Ranking and Search Effort in Matching

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Abstract

This paper studies the relationship between search effort and workers' ranking by employers. We propose a matching model in which employers have common preferences over a continuum of heterogeneous workers who choose a number of applications to send out. We show that in equilibrium, the relationship is hump-shaped for sufficiently high vacancy-to-worker ratios, that is highly-ranked and lowly-ranked workers send out fewer applications than workers of mid-range rank. This arises due to two opposing forces driving the incentives of applicants. Increasing the number of applications acts as insurance against unemployment, but is less effective when the probability of success for each application is low. This mechanism exacerbates the negative employment outcomes of low-rank workers. We discuss comparative statics with regards to the size of the vacancy pool and application cost, and show that, in contrast to the market equilibrium, in the social planner's solution the number of applications monotonically decrease in rank.

Keywords Simultaneous Search, Search Effort, Worker Heterogeneity

1 Introduction

Mismatch in labor markets with simultaneous applications is a widely-studied phenomenon. The idea is simple: without coordination many workers apply to the same firms while other firms do not receive any applications. This results in unemployment and affects workers' incentives to search in the first place. As Stigler (1962) first noted, endogenous search effort is an important factor for matching outcomes. This paper examines the behavior of workers who are commonly

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ranked by employers and choose a number of applications to send out. Though many papers have analyzed this problem for homogeneous workers (Albrecht, Gautier and Vroman (2003) and Kircher (2009)), this is, to our best knowledge, the first paper to study the case of heterogeneous workers.

Competition among identical workers differs substantially from competition between weak and strong applicants. When firms have a preference over workers, application incentives depend on the relative standing (rank) of workers as they need to consider the possibility of loosing job offers to more preferable candidates. The effect of rank on the number of applications is not obvious. On the one hand, because the success probability per application decreases with rank and applications are costly, a low-rank worker might be discouraged from applying as often as a high-rank worker. On the other hand, a low-rank worker might want to send out additional applications as a form of insurance, because each application is likely to fail.

Using our model we show that the relationship between rank and search effort is hump-shaped, so it first increases and then decreases in rank. As Shimer (2004) pointed out, traditional search models (e.g. Pissarides (2000)) generate the result that equilibrium search effort must increase in the baseline arrival rate.[1] This would suggest that high-rank workers search more due to a higher baseline rate. In contrast, our result is quite distinct, namely that the highest ranked workers apply among the least. Our result also implies that some lower-rank workers can have higher employment probability than higher-rank opponents, because they counteract a lower per-application success rate by sending more applications. For very low ranks, chances of employment can be low or zero, because workers both face an even lower per-application success rate and send out fewer or no applications.

The source of heterogeneity in our model is a worker characteristic over which employers have common preferences but cannot contract. Each employer offers the same wage to all workers and is not able to post type-specific wage schedules. The only way to discriminate between workers is by selecting a preferred candidate from the applicant pool. This assumption follows Peters (2010) and has the effect that differences in worker incentives are driven only by rank competition rather than employers' wage-posting. Because wages are fixed, workers cannot distinguish among vacancies and direct applications to specific firms to avoid applying to the same firm as others. This intensifies congestion and competition across types for the same jobs.

While assuming uniform wages is restrictive, it is appropriate for some labor markets. One prominent example is the market for entry-level public school teachers in the US (and possibly

[1]The arrival rate denotes the stochastic probability with which a worker receives a job offer in a given period. Search effort is often modeled as choosing a scaling parameter that is multiplied by the baseline rate.

other countries). Salaries and work conditions are negotiated by unions at district-level and only depend on years of experience and education. As a result, variation in the salary for entry-level positions is close to non-existent and wages are fixed within the local district's labor market.[2] Further, Boyd et al. (2005), who estimate an empirical matching model for the teaching market, find that when hiring a new teacher, schools have strong preferences over the selectivity of the candidates' college and performance in a basic knowledge test. It is reasonable to assume these qualifications as well as their distributions are public knowledge, such that there is a public common ranking of candidates.

This example extends readily to other parts of the public sector or markets with unionized wages. Another way to interpret our setting is that the worker characteristic cannot be priced because the value is difficult to quantify, e.g. soft skills or reference letters, or it is not legal to discriminate based on it, e.g. race, sex, motherhood status or attractiveness. Yet another possibility is that individuals with similar qualifications apply within a tier of employers/institutions in which wages are very similar, e.g. college graduates from top universities applying to entry-level consulting jobs. Overall, our model offers insights on worker search effort for all markets in which unequal searchers are in somewhat direct competition with one another.

In the literature, heterogeneous searchers have only been studied in a fixed search effort and directed search setting, such as in Peters (2010), Shimer (2005), Shi (2002), Burdett and Coles (1997). One exception is Lentz (2010) who finds a monotonic relationship between sequential search effort and worker type in his search model, confirming the prediction from Shimer (2004) mentioned earlier. Papers which model endogenous simultaneous search effort only consider firm heterogeneity and its role for wage dispersion and efficiency (e.g Albrecht, Gautier and Vroman (2003), Galenianos and Kircher (2009), Kircher (2009)) but not worker heterogeneity.

Modelling simultaneous search is complex because firms' and workers' actions are interdependent through a network of applications.[3] We rely on the assumption that the equilibrium matching be stable to obtain well-defined expressions for employment probability. This concept was first introduced by Kircher (2009) in the search and matching context, and has since been employed by a handful of papers (e.g. Gautier and Holzner (2017a, 2017b), Wolthoff (2018)). Here a matching is defined to be stable if no firm and worker pair can mutually be better off by matching with each other instead of their current partners.

[2]Boyd et al. (2005) also note that the majority of new teachers find employment in close proximity to where they grew up, suggesting it is plausible they do not search across several markets.

[3]To limit the number of strategic interactions, many papers restrict firms to make a single offer (e.g. Shimer (2004), Albrecht, Gautier and Vroman (2003), Galenianos and Kircher (2009)). In this setting some firms are left with open vacancies, even if they have unemployed applicants.

The paper proceeds as follows. In the next section we illustrate the idea using a simple example. Section 3 develops the full theoretical model and presents the main result. In Section 4 we provide an analysis of how market tightness and application costs affect application behavior and employment outcomes. Lastly, we discuss efficiency and the social planner solution in Section 5.

2 A Simple Discrete Example

Consider a labor market with M vacancies, each at a different firm, and three workers, $\{A, B, C\}$, ranked first, second, and third respectively by all firms. The workers decide how many applications to send to the firms. They apply to each firm at most once and can take at most one job.

Here, as well as for the full model, we make anonymity assumptions analogous to Kircher (2009). We assume workers do not know the identity of firms and applications are equivalent to random sampling of jobs without replacement. So, with m applications, a worker has $\binom{M}{m}$ ways of sampling firms. Workers also do not know where others apply, and might apply to the same job as higher-ranked competitors. When offered multiple employment opportunities, they randomly choose one without consideration for other applicants. This rules out coordination among workers to direct applications or purposeful selection of firm offers, e.g. a higher-rank worker chooses to take a specific job over another to leave the latter job to a lower-rank worker.

Note that worker A sends out at most one application, as this application lands a job with certainty. Worker B however, risks unemployment if he only sends out one application, since he might apply to the same firm as worker A. By sending two applications he gets a job with certainty since at least one firm he applies to cannot hire worker A.

For $M = 2$, worker B applies to the same job as worker A with probability $\frac{1}{2}$ if he sends one application. In this case worker C only gets a job if worker B applies to the same firm as worker A and if he sends an application to the firm to which worker A and B do not apply. We can see that each worker only needs information on higher-rank workers in order to evaluate his own decision. A worker always has priority over lower-rank workers and applying to the same firms as these workers is not a concern. Fixing worker A's number of applications to be one, the following table shows the employment probability of each worker as a function of the number of applications.

(B,C)	1	2
1	$(\frac{1}{2}, \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 0)$	$(\frac{1}{2}, \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0)$
2	$(1, 0)$	$(1, 0)$

The next table presents the marginal benefit of each additional applications for each worker. The third row and fourth row show worker C's marginal benefit when worker B sends out one or two applications respectively.

	MB (1)	MB (2)
A	1	0
B	$\frac{1}{2}$	$\frac{1}{2}$
C (1)	$\frac{1}{4}$	$\frac{1}{4}$
C (2)	0	0

As mentioned above, the marginal benefit for worker C is a function of worker B's applications but not vice versa.

We denote the flat cost of applications as c. The worker's expected utility is the probability of getting a job multiplied by the wage minus the cost of application. So the workers increase the number of their applications as long as the marginal benefit of the application exceeds the cost. Setting the wage to one for all jobs, we derive that for application cost less than $\frac{1}{2}$ worker A applies once, worker B applies twice, while worker C does not apply at all. This pattern of application illustrates our main result: the number of applications is hump-shaped in rank. Workers of mid-range rank might apply more to insure against losing against higher-rank competitors while low-rank workers might apply less because too many jobs are taken by higher-rank workers (due to their priority in rank and high search effort).

