial order.

Proof. Let $f : \mathcal{D}^N \to A$ be a simple and strategy-proof rule and let $x \in A$ be fixed, The following claim will be useful in this proof.

Claim 6: For each $y \neq c^x$, $f(y, c^x, \ldots, c^x) = c^x$. Assume otherwise; that is,

$$f(y, c^x, \ldots, c^x) = z \neq c^x.$$

By strategy-proofness, $f(z, c^x, \ldots, c^x) = z$, which means that $z \succ_0 c^x$, contradicting the fact that $c^x \in A \setminus A_0$.

Item (i): Let $t \in \{1, \ldots, n-2\}$ be given. We have to show that $\succeq_t^x$ is reflexive and antisymmetric.

Reflexivity: Let $y \in A_t^x$. Assume that $y \not\succeq_t^x y$. Then, $f(\underbrace{x, \ldots, x}_{t \text{ times}}, \underbrace{y, \ldots, y}_{n-t \text{ times}}) \equiv z \neq y$. Since $y \in A_t^x$, we have that $y \notin A_{t-1}^x$. Therefore, $f(\underbrace{x, \ldots, x}_{t-1 \text{ times}}, z, \underbrace{y, \ldots, y}_{n-t \text{ times}}) \equiv w \neq z$. Hence,

$$f(\underbrace{x, \ldots, x}_{t \text{ times}}, \underbrace{y, \ldots, y}_{n-t \text{ times}}) = z \; P_t^z \; w = f(\underbrace{x, \ldots, x}_{t-1 \text{ times}}, z, \underbrace{y, \ldots, y}_{n-t \text{ times}}),$$

and agent t manipulates f, contradicting strategy-proofness.

Antisymmetry: Let $y, z \in A_t^x$. Assume $y \succeq_t^x z$ and $z \succeq_t^x y$. Hence,

$$f(\underbrace{x, \ldots, x}_{t \text{ times}}, y, \underbrace{z, \ldots \ldots, z}_{n-t-1 \text{ times}}) = y \tag{11}$$

and

$$f(\underbrace{x, \ldots, x}_{t \text{ times}}, \underbrace{y, \ldots \ldots, y}_{n-t-1 \text{ times}}, z) = z. \tag{12}$$

By strategy-proofness, (12) implies

$$f(\underbrace{x, \ldots, x}_{t \text{ times}}, y, \underbrace{z, \ldots \ldots, z}_{n-t-1 \text{ times}}) = z. \tag{13}$$

In view of (11) and (13), it follows that $y = z$, and $\succeq_t^x$ is antisymmetric.

Item (ii): Fix $y \in A \setminus A_0$. By Claim 6,

$$f(\underbrace{c^x, \ldots, c^x}_{n-1 \text{ times}}, y) = c^x. \tag{14}$$

We claim that

$$f(x, \underbrace{c^x, \ldots, c^x}_{n-2 \text{ times}}, y) = c^x. \tag{15}$$

Assume otherwise, then

$$f(x, \underbrace{c^x, \ldots, c^x}_{n-2 \text{ times}}, y) = w \neq c^x. \tag{16}$$

By strategy-proofness,

$$f(w, \underbrace{c^x, \ldots, c^x}_{n-2 \text{ times}}, w) = w. \tag{17}$$

If $w \in [x, c^x]_{\succeq_0}$, $f(c^x, w, \ldots, w) = c^x$. By anonymity and a repeated use of strategy-proofness,

$$f(w, \underbrace{c^x, \ldots, c^x}_{n-2 \text{ times}}, w) = c^x,$$

contradicting (17). Therefore, $w \notin [x, c^x]_{\succeq 0}$ and by richness there is $R_1^x \in \mathcal{D}$ such that $c^x P_1^x w$. Hence, by (14) and (16),

$$f(\underbrace{c^x, \dots, c^x}_{n-1 \text{ times}}, y) = c^x P_1^x w = f(x, \underbrace{c^x, \dots, c^x}_{n-2 \text{ times}}, y),$$

contradicting strategy-proofness. Therefore, (15) holds. By anonymity, and repeating the previous argument $n-3$ times, if necessary, we obtain $f(\underbrace{x, \dots, x}_{n-2 \text{ times}}, c^x, y) = c^x$, which implies $c^x \succeq_{n-2}^x y$.

Item (iii): The proof consists of three steps.

Step 1: $\boldsymbol{c^x \notin \cup_{t=1}^{n-2} A_t^x}$. Assume otherwise; then, there is $\ell \in \{1, \dots, n-2\}$ such that $c^x \in A_\ell^x$. By definition of A_ℓ^x, $c^x \notin \cup_{t=1}^{\ell-1} A_t^x$ and there is $z \in A$ such that $z \succ_\ell^x c^x$. Hence,

$$f(\underbrace{x, \dots, x}_{\ell \text{ times}}, z, \underbrace{c^x, \dots\dots, c^x}_{n-\ell-1 \text{ times}}) = z. \tag{18}$$

By Claim 6,

$$f(\underbrace{c^x, \dots, c^x}_{n-1 \text{ times}}, z) = c^x. \tag{19}$$

We claim that

$$f(x, \underbrace{c^x, \dots, c^x}_{n-2 \text{ times}}, z) = c^x. \tag{20}$$

Assume otherwise, then

$$f(x, \underbrace{c^x, \dots, c^x}_{n-2 \text{ times}}, z) = w \neq c^x. \tag{21}$$

By strategy-proofness,

$$f(w, \underbrace{c^x, \dots, c^x}_{n-2 \text{ times}}, w) = w. \tag{22}$$

If $w \in [x, c^x]_{\succeq 0}$, $f(c^x, w, \dots, w) = c^x$. By anonymity and a repeated use of strategy-proofness,

$$f(w, \underbrace{c^x, \dots, c^x}_{n-2 \text{ times}}, w) = c^x,$$

contradicting (22). Therefore, $w \notin [x, c^x]_{\succeq 0}$ and by richness there is $R_1^x \in \mathcal{D}$ such that $c^x P_1^x w$. Hence, by (19) and (21),

$$f(\underbrace{c^x, \dots, c^x}_{n-1 \text{ times}}, z) = c^x P_1^x w = f(x, \underbrace{c^x, \dots, c^x}_{n-2 \text{ times}}, z),$$

contradicting strategy-proofness. Therefore, (20) holds. By anonymity, and repeating the previous argument ℓ times we obtain

$$f(\underbrace{x, \dots, x}_{\ell \text{ times}}, z, \underbrace{c^x, \dots\dots, c^x}_{n-\ell-1 \text{ times}}) = c^x,$$

which contradicts (18), and therefore $c^x \notin \cup_{t=1}^{n-2} A_t^x$.

Step 2: If $y \in (A \setminus A_0) \setminus \{c^x\}$, then $y \in \cup_{t=1}^{n-2} A_t^x$. If $y \in \cup_{t=1}^{n-3} A_t^x$ we are done, so assume $y \notin \cup_{t=1}^{n-3} A_t^x$. By Lemma 5 (ii), we have $c^x \succ_{n-2}^x y$. Therefore, $y \in A_{n-2}^x$. Thus, $y \in \cup_{t=1}^{n-2} A_t^x$.

Step 3: Concluding. By definition, we have that $A_k^x \cap A_{k'}^x = \emptyset$ if $k \neq k'$, so $\cup_{t=1}^{n-2} A_t^x$ is a pairwise disjoint union. Steps 1 and 2 show that c^x is the only element of $A \setminus A_0$ that does not belong to that union.

