Value-Free Reductions

David Pérez-Castrillo
Chaoran Sun

June 2020

Barcelona GSE Working Paper Series

Working Paper nº 1186

Value-Free Reductions*

David Pérez-Castrillo[†] and Chaoran Sun[‡]

June 18, 2020

Abstract

We introduce the *value-free (v-f) reductions*, which are operators that map a coalitional game played by a set of players to another "similar" game played by a subset of those players. We propose properties that v-f reductions may satisfy, we provide a theory of duality for them, and we characterize several v-f reductions (among which the value-free version of the reduced games propose by Hart and Mas-Colell, 1989, and Oishi et al., 2016). Unlike reduced games, which were introduced to characterize values in terms of consistency properties, v-f reductions are not defined in reference to values. However, a "path-independent" v-f reduction induces a value. We characterize v-f reductions that induce the Shapley value, the stand-alone value, and the Banzhaf value. Moreover, we can connect our approach to the literature on consistency because any value induced by a path-independent v-f reduction is consistent with that reduction.

Keywords: Coalitional Games, Reduced Games, Axiomatization, Consistency, Shapley Value, Duality

JEL Classification: C71

* We thank Ramon Caminal, Alfredo Contreras, Ezra Einy, Itzhak Gilboa, Roweno Heijmans, Ori Haimanko, Chenghong Luo, Inés Macho-Stadler, Andreu Mas-Colell, Jordi Massó, Abraham Neyman, David Wettstein, Eyal Winter, and seminar participants at MicroLab at Universitat Autònoma de Barcelona, ENTER Jamboree at Tilburg University and Brown bag seminar at Ben-Gurion University for helpful comments. We gratefully acknowledge financial support from Ministerio de Economía y Competitividad and Feder (PGC2018-094348-B-I00), Generalitat de Catalunya (2017SGR-711 and FI fellowship), and ICREA under the ICREA Academia programme.

[†]Universitat Autònoma de Barcelona and Barcelona GSE. Email: david.perez@uab.es

[‡]Universitat Autònoma de Barcelona. Email: chaoran.sun@barcelonagse.eu

1 Introduction

We consider environments where a set of participants can collaborate to obtain and share surplus, that is, we study coalitional games with transferable utility (TU games). In such environments, we study the consequences of removing some players from the game. In the new game faced by the remaining participants, the worth of each coalition of players is a function of the strategic possibilities of all the players in the original game.

This problem is relevant in many economic contexts. For instance, when a group of shareholders leave a company, the remaining shareholders reorganize the ownership among themselves. The process through which the shares of the leaving shareholders are acquired by the outstanding shareholders will determine the strategic environment where they will interact from then on, that is, the worth of each possible coalition in the new environment.

Thus, in this paper we look at TU games from a new perspective. We study "operators" that map a TU game played by a set N of players to another, similar but "reduced" game, played by a subset of N. We propose properties that such functions may satisfy and we use these properties to characterize several operators.

Our research question is different but related to the search for consistency properties of values for TU games.[1] Before continuing with the contribution of our paper, it is worthwhile discussing the relationship between this line of research and our approach. To that aim, we first briefly describe the consistency requirement. Consider a value for TU games, that is, a function that associates a payoff to every player in every game. Starting from a TU game with a set of players N, we can define a reduced game among the players of any $N' \subsetneq N$. The worth of a coalition in the reduced game takes into account the payoffs that the players in the coalition give, *according to the value*, to the players who are removed, that is, to the players in $N \setminus N'$. Hence, the characteristic function of the reduced game depends on the original characteristic function and the solution in question.[2] The value is consistent if a player in N' obtains the same payoff in the initial and in the reduced game.

There are several possibilities to define a reduced game depending on the way in which the removed players are compensated. In particular, Hart and Mas-Colell (1989)

[1] In this respect, the closest papers to ours are Hart and Mas-Colell (1989) and Oishi et al. (2016).
[2] See the survey on reduced games by Driessen (1991).

(HM) and Oishi et al. (2016) ($ONHF$) define two different reduced games. They use them to characterize the Shapley value as the only value that is consistent and standard for two-person games (that is, it divides the surplus equally between the two players).

In contrast to the previous literature on consistency, we study operators that reduce games without reference to any value. We refer to them as *value-free reductions* (*v-f reductions*, for short). For any TU game with a set of players N and for any $N' \subseteq N$, a v-f reduction generates a game played by N'. A simple example is the *subgame v-f reduction*, which assigns to each coalition in the reduced game the same worth as in the initial game.

Our interest lies in the analysis of the reduction processes, that is, in the v-f reductions. We propose properties that one may ask any such v-f reduction to satisfy. In this paper, we study v-f reductions that satisfy four properties. First, we request that a v-f reduction is "well defined," in the sense that the way in which players in $N \setminus N'$ are removed to arrive at a game with a set of players N' should not matter. The game played by the set N' should be the same if the players in $N \setminus N'$ have been removed one by one, all simultaneously, or in any other sequence. We call this property *path independence*. The second property is the *additivity* of the v-f reduction. Reducing two games through an additive v-f reduction and then summing the corresponding reduced games and directly reducing the sum of the games, gives the same result.

The other two properties are related to the presence of null players in the initial game. The contribution of a null player to any coalition is zero. Hence, it seems reasonable that they play no role in a v-f reduction. We require that if a player is a null player in the initial game, then he should still be a null player after a v-f reduction. We call this property the *permanent null player*. Moreover, if a null player is removed from the game, then the worth of the coalitions should not change, a property that we call the *null player out* property.

Path independence, additivity, permanent null player, and null player out do not suffice to identify a unique v-f reduction. But, by including alternative "invariance" properties, we characterize several v-f reductions. Each invariance property states how changes in the worth of coalitions of the same size affect the reduction of the game. First, we characterize the subgame v-f reduction using an axiom that requires that an increase in the worth of the grand coalition should not affect the reduction of a game, a property that we call *anti-efficiency*.

Second, we consider the four previous properties plus the invariant axiom that states that the reduced game is immune to changes in the strategic prospects of the players derived from an identical increase or decrease in all the stand-alone coalitions. That is, the axiom requires that if the strategic possibilities of the players change identically in the initial game because each stand-alone coalition, say, increases by the same amount, then this change should not affect how the game is reduced. Interestingly, these five axioms characterize a unique v-f reduction that corresponds to the *HM* v-f reduction, that is, the value-free version of the reduction method proposed by *HM*.

To continue our analysis of the properties of v-f reductions, we propose a duality theory for them. We define the dual of a v-f reduction as the v-f reduction of the dual of the game. We show that the *ONHF* v-f reduction (that is, the value-free version of the *ONHF* reduction method) is dual to that of the *HM* v-f reduction. We also show that our basic properties of path independence, additivity, permanent null player, and null player out are all self-dual properties, in the sense that they are satisfied by a v-f reduction if and only if they are satisfied by the dual of the v-f reduction. We use the duality theory to characterize the *ONHF* v-f reduction by using the invariance axiom that is dual of the one in the characterization of the *HM* v-f reduction. According to this new axiom, the reduction of a game should be immune to an identical increase or decrease in the maximum compensation that the rest of the players are willing to give to any player.

It is worth noting that path independence is a particularly interesting property. If a v-f reduction satisfies path independence, then any (initial) game can unambiguously be reduced to a game played by just one player, say player $i \in N$. We can interpret the worth of coalition $\{i\}$ (the only non-empty coalition) in this reduced game as the benefit or cost to be distributed to this player in the initial game. Repeating this process for every player in N allows us to define a value for the initial game. Thus, *a path-independent v-f reduction "induces" a value.* We show that the subgame v-f reduction induces the stand-alone value and, as one may expect, the *HM* and the *ONHF* v-f reductions induce the Shapley value. Moreover, we can connect our approach to the previous literature on consistency because any value induced by a path-independent v-f reduction is consistent with that reduction.

We use our approach to introduce and characterize other v-f reductions. First, we connect our approach to the theory of implementation. Indeed, we use the players' payoffs obtained at the Pérez-Castrillo–Wettstein bidding mechanism (a mechanism that implements the Shapley value, see Pérez-Castrillo and Wettstein, 2001) to propose a v-f reduction which is characterized by an alternative invariance axiom and also induces the Shapley value. Moreover, we can apply again our duality theory to characterize the dual of that v-f reduction. Second, we use the basic four axioms as part of the characterization of a v-f reduction which induces the Banzhaf value (Banzhaf, 1964).

Finally, we discuss the properties of *anonymity* and *linearity*. Anonymity of a v-f reduction requires that the name of the players does not matter in the reduction of the game. It has two implications: (a) the worth of the coalitions in the reduced game does not depend on the names of the players in the initial game but only on their contribution to coalitions, and (b) the v-f reduction itself depends not on the name of the removed players but only on their contribution. The notion of anonymity is unrelated to the other axioms. In fact, our basic properties do not imply anonymity. However, all the v-f reductions that we study satisfy the anonymity of the process. They also satisfy linearity, which is a more requiring property than additivity.

In addition to Hart and Mas-Colell (1989) and Oishi et al. (2016), several authors have used the consistency property to characterize values in TU games.[3] Among others, Sobolev (1975) defines a reduced game and characterizes, together with other properties, the prenucleolus. Peleg (1985) characterizes the core of cooperative games without transferable utility by proposing another reduced game. Moulin (1985) also uses a consistency property to characterize the "equal allocation of nonseparable costs."[4]

The analysis of our paper may shed light on the discussion on the use of consistency with respect to a reduced game when comparing different solutions for cooperative games. On that matter, Maschler (1990) suggests that the choice between two solution concepts that can be characterized by the same set of basic properties plus consistency relative to a reduced game (reduced games that are different for the two concepts) boils down to the examination of the reduced games. There are two strands of research related to this suggestion. The first strand is pursued by Chang and Hu (2007), who propose a criterion to "distinguish" two different solutions through two different reduced games. The second strand includes Driessen and Radzik (2003), Yanovskaya and Driessen (2001), and Yanovskaya (2004) which characterize reduced games directly. Our

[3] For an excellent introduction to the consistency principle in general, see Thomson (2011).

[4] See also Chang and Hu (2007).

approach is closer to the research in the second strand since we adopt a pure axiomatic approach.

The remainder of the paper is organized as follows. In Section 2, we recall basic concepts, including the definition of reduced games. In Section 3, we introduce our central concept of a value-free reduction, together with a list of properties that a v-f reduction may satisfy. In Section 4, we develop a duality theory for v-f reductions. In Section 5, we provide an axiomatic characterization of several v-f reductions. We also discuss the properties of anonymity and linearity. Logical independence of each property in the characterization of the *HM* v-f reduction is established in Section 6. In Section 7, we conclude the paper. All proofs are collected in the Appendix.

2 TU games, values, and reduced games

Let an infinite set $\mathcal{U}$ represent the universe of the players. We restrict attention to games where the set of players constitutes a finite subset of $\mathcal{U}$. We denote $\mathcal{P}_{\text{fin}}(\mathcal{U})$ the set of all finite subsets of $\mathcal{U}$.