3 The Model

Setting We now turn to the full model. We consider a bipartite matching problem of two sets of continuum of agents: workers $\mathcal{W}$, and firms $\mathcal{F}$. Let $\mathcal{W}$ be a unit mass of workers, indexed by $i \in [0, 1]$. Each worker is endowed with type $x \in [0, 1]$, and we denote workers' type distribution by F with support $[0, 1]$ and a pdf f. Define $X : [0, 1] \to [0, 1]$ which maps each worker to her type. Then F is induced by X, and Lebesgue measure λ over $\mathcal{W}$: $F(x) := \lambda(\{i : X(i) \leq x\})$. Let $\mathcal{F}$ be a positive mass $M(> 0)$ of firms indexed by $j \in [0, M]$.

As in our simple example, firms are identical with one vacancy each and we adopt the same anonymity assumptions as before. We now have large markets and add that firms cannot distinguish between workers of the same type, however the probability of receiving two applications from the same type is zero given the continuous type space. We define a strategy as a function k that maps from X into a discrete number of applications, which in turn are i.i.d. uniform draws from the firm pool.

We normalize each worker's value from matching with a firm to one and set the value of remaining unemployed to zero. A worker incurs a flat cost c for each application he sends. We assume a firm's payoff is equal to the matched worker's type or $-\varepsilon < 0$ if it remains unmatched. Hence, this implies firms strictly prefer to be matched with any worker than to remain unmatched. As we show later, worker types exactly correspond to their position in firm preference rankings which is the only object needed for equilibrium analysis.

Matching Process The application and hiring process begins with all workers simultaneously choosing $k \in \mathbb{Z}_+$. Then applications are realized, forming a network with 'links' from the worker to the firm pool. Given the network a stable matching, which we define in the following part, is formed.

We define a *matching* as a function μ: $\mathcal{W} \cup \mathcal{F} \to \mathcal{W} \cup \mathcal{F}$ such that

1. $\mu(i) = j \in \mathcal{F}$ if and only if $\mu(j) = i \in \mathcal{W}$,

2. If $\mu(i) \notin \mathcal{F}$, then $\mu(i) = i$,

3. If $\mu(j) \notin \mathcal{W}$, then $\mu(j) = j$.

For each $i \in \mathcal{W}$, and corresponding $k(i)$ applications i sent out in the first stage, define the set of firms receiving an application from i as $B(i) = \{j_1, j_2, \ldots, j_{k(i)}\}$. A matching μ in the second stage is *feasible* if $\mu(i) \in \{i\} \cup B(i)$ for all i. We require the second stage matching μ to be feasible, and *stable* in the following sense.

Definition 1. A feasible matching μ is stable if there is no pair (i, j) with $j \in B(i)$ such that $\mu(i) = i$ and either (i) $\mu(j) = j$, or (ii) $\mu(j) \neq j$ and $X(i) > X(\mu(j))$.

This is the notion of no blocking pair with both parties strictly better off. For any blocking pair (i, j) with $j \in B(i)$, since workers are indifferent over all firms, it must be $\mu(i) = i$. Furthermore, j strictly prefers i to his current match $\mu(j)$, himself or a worker with type lower than i.

In other words we require that workers only be unemployed, when all firms they applied to are able to hire higher-ranked workers. This lets us express the employment probability as a function of the mass of successfully employed higher-rank workers and derive a differential equation that characterizes how the latter changes with worker rank.

While technically there can be multiple stable matchings, these matchings all exhibit the same aggregate measures. This property follows from the continuum agent assumption. In particular, the mass of matched workers with ranks within any interval is constant across these matchings.

Employment Probability We first characterize employment probability by type for a given number of applications made by each worker. There are $1 - F(x)$ workers of higher rank for a type x worker. We define the mass of successful applicants of types higher than x as $A(x)$, here on referred to as the accumulation function. $A(x)$ takes a deterministic value at any x due to the continuous type space, i.e. the mass of employed mass of workers above a certain rank is determined with certainty for any application behavior. Since $A(x)$ is also the mass of of jobs occupied by higher-rank workers, a worker will only be successful if he does not apply to any of these jobs. His success probability per application is thus $1 - \frac{A(x)}{M}$. Note that when workers make more than one application, $A(x)$ is not equal to the mass of firms receiving applications from workers of ranks higher than x. Not all firms with applications can hire higher-rank workers, since those workers will choose only one of the available firms they applied to.

Rank x worker's employment probability given k applications is given by $1 - \left(\frac{A(x)}{M}\right)^k$, the probability that he applies to at least one firm that does not hire a higher-rank worker. Due to the large-market properties, the probabilities that each of the applications succeed are independent.

Assuming the derivative of f exists everywhere, and f and f' are bounded, we establish the following differential equation:

Proposition 1. The change of the accumulation function in a small neighborhood of x if all workers within this neighborhood apply k times is given by:

$$-A'(x) = f(x)\left(1 - \left(\frac{A(x)}{M}\right)^k\right)$$

The equation simply states that the change in A at x is equal to the expected mass of type x workers that succeeds in finding a job. The right-hand side is positive for $k > 0$ and $A(x) < M$; in this case the accumulation is strictly decreasing. For a better understanding of this equation the interested reader might turn to Appendix A.1 in which we provide an alternative derivation using the per-application success rate $\varphi(x) = 1 - \frac{A(x)}{M}$. We express it as a function of the effective queue length and success rate of higher-ranked competitors applying to the same job.

The differential equation does not apply for cutoff types $\bar{x}$ at which there is a discrete jump in the number of applications. Formally these are types at which in any arbitrarily small neighborhood there are at least two types sending out different numbers of applications from one another. At these types, $A(x)$ is not differentiable. However, note that $A(x)$ must be continuous, since $f(x)$ is continuous. This implies that the function $A(x)$ has a kink at $\bar{x}$, and the value $A(\bar{x})$ is the same as the left and the right limit of $A(x)$ as x goes to $\bar{x}$.

Worker Problem We now turn to the worker's utility maximization problem for a given level of accumulation A. His utility is his employment probability minus the cost of application. We restrict our attention to non-trivial costs $c \in (0, 1)$ so that at least some workers send non-zero applications and the number of applications is finite. The worker solves:

$$\max_{K \in \mathbb{Z}_+} (1 - \left(\frac{A}{M}\right)^K) - cK.$$

To analyze this problem, it is useful to define the gross marginal benefit of the $(k + 1)$th application MB_{k+1}. It is a function of $\frac{A}{M} \in [0, 1]$:

$$MB_{k+1}(\tfrac{A}{M}) = 1 - \left(\frac{A}{M}\right)^{k+1} - \left(1 - \left(\frac{A}{M}\right)^k\right) = \left(\frac{A}{M}\right)^k \left(1 - \frac{A}{M}\right).$$

The marginal benefit is equal to the increase in employment probability due to the $(k + 1)$th application. In other words it is the probability that the first k applications are not successful and the $k + 1$th application is. We graph the marginal benefit of the first three applications for $\frac{A}{M} \in [0, 1]$ in Figure 1 together with a cost of 0.07.

Note that, for a fixed $\frac{A}{M}$, the marginal benefit always decreases in k. With constant marginal cost c, the solution to the worker's problem is to increase the number of applications as long as the marginal benefit exceeds cost. Denote by $K(\frac{A}{M})$ the set of solutions to the worker's problem, who faces probability $\frac{A}{M}$ of his application not being successful. $K(\frac{A}{M})$ satisfies the following properties:

1. If $A = 0$, then $K(\frac{A}{M}) = 1$. This is because the employment probability is one for any K and additional applications only increase cost.

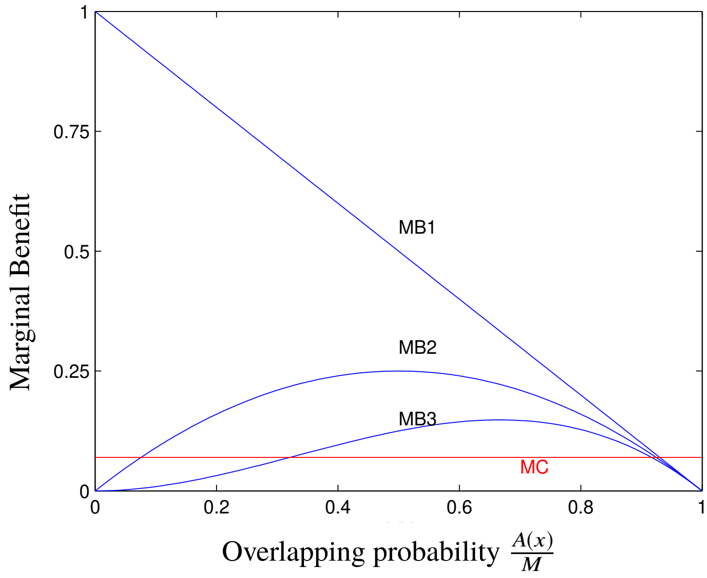

Figure 1: Marginal benefits of applications 1-3

2. For all $A \in (0, M(1-c))$:

   - If $MB_{k+1}(\frac{A}{M}) = (\frac{A}{M})^k(1-\frac{A}{M}) \neq c$ for any $k \in \mathbb{Z}_+$, then $K(\frac{A}{M}) = \hat{k}+1$ such that $(\frac{A}{M})^{\hat{k}}(1-\frac{A}{M}) > c$ and $(\frac{A}{M})^{\hat{k}-1}(1-\frac{A}{M}) < c$.
   - If $MB_{\tilde{k}+1}(\frac{A}{M}) = (\frac{A}{M})^{\tilde{k}}(1-\frac{A}{M}) = c$, the worker is indifferent between $\tilde{k}$ and $\tilde{k}+1$ applications, i.e. $K(\frac{A}{M})$ is then the set $\{\tilde{k}, \tilde{k}+1\}$.