Item (iv): Let $y, z \in A \setminus A_0$ and $t \in \{1, \ldots, n-2\}$ be such that $y \succ_t^x z$.

First, we show that $y \in (\cup_{k=t}^{n-2} A_k^x) \cup \{c^x\}$. To obtain a contradiction, and given Lemma 5 (iii), assume $y \in A_\ell^x$ for some $\ell \in \{1, \ldots, t-1\}$. Then, there is $w \in A$ such that $w \succ_\ell^x y$, *i.e.*, $w \notin A_0$ and

$$f(\underbrace{x, \ldots, x}_{\ell \text{ times}}, w, \underbrace{y, \ldots \ldots, y}_{n-\ell-1 \text{ times}}) = w. \tag{23}$$

Since $y \succ_t^x z$, $f(\underbrace{x, \ldots, x}_{t \text{ times}}, y, \underbrace{z, \ldots \ldots, z}_{n-t-1 \text{ times}}) = y$. Thus, by a repeated use of strategy-proofness, and as $t \geq \ell + 1$, we have

$$f(\underbrace{x, \ldots, x}_{\ell+1 \text{ times}}, \underbrace{y, \ldots \ldots, y}_{n-\ell-1 \text{ times}}) = y. \tag{24}$$

By Definition 2 (i), $c^x \overline{\succeq}^x x$. By Lemma 5 (ii), $c^x \succeq_{n-2}^x w$, which implies $w \overline{\succeq}^x c^x$. Then, by transitivity of $\overline{\succeq}^x$, $w \overline{\succeq}^x x$. Furthermore, as $w \succ_t^x y$, by Definition 2 (ii), $y \overline{\succ}^x w$. Thus, $y \notin [x, w]_{\overline{\succeq}^x}$ and by richness there is $R_{\ell+1}^x \in \mathcal{D}$ such that $w P_{\ell+1}^x y$. Hence, by (23) and (24),

$$f(\underbrace{x, \ldots, x}_{\ell \text{ times}}, w, \underbrace{y, \ldots \ldots, y}_{n-\ell-1 \text{ times}}) = w P_{\ell+1}^x y = f(\underbrace{x, \ldots, x}_{\ell+1 \text{ times}}, \underbrace{y, \ldots \ldots, y}_{n-\ell-1 \text{ times}}),$$

contradicting strategy-proofness. Thus, $y \in (\cup_{k=t}^{n-2} A_k^x) \cup \{c^x\}$.

Second, we show that $z \in (\cup_{k=1}^{t} A_k^x) \cup \{c^x\}$. To obtain a contradiction, and given Lemma 5 (iii), assume $z \in A_\ell^x$ for some $\ell \in \{t+1, \ldots, n-2\}$. As $\succeq_\ell^x$ is reflexive on A_ℓ^x, $z \succeq_\ell^x z$, *i.e.*,

$$f(\underbrace{x, \ldots, x}_{\ell \text{ times}}, \underbrace{z, \ldots, z}_{n-\ell \text{ times}}) = z.$$

By a repeated use of strategy-proofness, and since $\ell > t$, we have

$$f(\underbrace{x, \ldots, x}_{t+1 \text{ times}}, \underbrace{z, \ldots \ldots, z}_{n-t-1 \text{ times}}) = z. \tag{25}$$

Since $y \succ_t^x z$, it follows that

$$f(\underbrace{x, \ldots, x}_{t \text{ times}}, y, \underbrace{z, \ldots \ldots, z}_{n-t-1 \text{ times}}) = y \tag{26}$$

and, by Definition 2 (ii), $z\overline{\succ}^x y$. By Lemma 5 (ii), $c^x \succeq^x_{n-2} y$ and therefore, by Definition 2 (ii), $y\overline{\succeq}^x c^x$. By Definition 2 (i), $c^x\overline{\succeq}^x x$. Then, by transitivity of $\overline{\succeq}^x$, $y\overline{\succeq}^x x$. As $z \notin [x, y]_{\overline{\succeq}^x}$ by richness there is $R^x_{t+1} \in \mathcal{D}$ such that $yP^x_{t+1}z$. Hence, by (25) and (26),

$$f(\underbrace{x, \ldots, x}_{t \text{ times}}, y, \underbrace{z, \ldots \ldots, z}_{n-t-1 \text{ times}}) = yP^x z = f(\underbrace{x, \ldots, x}_{t+1 \text{ times}}, \underbrace{z, \ldots \ldots, z}_{n-t-1 \text{ times}}),$$

contradicting strategy-proofness. Thus, $z \in (\cup^t_{k=1} A^x_k) \cup \{c^x\}$.

Item (v): We have to show that $\overline{\succeq}^x$ is reflexive, antisymmetric and transitive.

Reflexivity: Let $y \in A$. If $y \in A_0$, by Lemma 1 (i), $y \succeq_0 y$ holds. If $y = c^x$, then $y \in \overline{A}_0$ and, by Lemma 1 (i), $y \succeq_0 y$ holds. Then, by Definition 2 (i), $y\overline{\succeq}^x y$ holds for $y \in A_0$ or $y = c^x$. If $y \in A \setminus A_0$ and $y \neq c^x$, then by Lemma 5 (iii), $y \in A^x_t$ for some $t = 1, \ldots, n-2$. Thus, by Lemma 5 (i) $y \succeq^x_t y$ holds. Then, by Definition 2 (ii), $y\overline{\succeq}^x y$ holds as well. Therefore, $\overline{\succeq}^x$ is reflexive.

Antisymmetry: Let $y, z \in A$ be such that $y\overline{\succeq}^x z$ and $z\overline{\succeq}^x y$. There are two cases to consider:

1. ***y* or *z* belong to $\boldsymbol{A_0}$.** Then, by Remark 1, $y, z \in A_0$, $y \succeq_0 z$ and $z \succeq_0 y$. Then, since $\succeq_0$ is antisymmetric by Lemma 1 (i), $y = z$.

2. **$\boldsymbol{y, z \in A \setminus A_0}$.** By $y\overline{\succeq}^x z$, there is t such that $z \succeq^x_t y$. By $z\overline{\succeq}^x y$, there is t' such that $y \succeq^x_{t'} z$. By Lemma 5 (iii), $t = t'$, and $y, z \in A^x_t$. Thus, since $\succeq^x_t$ is antisymmetric by Lemma 5 (i), $y = z$.

Therefore, $\overline{\succeq}^x$ is antisymmetric.

Transitivity: By Definition 2 (iii), $\overline{\succeq}^x$ is transitive.

This completes the proof of Lemma 5. □

Observe that by Lemma 5 (iii) the set $A \setminus A_0$ can be partitioned as the disjoint union $\{c^x\} \cup A^x_1 \cup \cdots \cup A^x_{n-2}$. Moreover, a simple and strategy-proof rule f and an alternative x induce uniquely the binary relations $\succeq_0$ and $\succeq^x_1, \ldots, \succeq^x_{n-2}$, and the partition of $A \setminus A_0$ identified in Lemma 5 (iii).

A.3 On the $\sup_{\overline{\succeq}^x}\{x, w\}$

Lemma 2 *Let* $f : \mathcal{D}^N \to A$ *be a simple and strategy-proof rule. For* $x \in A$, *let* $\overline{\succeq}^x$ *be the partially inverted extension over* A *relative to* x, *assume* $\mathcal{D}$ *is rich on* $(A, \overline{\succeq}^x)$ *and let* $w \in A$ *be such that* $w\overline{\not\succeq}^x x$. *Then,* $\sup_{\overline{\succeq}^x}\{x, w\}$ *exists. Moreover,* $\sup_{\overline{\succeq}^x}\{x, w\} = f(x, w, \ldots, w)$.