For $N \in \mathcal{P}_{\text{fin}}(\mathcal{U})$, a **coalitional game with transferable utility** (abbreviated as a **TU game**) with N as the set of players is a function $v : 2^N \to \mathbb{R}$ such that $v(\varnothing) = 0$. For $S \subseteq N$, $v(S)$ represents the worth of the coalition S in the game v. The class of all TU games with N as the set of players is denoted by $\mathcal{G}^N$. Thus, the set of all finite TU games is $\bigcup_{N \in \mathcal{P}_{\text{fin}}(\mathcal{U})} \mathcal{G}^N$.

A **subgame** of $v \in \mathcal{G}^N$ is a game $v|_{2^{N'}} \in \mathcal{G}^{N'}$ for some $N' \subseteq N$, where $v|_{2^{N'}}(S) = v(S)$ for all $S \subseteq N'$. Mathematically, a subgame is nothing more than a function restricted to a subdomain.

One particular class of games that we will use is the class of **unanimity games** $u_T \in \mathcal{G}^N$, for $T \in 2^N \setminus \{\varnothing\}$. The worth of the coalition $S \subseteq N$ in the unanimity game u_T is:

$$u_T(S) = \begin{cases} 1 & \text{if } S \supseteq T; \\ 0 & \text{otherwise.} \end{cases}$$

The class of unanimity games $(u_T)_{T \in 2^N \setminus \{\varnothing\}}$ constitutes a basis for the set of all TU games $\mathcal{G}^N$ with the fixed set of players N. Among the unanimity games, $u_N \in \mathcal{G}^N$ depicts a particularly simple situation: one unit of transferable utility is generated only when the grand coalition forms.

Cooperative game theory accords particular attention to the search of appealing solution concepts and their characterization through desirable properties from the mathematics and/or economics points of view. Single-valued solutions for TU games are called **values**. A value allocates a payoff to each player in a game, for every possible game. Thus, a value φ prescribes, for each $N \in \mathcal{P}_{\text{fin}}(\mathcal{U})$, for each TU game $v \in \mathcal{G}^N$, and for each $i \in N$, a payoff $\varphi_i(v) \in \mathbb{R}$.

The most prominent value is the **Shapley value** (Shapley, 1953), which is denoted by Sh henceforth:[5]

$$Sh_i(v) = \sum_{T \subseteq N \setminus \{i\}} \frac{t!(n-t-1)!}{n!} D^i v(T),$$

for any $v \in \mathcal{G}^N$ and for any $i \in N$, where $D^i v(T) \equiv v(T \cup \{i\}) - v(T)$ denotes the marginal contribution of player i to the coalition $T \subseteq N \setminus \{i\}$.

Another solution concept which will be discussed later is the **Banzhaf value** (see Banzhaf, 1964, and Owen, 1975) which we henceforth denote by Ban:

$$Ban_i(v) = \sum_{T \subseteq N \setminus \{i\}} \frac{1}{2^{n-1}} D^i v(T).$$

We notice that, in contrast to the Shapley value, the Banzhaf value is not efficient in the sense that the sum of the outcome obtained by the players need not be $v(N)$.

Two-player TU games constitute the most simple subclass of TU games. Unsurprisingly, several solution concepts for TU games prescribe the same payoff when restricted to this simple subclass. According to this prescription, in the game $v \in \mathcal{G}^{\{i,j\}}$ each player $k \in \{i, j\}$ is assigned, on top of his stand-alone value, half of the synergy generated from the collaboration:

$$\varphi_k(v) = v(\{k\}) + \frac{1}{2}[v(\{i,j\}) - v(\{i\}) - v(\{j\})]. \tag{1}$$

This is, in particular, the prescription of the Shapley value and the Banzhaf value for two-player games. Hence, it is commonly said that a value φ is **standard for two-player games** if for each game $v \in \mathcal{G}^{\{i,j\}}$, φ satisfies equation (1).

For TU games with more than two players, solution concepts may be pinned down by imposing consistency relative to some **reduced games**. In the literature, reduced

[5] We follow the convention by using uppercase letters to denote sets of players and letting the corresponding lowercase letters represent their cardinalities. For instance, the cardinality of N, N', and T are n, n', and t.

games are always associated with a solution concept as follows. Given a value φ, a **reduction** Ψ^{φ} is a function that associates each TU game $v \in \mathcal{G}^N$ with a reduced game $\Psi^{\varphi}_{NN'}(v) \in \mathcal{G}^{N'}$ for any two finite sets of players N, N' such that $N' \subsetneq N$.[6] That is, a reduction applied on a game with a set of players N specifies how to "reduce" the game if it were to be played only by a subset N' of N. Notice that the value φ appears in this function $\Psi^{\varphi}_{NN'} : \mathcal{G}^N \to \mathcal{G}^{N'}$ as a parameter, so that different values lead to different ways of "reducing" a game in $\mathcal{G}^N$ to a game in $\mathcal{G}^{N'}$.

Now we can formulate the definition of consistency of a value relative to some reduction:

Definition 1. *The value φ is **consistent** relative to the reduction Ψ^{φ} if for all $N, N' \in \mathcal{P}_{\mathit{fin}}(\mathcal{U})$ such that $N' \subsetneq N$, for all $v \in \mathcal{G}^N$, and for all $i \in N'$,*

$$\varphi_i\big(\Psi^{\varphi}_{NN'}(v)\big) = \varphi_i(v).$$

Consistency of φ means that any player $i \in N'$ is indifferent between playing the original game v and playing the reduced game $\Psi^{\varphi}_{NN'}(v)$ according to the value φ.

We close this section with two examples of reductions: the *HM* reduction (see Hart and Mas-Colell, 1989) and the *ONHF* reduction (see Oishi et al., 2016).

Definition 2. *Given a value φ, the HM reduction $\Psi^{HM\varphi}$ is defined by:*

$$\Psi^{HM\varphi}_{NN'}(v)(S) \equiv v(S \cup (N \setminus N')) - \sum_{i \in N \setminus N'} \varphi_i(v|_{2^{S \cup (N \setminus N')}}),$$

for all $S, N', N \in \mathcal{P}_{\mathit{fin}}(\mathcal{U})$ such that $S \subseteq N' \subsetneq N$ and for all $v \in \mathcal{G}^N$.

The interpretation of the *HM* reduction is as follows. Given a value φ, consider a game $v \in \mathcal{G}^N$ that is reduced to be played by players in $N' \subsetneq N$. If a coalition $S \subseteq N'$ is formed, then the players in S collaborate with all removed players in $N \setminus N'$, which yields a worth $v(S \cup (N \setminus N'))$. However, each removed player $i \in N \setminus N'$ is entitled to $\varphi_i(v|_{2^{S \cup (N \setminus N')}})$, his "fair" share of the worth of the coalition $S \cup (N \setminus N')$. Then, the coalition S has a claim to the residual, which defines the worth of coalition S in the *HM* reduced game.

[6] We call the operator Ψ^{φ} a reduction, even though the previous literature does not define such an operator. They propose the consistency property using reduced games, which are the images of a reduction.

Hart and Mas-Colell (1989) characterize the Shapley value as the unique value that is consistent relative to the *HM* reduction $\Psi^{HM\varphi}$ and that is standard for two-player games.

Oishi et al. (2016) obtain a different characterization of the Shapley value through a reduction à la Hart and Mas-Colell by exploiting the self-duality of the Shapley value. To define the *ONHF* reduction, we first introduce the following notation: given a TU game $v \in \mathcal{G}^N$ and $S \subsetneq N$, we denote by $v^S \in \mathcal{G}^{N\setminus S}$ the game defined by:

$$v^S(T) = v(T \cup S) - v(S), \tag{2}$$

for all $T \subseteq N \setminus S$.

Definition 3. *Given a value φ, the ONHF reduction $\Psi^{ONHF\varphi}$ is defined by:*

$$\Psi^{ONHF\varphi}_{NN'}(v)(S) \equiv v(S) - \sum_{i\in N\setminus N'} \varphi_i(v) + \sum_{i\in N\setminus N'} \varphi_i(v^S),$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subsetneq N$ and for all $v \in \mathcal{G}^N$.

In contrast to the *HM* reduction, the intuition of the *ONHF* reduced game (as acknowledged by Oishi et al., 2016) is more involved. To determine the worth of a coalition $S \subseteq N'$ in an *ONHF* reduced game, we consider all the players in S together. Forming the coalition S entitles the players in the coalition to offer their joint collaboration to the rest of the players to play a new TU game $v^S \in \mathcal{G}^{N\setminus S}$. As defined above, in this new game any coalition $T \subseteq N \setminus S$ is formed with the collaboration of S and T, which yields a worth $v(T \cup S)$. The coalition S is entitled to two payments. First, it receives $v(S)$ in forming this game. Second, it makes a swap agreement with the removed players: the coalition S pays $\varphi_i(v)$ to each player $i \in N \setminus N'$, which equals the amount i deserves in the original game, and it collects the sum of what these players receive in v^S, which adds up to $\sum_{i\in N\setminus N'} \varphi_i(v^S)$. The net payoff for S after the two payments corresponds to its worth in the *ONHF* reduced game.

Oishi et al. (2016) show that the Shapley value is the only value that is consistent relative to the *ONHF* reduction $\Psi^{ONHF\varphi}$ and that is standard for two-player games.

3 Value-free reductions: Definition and axioms

The existing literature takes the values as the main object of study and considers the reduced games associated with values to characterize particular values. By contrast, our approach takes the reductions as the primitive concept, analyzes properties of the reductions, characterizes some of them through the properties, and eventually uses the reductions to derive values.

To develop our approach, we first formally introduce the concept of a value-free reduction, that is, a reduction that does not make any reference to a value.

Definition 4. *A **value-free reduction (v-f reduction** for short) Ψ is a function that associates to each finite set of players N, each TU game $v \in \mathcal{G}^N$, and each subset $N' \subseteq N$, a TU game $\Psi_{NN'}(v) \in \mathcal{G}^{N'}$.*[7]

Because of the defining feature of v-f reductions, we must forsake the superscript φ from a generic v-f reduction.

To illustrate the concept, we provide a first example of a v-f reduction. Example 1 defines Ψ^{sub}, which we call the subgame v-f reduction.[8] According to this operator, the value of any subset in the reduced game is the same as its value in the original game.

Example 1. *We define the **subgame v-f reduction** Ψ^{sub} by:*

$$\Psi^{sub}_{NN'}(v)(S) \equiv v\left|_{2^{N'}}\right.(S) = v(S),$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$ and for all $v \in \mathcal{G}^N$.

We now propose and explain some properties that v-f reductions may satisfy. We see v-f reductions as a way to remove players from a game while keeping the remaining players' strategic prospect intact. Thus, we suggest properties that may be coherent with this view.