3. If $A > M(1-c)$, then $MB_1(\frac{A}{M}) < c$ and $K(\frac{A}{M}) = 0$. If $A = M(1-c)$, then $MB_1(\frac{A}{M}) = c$ and $K(\frac{A}{M}) = \{0, 1\}$.

After examining the accumulation function and the worker's solution separately, we describe their consistency in equilibrium.

Equilibrium Given cost c, firm mass M, and the distribution of worker types F, an equilibrium are functions (A, k) such that

- Given $A : [0,1] \to [0, M]$, the number of applications sent out by each worker $k : [0,1] \to \mathbb{Z}_+$ maximizes his respective utility. That is, for all x:

$$k(x) = \arg\max_{k \in \mathbb{Z}_+} 1 - \left(\frac{A(x)}{M}\right)^k - ck$$

- Given k, a continuous function A is a solution to the differential equation:

$$-A'(x) = f(x)(1 - \left(\frac{A(x)}{M}\right)^{k(x)}), \tag{1}$$

wherever applicable[4], with initial condition $A(1) = 0$.

An equilibrium exists, since each worker's solution only depends on the decisions of higher-ranked workers through $A(x)$ and we can solve for the equilibrium starting from the highest-ranked worker. In practice this involves identifying cutoff types and respective cutoff accumulations for which there is a discontinuity in $k(x)$ and kink point in $A(x)$. In the intervals between these cutoff types all worker types send out the same number of applications, so $A(x)$ follows the differential equation.

As we mentioned before it is possible for workers with a cutoff type to be indifferent between two adjacent numbers of applications. However, in equilibrium, varying the tie-breaking rule for these types can only affect a zero measure of workers[5]. For simplicity we restrict the analysis to left-continuous k for the rest of this paper. In summary:

Proposition 2. The equilibrium exists and is unique up to a measure zero set of workers.

As we can see from Figure 1, the marginal benefit of the first application ($MB_1(\frac{A}{M})$) is a monotone decreasing function of $\frac{A}{M}$, while the other marginal benefits are single peaked in $\frac{A}{M}$. Furthermore, as noted before, the marginal benefit decreases in k, so the curves are nested within each other, with higher numbers of applications being further on the inside. These properties of the marginal benefits and from the continuity and monotonicity of function $A : [0, 1] \to \mathbb{R}_+$, give rise to the main result.

Theorem 1. The equilibrium number of applications $k(x)$ is single-peaked in x.

We provide a detailed proof in Appendix A.2.

This result implies that for workers of types to the right of the peak, the number of applications decreases with rank. These workers have a lower per-application success rate $(1 - \frac{A}{M})$ than higher type competitors but tend to make up for it by sending out more applications. To the left of the peak, the number of applications increases with type and lower type workers apply less

[4]Refers to x within non-zero measure intervals of types sending the same number of application, for which the derivative exists.

[5]To elaborate, there are two different categories of cases where there is a tie. First of all, for types where $\frac{A(x)}{M} < 1 - c$, we have $k(x) > 0$ and $A(x)$ strictly decreasing. Worker types indifferent between k and $k + 1$ are of measure zero since $MB_{k+1}(\frac{A(x)}{M}) = (\frac{A(x)}{M})^k(1 - \frac{A(x)}{M}) = c$ is only satisfied for maximal two points in the type space.

Secondly, there may be types who are indifferent between $k = 0$ and $k = 1$. Let x_0 be a type with $MB_1(\frac{A(x_0)}{M}) = 1 - \frac{A(x_0)}{M} = c$. In equilibrium there cannot be a non-zero measure of types $x < x_0$ sending out a strictly positive number of applications, so all lower types must have marginal benefit equal to $MB_1(\frac{A(x_0)}{M})$. The proof is by contradiction: Assume there is an interval of worker types sending out a strictly positive number of applications, then $A(x)$ is strictly increasing over this interval. Take a type x' in the interior of the interval, then $MB_1(\frac{A(x')}{M}) < c$ since $A(x') > A(x_0)$. Hence a strictly positive number of applications cannot be optimal for this type.

because they are discouraged by the low per-application success rate. Therefore, the decline in employment probability is exacerbated for these workers, with the lowest type worker incurring the lowest probability.

Furthermore from the previous discussion on the workers' problem we know $k(0) = 0$ implies $A(0) < M(1-c)$ and $k(0) > 0$ implies $A(0) < M(1-c)$, so:

Corollary 1. If the lowest type workers send out zero applications then total employment $A(0)$ is given by $M(1-c)$. If the lowest type workers send out a strictly positive number of applications, total employment is strictly less than $M(1-c)$.

Normalization A property of our model is that the employment probability depends only on rank and not on the cardinality of types. Consequently, the equilibrium is invariant to the worker type distribution and we can express all objects of interests in terms of the worker's percentile ranking, i.e. the cdf $F(x)$, as transformed type.

Formally, for an equilibrium (A, K) with distribution F there exists an equivalent representation of an equilibrium $(\tilde{A}, \tilde{K})$ defined over types $\tilde{x} = F(x)$ satisfying the following:

$$-\tilde{A}'(\tilde{x}) = 1 - \left(\frac{\tilde{A}(\tilde{x})}{M}\right)^{\tilde{K}(\tilde{x})}$$
$$\tilde{K}(\tilde{x}) = \arg\max_k 1 - \left(\frac{\tilde{A}(\tilde{x})}{M}\right)^k - ck$$

This can be obtained by a straightforward change of variables as illustrated in Appendix A.3. This allows us to state the following results:

Corollary 2. Denote by (A^F, K^F) and (A^G, K^G) equilibria from two distributions F and G, respectively. For any pair $(x, x') \in [0,1]^2$ such that $F(x) = G(x')$, we have $A^F(x) = A^G(x')$.

Corollary 3. Total employment in the economy does not depend on F.

Without loss of generality, we restrict attention to a uniform distribution hereafter, by letting the type $\tilde{x}$ as x, and the equilibrium objects $(\tilde{A}, \tilde{K})$ as (A, K).

3.1 Example

Here we demonstrate how to find the equilibrium for a set of parameter values. Let $c = \frac{4}{27}$ and $M = 0.7$. For this cost at most two applications are sent out in equilibrium.

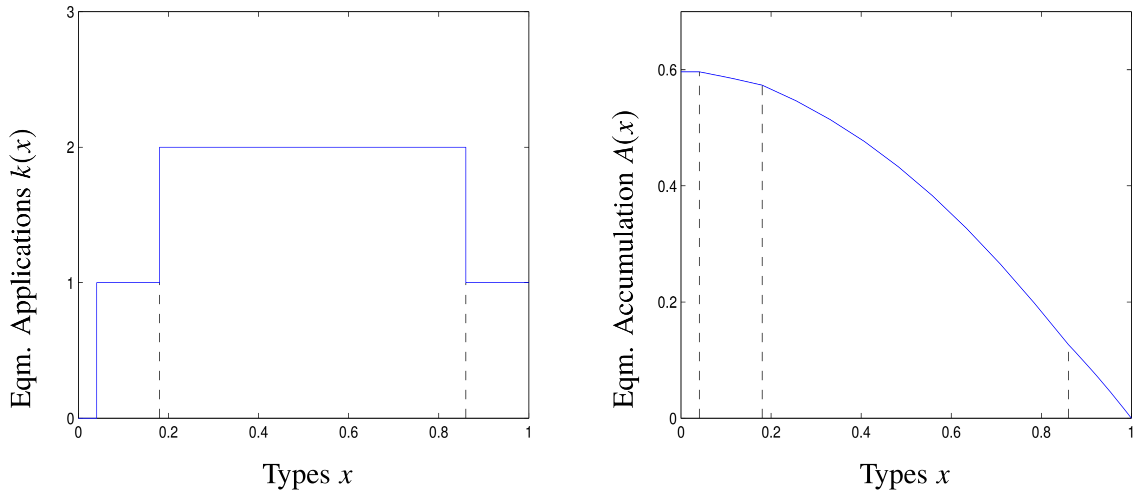

Figure 2: $k(x)$ and $A(x)$. Cutoff types indicated by dashed lines

For $k = 1, 2$, there is an analytic solution for $A(x)$ for all $x < \hat{x}$ with initial value $A(\hat{x}) = A$:

$$A(x) = M(1 - (1 - \frac{A}{M})e^{-\frac{\hat{x}-x}{M}}) \qquad (k = 1)$$

$$A(x) = M \tanh(\tanh^{-1}(\frac{A}{M}) + \frac{\hat{x} - x}{M}) \qquad (k = 2)$$

where tanh stands for hyperbolic tangent function. We can solve for the three cutoff types which satisfy $\frac{A(x)}{M}(1 - \frac{A(x)}{M}) = c$, and $1 - \frac{A(x)}{M} = c$. Figure 2 shows the equilibrium schedule of applications and $A(x)$. Note $A(x)$ is steeper for types that apply twice and has a maximum value of $M(1 - c) \approx 0.6$.