Proof. Let $x, w \in A$ be such that $w\overline{\not\succeq}^x x$. Before proving the result, we present two claims.

Claim 1: $\boldsymbol{w \in A_0}$. Assume not. By Lemma 5 (ii), $c^x \succeq^x_{n-2} w$. Since $w, c^x \in A \setminus A_0$, by Definition 2 (ii), $w\overline{\succeq}^x c^x$. By the definition of the cover, $c^x \succeq_0 x$ and, by Definition 2 (i), $c^x\overline{\succeq}^x x$. The transitivity of $\overline{\succeq}^x$ implies that $w\overline{\succeq}^x x$, contradicting our hypothesis.

Claim 2: $\boldsymbol{f(x, w, \ldots, w) \in [w, c^w]_{\succeq_0}}$. Let $z \equiv f(x, w, \ldots, w)$. By strategy-proofness, $z = f(z, w, \ldots, w)$. Therefore, $z \succeq_0 w$. Observe that, by Definition 2 (i), $\mathcal{D}$ rich on $(A, \overline{\succeq}^x)$ implies that $\mathcal{D}$ rich on $(A, \succeq_0)$. By Lemma 1 (iii), $c^z = c^w$. Since $c^z \succeq_0 z$, we obtain $c^w = c^z \succeq_0 z \succeq_0 w$.

To prove that $\sup_{\overline{\succeq}^x}\{x, w\}$ exists and that $\sup_{\overline{\succeq}^x}\{x, w\} = f(x, w, \ldots, w)$ holds, we distinguish between two cases:

1. $\boldsymbol{[x, c^x]_{\succeq_0} \cap [w, c^w]_{\succeq_0} \neq \emptyset}$. First, we show that

$$f(x, w, \ldots, w) \in [x, c^x]_{\succeq_0} \cap [w, c^w]_{\succeq_0}. \tag{27}$$

Let $z \equiv f(x, , \ldots, w)$ and assume $z \notin [x, c^x]_{\succeq_0} \cap [w, c^w]_{\succeq_0}$. Let $r \in [x, c^x]_{\succeq_0} \cap [w, c^w]_{\succeq_0}$. Since $r \succeq_0 w$, $r = f(r, w, \ldots, w)$. Since $z \notin [x, c^x]_{\succeq_0} \cap [w, c^w]_{\succeq_0}$, by Claim 2, $z \notin [x, c^x]_{\succeq_0}$. As $r \in [x, c^x]_{\succeq_0}$, it follows that $z \notin [x, r]_{\succeq_0}$. Then, by richness, there is $R_1^x \in \mathcal{D}$ such that $r P_1^x z$. Hence,

$$f(r, w, \ldots, w) = r P_1^x z = f(x, w, \ldots, w),$$

contradicting strategy-proofness. Thus, (27) holds and z is a $\succeq_0$−upper bound of $\{x, w\}$. Next, we show that $z = \sup_{\succeq_0}\{x, w\}$. If it is not the case, then there is another $\succeq_0$−upper bound of $\{x, w\}$, say p, such that $p \not\succeq_0 z$. Then, $p \succeq_0 x$ and $z \notin [x, p]_{\succeq_0}$. By richness, there is $R_1^x \in \mathcal{D}$ such that $p P_1^x z$. Since $p \succeq_0 w$ implies $p = f(p, w, \ldots, w)$, we have

$$f(p, w, \ldots, w) = p P_1^x z = f(x, w, \ldots, w),$$

contradicting strategy-proofness. Thus, $z = f(x, w, \ldots, w) = \sup_{\succeq_0}\{x, w\}$. Now, as by Definition 2 (i) we have $[x, z]_{\succeq_0} = [x, z]_{\overline{\succeq}^x}$ and $[w, z]_{\succeq_0} = [w, z]_{\overline{\succeq}^x}$. $\sup_{\succeq_0}\{x, w\} = \sup_{\overline{\succeq}^x}\{x, w\}$. Then, $f(x, w, \ldots, w) = \sup_{\overline{\succeq}^x}\{x, w\}$.

2. $\boldsymbol{[x, c^x]_{\succeq_0} \cap [w, c^w]_{\succeq_0} = \emptyset}$. We first show that $c^w \overline{\succeq}^x x$. By Claim 1, $w \in A_0$. Therefore, $c^w \in A \setminus A_0$. By Lemma 5 (ii), we have $c^x \succeq_{n-2}^x c^w$. Thus, by Definition 2 (ii), $c^w \overline{\succeq}^x c^x$ and, by Definition 2 (i) and (iii),

$$c^w \overline{\succeq}^x x. \tag{28}$$

We now show that

$$f(x, w, \ldots, w) = c^w. \tag{29}$$

Let $z \equiv f(x, w, \ldots, w)$ and assume $z \neq c^w$. Since $[x, c^x]_{\succeq_0} \cap [w, c^w]_{\succeq_0} = \emptyset$, by Claim 2,

$$z \notin [x, c^x]_{\succeq_0}. \tag{30}$$

By Claim 2, $c^w \succ_0 z$, and so $z \in A_0$. Notice that $z \overline{\not\succeq}^x c^x$; otherwise, by Definition 2 (i), $z \succeq_0 c^x$ and so $c^x \in A_0$, a contradiction. Then, $z \overline{\not\succeq}^x c^x$ together with (30) imply that

$z \notin [x, c^w]_{\overline{\succsim}^x}$. By richness, there is $R_1^x \in \mathcal{D}$ such that $c^w P_1^x z$. By definition of cover, $c^w = f(c^w, w, \ldots, w)$. Hence,

$$f(c^w, w, \ldots, w) = c^w P_1^x z = f(x, w, \ldots, w),$$

contradicting strategy-proofness. Thus, (29) holds, By (28), c^w is a $\overline{\succsim}^x$–upper bound of $\{x, w\}$. Next, we show that $c^w = \sup_{\overline{\succsim}^x}\{x, w\}$. Let p be another $\overline{\succsim}^x$–upper bound of $\{x, w\}$. Assume first that $p \in A \setminus A_0$. By Definition 2 (iv) we have that $p \overline{\succsim}^x w$ implies $p \overline{\succsim}^x c^w$. Hence, c^w is the smallest upper bound of $\{x, w\}$. Assume then that $p \in A_0$ and $p \overline{\not\succsim}^x c^w$. Since $c^w \notin [x, p]_{\overline{\succsim}^x}$, by richness, there is $R_1^x \in \mathcal{D}$ such that $p P_1^x c^w$. As $p \overline{\succsim}^x w$ and $p, w \in A_0$, it follows, by Definition 2 (i), that $p \succsim_0 w$ and $p = f(p, w, \ldots, w)$. Hence,

$$f(p, w, \ldots, w) = p P_1^x c^w = f(x, w, \ldots, w),$$

contradicting strategy-proofness. Thus, $c^w = \sup_{\overline{\succsim}^x}\{x, w\}$.