We first introduce a minimum requirement of a well-behaved v-f reduction, the path-independence property:

Axiom 1. *A v-f reduction Ψ is **path independent** if for all $N_1, N_2, N_3 \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N_3 \subseteq N_2 \subseteq N_1$, then*

$$\Psi_{N_2N_3} \circ \Psi_{N_1N_2} = \Psi_{N_1N_3}.^{9}$$

[7] We allow for the possibility that $N' = N$ for convenience.

[8] We refer to all the examples of value-free reductions as "v-f reductions" even though the use of "v-f" is not always necessary.

[9] The symbol "$\circ$" denotes the composition of two functions: for $f : X \to Y$ and $g : Y \to Z$, $g \circ f(x) = g\big(f(x)\big) \in Z$ for all $x \in X$.

Path independence means that, for any game $v \in \mathcal{G}^N$, the way players in $N \setminus N'$ are removed to reach the v-f reduced game of v with N' as the remaining players should be irrelevant. In particular, it should not matter whether a player's removal precedes another player's or if they are removed simultaneously. The only relevant information is the set of players who remain at the end.

One important merit of a path-independent v-f reduction is that it induces unambiguous one-player v-f reduced games. That is, a game $v \in \mathcal{G}^N$ can be unambiguously reduced to n games $\Psi_{N\{i\}}(v)$, for $i \in N$. This procedure provides the possibility of identifying the value of a player i in the game v as the worth of the coalition $\{i\}$ in the v-f reduced game consisting of this player only. Hence, we can propose the following definition:

Definition 5. *The value φ^{Ψ} induced by a path-independent v-f reduction Ψ is, for $v \in \mathcal{G}^N$,*

$$\varphi^{\Psi}(v) \equiv (\Psi_{N\{i\}}(v)(\{i\}))_{i \in N}.$$

For instance, the value induced by the subgame v-f reduction (which is trivially path-independent) is the stand-alone value:

$$\varphi^{\Psi^{sub}}(v) = (\Psi^{sub}_{N\{i\}}(v)(\{i\}))_{i \in N} = v(\{i\})_{i \in N},$$

because the prescribed payoff of the value induced by the subgame v-f reduction for all $i \in N$ is $v|_{2^{\{i\}}}(\{i\}) = v(\{i\})$.

Reduced games were introduced in the literature to study the consistency of values. Then, it is natural to ask about the consistency of the value induced by a path-independent v-f reduction with respect to that reduction. Although Definition 1 refers to consistency relative to a reduced game (and not to v-f reduced games), the definition can be easily accommodated. Proposition 1 addresses the previous question.

Proposition 1. *The value φ^{Ψ} induced by a path-independent v-f reduction Ψ is consistent with respect to Ψ.*

Our second axiom on v-f reductions is additivity:

Axiom 2. *A v-f reduction Ψ is **additive** if for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subseteq N$, for all $v_1, v_2 \in \mathcal{G}^N$, then*

$$\Psi_{NN'}(v_1 + v_2) = \Psi_{NN'}(v_1) + \Psi_{NN'}(v_2).$$

To put it in words, additivity means that if game v is the sum of two games v_1 and v_2, then directly reducing v, and reducing v_1 and v_2 and then summing the corresponding reduced games, give the same result.

We will use additivity in our characterizations. Since we use the concept of linear v-f reductions later, we introduce the stronger concept of linearity here:

Axiom 3. *A v-f reduction Ψ is **linear** if for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subseteq N$, for all $v_1, v_2 \in \mathcal{G}^N$ and for all $\alpha \in \mathbb{R}$, then*

$$\Psi_{NN'}(\alpha v_1 + v_2) = \alpha \Psi_{NN'}(v_1) + \Psi_{NN'}(v_2).$$

Compared with additivity, linearity of a v-f reduction Ψ has one extra implication: the scale in which we measure the worth of the coalitions in a TU game does not influence how the game is reduced.

Our next two axioms concern the consequences of the presence of "null players" in the game, that is, players who do not contribute to any coalition, on the reduced game. Before introducing the axioms, we formally define null players.

Definition 6. *A player $i \in N$ is a **null player** in a TU game $v \in \mathcal{G}^N$ if $D^i v(S) = 0$ for all $S \subseteq N \setminus \{i\}$.*

Given that null players have no impact on the worth of any coalition, it may seem reasonable that they also have no impact on the reduction of games. Thus, we propose the following property:

Axiom 4. *A v-f reduction Ψ satisfies the **null player out** property if for all $N \in \mathcal{P}_{fin}(\mathcal{U})$, for all $i \in N$, and for all $v \in \mathcal{G}^N$ such that player i is a null player in v, then*

$$\Psi_{N(N\setminus\{i\})}(v) = v|_{2^{N\setminus\{i\}}} .$$

The null player out property means that if a null player is removed from the game, then his removal has no effect on the worth of coalitions in the game without him. The axiom reflects the idea that given that a null player has no influence on the game, the worth of any coalition should not change if the game is reduced because he is removed.

Moreover, a null player should gain no influence after a reduction:

Axiom 5. *A v-f reduction Ψ satisfies the **permanent null player** property if for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subseteq N$, for all $i \in N'$, and for all $v \in \mathcal{G}^N$ such that player i is a null player in v, then player i is also a null player in $\Psi_{NN'}(v)$.*

The interpretation of the permanent null player property is that if a player is a null player in the original game, then he is still a null player after the removal of some other arbitrary players.

In general, null player out and permanent null player properties reflect the rationale perceiving null players as irrelevant or redundant. Still, they are distinct axioms, as we will show in Section 6, where we analyze the logical independence of the axioms.

Our last set of axioms provides alternative views of how the reduction of a game is affected by changes in the worth of coalitions of the same size. Indeed, it is conventional to postulate the monotonocity principle that a player's strategic perspective should be monotonic with respect to the worth of the coalitions containing him (see, e.g., Young, 1985). In line with this principle, if we consider, for example, a symmetric game and we increase the worth of all coalitions of the same size by the same amount, then the enhancing strategic effects for the players may be entirely canceled out. This reasoning is akin to the disagreement convexity in Peters and van Damme (1991) in the context of the bargaining problem: if each player's disagreement point is increased properly, then the solution should not be changed.

Our version of addition invariance properties borrows from ideas developed by Béal et al. (2015). In our formulation, we follow the terminology used in that paper, which we introduce here:

Definition 7. *Given the set of players N, for all $k \in \mathbb{Z}_+$ such that $k \leq n$ and $\alpha \in \mathbb{R}$, the game $w_{(k,\alpha)} \in \mathcal{G}^N$ is defined as follows: for all $S \subseteq N$,*

$$w_{(k,\alpha)}(S) = \begin{cases} \alpha & \textit{if } |S| = k; \\ 0 & \textit{otherwise.} \end{cases}$$

The game $w_{(k,\alpha)}$ is a useful tool to express an identical increase or decrease in the worth of all coalitions of size k in a TU game v as the addition of v and $w_{(k,\alpha)}$.

Our first invariance axiom suggests that if the worth of every coalition of size one in the unanimity game $u_N \in \mathcal{G}^N$ is increased or decreased by the same amount, then the reduction of the game u_N should not change. A rationale for this axiom is that the reduction of the game should be the same because the relative strategic possibilities of all the players remain intact after the modification. Indeed, $u_N(\{i\})$ can be interpreted as an opportunity cost for player i to play the unanimity game u_N. An increase in his stand-alone worth, ceteris paribus, is supposed to improve his strategic prospect. But an identical increase for all players' opportunity cost for this game should not change any of their strategic positions.

Axiom 6. *A v-f reduction Ψ satisfies* ***1-addition invariance*** *if for all $\alpha \in \mathbb{R}$ and for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subsetneq N$, then*

$$\Psi_{NN'}(u_N + w_{(1,\alpha)}) = \Psi_{NN'}(u_N).$$

Our second invariance axiom proposes an alternative property, in the same spirit as the previous one. It prescribes what happens after an increase or decrease in the worth of every coalition except the grand coalition, where the change in the worth is proportional to the number of players in the coalition. The axiom requires that the change does not affect the reduction of the unanimity game.

Axiom 7. *A v-f reduction Ψ satisfies* ***proportional addition invariance*** *if for all $\alpha \in \mathbb{R}$ and for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subsetneq N$, then*

$$\Psi_{NN'}(u_N + \sum_{k=1}^{n-1} w_{(k,k\alpha)}) = \Psi_{NN'}(u_N).$$

The previous two axioms share the view that the players are bargaining over the worth of the grand coalition and that the worth of smaller coalitions are important only because they set the players' strategic possibilities, that is, their outside options. Our third invariance axiom takes the opposite view. It postulates that the players will be unable to coordinate and form the grand coalition; hence, an increase in the worth of the grand coalition should not affect the reduction of the unanimity game.

Axiom 8. *A v-f reduction Ψ satisfies* ***anti-efficiency*** *if for all $\alpha \in \mathbb{R}$ and for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subsetneq N$, then*

$$\Psi_{NN'}(u_N + w_{(n,\alpha)}) = \Psi_{NN'}(u_N).$$

Before we turn to the characterization of several v-f reductions in Section 5, we first propose a duality theory for v-f reductions in Section 4. We adapt the approach of Oishi et al. (2016). The main difference of our approach is that we take the v-f reductions as primitive, while Oishi et al. (2016) stick to the conventional view that takes the solution concepts as primitive and uses reduced games to characterize solutions through consistency properties. We use our duality theory in two characterizations of Section 5.

4 Duality theory for value-free reductions

We first remember the definition of the dual of a game and the dual of a value. For a TU game $v \in \mathcal{G}^N$, the dual of v is the game $v^* \in \mathcal{G}^N$, defined by:

$$v^*(S) \equiv v(N) - v(N \setminus S), \tag{3}$$

for all $S \subseteq N$. For a value φ, the dual φ^* of φ is defined by the value:

$$\varphi^*(v) = \varphi(v^*), \tag{4}$$

for all $v \in \mathcal{G}^N$ and for all $N \in \mathcal{P}_{fin}(\mathcal{U})$.

A value is self-dual if $\varphi = \varphi^*$. Examples of self-dual values include the Shapley value and the Banzhaf value.

We now define the dual of a v-f reduction:

Definition 8. *The* ***dual*** Ψ^* ***of a v-f reduction*** Ψ *is defined as, for any* $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ *such that* $N' \subseteq N$*, and for any* $v \in \mathcal{G}^N$*,*

$$\Psi^*_{NN'}(v) = \big(\Psi_{NN'}(v^*)\big)^*.$$

That is, consider a v-f reduction Ψ and a game v. The dual v-f reduction of v consists in first, applying Ψ to the dual of v, and then taking the dual of the reduced game.

We already know that the dual operator for TU games is reflexive because $(v^*)^* = v$. The dual operator for v-f reductions is also reflexive, that is, $(\Psi^*)^* = \Psi$.[10]

Both, the dual operator for TU games and the dual operator for v-f reductions are reflexive because $(v^*)^* = v$ and $(\Psi^*)^* = \Psi$.