As Figure 3 shows, the equilibrium employment probability jumps when the number of applications changes. Furthermore worker types in $[0.65, 0.87]$ have a similar probability to types in $(0.87, 0.97]$, while types below 0.65 are consistently less likely to find a job the lower their type.

4 Comparative Statics

In this section we examine how equilibrium outcomes relate to the market tightness M and application cost c.

Simple Example Revisited We illustrate our results with a modified version of our simple example. Assume there are now $M = 3$ jobs. The employment probability is:

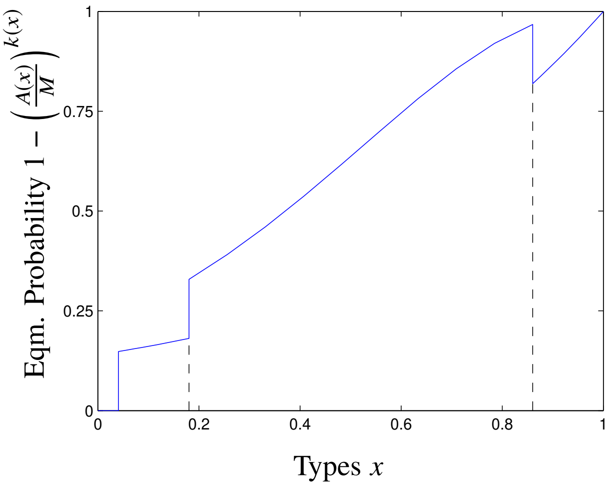

Figure 3: Equilibrium Employment Probability

(B,C)	1	2	3
1	$(\frac{2}{3}, \frac{2}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{2}{3})$	$(\frac{2}{3}, \frac{2}{3}\cdot\frac{2}{3}+\frac{1}{3}\cdot 1)$	$(\frac{2}{3}, 1)$
2	$(1, \frac{1}{3})$	$(1, \frac{1}{3}\cdot 0+\frac{2}{3})$	$(1, 1)$
3	$(1, \frac{1}{3})$	$(1, \frac{1}{3}\cdot 0+\frac{2}{3})$	$(1, 1)$

Compared to the case with $M = 2$, the payoffs uniformly increased for both workers, which is intuitive. The marginal benefits are:

	MB (1)	MB (2)	MB (3)
A	1	0	0
B	$\frac{2}{3}$	$\frac{1}{3}$	0
C (1)	$\frac{4}{9}$	$\frac{3}{9}$	$\frac{2}{9}$
C (2)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Observe that a higher M has an ambiguous effect on the marginal benefit of applications. Worker B's first application is more valuable now, while the reverse is true for the second. For instance, for c between $\frac{1}{3}$ and $\frac{4}{9}$, worker B applies twice if there are two jobs and only once when there are three. Worker C, however, does not apply with two jobs and applies once with three. This suggest, that with higher M, workers ranked in the mid-range decrease their search effort, due to diminished insurance incentives, while low-rank workers increase their search effort, encouraged by greater availability of jobs.

We can further note that decreasing the cost of application might hurt worker C in equilibrium. For the above range of c the worker C has expected utility $\frac{2}{3} - c < \frac{1}{3}$, and worker B has $\frac{2}{3}\frac{1}{3} + \frac{1}{3}\frac{2}{3} - c = \frac{4}{9} - c < \frac{1}{9}$. However, if c is slightly below $\frac{1}{3}$, worker B sends out two applications, and worker C sends three applications (see fourth row of the table). The utility is $1 - 2c > \frac{1}{3}$ for worker B, and $1 - 3c > 0$ for worker C. For c close enough to $\frac{1}{3}$, worker B gains from a lower cost, while worker C is worse off. Next, we provide a formal analysis using the full model.

4.1 Market Tightness

Figure 4 depicts the marginal benefits for two sizes of job pools, $M = 1$ and $M' = 0.8$, represented by the blue and red graph respectively. The marginal benefit of the first application $1 - \frac{A(x)}{M}$ is uniformly lower for smaller M, while the marginal benefits of additional applications increase for small A, and decrease for large A.

This suggests that, for fewer available jobs, higher type workers might increase and lower types might decrease their number of applications. Indeed we demonstrate that this is the case by showing that equilibrium objects can be normalized with respect to M. The normalization further allows us to determine that the expected quality of a worker hired by firms in equilibrium is decreasing in the M.

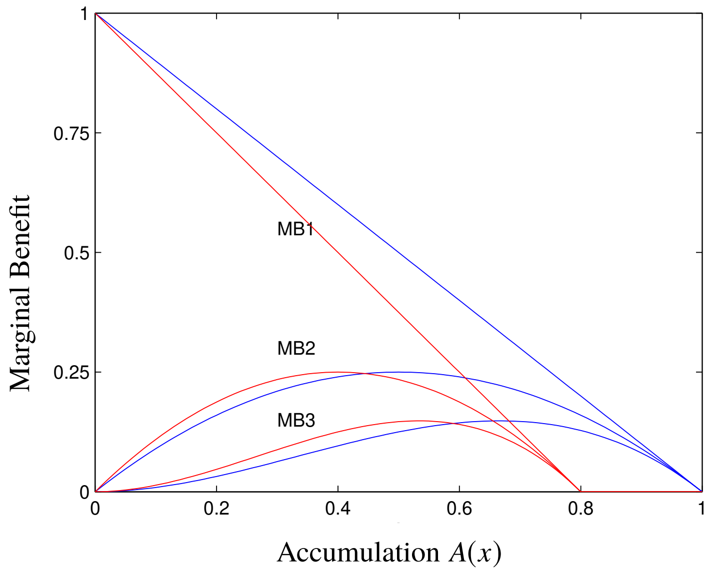

Figure 4: Marginal Benefit for Applications 1-3

Define $\hat{A}(\frac{1-x}{M}) = \frac{A(x)}{M}$. The equilibrium can be restated with the new transformed variable $\hat{x} = \frac{1-x}{M}$ and equilibrium functions $(\hat{A}, \hat{K})$, $\hat{A} : [0, \frac{1}{M}] \to \mathbb{R}_+$, and $\hat{K} : [0, \frac{1}{M}] \to \mathbb{Z}_+$:

$$\hat{A}'(\hat{x}) = 1 - \hat{A}(\hat{x})^{\hat{K}(\hat{x})}, \quad \hat{A}(0) = 0$$
$$\hat{K}(\hat{x}) \in K(\hat{A}(\hat{x}))$$

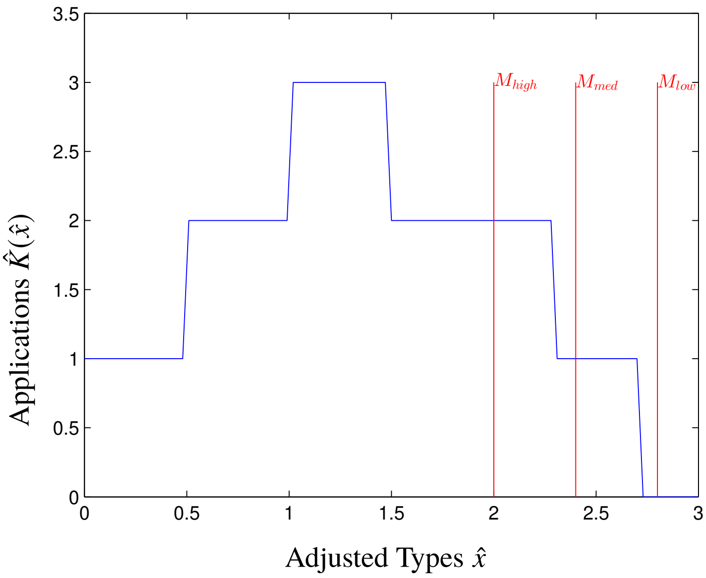

Figure 5: Observed Investment Behavior for Different M

where $K(\cdot)$ maps from the accumulation that a worker faces to the set of the worker's best response.

These equilibrium objects do not depend on M except through changes in the domain. Figure 5 depicts an example application schedule for transformed types and different M, with $\hat{x} = 0$ as the highest type $x = F(1)$. A high M corresponds to a smaller domain, since the red lines indicate the number of applications sent by the lowest type worker, $\hat{x} = \frac{1}{M}$, which is weakly higher for larger M. The observed application pattern (represented by the blue line) is cropped by the red lines. We further see that the maximum number of applications sent out by any worker can only decrease with larger M.

The equivalence of the original and the newly transformed equilibrium definition implies that the objects of interest are scaled by M. Let us denote by a 'cutoff type' x^j who is indifferent between sending out j and $j + 1$ applications. In particular, on the rescaled type space, the cutoff types $\hat{x}^j$ and accumulations $\hat{A}(\hat{x}^j)$ are invariant in M. Therefore, for two different levels of market tightness M_0, M_1, with $\gamma = \frac{M_1}{M_0}$ we have

$$\hat{x}_0^j = \frac{1 - x_0^j}{M_0} = \frac{1 - x_1^j}{M_1} = \hat{x}_1^j, \quad \forall j,$$

$$\gamma(1 - \hat{x}_0^j) = 1 - \hat{x}_1^j.$$

We now describe how M relates to total employment and firms' hiring probability defined as total employment over job pool size. Without loss of generality assume $\gamma < 1$. The following proposition follows immediately from the normalization and Corollary 1.