This completes the proof that $\sup_{\overline{\succsim}^x}\{x, w\}$ exists and $\sup_{\overline{\succsim}^x}\{x, w\} = f(x, w, \ldots, w)$. □

A.4 Proof of Theorem 1

Theorem 1 *Let $f : \mathcal{D}^N \to A$ be a simple and strategy-proof rule and assume that $\mathcal{D}$ is rich relative to f. Then, $\mathcal{D} \subseteq \mathcal{LWSP}(f)$.*

Proof. Let $f : \mathcal{D}^N \to A$ be a simple and strategy-proof rule and assume $\mathcal{D}$ is rich relative to f. Let $x \in A$ and $R^x \in \mathcal{D}$ be arbitrary. To show that R^x is local and weakly single-peaked over $(A, \overline{\succsim}^x)$, we need to see that both (LWSP.1) and (LWSP.2) hold.

(LWSP.1) Let $y, z \in A$ be such that $z \overline{\succsim}^x y \overline{\succsim}^x x$. If $y = x$, $y = z$ or $x = z$, $y R^x z$ holds trivially. Accordingly, assume $z \overline{\succ}^x y \overline{\succ}^x x$. There are two cases to consider:

1. $\boldsymbol{z \in A_0}$. By Remark 1, $x, y \in A_0$, and by Definition 2 (i), $z \succ_0 y$ and $y \succ_0 x$. The latter means that $f(y, x, \ldots, x) = y$. By a repeated use of strategy-proofness and anonymity, $f(x, y, \ldots, y) = y$. As $z \succ_0 y$, $f(z, y, \ldots, y) = z$. Then, by strategy-proofness,

$$y = f(x, y, \ldots, y) R^x f(z, y, \ldots, y) = z,$$

as desired.

2. $\boldsymbol{z \in A \setminus A_0}$. There are two subcases to consider:

   2.1. $\boldsymbol{y \in A_0}$. By Remark 1, $x \in A_0$, and by Definition 2 (i), $y \succ_0 x$. Then, $f(y, x, \ldots, x) = y$ and by a repeated use of strategy-proofness and anonymity, $f(x, y, \ldots, y) = y$. As $y \in A_0$, $f(c^y, y, \ldots, y) = c^y$. By Lemma 1 (iii), $c^y = c^x$. Therefore, by strategy-proofness,

$$y = f(x, y, \ldots, y) R^x f(c^x, y, \ldots, y) = c^x.$$

Then,

$$yR^x c^x. \tag{31}$$

As $z \in A \setminus A_0$, by Lemma 5 (iii), either $z = c^x$ or there is $t \in \{1, \ldots, n-2\}$ such that $z \in A_t^x$. If the former holds, (31) implies that $yR^x z$. Assume now that $z \neq c^x$ and the latter holds. We claim that

$$f(\underbrace{x, \ldots, x}_{t-1 \text{ times}}, \underbrace{z, \ldots, z}_{n-t \text{ times}}, c^x) = z. \tag{32}$$

Otherwise, by anonymity, $f(\underbrace{x, \ldots, x}_{t-1 \text{ times}}, c^x, \underbrace{z, \ldots, z}_{n-t \text{ times}}) = w \neq z$ and strategy-proofness imply

$$f(\underbrace{x, \ldots, x}_{t-1 \text{ times}}, w, \underbrace{z, \ldots, z}_{n-t \text{ times}}) = w,$$

and therefore $w \succ_{t-1}^x z$. By Lemma 5 (iv), $z \in \cup_{k=1}^{t-1} A_k^x$, contradicting that $z \in A_t^x$. Hence, (32) holds. As $f(\underbrace{x, \ldots, x}_{n-1 \text{ times}}, c^x) = c^x$, by strategy-proofness and anonymity, it follows that

$$c^x = f(\underbrace{x, \ldots, x}_{n-1 \text{ times}}, c^x) R^x f(z, \underbrace{x, \ldots, x}_{n-2 \text{ times}}, c^x) = f(\underbrace{x, \ldots, x}_{n-2 \text{ times}}, z, c^x).$$

Applying strategy-proofness again,

$$f(\underbrace{x, \ldots, x}_{n-2 \text{ times}}, z, c^x) R^x f(z, \underbrace{x, \ldots, x}_{n-3 \text{ times}}, z, c^x).$$

By iterating this substitution of one x by one z, a repeated use of anonymity and strategy-proofness, and (32), we obtain

$$f(\underbrace{x, \ldots, x}_{t \text{ times}}, \underbrace{z, \ldots\ldots, z}_{n-t-1 \text{ times}}, c^x) R^x f(z, \underbrace{x, \ldots, x}_{t-1 \text{ times}}, \underbrace{z, \ldots\ldots, z}_{n-t-1 \text{ times}}, c^x) = z.$$

Then, by transitivity of R^x,

$$c^x R^x z. \tag{33}$$

Finally, by (31), (33) and the transitivity of R^x, $yR^x z$.

2.2. $\boldsymbol{y \in A \setminus A_0}$. Since $z \overline{\succ}^x y$, by Definition 2 (ii), there is $t \in \{1, \ldots, n-2\}$ such that $y \succ_t^x z$. Then,

$$f(\underbrace{x, \ldots, x}_{t \text{ times}}, y, \underbrace{z, \ldots\ldots, z}_{n-t-1 \text{ times}}) = y. \tag{34}$$

We claim that

$$f(y, z, \ldots, z) = z. \tag{35}$$

Otherwise, $f(y,z,\ldots,z) = w \neq z$ would imply, together with strategy-proofness, that $f(w,z,\ldots,z) = w$, and so $w \succ_0 z$, contradicting the fact that $z \in A \setminus A_0$. Hence, (35) holds. Now, by strategy-proofness and anonymity, it follows from (34) that

$$y = f(\underbrace{x,\ldots,x}_{t \text{ times}}, y, \underbrace{z,\ldots\ldots,z}_{n-t-1 \text{ times}}) R^x f(z, \underbrace{x,\ldots,x}_{t-1 \text{ times}}, y, \underbrace{z,\ldots\ldots,z}_{n-t-1 \text{ times}}) = f(\underbrace{x,\ldots,x}_{t-1 \text{ times}}, y, \underbrace{z,\ldots,z}_{n-t \text{ times}}).$$

If $t = 1$, this and (35) imply $yR^x z$. Assume $t > 1$. Applying strategy-proofness again,

$$f(\underbrace{x,\ldots,x}_{t-1 \text{ times}}, y, \underbrace{z,\ldots,z}_{n-t \text{ times}}) R^x f(z, \underbrace{x,\ldots,x}_{t-2 \text{ times}}, y, \underbrace{z,\ldots,z}_{n-t \text{ times}}).$$

By iterating this substitution of one x by one z, a repeated use of anonymity and strategy-proofness, and (35),

$$f(x, y, \underbrace{z,\ldots,z}_{n-2 \text{ times}}) R^x f(z, y, \underbrace{z,\ldots,z}_{n-2 \text{ times}}) = z.$$

Then, by transitivity of R^x, $yR^x z$.

(LWSP.2) Assume $w \in A$ is such that $w \overline{\not\succsim}^x x$. Then, by Lemma 2, $\sup_{\succsim^x}\{x,w\} = f(x,w,\ldots,w)$. By strategy-proofness and unanimity,

$$\sup_{\succsim^x}\{x,w\} = f(x,w,\ldots,w) R^x f(w,\ldots,w) = w.$$

Hence, $\sup_{\succsim^x}\{x,w\} R^x w$, as desired. □

A.5 Proof of Proposition 1

First, we state and prove a lemma that is essential to show that our maximality result holds.