If the v-f reduction is path-independent, then we can relate the concepts of duality for values and for v-f reductions. Indeed, by recognizing that a one-player TU game coincides with its dual, we obtain the result that the concept of the dual of a value is compatible with the concept of the dual of a v-f reduction:

Proposition 2. *The value induced by a path-independent v-f reduction is dual to the value induced by the dual v-f reduction:*

$$\big(\varphi^{\Psi}\big)^* = \varphi^{(\Psi^*)}. \tag{5}$$

[10] This property holds because for any $v \in \mathcal{G}^N$ and for any $N' \subseteq N$: $(\Psi^*)^*_{NN'}(v) = (\Psi^*_{NN'}(v^*))^* = ((\Psi_{NN'}((v^*)^*))^*)^* = \Psi_{NN'}(v)$, where the last equality uses twice that the dual operator for TU games is reflexive.

An immediate corollary of Proposition 2 is the following:

Corollary 1. *The value induced by a path-independent v-f reduction is self-dual if and only if it is also induced by the dual of the v-f reduction.*

We also define dual properties, or axioms, of v-f reductions.

Definition 9. *Consider two properties $\mathcal{P}$ and $\mathcal{P}^*$ regarding v-f reductions. We say that property $\mathcal{P}$* ***is dual to property*** *$\mathcal{P}^*$ if for all v-f reduction Ψ,*

$$\Psi \text{ satisfies } \mathcal{P} \iff \Psi^* \text{ satisfies } \mathcal{P}^*.$$

We say that a property is self-dual if it is satisfied by a v-f reduction if and only if it is satisfied by the dual of the v-f reduction:

Definition 10. *$\mathcal{P}$ is* ***self-dual*** *if $\mathcal{P}$ is dual to itself.*

An important result, very helpful in the characterization of v-f reductions, is that the basic axioms that we use are all self-dual, as Proposition 3 states.

Proposition 3. *The axioms of additivity, null player out, permanent null player, and path independence of v-f reductions are all self-dual properties.*

5 Characterization of several value-free reductions

In this section, we use the axioms of additivity, null player out, permanent null player, and path independence to characterize several v-f reductions. Each characterization of a v-f reduction uses an additional invariance axiom.

Before presenting our characterizations, we state an intuitive property that is common to the v-f reductions that are path independence and satisfy the axiom of null player out: a game will be unchanged after a reduction where no player is removed. We state this property in Remark 1, which we will use in the proofs of the characterizations.

Remark 1. *If a v-f reduction Ψ satisfies null player out and path independence, then for all $N \in \mathcal{P}_{fin}(\mathcal{U})$ and for all $v \in \mathcal{G}^N$,*

$$\Psi_{NN}(v) = v.$$

5.1 Characterization of the subgame value-free reduction

The subgame v-f reduction Ψ^{sub}, defined in Example 1, satisfies our four basic axioms. Moreover, it is characterized with the help of the axiom of anti-efficiency (Axiom 8), which postulates that changes in the worth of the grand coalition should not influence the way in which the unanimity game is reduced.

Theorem 1. *A v-f reduction Ψ satisfies additivity, null player out, permanent null player, path independence, and anti-efficiency if and only if:*

$$\Psi = \Psi^{sub}.$$

Given that the axiom of anti-efficiency emphasizes how difficult coordination is for players striving to achieve the worth of the grand coalition, it is reasonable that it leads to the characterization of a v-f reduction where those outside the reduced set of players have no role: the worth of any subgame coincides with that in the original game.

5.2 Characterization of the *HM* value-free reduction

Next, we study the consequences of including the axiom of 1-addition invariance. The intuition of this axiom is that an identical increase or decrease in the worth of all the one-player coalitions in a game should not affect the reduction of the game. Interestingly, 1-addition invariance together with our four basic axioms characterize the value-free version of the most popular reduced game, the *HM* reduction (see Definition 2). We call this v-f reduction the *HM* v-f reduction and we denote it Ψ^{HM}. We construct the *HM* v-f reduction by substituting $\varphi = Sh$ in $\Psi^{HM^{\varphi}}$.

Example 2. *We define the **HM v-f reduction** Ψ^{HM} by:*[11]

$$\begin{aligned}\Psi^{HM}_{NN'}(v)(S) \equiv & v(S \cup (N \setminus N')) - \sum_{i \in N \setminus N'} Sh_i(v|_{2^{S \cup (N \setminus N')}}) \\ = & \sum_{i \in S} Sh_i(v|_{2^{S \cup (N \setminus N')}}),\end{aligned}$$

for all $S, N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$ and for all $v \in \mathcal{G}^N$.

Theorem 2 states the characterization. It also stresses that, as one could expect, the *HM* v-f reduction induces the Shapley value.

[11] The second equality is implied by the efficiency of the Shapley value.

Theorem 2. *A v-f reduction Ψ satisfies additivity, null player out, permanent null player, path independence, and* 1*-addition invariance if and only if:*

$$\Psi = \Psi^{HM}.$$

Moreover, Ψ^{HM} induces the Shapley value.

Theorem 2 provides a characterization of Ψ^{HM} that is particularly interesting because it is based on a property (the 1-addition invariance) which seems unrelated to the definition of the reduction. On the one hand, the idea behind the reduction of a game $v \in \mathcal{G}^N$ to $\Psi^{HM}_{NN'}(v)$ is that the worth of a coalition $S \subseteq N'$ in $\Psi^{HM}_{NN'}(v)$ is computed taking into account that the players in S profit from the collaboration with all removed players in $N \setminus N'$, who are entitled to a compensation of $Sh_i(v|_{2^{S\cup(N\setminus N')}})$. On the other hand, the notion of 1-addition invariance concerns the effect of identical changes in the worth of the one-player coalitions.[12] Therefore, Theorem 2 highlights that a characteristic property of the *HM* v-f reduction is that it is immune to changes in the strategic prospects of the players derived from the changes in their stand-alone worth, as long as the changes are identical for every player.

5.3 Characterization of the *ONHF* value-free reduction

In the previous subsection, we define the value-free version of the *HM* reduction. We can use the same method to define the value-free version of the *ONHF* reduction, Ψ^{ONHF}, which we will refer to as the *ONHF* v-f reduction:

Example 3. *We define the* ***ONHF v-f reduction*** Ψ^{ONHF} *by:*[13]

$$\begin{aligned}\Psi^{ONHF}_{NN'}(v)(S) \equiv& v(S) - \sum_{i\in N\setminus N'} Sh_i(v) + \sum_{i\in N\setminus N'} Sh_i(v^S) \\ =& \sum_{i\in N'} Sh_i(v) - \sum_{i\in N'\setminus S} Sh_i(v^S),\end{aligned} \tag{6}$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$, and for all $v \in \mathcal{G}^N$.

[12] 1-addition invariance together with additivity imply that $\Psi_{NN'}(v + w_{(1,\alpha)}) = \Psi_{NN'}(v)$ for any $v \in \mathcal{G}^N$.

[13] The two expressions for Ψ^{ONHF} are equivalent because $\sum_{i\in N\setminus N'} Sh_i(v) = v(N) - \sum_{i\in N'} Sh_i(v)]$, $\sum_{i\in N\setminus N'} Sh_i(v^S) = v^S(N \setminus S) - \sum_{i\in N'\setminus S} Sh_i(v^S)]$, and $v^S(N \setminus S) = v(N) - v(S)$.

The *ONHF* reduced game is dual to the *HM* reduced game. Hence, it is no surprise that Ψ^{ONHF} is the dual v-f reduction of Ψ^{HM}. Indeed,[14]

$$\begin{aligned}
\left(\Psi^{ONHF}_{NN'}(v^*)\right)^*(S) =& \Psi^{ONHF}_{NN'}(v^*)(N') - \Psi^{ONHF}_{NN'}(v^*)(N' \setminus S) \\
=& \sum_{i \in N'} Sh_i(v^*) - \sum_{i \in N' \setminus N'} Sh_i(v^{*N'}) - \sum_{i \in N'} Sh_i(v^*) + \sum_{i \in N' \setminus (N' \setminus S)} Sh_i(v^{*N' \setminus S}) \\
=& \sum_{i \in S} Sh_i(v^{*N' \setminus S}) = \sum_{i \in S} Sh_i(v\,|_{2^{S \cup (N \setminus N')}}{}^*) = \sum_{i \in S} Sh_i(v\,|_{2^{S \cup (N \setminus N')}}).
\end{aligned}$$

As we proved in Section 4, additivity, null player out, permanent null player, and path independence are all self-dual properties. Given that they are satisfied by Ψ^{HM}, Ψ^{ONHF} also satisfies these axioms. On the other hand, the property of 1-addition invariance, which is the additional axiom that characterizes Ψ^{HM}, is not self-dual.

Proposition 4 states that the dual property of the 1-addition invariance is the $(n-1)$-addition invariance axiom, defined as follows:

Axiom 9. *A v-f reduction Ψ satisfies **(n-1)-addition invariance** if for all $\alpha \in \mathbb{R}$, for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subsetneq N$, and for all $v \in \mathcal{G}^N$,*

$$\Psi_{NN'}(u_N + w_{(n-1,\alpha)}) = \Psi_{NN'}(u_N).$$

Proposition 4. *The dual of the* 1*-addition invariance axiom is the* $(n-1)$*-addition invariance axiom.*

In conjunction with the interpretation of 1-addition invariance property provided in the previous section, there is a dual interpretation of the $(n-1)$-addition invariance property. For any player $i \in N$, the difference $v(N) - v(N \setminus \{i\})$ can be interpreted as the maximum amount of compensation the rest of the players are willing to pay for player i's participation. Then, $(n-1)$-addition invariance means that if each player's maximum compensation is changed by the same amount in u_N, their strategic prospects stay unchanged.

[14] The first equality follows the definition of a dual game, the second one from the defining equation (6) of Ψ^{ONHF}, and the last equality from the self-duality of the Shapley value. To check the fourth equality notice that, for all $T \subseteq S \cup (N \setminus N')$, on the one hand, $v^{*N' \setminus S}(T) = v^*(T \cup (N' \setminus S)) - v^*(N' \setminus S) = [v(N) - v(N \setminus (T \cup (N' \setminus S)))] - [v(N) - v(N \setminus (N' \setminus S))] = v(N \setminus (N' \setminus S)) - v(N \setminus (T \cup (N' \setminus S))) = v(S \cup (N \setminus N')) - v((S \cup (N \setminus N')) \setminus T)$; on the other hand, $(v\,|_{2^{S \cup (N \setminus N')}})^*(T) = v\,|_{2^{S \cup (N \setminus N')}}\,(S \cup (N \setminus N')) - v\,|_{2^{S \cup (N \setminus N')}}\,((S \cup (N \setminus N')) \setminus T) = v(S \cup (N \setminus N')) - v((S \cup (N \setminus N')) \setminus T)$. Thus $v^{*N' \setminus S} = (v\,|_{2^{S \cup (N \setminus N')}})^*$.