Proposition 3. If $K_0(0) = 0$ and $K_1(0) = 0$, then $A_1(0) = \gamma A_0(0)$, that is, total employment is proportional to M. Firms' hiring probability is the same regardless of M.

If $K_0(0) > 0$ or $K_1(0) > 0$ or both, then $\gamma A_0(0) < A_1(0)$, that is, the difference in total employment is less than proportional to the difference in vacancy pool size. Firms' hiring probability is lower for higher M.

How application behavior differs by M is not obvious. Given a tighter market some workers might try harder to secure a job and some workers might be discouraged and apply less. It turns out that for smaller M, high types tend to apply more and lower types less:

Proposition 4. There exist a cutoff type such that all workers with higher types apply weakly more with M_0 compared to M_1 and the converse is true for lower types.

Proof. Take the type to be the lowest type that applies strictly more for $M = M_1$. □

Since all workers' normalized type $\hat{x}$ is higher for smaller M, this result reflects the non-monotonicity of application behavior in type from our main theorem.

Note that worker types who apply more with small M do not need to have a higher employment probability compared to the case with large M, since the fraction of available jobs is also smaller.

Worker Quality and Firm Entry We now show that the average quality of hired workers decrease with M. The unconditional expected quality of a worker hired by a firm is given by:

$$E[x|\text{hired}]Pr(\text{hired})\frac{1}{M} = \int_{\max\{\underline{x},0\}}^{1} Pr(x \text{ is hired})x\, dx \frac{A(0)}{M} \qquad = \int_{\max\{\underline{x},0\}}^{1} -\frac{A'(x)}{M} x\, dx.$$

It turns out that first order stochastic dominance in the worker type distribution over M carries over to $\frac{A(x)}{M}$ and we can establish the following:

Proposition 5. For equilibria of two economies with (M_1, F_1) and (M_2, F_2), if $\frac{1-F_1(x)}{M_1} \leq \frac{1-F_2(x)}{M_2}$ for all x, then the expected quality of hired workers in economy 2 is weakly higher than in economy 1.

The proof can be found in A.4.

Therefore, if either $1 - F_2$ first order stochastically dominates $1 - F_2$ and $M_1 = M_2$, or $F_1 = F_2$ and $M_2 < M_1$, the expected quality of hired workers will be higher in economy 2. If we consider an initial stage in which firms can choose to enter at some cost $v > 0$, it immediately follows that there is a unique M that solves the free entry condition:

$$E[x] - v = E[x|\text{hired}]Pr(\text{hired})\frac{1}{M} - v = 0$$

4.2 Application Costs

Now we examine the relationship between application behavior and cost c. Note that a lower c is equivalent to a higher wage since the application decision is determined by the condition $w\left(\frac{A(x)}{M}\right)^k(1 - \frac{A(x)}{M}) \geq c$, which only depends on the ratio of cost to wage. There are two forces to consider for a lower c. On the one hand, workers have a higher net-return on each application. On the other hand, because higher-rank workers might apply more, low-rank workers might have a lower per-application success rate. Hence, it is not clear how a worker's application choice and utility relate to the cost. Counter-intuitively some workers might apply fewer times and be worse off with lower cost.

This can be seen from the results of a numerical exercise for $c \in \{0.148, 0.198, 0.248\}$ and $M = 0.7$. Figure 6 depicts equilibrium applications for different values of c. Some worker types around 0.1 do not apply for $c = 0.198$ but apply for the other cost values. We see from Figure 7 which plots worker utility, that utility for these workers is higher for $c = 0.248$ than $c = 0.148$.

Alternatively, we can consider a frictionless market with $c \approx 0$ and perfectly assortative matching. All workers in the top M-percentile find a job with certainty. So if $M < 1$, then workers of types $x < 1 - M$ remain unemployed. Now turn to the case with a slightly higher cost. We can conjecture that the workers in the top M-th percentile get hired with probability slightly less than one and some workers with types just below $1 - M$ now apply and have a strictly positive employment probability.

5 Efficiency

Note that each worker's search imposes a negative externality on lower type workers, since they 'take away' jobs from the vacancy pool. When evaluating whether to make an additional application, workers weigh the increase in employment probability against c but do not consider the effect on lower-rank workers, or specifically, that a newly found job could have otherwise been taken by another lower type worker.

To illustrate the discrepancy between individual and social welfare, we can consider the

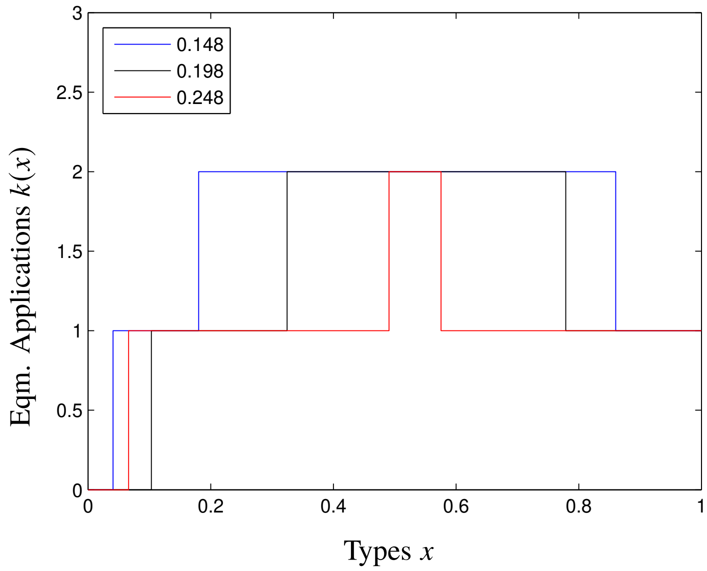

Figure 6: Equilibrium Applications for Different Levels of c

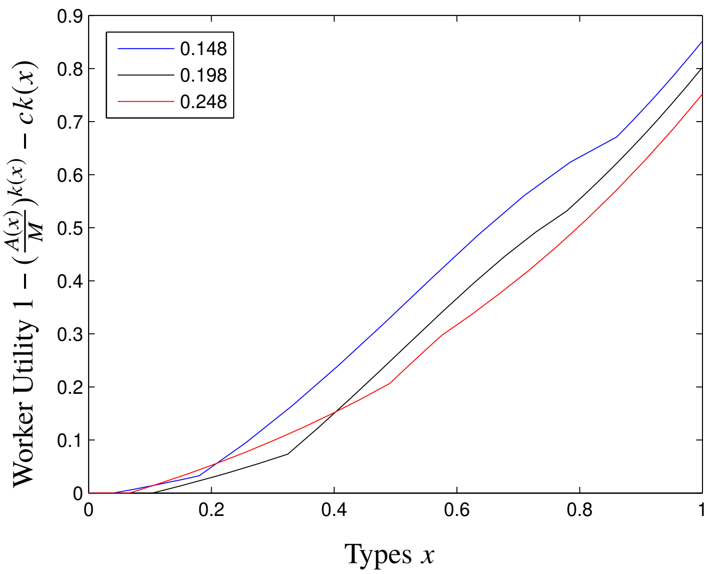

Figure 7: Worker Utility for Different Levels of c

problem of a social planner whose objective is to maximize total worker utility, which is equal to total employment minus total application costs. Assume that the planner can dictate each worker type to send out a particular number of applications but is still constrained by the anonymity assumption and application friction. The planner solves:

$$\max_{k\in\mathcal{K}} \int_0^1 1-\left(\frac{A(x)}{M}\right)^{k(x)} - ck(x)\,dx$$

$$s.t. \quad -A'(x) = 1-\left(\frac{A(x)}{M}\right)^{k(x)}, \; A(1)=0, \;\; \forall x.$$

The planner's solution satisfies two conditions:

Proposition 6. In the planner's solution:

1. If all workers send out applications, the number of applications is decreasing in type.

2. If some workers send zero applications, then all workers send at most one application.

The formal proof can be found in the Appendix A.5. We show that any problem that maximizes total employment for a fixed cost has a solution of this form. For a fixed number of total applications (and hence fixed total cost), if some interval of worker types sends more applications than an adjacent interval of lower types, then employment can always be increased by switching the number of applications for these two type intervals.

Note that this result must not apply if the planner's objective depends on cardinal worker types. However, our formulation of the planner's objective can be seen as the limit case when worker types are arbitrarily close to each other while preserving the rank. In the limit, the match quality is irrelevant for the social point of view.

Numerical Example Revisited We demonstrate that the market solution in our numerical example from Section 3.1 is not welfare maximizing. Figure 8 depicts a solution (in red) that achieves the same level of employment as the market equilibrium (in blue) with lower costs. From the upper graph, we see applications decrease from 2 to 1 at around $x = 0.5$, and everyone sends out nonzero applications. In the lower graph we see that $A(0)$ is the same for both solutions, but the employment accumulation is higher for most types in the market equilibrium. In the full solution to the planner's problem, total employment would be lower than in the cost-minimizing solution given here.