Lemma 6 *Let $f : \mathcal{D}^N \to A$ be a simple and strategy-proof rule and assume $\mathcal{D}$ is a basic and rich domain relative to f. Then, the voting scheme $\widehat{f} : A^n \to A$ induced by f is strategy-proof on $\mathcal{LWSP}(f)$.*

Proof. Suppose, by contradiction, that $\widehat{f}$ is not strategy-proof on $\mathcal{LWSP}(f)$. Then, there exist $i \in N$, $(x, t_{-i}) \in A^n$, $y \in A$ and $R_i^x \in \mathcal{LWSP}(f)$ such that

$$\widehat{f}(y, t_{-i}) P_i^x \widehat{f}(x, t_{-i}). \tag{36}$$

Fix any $R_i^y \in \mathcal{D}$ and any $R_{-i} \in \mathcal{D}^{N\setminus\{i\}}$ such that $t(R_j) = t_j$ for all $j \neq i$, and let $z \equiv \widehat{f}(x, t_{-i})$ and $w \equiv \widehat{f}(y, t_{-i})$. Thus, by (36), $z \neq w$, $z \neq x$ and $y \neq x$. Notice that we also have $w \neq x$. Otherwise, $w = x$ would imply, by (36) and tops-onlyness, that $x = f(R_i^y, R_{-i}) \overline{P}_i^x f(\overline{R}_i^x, R_{-i}) = z$ holds for any $\overline{R}_i^x \in \mathcal{D}$, contradicting that f is strategy-proof on $\mathcal{D}$. There are two cases to consider:

1. $\boldsymbol{w}\overline{\succ}^{\boldsymbol{x}}\boldsymbol{x}$. By (LWPS.1), $z \notin [x,w]_{\succeq^x}$. By richness, there is $\overline{R}_i^x \in \mathcal{D}$ such that $w\overline{P}_i^x z$. By tops-onlyness,

$$w = f(R_i^y, R_{-i})\overline{P}_i^x f(\overline{R}_i^x, R_{-i}) = z,$$

contradicting that f is strategy-proof on $\mathcal{D}$.

2. $\boldsymbol{w}\overline{\not\succ}^{\boldsymbol{x}}\boldsymbol{x}$. First, we claim that $z\overline{\succ}^x x$. Otherwise, $z\overline{\not\succ}^x x$ and Lemma 2 imply that $\sup_{\succeq^x}\{x,z\} = \widehat{f}(x,z,\ldots,z)$. As $z = \widehat{f}(x,t_{-i})$, repeated use of strategy-proofness of $\widehat{f}$ on $\mathcal{D}$ implies $z = \widehat{f}(x,z,\ldots,z)$. Hence, we obtain $\sup_{\succeq^x}\{x,z\} = z$, contradicting $z\overline{\not\succ}^x x$. So $z\overline{\succ}^x x$, proving our claim. Let $r \equiv \sup_{\succeq^x}\{x,w\}$, that exists by Lemma 2. Notice that, by (LWSP.2) and (36), $z \neq r$. Furthermore, by Lemma 2, $r = \widehat{f}(x,w,\ldots,w)$ and, using strategy-proofness of $\widehat{f}$ on $\mathcal{D}$,

$$\widehat{f}(r,w,\ldots,w) = r. \tag{37}$$

There are two subcases to consider:

2.1. $\boldsymbol{x}\overline{\succ}^{\boldsymbol{x}}\boldsymbol{w}$. Since $z\overline{\succ}^x x$, $z \notin [w,x]_{\succeq^x}$ and the argument follows a similar one to that already used in Case 1.

2.2. $\boldsymbol{x}\overline{\not\succ}^{\boldsymbol{x}}\boldsymbol{w}$. Let $R_i^r \in \mathcal{D}$ and define $\ell \equiv \widehat{f}(r,t_{-i})$. Then, $\ell \in [w,r]_{\succeq^x}$. Otherwise, by richness and tops-onlyness, we can assume that R_i^r is such that $wP_i^r\ell$. Therefore, $\widehat{f}(y,t_{-i}) = wP_i^r\ell = \widehat{f}(r,t_{-i})$, contradicting the strategy-proofness of $\widehat{f}$ on $\mathcal{D}$. Next, define

$$N_{-i}^w \equiv \{j \in N \setminus \{i\} \mid t_j = w\}.$$

We want to show that $\ell = r$. Notice that if $N_{-i}^w = N \setminus \{i\}$, then the result follows from tops-onlyness and (37). Assume then that $N_{-i}^w \subsetneq N \setminus \{i\}$ and $\ell \neq r$. Let $j \in (N \setminus \{i\}) \setminus N_{-i}^w$ and consider $R_j^w \in \mathcal{D}$. Then, we claim that $\widehat{f}(r,w,t_{-\{i,j\}}) \in [w,\ell]_{\succeq^x}$. To see this, notice that if $\widehat{f}(r,w,t_{-\{i,j\}}) \notin [w,\ell]_{\succeq^x}$, then by richness and tops-onlyness we can assume that R_j^w is such that $\ell P_j^w \widehat{f}(r,w,t_{-\{i,j\}})$. Since $\ell = \widehat{f}(r,t_{-i})$, agent j could manipulate $\widehat{f}$ at $(r,w,t_{-\{i,j\}})$ by voting for t_j instead, a contradiction with the strategy-proofness of $\widehat{f}$ on $\mathcal{D}$. Thus, $\widehat{f}(r,w,t_{-\{i,j\}}) \in [w,\ell]_{\succeq^x}$. Repeating this argument, if necessary, there is an agent $s \in (N \setminus \{i\}) \setminus N_{-i}^w$ and a profile $(R_i^r, R_s, R_{-\{i,s\}}^w) \in \mathcal{D}^N$ such that, if $\ell' \equiv f(R_i^r, R_s, R_{-\{i,s\}}^w)$ then $\ell' \in [w,\ell]_{\succeq^x}$. As $\ell' \in [w,\ell]_{\succeq^x}$ and $\ell \in [w,r]_{\succeq^x}$ imply $r \notin [w,\ell']_{\succeq^x}$, by richness and tops-onlyness there is $R_s^w \in \mathcal{D}$ such that $\ell' P_s^w r$. By tops-onlyness and (37), $f(R_i^r, R_{-i}^w) = r$. Therefore,

$$\ell' = f(R_i^r, R_s, R_{-\{i,s\}}^w) P_s^w f(R_i^r, R_{-i}^w) = r,$$

contradicting the strategy-proofness of f on $\mathcal{D}$. Hence, $\ell = r$. There are two subcases to consider:

2.2.1. $\boldsymbol{z} \notin [\boldsymbol{x},\boldsymbol{r}]_{\succeq^{\boldsymbol{x}}}$. By richness, there exists $\overline{R}_i^x \in \mathcal{D}$ such that $r\overline{P}_i^x z$. Therefore, $\widehat{f}(r,t_{-i}) = r\overline{P}_i^x z = \widehat{f}(x,t_{-i})$, contradicting the strategy-proofness of $\widehat{f}$ on $\mathcal{D}$.

2.2.2. $\boldsymbol{z \in [x, r]_{\succsim^x}}$. By (LWSP.1), $zR_i^x r$. By (LWSP.2), since $r = \sup_{\succsim^x}\{x, w\}$, $rR_i^x w$. Therefore, by transitivity, $zR_i^x w$. This contradicts (36).