Theorem 3 provides our characterization of Ψ^{ONHF}. It can be thought of as a dual theorem to Theorem 2 as it gives a characterization of the dual of Ψ^{HM} through the dual properties of the axioms used in Theorem 2. The theorem also states that the Ψ^{ONHF} v-f reduction induces the Shapley value.

Theorem 3. *A v-f reduction Ψ satisfies additivity, null player out, permanent null player, path independence, and $(n-1)$-addition invariance if and only if*

$$\Psi = \Psi^{ONHF}.$$

Moreover, Ψ^{ONHF} induces the Shapley value.

Theorems 2 and 3 together reveal a distinctive difference between Ψ^{HM} and its dual, Ψ^{ONHF}. Whereas the *HM* v-f reduction postulates that the strategic prospects of the agents should not change after an identical modification in the worth of every stand-alone coalition (the 1-addition invariance property), the *ONHF* v-f reduction considers that the players' strategic prospects should not change after an identical modification in each player's maximum compensation (the $(n-1)$-addition invariance property).

5.4 Value-free reductions inspired by the bidding mechanism

In the previous subsections, we characterized v-f reductions that have some relationship with existing reduced games. In the current subsection, we propose and characterize two new v-f reductions. They link our approach to the theory of implementation. Indeed, the first v-f reduction is based on the out-of-equilibrium payoffs obtained at the Pérez-Castrillo–Wettstein (PW) bidding mechanism (see, Pérez-Castrillo and Wettstein, 2001), which implements the Shapley value. The second v-f reduction is the dual of the first. Thus, we start by explaining the bidding mechanism, and its equilibrium.

In the PW bidding mechanism, each player $j \in N$ in a game $v \in \mathcal{G}^N$ makes a bid $b_i^j \in \mathbb{R}$ to each player $i \in N \setminus \{j\}$. The player with the highest total net bid (the difference between a player's total bid to the others minus the sum of the bids the others make to him) is chosen as the proposer. He pays the bids to the rest of the players and makes them an offer to join him. If the proposal is rejected, then the proposer is removed from the game and the rest of the players keep the bids and play the same game again.

At the subgame perfect equilibrium of the bidding mechanism, any player $j \in N$ bids $b_i^j = Sh_i(v) - Sh_i(v\,|_{2^{N\setminus\{j\}}})$ to each player $i \in N \setminus \{j\}$ and the proposer (let's denote him by α) makes an offer that is accepted. The offer submitted to the players in $N \setminus \{\alpha\}$ makes them indifferent between accepting the offer and playing the new game among them (because this is the continuation outcome of the mechanism in case of rejection). That is, the offer to each player is the payoff that this player would obtain in the "reduced game" where the set of players is $N \setminus \{\alpha\}$. In this reduced game, the assets of any coalition $S \subseteq N \setminus \{\alpha\}$ are composed by two elements: the worth of the coalition and the sum of the bids that the players in S collect from α, that is, $v(S) + \sum_{i\in S} b_i^\alpha = v(S) + \sum_{i\in S}(Sh_i(v) - Sh_i(v\,|_{2^{N\setminus\{\alpha\}}})$.

If we continue deleting players, we obtain the extension of the previous formulae for the reduced game played by any $N' \subsetneq N$ (which corresponds to a situation where the players in $N \setminus N'$ were proposers in the bidding mechanism and their proposals were rejected). This way, we define the following v-f reduction:

Example 4. *We define the* ***PW v-f reduction*** Ψ^{PW} *by:*

$$\Psi^{PW}_{NN'}(v)(S) \equiv v(S) - \sum_{i\in S} Sh_i(v\,|_{2^{N'}}) + \sum_{i\in S} Sh_i(v), \tag{7}$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ *such that* $S \subseteq N' \subseteq N$ *and for all* $v \in \mathcal{G}^N$.

Theorem 4 shows that Ψ^{PW} is characterized in a similar way to theorems 1, 2, and 3. It uses the alternative property of proportional addition invariance, which we have described in Section 3 (see Axiom 7).

Theorem 4. *A v-f reduction* Ψ *satisfies additivity, null player out, permanent null player, path independence, and proportional addition invariance if and only if*

$$\Psi = \Psi^{PW}.$$

Moreover, Ψ^{PW} *induces the Shapley value.*

Theorem 4 also identifies the value $\varphi^{\Psi^{PW}}$ induced by the path-independent v-f reduction Ψ^{PW}. Given that the PW bidding mechanism implements the Shapley value, it is unsurprising that the value induced by the reduction is also the Shapley value. On the other hand, nothing in the bidding mechanism suggests that the equilibrium bids are related to the size of the coalitions. Therefore, the characterization of the PW

v-f reduction thanks to the axiom of proportional addition invariance provides a new perspective on the out-of-equilibrium payoffs of the players in the bidding mechanism.

We now use the duality theory developed in the previous section to provide and characterize another v-f reduction, the dual of Ψ^{PW}, which we denote Ψ^{PW^*}. To that end, we first identify the dual of the proportional addition invariance (since the other axioms used in the characterization of Theorem 4 are self-dual). The proportional addition invariance prescribes that a change in the worth of every coalition (except for the grand coalition) that is proportional to the size of the coalition, should not affect the strategic possibilities of the players, hence it should also not affect the reduction of the unanimity game. The reverse-proportional additional invariance axiom proposes that the reduction should not be affected if the worth of every coalition is changed in reverse proportion to their size.

Axiom 10. *A v-f reduction Ψ satisfies* ***reverse-proportional addition invariance*** *if for all $\alpha \in \mathbb{R}$ and for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subsetneq N$, then*

$$\Psi_{NN'}(u_N + \sum_{k=1}^{n-1} w_{(k,(n-k)\alpha)}) = \Psi_{NN'}(u_N).$$

Proposition 5. *The dual of the proportional addition invariance axiom is the reverse-proportional addition invariance axiom.*

Theorem 5 provides the characterization of Ψ^{PW^*}, which we formally define in Example 5:[15]

Example 5. *We define the* ***PW* v-f reduction*** *Ψ^{PW^*} by:*

$$\Psi^{PW^*}_{NN'}(v)(S) \equiv v(S \cup (N \setminus N')) - v(N \setminus N') - \sum_{i \in S} Sh_i(v^{N \setminus N'}) + \sum_{i \in S} Sh_i(v), \qquad (8)$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$ and for all $v \in \mathcal{G}^N$.

Theorem 5. *A v-f reduction Ψ satisfies additivity, null player out, permanent null player, path independence, and reverse-proportional addition invariance if and only if*

$$\Psi = \Psi^{PW^*}.$$

Moreover, Ψ^{PW^*} induces the Shapley value.

[15]See the Appendix for the derivation of the expression for Ψ^{PW^*}.

Taken together, theorems 2 to 5 provide additional evidence that the Shapley value is a solution concept with strong properties. Indeed, it is induced by v-f reductions that are characterized by very diverse invariance properties. We can use an operator that reduces a game so as to keep the same players' strategic possibilities after an identical change in the worth of all the one-player coalitions or of all maximum possible compensations; or after a change that is proportional to the number of players in any subcoalition, or that is reverse to the number of players in any subcoalition. The Shapley value is attained after any of those different reductions.

5.5 A value-free reduction inducing the Banzhaf value

The objective of this subsection is to illustrate how to use our approach to characterize v-f reductions that induce solution concepts different from the Shapley value, or the stand-alone value. In particular, we propose a v-f reduction that induces the Banzhaf value, which we introduced in Section 2.

Dragan (1996) proposes a reduced game which is implicitly defined by a functional equation to axiomatize the Banzhaf value.[16] In contrast, we propose a v-f reduction that is based on the same basic axioms used in our previous characterizations, to which we add a new axiom that we call the "maximum ignorance" property:

Axiom 11. *A v-f reduction Ψ satisfies* ***maximum ignorance*** *if for all $N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $|N| \geq 2$, for all $i \in N$, for all $\alpha \in \mathbb{R}$ and for all $S \subseteq N \setminus \{i\}$,*

$$\Psi_{N(N\setminus\{i\})}(\alpha u_N)(S) = \frac{\alpha}{2} u_N(S \cup \{i\}).$$

The maximum ignorance property takes the view that when player i is removed from the scene, he is still able to exert influence on the rest of the players, but his influence is uncertain. The resulting reduced game is a game of the remaining players contingent on the removed players' behavior. However, unlike for instance the *HM* v-f reduction, the model analyst is totally ignorant of the removed players' behavior. So the

[16]The reduced game Ψ^{φ} proposed by Dragan (1996) is implicitly defined as follows: for $S, N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$,

$$\sum_{i \in S} Ban_i(\Psi^{\varphi}_{NN'}(v)\Big|_{2^S}) = \sum_{i \in S \cup (N \setminus N')} Ban_i(v|_{2^{S \cup (N\setminus N')}}) - \sum_{i \in N \setminus N'} \varphi_i(v|_{2^{S\cup(N\setminus N')}}). \qquad (9)$$

predicted distribution should be the one with the maximum entropy, which is, player i independently chooses to join or leave with equal probability (for an introduction to the principle of maximum entropy, see e.g., chapter 11 of Jaynes, 2003). Then $\Psi_{NN'}(u_N)$ can be interpreted as the resulting expected game.

The reduction that we propose is given in the next example. We call it the Banzhaf v-f reduction.

Example 6. *We define the* ***Banzhaf v-f reduction*** Ψ^{Ban} *by:*

$$\Psi^{Ban}_{NN'}(v)(S) \equiv \sum_{T\subseteq N\setminus N'} \frac{1}{2^{n-n'}}[v(S\cup T) - v(T)], \tag{10}$$

for all $N, N', S \in \mathcal{P}_{fin}(\mathcal{U})$ *such that* $S \subseteq N' \subseteq N$ *and for all* $v \in \mathcal{G}^N$*.*

We can interpret the Banzhaf v-f reduction as follows. Consider a game $v \in \mathcal{G}^N$ that is reduced to be played by players in $N' \subseteq N$. The players in the coalition $S \subseteq N'$ can collaborate with any subset T of the set of removed players $N \setminus N'$. Then, they obtain a worth of $v(S \cup T)$ but they have to compensate the players in T with the worth of their coalition $v(T)$. Each of the possible coalitions $T \subseteq N \setminus N'$ has the same probability of being available. Therefore, the worth of a coalition $S \subseteq N'$ in $\Psi^{Ban}_{NN'}$ is the simple average of the marginal worth that S can add to the worth of the coalitions $T \subseteq N \setminus N'$.

Theorem 6 provides an axiomatic characterization of Ψ^{Ban}. It also postulates that $\varphi^{\Psi^{Ban}} = Ban$.

Theorem 6. *A v-f reduction* Ψ *satisfies additivity, null player out, permanent null player, path independence, and the maximum ignorance property if and only if*

$$\Psi = \Psi^{Ban}.$$

Moreover, Ψ^{Ban} *induces the Banzhaf value.*

5.6 The axioms of anonymity and linearity

In this section, we discuss two additional properties that v-f reductions can satisfy: anonymity and linearity.