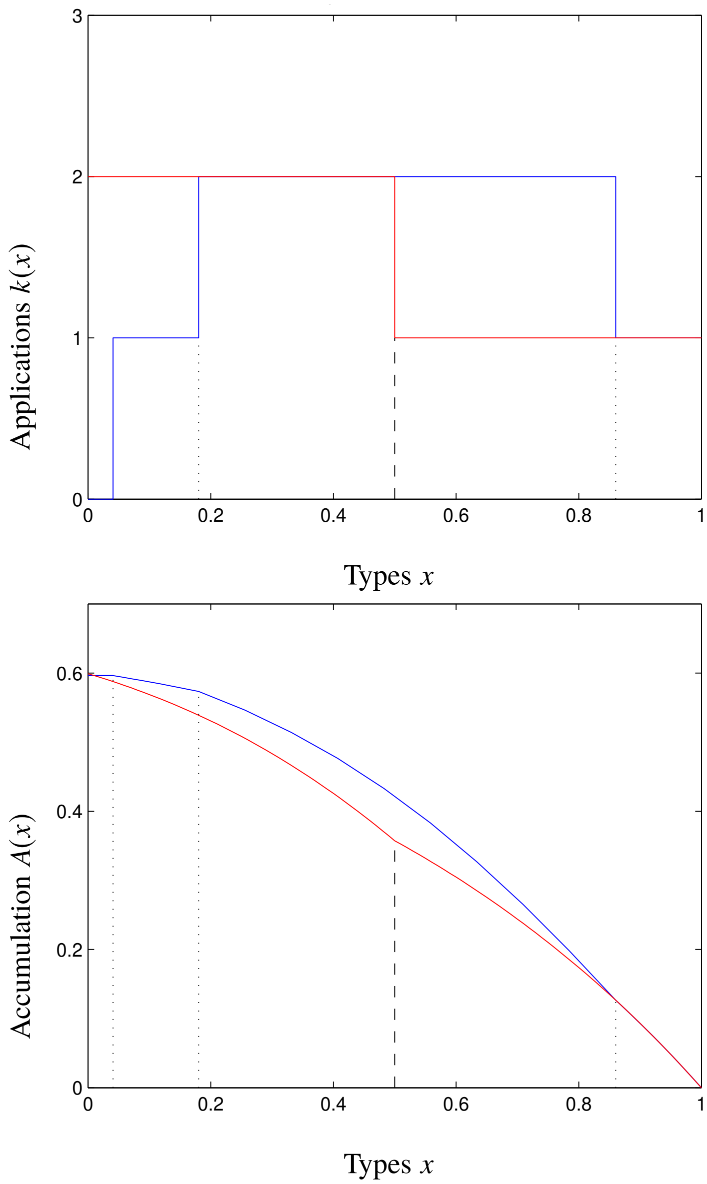

Figure 8: Cost-minimizing solution(red) along with the market solutioin(blue), dashed lines indicate cutoff types

6 Conclusion

We examined how heterogeneity in ranking of workers by firms, translates into differences in search effort and employment probability. We find that if a worker's rank is sufficiently high, he applies more than higher-ranked competitors to insure against the event of firms choosing someone else. However if a worker's rank is rather low, his poor chances of success per application discourage him from applying as much as more highly-ranked counterparts. In this case, the disadvantage from the lower per-application success rate is amplified by lower search effort. This offers a novel insight to unequal labor market matching outcomes: higher unemployment among low-skilled workers can arise due to search effort choices.

We further showed that workers' equilibrium number of applications and utility can depend ambiguously on the size of the vacancy pool and application costs. Lastly, we showed that in a solution that maximizes worker utility the number of applications must decrease in type.

References

Albrecht, James, Pieter A Gautier, and Susan Vroman. 2006. “Equilibrium directed search with multiple applications.” *The Review of Economic Studies*, 73(4): 869–891.

Albrecht, James, Serene Tan, Pieter Gautier, and Susan Vroman. 2004. “Matching with multiple applications revisited.” *Economics Letters*, 84(3): 311–314.

Albrecht, James W, Pieter A Gautier, and Susan B Vroman. 2003. “Matching with multiple applications.” *Economics Letters*, 78(1): 67–70.

Azevedo, Eduardo M, and Jacob D Leshno. 2016. “A supply and demand framework for two-sided matching markets.” *Journal of Political Economy*, 124(5): 1235–1268.

Boyd, Donald, Lankford Hamilton, Susanna Loeb, and James Wyckoff. 2005. “Explaining the short careers of high-achieving teachers in schools with low-performing students.” *American Economic Review*, 92(2): 166–171.

Burdett, Ken, and Melvyn G Coles. 1997. “Marriage and class.” *The Quarterly Journal of Economics*, 112(1): 141–168.

Burdett, Kenneth, and Melvyn G Coles. 1999. “Long-term partnership formation: Marriage and employment.” *The Economic Journal*, 109(456): 307–334.

Burdett, Kenneth, Shi Shouyong, and Randall Wright. 2001. “Pricing and matching with frictions.” *Journal of Political Economy*, 109(5): 1060–1085.

Chade, Hector. 2001. “Two-sided search and perfect segregation with fixed search costs.” *Mathematical Social Sciences*, 42(1): 31–51.

Chade, Hector, and Lones Smith. 2006. “Simultaneous search.” *Econometrica*, 74(5): 1293–1307.

Chade, Hector, Gregory Lewis, and Lones Smith. 2014. “Student portfolios and the college admissions problem.” *The Review of Economic Studies*, 81(3): 971–1002.

Galenianos, Manolis, and Philipp Kircher. 2009. “Directed search with multiple job applications.” *Journal of Economic Theory*, 144(2): 445–471.

Gautier, Pieter A, and Christian Holzner. 2017*a*. "Maximum matching in the labor market under incomplete information." Working Paper.

Gautier, Pieter A, and Christian Holzner. 2017*b*. "Simultaneous Search, maximum matching and Efficiency." *American Economic Journal: Microeconomics*, 9(3): 245–282.

Hornstein, Andreas, and Marianna Kudlyak. 2015. "Estimating Matching Efficiency with Variable Search Effort." Working Paper.

Kircher, Philipp. 2009. "Efficiency of simultaneous search." *Journal of Political Economy*, 117(5): 861–913.

Lentz, Rasmus. 2010. "Sorting by search intensity." *Journal of Economic Theory*, 145(4): 1436–1452.

Peters, Michael. 2010. "Noncontractible heterogeneity in directed search." *Econometrica*, 78(4): 1173–1200.

Petrongolo, Barbara, and Christopher A Pissarides. 2001. "Looking into the black box: A survey of the matching function." *Journal of Economic Literature*, 39(2): 390–431.

Pissarides, Christopher A. 2000. *Equilibrium Unemployment Theory, 2nd Edition.* The MIT Press.

Shimer, Robert. 2004. "Search intensity." Working Paper.

Shimer, Robert. 2005. "The assignment of workers to jobs in an economy with coordination frictions." *Journal of Political Economy*, 113(5): 996–1025.

Shi, Shouyong. 2002. "A directed search model of inequality with heterogeneous skills and skill-biased technology." *The Review of Economic Studies*, 69(2): 467–491.

Stigler, George J. 1962. "Information in the labor market." *Journal of Political Economy*, 70(5, Part 2): 94–105.

Wolthoff, Ronald P. 2018. "Applications and Interviews: Firms' Recruiting Decisions in a Frictional Labor Market." *The Review of Economic Studies*, 85(2): 1314–1351.

A Appendix

A.1 Alternative Derivation of DE

Assume a large economy, and then let $\varphi(x) \in [0, 1]$ be the deterministic probability that type x's application results in an offer. We now derive $\varphi(x)$ by examining under which conditions a particular firm is willing to hire worker x.

Fix a worker type x, and assume that there are N different segments in the type space $[x, 1]$, and all types in one segment choose identical number of applications. That is, the type space $[x, 1]$ can be segmented by N different numbers $x = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = 1$ such that all types in (x_{i-1}, x_i) send out the same number of applications. This restriction is not necessary for the proof, but does hold in any equilibrium.

We refer to the types in (x_{i-1}, x_i) as the i-th group. Denote by $k_i \in \mathbb{Z}_+$ the applications sent out by group i, and $\lambda_i = F(x_i) - F(x_{i-1})$ the mass of group i. Since the applications are uniformly randomly distributed over firms, the total number of applications received by a single firm is a random Poisson variable. For the collection $(k_i, \lambda_i)_{i=1}^N$, the Poisson arrival rate of applications from types higher than x is given by $\frac{\lambda(x)}{M}$, where

$$\lambda(x) = \sum_{i=1}^{N} k_i \lambda_i = \sum_{i=1}^{N} k_i (F(x_i) - F(x_{i-1}))$$

This object, divided by the mass of firms M, is the 'gross queue length' of competing applications from higher types for an individual job.