We conclude that $\widehat{f}$ is strategy-proof on $\mathcal{LWSP}(f)$. □

Proposition 1 *Let $\mathcal{D}$ be a basic domain and let $\mathcal{F} \subseteq \{f : \mathcal{D}^N \to A \mid f$ is simple and strategy-proof on $\mathcal{D}\}$. If $\mathcal{D}$ is rich relative to f for each $f \in \mathcal{F}$, then $\bigcap_{f\in\mathcal{F}} \mathcal{LWSP}(f)$ is maximal for $\mathcal{F}$.*

Proof. Let $\mathcal{D}$ be a basic domain, let $\mathcal{F} \subseteq \{f : \mathcal{D}^N \to A \mid f$ is simple and strategy-proof on $\mathcal{D}\}$, and assume that $\mathcal{D}$ is rich relative to f for each $f \in \mathcal{F}$. In order to see that $\bigcap_{f\in\mathcal{F}} \mathcal{LWSP}(f)$ is maximal for $\mathcal{F}$, we need to prove:

(i) each $f \in \mathcal{F}$ is strategy-proof on $\bigcap_{g\in\mathcal{F}} \mathcal{LWSP}(g)$, and

(ii) for each preference $R \notin \bigcap_{f\in\mathcal{F}} \mathcal{LWSP}(f)$ with $|t(R)| = 1$, there is $f \in \mathcal{F}$ such that its induced voting scheme $\widehat{f}$ is not strategy-proof on $\bigcap_{g\in\mathcal{F}} \mathcal{LWSP}(g) \cup \{R\}$.

To see (i), let $f \in \mathcal{F}$. By Lemma 6, f is strategy-proof on $\mathcal{LWSP}(f)$. As $\bigcap_{g\in\mathcal{F}} \mathcal{LWSP}(g) \subseteq \mathcal{LWSP}(f)$, f is strategy-proof on $\bigcap_{g\in\mathcal{F}} \mathcal{LWSP}(g)$.

To see (ii), consider any preference $R \notin \bigcap_{f\in\mathcal{F}} \mathcal{LWSP}(f)$ with $|t(R)| = 1$. This means that there is $x \in A$ such that $t(R) = x$. From now on, we write R^x instead of R. Since $R^x \notin \bigcap_{f\in\mathcal{F}} \mathcal{LWSP}(f)$, there is $f \in \mathcal{F}$ such that $R^x \notin \mathcal{LWSP}(f) = \mathcal{LWSP}(\{\succsim^{f,r}\}_{r\in A})$; namely, $R^x \notin \mathcal{LWSP}(\succsim^{f,x})$. To obtain a contradiction, suppose that the voting scheme $\widehat{f} : A^n \to A$ induced by f is strategy-proof on $\mathcal{LWSP}(\{\succsim^{f,r}\}_{r\in A}) \cup \{R^x\}$. By the definition of local and weak single-peakedness, there are two cases to consider.

1. **(LWSP.1) does not hold**. Namely, there are $y, z \in A$ with $z\succsim^{f,x} y \succ^{f,x} x$ and $zP^x y$. There are two subcases to consider:

1.1. $\boldsymbol{z \in A_0^f}$. By Remark 1, $x, y \in A_0^f$, and by Definition 2 (i), $z \succ_0^f y$ and $y \succ_0^f x$. The latter means that $\widehat{f}(y, x, \ldots, x) = y$. By anonymity, tops-onlyness and a repeated use of strategy-proofness of $\widehat{f}$ on $\mathcal{LWSP}(\{\succsim^{f,r}\}_{r\in A}) \cup \{R^x\}$, $\widehat{f}(x, y, \ldots, y) = y$. As $z \succ_0^f y$, $\widehat{f}(z, y, \ldots, y) = z$. Then,

$$z = \widehat{f}(z, y, \ldots, y) P^x \widehat{f}(x, y, \ldots, y) = y,$$

which contradicts that $\widehat{f}$ is strategy-proof on $\mathcal{LWSP}(\{\succsim^{f,r}\}_{r\in A}) \cup \{R^x\}$.

1.2. $\boldsymbol{z \in A \setminus A_0^f}$. There are two subcases to consider:

1.2.1. $\boldsymbol{y \in A \setminus A_0^f}$. Since $z\succsim^{f,x} y$, by Definition 2 (ii), there is $t \in \{1, \ldots, n-2\}$ such that $y \succ_t^{f,x} z$. Then,

$$\widehat{f}(\underbrace{x, \ldots, x}_{t \text{ times}}, y, \underbrace{z, \ldots\ldots, z}_{n-t-1 \text{ times}}) = y. \tag{38}$$

By strategy-proofness, anonymity, and (38), it follows that

$$y = \widehat{f}(\underbrace{x, \ldots, x}_{t \text{ times}}, y, \underbrace{z, \ldots\ldots, z}_{n-t-1 \text{ times}}) R^x \widehat{f}(\underbrace{x, \ldots, x}_{t-1 \text{ times}}, y, \underbrace{z, \ldots, z}_{n-t \text{ times}}) R^x \cdots R^x \widehat{f}(x, y, \underbrace{z, \ldots, z}_{n-2 \text{ times}}).$$

Hence, $yR^x \widehat{f}(x, y, \underbrace{z, \ldots, z}_{n-2 \text{ times}})$. As $zP^x y$, by transitivity,

$$zP^x \widehat{f}(x, y, \underbrace{z, \ldots, z}_{n-2 \text{ times}}). \tag{39}$$

We now show that

$$\widehat{f}(y, z, \ldots, z) = z \tag{40}$$

holds. Otherwise, $\widehat{f}(y, z, \ldots, z) = w \neq z$ would imply, by strategy-proofness, that $\widehat{f}(w, z, \ldots, z) = w$. Hence, $w \succ_0^f z$, contradicting the fact that $z \in A \setminus A_0^f$. Therefore, (40) holds. By (39), (40), and anonymity,

$$z = \widehat{f}(z, y, \underbrace{z, \ldots, z}_{n-2 \text{ times}}) P^x \widehat{f}(x, y, \underbrace{z, \ldots, z}_{n-2 \text{ times}}),$$

which contradicts that $\widehat{f}$ is strategy-proof on $\mathcal{LWSP}(\{\succsim^{f,r}\}_{r \in A}) \cup \{R^x\}$.

1.2.2. $\boldsymbol{y \in A_0^f}$. Since $y \succ^{f,x} x$ and $y \in A_0^f$, $x \in A_0^f$ by Remark 1. By Definition 2 (i), $y \succ_0 x$. Hence, $f(y, x, \ldots, x) = y$ and by a repeated use of strategy-proofness and anonymity, $f(x, y, \ldots, y) = y$. As $y \in A_0^f$, $f(c^y, y, \ldots, y) = c^y$. Therefore, by strategy-proofness,

$$y = f(x, y, \ldots, y) R^x f(c^y, y, \ldots, y) = c^y.$$

Then, $yR^x c^y$. By hypothesis, $zP^x y$ and transitivity implies $zP^x c^y$. Hence, we have $z \succ^{f,x} c^y \succ^{f,x} x$ and $zP^x c^y$. The proof continues following the same argument used in Case 1.2.1 replacing y by c^y.

2. **(LWSP.2) does not hold**. Namely, there is $w \in A$ such that $w \not\succsim^{f,x} x$, $\sup_{\succsim^{f,x}}\{x, w\}$ exists and $wP^x \sup_{\succsim^{f,x}}\{x, w\}$. First, notice that $x \not\succsim^{f,x} w$ as well. Otherwise $wP^x x$, contradicting that $t(R^x) = x$. By Lemma 2, $\sup_{\succsim^{f,x}}\{x, w\} = \widehat{f}(x, w, \ldots, w)$. By unanimity, $\widehat{f}(w, \ldots, w) = w$. Therefore,

$$\widehat{f}(w, \ldots, w) = wP^x \sup_{\succsim^{f,x}}\{x, w\} = \widehat{f}(x, w, \ldots, w),$$

which contradicts that $\widehat{f}$ is strategy-proof on $\mathcal{LWSP}(\{\succsim^{f,r}\}_{r \in A}) \cup \{R^x\}$.