One sensible property that many values satisfy is anonymity, which requires that the players' names are irrelevant for the value they obtain in the game. We can propose an axiom for v-f reductions in the same spirit. The axiom of anonymity for v-f reductions requires that the name of the players does not matter in the reduction of the game. To formally define the axiom, let $\sigma : N \to \mathcal{U}$ be an injection. For $v \in \mathcal{G}^N$, we define $\sigma v \in \mathcal{G}^{\sigma[N]}$ by $\sigma v(T) \equiv v(\sigma^{-1}(T))$ for all $T \subseteq \sigma[N]$.

Axiom 12. *A v-f reduction Ψ satisfies* ***anonymity*** *if for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$, for all $v \in \mathcal{G}^N$, and for all injections $\sigma : N \to \mathcal{U}$, then*

$$\Psi_{\sigma[N]\sigma[N']}(\sigma v)(\sigma[S]) = \Psi_{NN'}(v)(S). \tag{11}$$

Anonymity of a v-f reduction implies that the contribution of a player in the reduced game depends not on his name but on his contributions in the initial game. It also implies that if two players in the initial game are identical in terms of their contribution, then the reduced game if one of them is removed should be the same if the other is removed.

We notice that although anonymity refers to the way games are reduced according to v-f reductions, it has implications for the prescribed payoff that equal players obtain in the induced value. In fact, if we substitute both N' and S with $\{i\}$ in Axiom 12, we have $\Psi_{N\{i\}}(v)(\{i\}) = \Psi_{\sigma[N]\{\sigma(i)\}}(\sigma v)(\{\sigma(i)\})$, which is, $\varphi_i^{\Psi}(v) = \varphi_{\sigma(i)}^{\Psi}(\sigma v)$. Therefore, anonymity of a v-f reduction Ψ implies anonymity of its induced value φ^{Ψ}. We state this result in Proposition 6.

Proposition 6. *If a v-f reduction Ψ satisfies anonymity, then the induced value φ^{Ψ} satisfies anonymity as well.*

None of the axioms used in the characterizations provided in theorems 1 to 6 is related to the idea of anonymity. However, Proposition 7, whose proof is immediate, shows that all of the v-f reductions characterized in our paper satisfy the axiom of anonymity.

Proposition 7. *The v-f reductions Ψ^{Sub}, Ψ^{HM}, Ψ^{ONHF}, Ψ^{PW}, Ψ^{PW^*}, and Ψ^{Ban} satisfy anonymity.*

Given that all the characterizations use the axioms of additivity, null player out, permanent null player, and path independence, one may think that these axioms imply anonymity. Moreover, like the previous axioms, we can easily check that anonymity is a self-dual property. However, Example 7 satisfies our four basic properties although it is not anonymous.

Example 7. *Given $X \subseteq \mathcal{U}$, the v-f reduction Ψ^X is defined by*

$$\Psi^X_{NN'}(v)(S) = \sum_{i\in S} Sh_i(v|_{2^{S\cup((N\setminus N')\cap X)}}) - \sum_{i\in S} Sh_i(v|_{2^{N'\cup((N\setminus N')\cap X)}}) + \sum_{i\in S} Sh_i(v), \qquad (12)$$

for all $S, N', N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$, for all $v \in \mathcal{G}^N$.

Proposition 8. *The v-f reduction Ψ^X satisfies additivity, null player out, permanent nullplayer, and path independence for any $X \subseteq \mathcal{U}$. However, it does not satisfy anonymity.*

Finally, let us mention that all the v-f reductions that we have characterized also satisfy the axiom of linearity (see Axiom 3), which is stronger than that of additivity that we have used in the characterizations. Moreover, as anonymity, linearity is not implied by our four basic axioms. The construction of an example requires the use of the Hamel basis and we provide it in the Appendix.

6 Logical independence

In this section, we show that our characterization of the *HM* v-f reduction is minimal in the sense that none of the characterizing properties can be deduced from the rest. Each time we leave out one axiom, we can find examples of v-f reductions satisfying the remaining four properties.

First, as we have already shown in theorems 1, 3 and 4, the subgame v-f reduction, the *ONHF* v-f reduction and the PW v-f reduction satisfy all the axioms but 1-addition property. Examples 8, 9, 10, and 11 show that the axioms of null player out, permanent null player, additivity, and path independence are not redundant either.

Example 8 (No null player out)**.** *Let $\Psi^{\neg NPO}$ be the v-f reduction defined by:*

$$\Psi^{\neg NPO}_{NN'}(v)(S) = 0,$$

for all $N, N', S \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$ and for all $v \in \mathcal{G}^N$. The v-f reduction $\Psi^{\neg NPO}$ satisfies additivity, permanent null player, path independence and 1*-addition invariance, but it does not satisfy null player out.*

Example 9 (No permanent null player). *Let $\Psi^{\neg PNP}$ be the v-f reduction defined by:*

$$\Psi^{\neg PNP}_{NN'}(v)(S) = \begin{cases} 0 & S = \varnothing \\ v(S \cup (N \setminus N')) & \textit{otherwise}, \end{cases}$$

for all $N, N', S \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$ and for all $v \in \mathcal{G}^N$. The v-f reduction $\Psi^{\neg PNP}$ satisfies additivity, null player out, path independence and 1-addition invariance, but it does not satisfy permanent null player.

Example 10 (No additivity). *Let $\Psi^{\neg A}$ be the v-f reduction defined by:*

$$\Psi^{\neg A}_{NN'}(v) = \begin{cases} \Psi^{HM}_{NN'}(v) & \textit{if } Sh_i(v) = 0 \textit{ for all } i \in N \setminus N' \\ \Psi^{\neg NPO}_{NN'}(v) & \textit{otherwise}, \end{cases}$$

for all $N, N' \in \mathcal{P}_{fin}(\mathcal{U})$ such that $N' \subseteq N$ and for all $v \in \mathcal{G}^N$. The v-f reduction $\Psi^{\neg A}$ satisfies null player out, permanent null player, path independence and 1-addition invariance, but it does not satisfy additivity.

Example 11 (No path independence). *Let $\Psi^{\neg PI}$ be the v-f reduction defined by:*

$$\Psi^{\neg PI}_{NN'}(v)(S) = \begin{cases} 2v(S) & \textit{if } n = n' = 1 \\ \Psi^{HM}_{NN'}(v)(S) & \textit{otherwise}, \end{cases}$$

for all $N, N', S \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N' \subseteq N$ and for all $v \in \mathcal{G}^N$. The v-f reduction $\Psi^{\neg PI}$ satisfies additivity, null player out, permanent null player, and 1-addition invariance, but it does not satisfy path independence.

7 Conclusion

In this paper, we introduce the notion of the value-free reduction of a coalitional game with transferable utility. A v-f reduction of a game describes the change in the worth of the coalitions in a TU game when some players leave the game. Thus, this new concept allows us to study TU games from a different perspective, focusing on the properties that a v-f reduction may or may not satisfy. A particularly appealing property of the v-f reductions that we analyze is path independence, because a path-independent v-f reduction induces a value. One may say that the value somehow reflects the properties of the v-f reductions that induce it.

In addition to path independence, we consider v-f reductions that are additive and satisfy properties that indicate that dummy players must still be treated as dummy players when any such reduction is applied. These properties by themselves do not pin down a unique v-f reduction. Moreover, they also do not identify a unique value induced by the reductions. We define v-f reductions that satisfy all the previous properties and induce either the Shapley value, or the Banzhaf value, or the stand-alone value.

To characterize each of the examples of v-f reductions that we have defined, we use an additional axiom that ensures that the players remaining in the reduced game keep the same strategic perspective as in the original game after a change in the worth of some particular coalitions. These are invariance properties. The exercises suggest that the Shapley value is a resilient value as it is induced by several v-f reductions, each characterized by a different invariance axiom. A duality theory for v-f reductions, which is also developed in this paper, helps in the proof of some of the characterizations.

Our approach finds its root in the literature that proposes the use of reduced games to study the internal consistency of values (starting from Davis and Maschler, 1965, and Sobolev, 1975). In particular, two of the v-f reductions that we characterize are value-free versions of reduced games proposed by Hart and Mas-Colell (1989) and Oishi et al. (2016). We think that further research on v-f reductions can also contribute to this literature by characterizing values using consistency properties based on v-f reductions, instead of reduced games.

Appendix

Proof of Proposition 1. We prove that the value φ^{Ψ} induced by a path-independent v-f reduction Ψ is consistent with respect to Ψ. For a given N we have $\Psi_{N'\{i\}} \circ \Psi_{NN'} = \Psi_{N\{i\}}$ for all $N' \subseteq N$ and for all $i \in N'$, by path independence. Therefore, for any $v \in \mathcal{G}^N$, given that $\Psi_{NN'}(v) \in \mathcal{G}^{N'}$, we have $\varphi_i^{\Psi}\big(\Psi_{NN'}(v)\big) = \Psi_{N'\{i\}}\big(\Psi_{NN'}(v)\big)(\{i\}) = \Psi_{N'\{i\}} \circ \Psi_{NN'}(v)(\{i\}) = \Psi_{N\{i\}}(v)(\{i\}) = \varphi_i^{\Psi}(v)$. Hence, φ^{Ψ} is consistent with respect to Ψ. $\square$

To prove Proposition 3, as well as propositions 4 and 5 later, some properties of the mapping $v \mapsto v^*$ are useful, which we state in Lemma 1:

Lemma 1. *The mapping $v \mapsto v^*$ is additive. Moreover, if $i \in N$ is a null player in $v \in \mathcal{G}^N$, then $i \in N$ is a null player in v^*.*

Proof of Lemma 1. We check that $v \mapsto v^*$ is additive: for all $v, w \in \mathcal{G}^N$ and for all $S \subseteq N$, then $(v+w)^*(S) = (v+w)(N) - (v+w)(N \setminus S) = (v(N) + w(N)) - (v(N \setminus S) + w(N \setminus S)) = (v(N) - v(N \setminus S)) + (w(N) - w(N \setminus S)) = v^*(S) + w^*(S)$.

To see that if i is a null player in v, then i is also a null player in v^* we have that for all $S \subseteq N \setminus \{i\}$, $v^*(S \cup \{i\}) - v^*(S) = (v(N) - v(N \setminus (S \cup \{i\}))) - (v(N) - v(N \setminus S)) = v(N \setminus S) - v(N \setminus (S \cup \{i\})) = 0$. □

Proof of Proposition 3. To verify that additivity is self-dual, we show that the mapping $v \mapsto \Psi^*_{NN'}(v)(S)$ is additive if the mapping $v \mapsto \Psi_{NN'}(v)(S)$ is additive. Indeed, $\Psi^*_{NN'}(v+w)(S) = \big(\Psi_{NN'}((v+w)^*)\big)^*(S) = \big(\Psi_{NN'}(v^* + w^*)\big)^*(S) = \big(\Psi_{NN'}(v^*) + \Psi_{NN'}(w^*)\big)^*(S) = \big(\Psi_{NN'}(v^*)\big)^*(S) + \big(\Psi_{NN'}(w^*)\big)^*(S) = \Psi^*_{NN'}(v)(S) + \Psi^*_{NN'}(w)(S)$, where the first equality follows from Definition 8, the second and fourth from the additivity of $v \mapsto v^*$ (Lemma 1 in the Appendix), and the third from the additivity of Ψ. Therefore, additivity is self-dual.