An application by x generates an offer from the firm when either (1) the firm has no other applications or (2) there are other applications but all competitors choose other offers. For (2), the probability that a competing application is from i-th group is $\frac{k_i \lambda_i}{\sum k_i \lambda_i}$. A competitor $z_i \in (x_{i-1}, x_i)$ randomizes equally over his offers if there are multiple. The probability that z_i picks an offer other than the one at the firm considered is given by:

$$P_i(z_i; \varphi) = \sum_{j=0}^{k_i - 1} \frac{j}{j+1} \binom{k_i - 1}{j} \varphi(z_i)^j (1 - \varphi(z_i))^{k_i - 1 - j}.$$

The term within the summation states the following. For the $k_i - 1$ applications the competitors sends out to other firms, the worker generates j offers with a probability that depends on *his* probability of success $\varphi(z_i)$. The worker chooses on of the j offers with probability $\frac{j}{j+1}$.

This is true for any $z_i \in (x_{i-1}, x_i)$, the probability that the competitor does not take the job at this specific firm conditional on there being one from group i is given by:

$$\int_{x_{i-1}}^{x_i} P_i(z;\varphi)\frac{f(z)}{\lambda_i}dz := p_i(\varphi)$$

Therefore, the overall probability that the job is not taken by a competitor from any group is:

$$J(x) = \sum_i \frac{k_i\lambda_i}{\lambda(x)}p_i(\varphi) = \sum_i \frac{k_i}{\lambda(x)}\int_{x_{i-1}}^{x_i} P_i(z;\varphi)f(z)dz$$

From the large market assumption, the probability is independent across competitors and with $n > 1$ competitors, the probability that the job is not taken is simply $J(x)^n$.

Since the number of competitors (random variable N) is a Poisson random variable with gross queue length as parameter, the expectation summarizes to:

$$\varphi(x) = \mathbb{E}_N\left[J(x)^N\right] = \sum_{n=0}^{\infty} \frac{(\frac{\lambda(x)}{M})^n e^{-\frac{\lambda(x)}{M}}}{n!}J(x)^n = e^{-\lambda(x)(1-J(x))}$$

The last equality follows from the fact that $E[a^X] = \exp(\lambda(a-1))$ for $X \sim Pois(\lambda)$ and $a > 0$.

This expression is similar to the one in Kircher (2009). The difference is that in our case, the probability of generating an offer is defined recursively as an expectation over the probability of types higher than x.

If $\varphi(x)$ is differentiable at x, taking the log and differentiating with respect to x yields:

$$\frac{\varphi'(x)}{\varphi(x)} = \frac{d}{dx}(-\frac{\lambda(x)}{M}(1 - J(x)))$$

Note that from the definition of $J(x)$,

$$\lambda(x)J(x) = \sum_{i=2}^{N} k_i \int_{x_{i-1}}^{x_i} P_i(z;\varphi)f(z)\,dz + k_1\int_x^{x_1} P_1(z;\varphi)f(z)\,dz$$

so that

$$\frac{d}{dx}(\frac{\lambda(x)}{M}J(x)) = -k_1P_1(x;\varphi)\frac{f(x)}{M}.$$

Furthermore:

$$\frac{d}{dx}\big(\lambda(x)\big) = -k_1 f(x),$$

We can rewrite:

$$\frac{\varphi'(x)}{\varphi(x)} = -k_1 \frac{f(x)}{M}(P_1(x;\varphi) - 1)$$

Taking the complement of the binomial sum:

$$\begin{aligned} P_1(x;\varphi) &= \sum_{j=0}^{k_1-1} \frac{j}{j+1}\binom{k_1-1}{k_1-1-j}\varphi(x)^j(1-\varphi(x))^{k_1-1-j} \\ &= 1 - \sum_{j=0}^{k_1-1} \frac{1}{j+1}\binom{k_1-1}{k_1-1-j}\varphi(x)^j(1-\varphi(x))^{k_1-1-j} \end{aligned}$$

Using this result and changing indices, we get the sum:

$$\begin{aligned} \varphi(x)k_1(P_1(x;\varphi) - 1) &= -\sum_{j=0}^{k_1-1} \frac{k_1}{j+1}\binom{k_1-1}{k_1-1-j}\varphi(x)^{j+1}(1-\varphi(x))^{k_1-1-j} \\ &= -\sum_{l=1}^{k_1}\binom{k_1}{l}\varphi(x)^l(1-\varphi(x))^{k_1-l} \\ &= -(1-(1-\varphi(x))^{k_1}) \end{aligned}$$

So we see that:

$$\varphi'(x) = \frac{f(x)}{M}(1-(1-\varphi(x))^{k_1}))$$

$1 - \varphi(x)$ the proportion of jobs that are already taken up by types higher than x. That is, by definition $1 - \varphi(x) = \frac{A(x)}{M}$, and we arrive at the DE defined in the main part of the paper:

$$-A'(x) = f(x)(1 - (\frac{A(x)}{M})^k).$$

A.2 Proof of Theorem 1

Consider follwing possibilities:

- $A(x) < M(1-c)$ for all x: In this case, everyone sends out 1 or more applications and A is strictly decreasing everywhere.

  Define A_l as the cutoff accumulation for which $MB_{l+1}(\frac{A_l}{M}) = \left(\frac{A_l}{M}\right)^l(1 - \frac{A_l}{M}) = c$ for $l \in \mathbb{Z}_+\backslash\{0\}$. In general, A_l has either two or zero real solutions due to single-peakedness of the marginal benefits. Denote the two solutions $(\underline{A}_l, \bar{A}_l)$. The types x_l for which $A(x_l) = \underline{A}_l$ or $A(x_l) = \bar{A}_l$ are indifferent between sending out $l+1$ applications and l

applications. A is continuous, strictly decreasing, and we assumed that types that are indifferent are choosing the smaller number of application. Therefore, the set of types who send out l or more applications

$$X_l = \{x \in [0,1] : A(x) \in (\underline{A}_l, \bar{A}_l]\}$$

is a connected interval on $[0,1]$. Denote this interval by $[x_l^{low}, x_l^{high}]$. From the decreasing marginal benefit, it is true that the sets are nested: for all $l \geq 1$

$$X_l = [x_l^{low}, x_l^{high}] \supseteq [x_{l+1}^{low}, x_{l+1}^{high}] = X_{l+1}$$

Mapping the nested set of intervals to the respective number of applications, we get a single-peaked graph.

- $A(x) = M(1-c)$ for some x: In this case, there is an interval $[0, \bar{x}]$ where $K(\frac{A(\bar{x})}{M}) = 0$. For $x > \bar{x}$, $K(\frac{A(x)}{M}) > 0$ and $A'(x) < 0$; this implies that the above observation applies and the equilibrium applications is single-peaked. For $x < \bar{x}$, the applications is 0.

A.3 Proof of Normalization

We change variables and define $\tilde{x} = F(x)$, and $F^{-1}(\tilde{x}) = x$. Assuming F is continuous, differentiable and one-to-one, its inverse is well-defined and the equilibrium conditions can be rewritten as

$$-A'(F^{-1}(\tilde{x})) = F'(F^{-1}(\tilde{x}))(1 - \Big(\frac{A(F^{-1}(\tilde{x}))}{M}\Big)^{k(F^{-1}(\tilde{x}))}),$$

$$k(F^{-1}(\tilde{x})) = K(A(F^{-1}(\tilde{x})))$$

$F'(F^{-1}(\tilde{x})) = \frac{1}{\frac{dF^{-1}(\tilde{x})}{d\tilde{x}}}$, so that

$$-A'(F^{-1}(\tilde{x}))\frac{dF^{-1}(\tilde{x})}{d\tilde{x}} = 1 - \Big(\frac{A(F^{-1}(\tilde{x}))}{M}\Big)^{k(F^{-1}(\tilde{x}))}.$$

Let $A(F^{-1}(\tilde{x})) = \tilde{A}(\tilde{x})$, and $k(F^{-1}(\tilde{x})) = \tilde{K}(\tilde{x})$. Then:

$$-\tilde{A}'(\tilde{x}) = 1 - \Big(\frac{\tilde{A}(\tilde{x})}{M}\Big)^{\tilde{K}(\tilde{x})},$$

$$\tilde{K}(\tilde{x}) \in K(\frac{\tilde{A}(\tilde{x})}{M})$$

A.4 Proof for Proposition 5

We show that $\frac{1-F_1(x)}{M_2}$ FOSD $\frac{1-F_1(x)}{M_1}$ implies that $\frac{A_1(x)}{M_1} \leq \frac{A_2(x)}{M_2}$ for all x:

$$\frac{A_1(x)}{M_1} \leq \frac{A_2(x)}{M_2} \iff \int_x^1 -\frac{A_1'(y)}{M_1}\,dy \leq \int_x^1 -\frac{A_2'(y)}{M_2}\,dy$$

$$\iff \int_x^1 (1 - \Big(\frac{A_1(y)}{M_1}\Big)^{k_1(y)})\frac{f_1(y)}{M_1}\,dy \leq \int_x^1 (1 - \Big(\frac{A_2(y)}{M_2}\Big)^{k_2(y)})\frac{f_2(y)}{M_2}\,dy$$

$$\iff \int_0^{\frac{1-F_1(x)}{M_1}} (1 - \tilde{A}(z)^{\tilde{K}(z)})dz \leq \int_0^{\frac{1-F_2(x)}{M_2}} (1 - \tilde{A}(z)^{\tilde{K}(z)})dz$$

Last line by the change of variables $z = \frac{1-F_i(x)}{M_i}$.