This completes the proof of Proposition 1. □

A.6 Relationship between $\mathcal{SSP}(\succeq)$ and $\mathcal{LWSP}(f)$

Let $(A, \succeq)$ be a semilattice and $f : \mathcal{SSP}(\succeq)^N \to A$ be a simple and strategy-proof rule. By Bonifacio and Massó (2021), f is either the $\sup_{\succeq}$ rule or a generalized quota-supremum rule.

To establish the relationship between $\mathcal{SSP}(\succeq)$ and $\mathcal{LWSP}(f)$ we first check that, for each $w \in A$, $\mathcal{SSP}(\succeq)$ is rich on $(A, \overline{\succeq}^{f,w})$.[17] There are two cases to consider:

1. **f is the supremum rule.** There are two subcases to consider:

   1.1. $\sup_{\succeq} A$ does not exist. Since $\succeq$ is a semilattice, for each $y \in A$ there exists $z \in A \setminus \{y\}$ such that $z \succ y$. Hence, since $\succeq$ coincides with $\succeq_0^f$ over A, $A_0^f = A$ and no alternative has a cover. Moreover, since $A \setminus A_0^f = \emptyset$, for every $w \in A$ and $t \in \{1, \ldots, n-2\}$, $\succeq_t^{f,w}$ does not relate any pair of alternatives and $A_t^{f,w} = \emptyset$. Thus, for each $w \in A$, $\overline{\succeq}^{f,w}$ is equal to $\succeq_0^f$ and, in turn, equal to $\succeq$. Then, by Bonifacio and Massó (2020), $\mathcal{SSP}(\succeq)$ is rich on $(A, \overline{\succeq}^{f,w})$.

   1.2. $\sup_{\succeq} A$ does exist. Define $\sup_{\succeq} A = \alpha$. Fix any $w \in A$. Then, $\alpha \succeq_0^f w$. Hence, $A_0^f = A \setminus \{\alpha\}$, which means that $A \setminus A_0^f = \{\alpha\}$ and, for all $t \in \{1, \ldots, n-2\}$, $A_t^{f,w} = \emptyset$. Thus, $\overline{\succeq}^{f,w}$ is equal to $\succeq_0^f$ and, in turn, equal to $\succeq$. Again, by Bonifacio and Massó (2020), $\mathcal{SSP}(\succeq)$ is rich on $(A, \overline{\succeq}^{f,w})$.

2. **f is a generalized quota-supremum rule.** Then, $\sup_{\succeq} A$ does exist and $A^\star(\succeq) \neq \emptyset$. Define $\sup_{\succeq} A = \alpha$. Let $q = \{q^{x'}\}_{x' \in A^\star(\succeq)}$ be the quota system associated to f. Fix $w \in A$ and let $x, y, z \in A$ be such that $y \overline{\succeq}^{f,w} x$ and $z \notin [x, y]_{\overline{\succeq}^{f,w}}$. We need to show that

$$\text{there is } R^x \in \mathcal{SSP}(\succeq) \text{ such that } y P^x z, \tag{41}$$

and

$$\text{there is } R^y \in \mathcal{SSP}(\succeq) \text{ such that } x P^y z. \tag{42}$$

If $x = y$, (41) and (42) follow trivially. Assume $x \neq y$. There are two subcases to consider:

   2.1. There is $x' \in A^\star(\succeq)$ such that $q^{x'} = 1$. This implies, by (QS.2) and $1 \leq q^{y'} \leq n$, that $q^{y'} = n$ for each $y' \in A^\star(\succeq) \setminus \{x'\}$. Then, for each $z' \in A \setminus \{x'\}$, $f(x', z', \ldots, z') = x'$ and therefore $x' \succ_0^f z'$ and $c^{z'} = x'$. Hence, $A_0^f = A \setminus \{x'\}$. Also, by the definition of a generalized quota-supremum rule, for $z', w' \in A \setminus \{x'\}$, $f(z', w', \ldots, w') = z'$ if and only if $z' \succeq w'$, which means that $z' \succeq_0^f w'$ if and only if $z' \succeq w'$. Hence, $\succeq_0^f$ coincides with $\succeq$ except that $x' \succ_0^f \alpha$ and $\overline{A}_0^f = A$. Therefore, by Definition 2 (i), for each $r \in A$, $\overline{\succeq}^{f,r}$ coincides with $\succeq_0^f$. We consider two subcases:

[17] By Corollary 2 in Bonifacio and Massó (2020), $\mathcal{SSP}(\succeq)$ is rich on $(A, \succeq)$.

2.1.1. $\boldsymbol{y = x'}$. Then, by our hypothesis, $z \notin [x,x']_{\succsim^{f,w}}$. Moreover, by definition of a generalized quota-supremum rule, $f(\alpha, x, \ldots, x) = \alpha$, and so $\alpha \succsim_0^f x$. Hence, $\alpha \in [x,x']_{\succsim^{f,w}}$ implies that $z \notin [x,\alpha]_{\succsim^{f,w}} = [x,\alpha]_{\succeq}$. Since $x' \in A^\star(\succeq)$ and $\sup_{\succeq}\{x,x'\} = \alpha$ we have that $x \not\succeq x'$, $x' \not\succeq x$ and $z \notin [x, \sup_{\succeq}\{x,x'\}]_{\succeq}$. By Remark 5 in Bonifacio and Massó (2020), there is $R^x \in \mathcal{SSP}(\succeq)$ such that $x'P^x z$, and this is condition (41). Similarly, as $x \not\succeq x'$, $x' \not\succeq x$ and $z \notin [x', \sup_{\succeq}\{x,x'\}]_{\succeq}$, again by Remark 5 in Bonifacio and Massó (2020), there is $R^{x'} \in \mathcal{SSP}(\succeq)$ such that $xP^{x'}z$, and this is condition (42).

2.1.2. $\boldsymbol{y \neq x'}$. Thus, $z \notin [x,y]_{\succsim^{f,w}} = [x,y]_{\succeq}$. Therefore, (41) and (42) follow from richness of $\mathcal{SSP}(\succeq)$ on $(A,\succeq)$.

2.2. **For all** $\boldsymbol{x' \in A^\star(\succeq), 1 < q^{x'}}$. Let $A^{\star\star} = \{x' \in A^\star(\succeq) \mid 1 < q^{x'} \leq n-1\}$. By (QS.1), $A^{\star\star} \neq \emptyset$. Hence, $n > 2$. Let $x' \in A^{\star\star}$. For each $y' \in A \setminus \{x'\}$, $f(y', x', \ldots, x') = x'$ and so $y' \not\succ_0^f x'$ and $x' \notin A_0^f$. Moreover, for each $y' \in A \setminus \{\alpha\}$, $f(y', \alpha, \ldots, \alpha) = \alpha$ and so $y' \not\succ_0^f \alpha$ and $\alpha \notin A_0^f$. Hence, $y' \in A^{\star\star} \cup \{\alpha\}$ implies $y' \notin A_0^f$. Therefore, $A_0^f \subset A \setminus (A^{\star\star} \cup \{\alpha\})$. Moreover, for each $y' \in A \setminus (A^{\star\star} \cup \{\alpha\})$, $f(\alpha, y', \ldots, y') = \alpha$ and so $\alpha \succ_0^f y'$ and $y' \in A_0^f$. Hence, $A \setminus (A^{\star\star} \cup \{\alpha\}) \subset A_0^f$. Therefore, $A_0^f = A \setminus (A^{\star\star} \cup \{\alpha\})$. Since $c^{z'} = \alpha$ for each $z' \in A_0^f$, we have that $\overline{A}_0^f = A_0^f \cup \{\alpha\}$. Therefore, $A = \overline{A}_0^f \cup A^{\star\star}$, where the union is disjoint. We now relate the order $\succsim^{f,w}$ with the original order $\succeq$ over A. Let $y', z' \in \overline{A}_0^f$. Then,

$$y' \succsim_0^f z' \Leftrightarrow f(y', z', \ldots, z') = y' \Leftrightarrow \sup_{\succeq}\{y', z'\} = y' \Leftrightarrow y' \succeq z'.$$