We now check that null player out is self-dual. We show that if Ψ satisfies the null player out axiom, then $\Psi^*_{N(N\setminus\{i\})}(v)(S) = v(S)$ for all $S \subseteq N \setminus \{i\}$ if $i \in N$ is a null player in v. Indeed, we have $\Psi^*_{N(N\setminus\{i\})}(v)(S) = \big(\Psi_{N(N\setminus\{i\})}(v^*)\big)^*(S) = (v^* \mid_{2^{N\setminus\{i\}}})^*(S) = v^* \mid_{2^{N\setminus\{i\}}} (N \setminus \{i\}) - v^* \mid_{2^{N\setminus\{i\}}} ((N \setminus \{i\}) \setminus S) = v^*(N \setminus \{i\}) - v^*(N \setminus (S \cup \{i\})) = v^*(N) - v^*(N \setminus S) = v(S)$, where the first equality follows from Definition 8, the second one holds because i is a null player in v^* according to Lemma 1, the third from the definition of the dual of a game, and the penultimate equality follows again from the fact that i is a null player in v^*. Therefore, null player out is self-dual.

We verify that the permanent null player property is self-dual by proving that if Ψ satisfies the permanent null player property and $i \in N'$ is a null player in v, then i is a null player in $\Psi^*_{NN'}(v)$ as well. Let $i \in N'$ be a null player in v. Then, from Lemma 1, $i \in N'$ is a null player in v^* and, by the permanent null player property of Ψ, he is also a null player in $\Psi_{NN'}(v^*)$. Using Lemma 1 again, i is a null player in $\big(\Psi_{NN'}(v^*)\big)^*$, that is, in $\Psi^*_{NN'}(v)$. Therefore, the permanent null player property is self-dual.

Finally, we prove that path independence is self-dual by proving $\Psi^*_{N_2N_3}(\Psi^*_{N_1N_2}(v)) = \Psi^*_{N_1N_3}(v)$ if Ψ is path-independent: $\Psi^*_{N_2N_3}(\Psi^*_{N_1N_2}(v)) = \big(\Psi_{N_2N_3}(((\Psi_{N_1N_2}(v^*))^*)^*)\big)^* = \big(\Psi_{N_2N_3}(\Psi_{N_1N_2}(v^*))\big)^* = \big(\Psi_{N_1N_3}(v^*)\big)^* = \Psi^*_{N_1N_3}(v)$, where the first and last equalities follow from Definition 8, the second from $v^{**} = v$, and the third from the assumption of the path-independence of Ψ. Therefore, path independence is self-dual. □

Proof of Remark 1. Given $N \in \mathcal{P}_{fin}(\mathcal{U})$ and $v \in \mathcal{G}^N$, take any $i \in \mathcal{U} \setminus N$. Define $w \in \mathcal{G}^{N\cup\{i\}}$ by $w(S) \equiv v(S \setminus \{i\})$ for all $S \subseteq N \cup \{i\}$. Notice that player i is a null player in w and that the subgame of w restricted to N is v. Then for any v-f reduction Ψ satisfying null player out and path independence, $\Psi_{NN}(v) = \Psi_{NN}(\Psi_{(N\cup\{i\})N}(w)) = \Psi_{(N\cup\{i\})N}(w) = v$, where the first and the third equality follow from null player out and the second from path independence. Therefore, Ψ_{NN} must be an identity function if Ψ satisfies null player out and path independence. □

Since every v-f reduction we will present satisfies null player out and path independence, we will not repeat the property established in Remark 1 in the proof of their corresponding theorems below.

Proof of Theorem 1. It is immediate that the subgame v-f reduction satisfies all the stated properties.

We now prove that if the v-f reduction Ψ satisfies the five properties, then $\Psi = \Psi^{Sub}$. Notice first that, under path independence, it suffices to show the equality restricted to one-player operators $\left(\Psi_{N(N\setminus\{i\})}\right)$, for any $N \in \mathcal{P}_{fin}(\mathcal{U})$ and $i \in N$.

Second, by additivity, it suffices to establish the equality for each operator $\Psi_{N(N\setminus\{i\})}$ restricted to the set of all scalar multiples of elements in a basis of $\mathcal{G}^N$. We choose the set of all scalar multiples of all unanimity games $\left(\alpha u_T\right)_{T\in 2^N\setminus\{\varnothing\},\alpha\in\mathbb{R}}$.

We show that $\Psi_{N(N\setminus\{i\})}(\alpha u_T) = \Psi^{Sub}_{N(N\setminus\{i\})}(\alpha u_T)$ for all $T \in 2^N \setminus \{\varnothing\}$, for all $\alpha \in \mathbb{R}$ and for all $i \in N$ by induction on n. We notice that since $\alpha u_N = w_{(n,\alpha)}$, additivity and anti-efficiency imply that $\Psi_{N(N\setminus\{i\})}(\alpha u_N) = \mathbf{0} = \alpha u_N \mid_{2^{(N\setminus\{i\})}} = \Psi^{Sub}_{N(N\setminus\{i\})}(\alpha u_N)$, where $\mathbf{0} \in \mathcal{G}^{N\setminus\{i\}}$ is defined as $\mathbf{0}(S) = 0$ for all $S \subseteq N \setminus \{i\}$. Thus, we only need to check the equality of the remaining scalar multiples of elements in the basis, i.e., $\left(\alpha u_T\right)_{T\in 2^N\setminus\{\varnothing,N\},\alpha\in\mathbb{R}}$.

Consider $N = \{i, j\}$, that is, $|N| = 2$. (a) When $T = \{j\}$, then $\Psi_{\{i,j\}\{j\}}(\alpha u_{\{j\}})(\{j\}) = \alpha u_{\{j\}} \mid_{2^{\{j\}}} (\{j\})$ by null player out, since i is a null player in $\alpha u_{\{j\}}$. (b) When $T = \{i\}$, then $\Psi_{\{i,j\}\{j\}}(\alpha u_{\{i\}})(\{j\}) = 0 = \alpha u_{\{i\}} \mid_{2^{\{j\}}} (\{j\})$ by permanent null player, since j is a null player in $\alpha u_{\{i\}}$. Therefore, $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \alpha u_T \mid_{2^{N\setminus\{i\}}} (S) = \Psi^{Sub}_{N(N\setminus\{i\})}(\alpha u_T)(S)$ for any $S \subseteq N \setminus \{i\}$, for any T with $|T| = 1$, for any $\alpha \in \mathbb{R}$, and for any N with $|N| = 2$.

Now we proceed to consider any N, and suppose that the induction property holds for any set with fewer than n players. (a) When $i \notin T$ then i is a null player in αu_T, hence $\Psi_{N(N\setminus\{i\})}(\alpha u_T) = \alpha u_T\,|_{2^{N\setminus\{i\}}}$ by null player out. (b) We show that $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \alpha u_T\,|_{2^{N\setminus\{i\}}}\,(S)$ for all $S \subseteq N\setminus\{i\}$ when $i \in T$ and $T \subsetneq N$. Take any player $j \in N\setminus T$. Then, j is a null player in αu_T. Moreover, by the permanent null player property, j is also a null player in $\Psi_{N(N\setminus\{i\})}(\alpha u_T)$. We consider two possibilities. (b1) First, if $S \subseteq N\setminus\{i,j\}$, then

$$\begin{aligned}\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) &= \Psi_{N(N\setminus\{i\})}(\alpha u_T)\,|_{2^{N\setminus\{i,j\}}}\,(S) = \Psi_{(N\setminus\{i\})(N\setminus\{i,j\})}\big(\Psi_{N(N\setminus\{i\})}(\alpha u_T)\big)(S)\\ &= \Psi_{(N\setminus\{j\})(N\setminus\{i,j\})}\big(\Psi_{N(N\setminus\{j\})}(\alpha u_T)\big)(S) = \Psi_{(N\setminus\{j\})(N\setminus\{i,j\})}(\alpha u_T\,|_{2^{N\setminus\{j\}}})(S),\end{aligned} \tag{13}$$

where the first equality holds because $S \subseteq N\setminus\{i,j\}$; the second by null player out, given that j is a null player in $\Psi_{N(N\setminus\{i\})}(\alpha u_T)$; the third by path independence; and the fourth by null player out, given that j is a null player in αu_T. We apply the induction argument to state that the last expression (which involves a reduction from a set of $n-1$ players) is equal to $\Psi^{Sub}_{(N\setminus\{j\})(N\setminus\{i,j\})}(\alpha u_T\,|_{2^{N\setminus\{j\}}})(S) = \alpha u_T\,|_{2^{N\setminus\{i,j\}}}\,(S) = \alpha u_T(S)$, where the last equality holds because $S \subseteq N\setminus\{i,j\}$. (b2) Second, if $j \in S$, then $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \Psi_{N(N\setminus\{i\})}(\alpha u_T)(S\setminus\{j\})$ because j is a null player in $\Psi_{N(N\setminus\{i\})}(\alpha u_T)$. Now we apply equation (13) to $S\setminus\{j\}$ and, by the same argument as in (b1), $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \alpha u_T(S\setminus\{j\})$, which is equal to $\alpha u_T(S)$ since j is a null player in αu_T.

Thus, if a v-f reduction satisfies the five properties, then it is equal to Ψ^{Sub}. □

Proof of Theorem 2. We verify the stated properties of Ψ^{HM}. First, Ψ^{HM} is the composition of three functions: the restriction operator, the Shapley value, and the summation operator. It is easy to check that the three functions are additive. Therefore, Ψ^{HM} is additive.

Second, to verify that Ψ^{HM} satisfies null player out, let $i \in N$ be a null player in $v \in \mathcal{G}^N$. Then, $\Psi^{HM}_{N(N\setminus\{i\})}(v)(S) = \sum_{j\in S} Sh_j(v\,|_{2^{S\cup(N\setminus(N\setminus\{i\}))}}) = \sum_{j\in S} Sh_j(v\,|_{2^{S\cup\{i\}}}) = v(S\cup\{i\}) - Sh_i(v|_{2^{S\cup\{i\}}}) = v(S\cup\{i\}) = v(S)$, where the third equality follows from the efficiency of the Shapley value, the fourth from the null player property of the Shapley value, and the fifth holds because i is a null player in v.