It is then straightforward to show that first order stochastic dominance in $\frac{A(x)}{M}$ implies higher expected quality. Since the endpoints are the same for all equilibria: $A_1(1) = A_2(1) = 0$, by the fundamental theorem of calculus:

$$\frac{A_1(x)}{M_1} \leq \frac{A_2(x)}{M_2} \iff 0 \leq \int_x^1 -\frac{A_1'(y)}{M_1}dy \leq \int_x^1 -\frac{A_2'(y)}{M_2}dy$$

Applying integration by parts, for all x, we obtain that

$$\begin{aligned}\int_x^1 -\frac{A_1'(y)}{M_1}y\,dy &= \frac{A_1(x)}{M_1}x + \int_x^1 \frac{A_1(y)}{M_1}\,dy \\ &\leq \frac{A_2(x)}{M_2}x + \int_x^1 \frac{A_2(y)}{M_2}\,dy \\ &= \int_x^1 -\frac{A_2'(y)}{M_2}y\,dy\end{aligned}$$

A.5 Proof for Section 6

First of all, it is without loss of generality to focus on piecewise continuous k, since any measure zero change of k's does not affect the objective. Secondly, we can also rule out any nonzero gap in the support of $k(x)$, where $k(x)$ takes a nonzero value, because one can find a k with connected support that obtains the same value. Formally:

Lemma 1. For $n > m > 0$ number of applications, assume that in an equilibrium, there is a lower type applies less than a higher type, i.e., there are types $x' > x$ such that $k(x') = n$ and $k(x) = m$. Then the planner can increase the total employment without affecting the cost.

Proof. There exists small interval $\Delta > 0$ and threshold x such that k takes value $n > m > 0$ for

$z \in (x, x+\Delta]$ and $m > 0$ for $z \in [x-\Delta, x)$. For any value $A = A(x+\Delta)$, the accumulation up until type $x+\Delta$, the choice of k implies

$$A(x-\Delta) = \int_{x-\Delta}^{x} 1 - \Big(\frac{A(z)}{M}\Big)^m dz + \int_{x}^{x+\Delta} 1 - \Big(\frac{A(z)}{M}\Big)^n dz + A(x+\Delta)$$

Now consider an application schedule of the planner k^* which differs from k only for types in $[x-\Delta, x+\Delta]$. In particular, $k(z) = m$ for $z \in (x, x+\Delta]$ and $k(z) = n$ for $z \in [x-\Delta, x)$. Since $k = k^*$ over $[x+\Delta, 1]$, the accumulations A and A^*, are also equivalent for these types, however,

$$A^*(x-\Delta) = \int_{x-\Delta}^{x} 1 - \Big(\frac{A(z)}{M}\Big)^n dz + \int_{x}^{x+\Delta} 1 - \Big(\frac{A(z)}{M}\Big)^m dz + A(x+\Delta)$$

Let $A = A(x+\Delta)$. For small Δ, the integrals are approximated by:

$$\Big(1 - \Big(\frac{A + (1-(\frac{A}{M})^n)\Delta}{M}\Big)^m\Big)\Delta + \Big(1 - \Big(\frac{A}{M}\Big)^n\Big)\Delta + A,$$

$$\Big(1 - \Big(\frac{A + (1-(\frac{A}{M})^m)\Delta}{M}\Big)^n\Big)\Delta + \Big(1 - \Big(\frac{A}{M}\Big)^m\Big)\Delta + A$$

We show that we can find small Δ around x such that the second term is greater than the first. If so, then by switching the numbers of application for these Δ interval of types, the planner achieves higher total employment without affecting the cost. Expanding the two sums of polynomials, $1 - (\frac{A}{M})^m + 1 - (\frac{A}{M})^n$ term is common for both sums. Cancelling them out, we have to compare:

$$-\Big(\frac{A + (1-(\frac{A}{M})^n)\Delta}{M}\Big)^m + \Big(\frac{A}{M}\Big)^m \tag{a}$$

$$-\Big(\frac{A + (1-(\frac{A}{M})^m)\Delta}{M}\Big)^n + \Big(\frac{A}{M}\Big)^n \tag{b}$$

We can compare the coefficients of the same order of $\frac{\Delta}{M}$. Expansion yields that the k-th order terms ($k \leq m < n$) are

$$-\binom{m}{k}\Big(\frac{A}{M}\Big)^{m-k}\Big(1 - \Big(\frac{A}{M}\Big)^n\Big)^k\Big(\frac{\Delta}{M}\Big)^k \tag{a-k}$$

$$-\binom{n}{k}\Big(\frac{A}{M}\Big)^{n-k}\Big(1 - \Big(\frac{A}{M}\Big)^m\Big)^k\Big(\frac{\Delta}{M}\Big)^k \tag{b-k}$$

respectively. Division yields their ratio:

$$\frac{m(m-1)\dots(m-k+1)\Big(\frac{A}{M}\Big)^{m-1}}{n(n-1)\dots(n-k+1)\Big(\frac{A}{M}\Big)^{n-1}}\Big(\frac{1-(\frac{A}{M})^n}{1-(\frac{A}{M})^m}\Big)^k$$

Collecting terms, this ratio can be written as:

$$\underbrace{\frac{\frac{m(\frac{A}{M})^{m-1}}{1-(\frac{A}{M})^m}}{\frac{n(\frac{A}{M})^{n-1}}{1-(\frac{A}{M})^n}}}_{(1)}\underbrace{\Big(\prod_{z=1}^{k-1}\frac{m-z}{n-z}\Big)\Big(\frac{1-(\frac{A}{M})^n}{1-(\frac{A}{M})^m}\Big)^{k-1}}_{(2)}$$

Note that for the first order term ($k = 1$), the part (2) vanishes. The following claim shows that (1) is greater than 1, from which follows that the magnitude of the first order effect in (a) is greater than the effect in (b).

Claim 1. For $A < M$, part (1) is greater than 1.

Proof of claim. For simplicity, denote $\frac{A}{M}$ by $x \in [0, 1]$. Since $n > m$, it is true that $\frac{mx^{m-1}}{nx^{n-1}} = \frac{m}{n}\frac{1}{x^{n-m}}$ is strictly decreasing in x. Hence, for all $y > x$,

$$mx^{m-1}ny^{n-1} > nx^{n-1}my^{m-1},$$

Integrating both sides from x to 1:

$$\frac{mx^{m-1}}{1-x^m} > \frac{nx^{n-1}}{1-x^n}. \tag{A}$$

Note that this is true for any $x < 1$, and in the limit as x goes to 1, the two expressions are equal. □

Therefore, for $k = 1$, (a-k) is strictly smaller than (b-k). The signs of the differences of higher order terms are ambiguous, but their effects are dominated by the first order effect, for small Δ.

Formally, if we denote by $B = \frac{mx^{m-1}}{1-x^m} - \frac{nx^{n-1}}{1-x^n} > 0$, the difference in first order term, then, (b)-(a) is given by:

$$B\frac{\Delta}{M} - O((\frac{\Delta}{M})^2)$$

Furthermore, the big-O term is bounded above by (the most conservative bound) $n!\left(\frac{\Delta}{M}\right)^2$, hence, the difference is

$$B\frac{\Delta}{M} - o(\frac{\Delta}{M}).$$

In fact, this is true independent of A as long as $\frac{A}{M} < 1$. Hence, even if A varies with Δ the result holds. For small enough Δ, the expression is strictly positive and the planner's solution outperforms the market equilibrium. $\square$

For the second part of the proposition, we prove the following lemma:

Lemma 2. In the planner's solution, if some workers send out more than one application, then all workers send out at least one application.

Proof. Suppose, on the contrary, that there are some workers who send out $m \geq 2$ applications, while some other workers send out zero applications. From the previous lemma, we see that the planner's solution has to be monotone. Hence, the only possibility is that the workers increase the number of applications up until some threshold type, and all types below the threshold do not apply.

We show that the planner can improve this outcome. Denote by A the total accumulation up until the cutoff type x. Then, there exists an interval $[x, x + \Delta]$ of types who send out m applications. Take a small neighborhood $\varepsilon < \Delta$ of x and let $[x, x+\varepsilon]$ workers instead send $m-1$ applications and let the $[x - \varepsilon, x]$ workers send one application. By letting $\tilde{A} = A(x + \varepsilon)$, the high types $[x, x + \varepsilon]$ lose

$$(\frac{\tilde{A}}{M})^{m-1}(1 - \frac{\tilde{A}}{M})\varepsilon.$$

The gain by the low types $[x - \varepsilon, x]$ is

$$\left(1 - \frac{\tilde{A} + (1 - (\frac{\tilde{A}}{M})^{m-1})\varepsilon}{M}\right)\varepsilon = (1 - \frac{\tilde{A}}{M})\varepsilon - (1 - \frac{\tilde{A}}{M})^{m-1}\frac{\varepsilon^2}{M}$$

The difference between them is:

$$(1 - (\tfrac{\tilde{A}}{M})^{m-1})(1 - \tfrac{\tilde{A}}{M})\varepsilon - (1 - (\tfrac{\tilde{A}}{M})^{m-1})\tfrac{1}{M}\varepsilon^2.$$

Again, as long as $\frac{\tilde{A}}{M}$ remains strictly bounded away from 1, for ε small enough, the switch increases employment. $\square$