Therefore, by Definition 2 (i), $\succsim^{f,w}$ coincides with $\succeq$ over $\overline{A}_0^f$. To proceed establishing the relationship between $\succsim^{f,w}$ and $\succeq$ and proving that (41) and (42) hold, we distinguish between two subcases:

2.2.1. $\boldsymbol{w \in \overline{A}_0^f}$. Let $x' \in A^{\star\star}$. Then, $f(\underbrace{w, \ldots, w}_{n-q^{x'} \text{ times}}, \alpha, \underbrace{x', \ldots, x'}_{q^{x'}-1 \text{ times}}) = \alpha$ implies $\alpha \succ_{n-q^{x'}}^{w} x'$ and so, by Definition 2 (ii), $x' \succ^{f,w} \alpha$ since $x', \alpha \in A \setminus A_0^f$. Therefore, the only difference between $\succsim^{f,w}$ and $\succeq$ is that the elements of $A^{\star\star}$ are above α according to $\succsim^{f,w}$. Remember that, in order to prove that $\mathcal{SSP}(\succeq)$ is rich on $(A, \succsim^{f,w})$, we assumed that $z \notin [x,y]_{\succsim^{f,w}}$. There are two cases to consider: (i) $y = x'$ for some $x' \in A^{\star\star}$, and (ii) $y \preceq \alpha$. The arguments to prove that (41) and (42) hold in cases (i) and (ii) follow reasonings similar to those already used in Subcases 2.1.1 and 2.1.2 above, respectively.

2.2.2. $\boldsymbol{w \in A^{\star\star}}$. Let $y' \in A^{\star\star} \setminus \{w\}$. Then, $f(\underbrace{w, \ldots, w}_{q^w-1 \text{ times}}, \alpha, \underbrace{y', \ldots, y'}_{n-q^w \text{ times}}) = \alpha$ because, by (QS.2), $n - q^w < q^{y'}$. Hence, $\alpha \succ_{q^w-1}^{f,w} y'$ and therefore, since $y', \alpha \notin A_0^f$, by Definition 2 (ii), $y' \succ^{f,w} \alpha$. Furthermore, $f(\underbrace{w, \ldots, w}_{q^w \text{ times}}, \underbrace{\alpha, \ldots, \alpha}_{n-q^w \text{ times}}) = w$ implies

$w \succ^{f,w}_{q^w-1} \alpha$ and thus, since $w, \alpha \notin A^f_0$, by Definition 2 (ii), $\alpha \overline{\succ}^{f,w} w$. Therefore, the only difference between $\overline{\succeq}^{f,w}$ and $\succeq$ is that the elements of $A^{\star\star} \setminus \{w\}$ (if any) are above α according to $\overline{\succeq}^{f,w}$. Again we have two cases to consider: (i) $y = x'$ for some $x' \in A^{\star\star} \setminus \{w\}$, and (ii) $y \preceq \alpha$. The arguments to prove that (41) and (42) hold in cases (i) and (ii) follow reasonings similar to those already used in Subcases 2.1.1 and 2.1.2 above, respectively.

Therefore, $\mathcal{SSP}(\succeq)$ is rich relative to f and, by Theorem 1, $\mathcal{SSP}(\succeq) \subseteq \mathcal{LWSP}(f)$.

We now identify the special subfamily of simple and strategy-proof rules $f : \mathcal{SSP}(\succeq)^N \to A$ for which $\mathcal{SSP}(\succeq)$ is a strict subset of $\mathcal{LWSP}(f)$. There are two cases to consider.

1. ***f* is the supremum rule**. Let $w \in A$ be arbitrary. We have already established, when proving that $\mathcal{SSP}(\succeq)$ is rich on $(A, \overline{\succeq}^{f,w})$, that $\overline{\succeq}^{f,w}$ is equal to $\succeq$. Hence, $\mathcal{LWSP}(f) = \mathcal{SSP}(\overline{\succeq}^{f,w}) = \mathcal{SSP}(\succeq)$.

2. ***f* is a generalized quota-supremum rule**. Then, $\sup_{\succeq} A$ does exist and $A^\star(\succeq) \neq \emptyset$. Define $\sup_{\succeq} A = \alpha$. Let $q = \{q^x\}_{x \in A^\star(\succeq)}$ be the quota system associated to f.

   2.1. There is $x \in A^\star(\succeq)$ such that $q^x = 1$. We know, by Subcase 2.1 above, that $\overline{\succeq}^{f,x}$ coincides with $\succeq$ except that $x \overline{\succ}^{f,x} \alpha$. Moreover, since $x \in A^\star(\succeq)$ and $\succeq$ is a semilattice, so is $\overline{\succeq}^{f,x}$. Let $y \in A \setminus \{\alpha, x\}$. As $x \overline{\succ}^{f,x} \alpha$, by Remark 4 (i) in Bonifacio and Massó (2020), there is $R^x \in \mathcal{SSP}(\overline{\succeq}^{f,x})$ such that $y P^x \alpha$ (remember that $R^x \in \mathcal{SSP}(\overline{\succeq}^{f,x})$ implies that $R^x \in \mathcal{LWSP}(\overline{\succeq}^{f,x})$ and so $R^x \in \mathcal{LWSP}(f)$). However, $R^x \notin \mathcal{SSP}(\succeq)$ since (LWSP.2) does not hold because $\sup_{\succeq}\{x, y\} = \alpha$. Therefore, $\mathcal{SSP}(\succeq) \subsetneq \mathcal{LWSP}(f)$.

   2.2. For all $x \in A^\star(\succeq)$, $1 < q^x$. Similarly as we did in Subcase 2.2 above, $A^{\star\star} \neq \emptyset$ and $A = \overline{A}^f_0 \cup A^{\star\star}$, where the union is disjoint. Let $x \in A^{\star\star}$ and $y \in A \setminus \{x\}$. There are two subcases to consider:

      2.2.1. $y \in \overline{A}^f_0$. Then, the only difference between $\overline{\succeq}^{f,y}$ and $\succeq$ is that the elements of $A^{\star\star}$ are above α according to $\overline{\succeq}^{f,y}$. Hence,

$$R^y \in \mathcal{LWSP}(f) \text{ if and only if } R^y \in \mathcal{SSP}(\succeq). \tag{43}$$

      2.2.2. $y \in A^{\star\star}$. Then, the only difference between $\overline{\succeq}^{f,y}$ and $\succeq$ is that the elements of $A^{\star\star} \setminus \{y\}$ are above α according to $\overline{\succeq}^{f,y}$. Then, (43) also holds in this case.

Thus, we have that $\mathcal{SSP}(\succeq) = \mathcal{LWSP}(f)$.