Third, we check that Ψ^{HM} satisfies permanent null player. Let $i \in N'$ be a null player in $v \in \mathcal{G}^N$. Then, for all $S \subseteq N'\setminus\{i\}$, $D^i(\Psi^{HM}_{NN'}(v))(S) = \Psi^{HM}_{NN'}(v)(S\cup\{i\}) - \Psi^{HM}_{NN'}(v)(S) = [\sum_{j\in S\cup\{i\}} Sh_i(v|_{2^{(S\cup\{i\})\cup(N\setminus N')}})] - [\sum_{j\in S} Sh_i(v|_{2^{S\cup(N\setminus N')}})] = [\sum_{j\in S} Sh_i(v\,|_{2^{(S\cup\{i\})\cup(N\setminus N')}})] - [\sum_{j\in S} Sh_i(v\,|_{2^{S\cup(N\setminus N')}})] = [\sum_{j\in S} Sh_i(v\,|_{2^{S\cup(N\setminus N')}})] - [\sum_{j\in S} Sh_i(v\,|_{2^{S\cup(N\setminus N')}})] = 0,$

where the third equality follows from the null player property of the Shapley value, and the fourth from null player out of the Shapley value (see Derks and Haller, 1999).

Fourth, we prove the path independence axiom. For any $T \subseteq S$, we can write $\Psi^{HM}_{(S\cup(N\setminus N'))S}(v\,|_{2^{S\cup(N\setminus N')}})(T) = \sum_{i\in T} Sh_i(v\,|_{2^{T\cup(N\setminus N')}}) = \Psi^{HM}_{NN'}(v)(T) = \Psi^{HM}_{NN'}(v)\,|_{2^S}\,(T)$, where the first equality holds because $v\,|_{2^{S\cup(N\setminus N')}}|_{2^{T\cup(S\cup(N\setminus N'))\setminus S)}} = v\,|_{2^{T\cup((S\cup(N\setminus N'))\setminus S)}} = v\,|_{2^{T\cup(N\setminus N')}}$. Therefore:

$$\Psi^{HM}_{NN'}(v)\,|_{2^S} = \Psi^{HM}_{(S\cup(N\setminus N'))S}(v\,|_{2^{S\cup(N\setminus N')}}) \tag{14}$$

$$\Psi^{HM}_{NN'}(v)(S) = \Psi^{HM}_{(S\cup(N\setminus N'))S}(v\,|_{2^{S\cup(N\setminus N')}})(S). \tag{15}$$

We now claim that, given (14) and (15), the verification of path independence, that is, $\Psi^{HM}_{N_2N_3}\big(\Psi^{HM}_{N_1N_2}(v)\big)(S) = \Psi^{HM}_{N_1N_3}(v)(S)$ for all $N_1, N_2, N_3, S \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N_3 \subseteq N_2 \subseteq N_1$, is equivalent to verifying the condition only for $S = N_3$, i.e.,

$$\Psi^{HM}_{N_2N_3}\big(\Psi^{HM}_{N_1N_2}(v)\big)(N_3) = \Psi^{HM}_{N_1N_3}(v)(N_3). \tag{16}$$

To prove the equivalence, we use (15), where we substitute N, N' and v by N_2, N_3 and $\Psi^{HM}_{N_1N_2}(v)$, to obtain

$$\Psi^{HM}_{N_2N_3}\big(\Psi^{HM}_{N_1N_2}(v)\big)(S) = \Psi^{HM}_{(S\cup(N_2\setminus N_3))S}\big(\Psi^{HM}_{N_1N_2}(v)\,|_{2^{S\cup(N_2\setminus N_3)}}\,\big)(S). \tag{17}$$

Similarly, we substitute N, N' and S by N_1, N_2 and $S \cup (N_2 \setminus N_3)$ in (14), to obtain

$$\Psi^{HM}_{N_1N_2}(v)\,|_{2^{S\cup(N_2\setminus N_3)}} = \Psi^{HM}_{(S\cup(N_2\setminus N_3)\cup(N_1\setminus N_2))(S\cup(N_2\setminus N_3))}(v\,|_{2^{S\cup(N_2\setminus N_3)\cup(N_1\setminus N_2)}}),\ \text{i.e.,}$$

$$\Psi^{HM}_{N_1N_2}(v)\,|_{2^{S\cup(N_2\setminus N_3)}} = \Psi^{HM}_{(S\cup(N_1\setminus N_3))(S\cup(N_2\setminus N_3))}(v\,|_{2^{S\cup(N_1\setminus N_3)}}). \tag{18}$$

Using (18) in equation (17), we have

$$\Psi^{HM}_{N_2N_3}\big(\Psi^{HM}_{N_1N_2}(v)\big)(S) = \Psi^{HM}_{(S\cup(N_2\setminus N_3))S}\big(\Psi^{HM}_{(S\cup(N_1\setminus N_3))(S\cup(N_2\setminus N_3))}(v\,|_{2^{S\cup(N_1\setminus N_3)}})\big)(S).$$

Then, the worth of coalition $S \subseteq N_3$ in the game resulting from two sequential reductions of v (from N_1 to N_2, then from N_2 to N_3) is equal to the worth of the grand coalition S in the game resulting from two reductions of $v\,|_{S\cup(N_1\setminus N_3)}$ (from $S\cup(N_1\setminus N_3)$ to $S \cup (N_2 \setminus N_3)$, then from $S \cup (N_2 \setminus N_3)$ to S). This property means that it suffices to verify that the worth of the grand coalition satisfies path independence, that is, that equation (16) holds for all possible games. To prove (16), we use the definition of Ψ^{HM}:

$$\Psi^{HM}_{N_2N_3}\big(\Psi^{HM}_{N_1N_2}(v)\big)(N_3) = \sum_{i\in N_3} Sh_i(\Psi^{HM}_{N_1N_2}(v)) = \sum_{i\in N_3} Sh_i(v) = \Psi^{HM}_{N_1N_3}(v)(N_3).$$

Therefore, Ψ^{HM} is path-independent.

Finally, we verify the 1-invariance property of Ψ^{HM}. The axiom of additivity implies that $\Psi^{HM}_{NN'}(u_N + w_{(1,\alpha)}) = \Psi^{HM}_{NN'}(u_N)$ if and only if $\Psi^{HM}_{NN'}(w_{(1,\alpha)}) = 0$. We show that $\Psi^{HM}_{NN'}(w_{(1,\alpha)})(S) = 0$ for all $S \subseteq N'$. By definition of Ψ^{HM}, $\Psi^{HM}_{NN'}(w_{(1,\alpha)})(S) = \sum_{i\in S} Sh_i(w_{(1,\alpha)}|_{2^{S\cup(N\setminus N')}})$. Notice that $w_{(1,\alpha)}|_{2^{S\cup(N\setminus N')}} \in \mathcal{G}^{S\cup(N\setminus N')}$ is a game where each player is symmetric with each other. Then, the Shapley value prescribes an equal share of the worth of the grand coalition $S\cup(N\setminus N')$. Thus, $\sum_{i\in S} Sh_i(w_{(1,\alpha)}|_{2^{S\cup(N\setminus N')}}) = \sum_{i\in S} \frac{1}{|S\cup(N\setminus N')|} w_{(1,\alpha)}(S\cup(N\setminus N')) = 0$. Therefore, the HM v-f reduction satisfies the 1-addition invariance property.

To show the reverse implication of the theorem, we first prove the following lemma:

Lemma 2. *For all $N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $|N| > 2$, the set $\{u_T \mid T \subsetneq N, T \neq \varnothing\} \cup \{w_{(1,1)}\}$ forms a basis of $\mathcal{G}^N$.*

Proof of Lemma 2. Take any $N \in \mathcal{P}_{fin}(\mathcal{U})$. To prove Lemma 2, we start by showing the following equality between games in $\mathcal{G}^N$:

$$(-1)^n n u_N = -w_{(1,1)} + \sum_{S\in 2^N\setminus\{\varnothing,N\}} (-1)^{s-1} s u_S. \tag{19}$$

We show that the two functions in equation (19) are equal when evaluated at any $T \subseteq N$, by considering three different cases: (a) If $T \subsetneq N$ and $|T| = 1$, then $-w_{(1,1)}(T) + \sum_{S\in 2^N\setminus\{\varnothing,N\}}(-1)^{s-1} s u_S(T) = -1 + u_T(T) = 0 = u_N(T) = (-1)^n n u_N(T)$. For the other two cases, we use the following formula:

$$\sum_{S\in 2^T\setminus\{\varnothing\}} s(-1)^{s-1} = 0, \tag{20}$$

for any T such that $|T| > 1$.[17] Then, (b) for $T \subsetneq N$ such that $|T| > 1$, we can write: $-w_{(1,1)}(T) + \sum_{S\in 2^N\setminus\{\varnothing,N\}}(-1)^{s-1} s u_S(T) = \sum_{S\in 2^T\setminus\{\varnothing\}} s(-1)^{s-1} = 0 = u_N(T) = (-1)^n n u_N(T)$. Finally, (c) for $T = N$, $-w_{(1,1)}(T) + \sum_{S\in 2^N\setminus\{\varnothing,N\}}(-1)^{s-1} s u_S(T) = \sum_{S\in 2^N\setminus\{\varnothing,N\}}(-1)^{s-1} s = (-1)^n n + \sum_{S\in 2^N\setminus\{\varnothing\}}(-1)^{s-1} s = (-1)^n n = (-1)^n n u_N(T)$.

Given that equation (19) holds and the set $\{u_T \mid T \subseteq N, T \neq \varnothing\}$ forms a basis of $\mathcal{G}^N$, then the set resulting from replacing u_N with $w_{(1,1)}$ in this basis spans $\mathcal{G}^N$, which proves Lemma 2. $\square$

[17] We check that (20) holds: $\sum_{S\in 2^T\setminus\{\varnothing\}} s(-1)^{s-1} = \left[\sum_{S\in 2^T\setminus\{\varnothing\}} s x^{s-1}\right]_{x=-1} = \left[\sum_{S\in 2^T\setminus\{\varnothing\}} \frac{dx^s}{dx}\right]_{x=-1} = \left[\frac{d(\sum_{S\in 2^T\setminus\{\varnothing\}} x^s)}{dx}\right]_{x=-1} = \left[\frac{d(\sum_{s=1}^{t}\binom{t}{s}x^s)}{dx}\right]_{x=-1} = \left[\frac{d((1+x)^t-1)}{dx}\right]_{x=-1} = \left[t(1+x)^{t-1}\right]_{x=-1} = 0$.

We now continue with the reverse implication of Theorem 2. We prove that $\Psi = \Psi^{HM}$ if the v-f reduction Ψ satisfies the five properties, using the same procedure as in the proof of Theorem 1. Thus, using path independence and additivity, it suffices to show that $\Psi_{N(N\setminus\{i\})}(v) = \Psi^{HM}_{N(N\setminus\{i\})}(v)$ for all $i \in N$, and for all $v \in \{\alpha u_T \mid T \subsetneq N, T \neq \varnothing, \alpha \in \mathbb{R}\} \cup \{w_{(1,\alpha)} \mid \alpha \in \mathbb{R}\}$ (see Lemma 2).

First, if $v = w_{(1,\alpha)}$, then additivity and 1-addition invariance imply $\Psi_{N(N\setminus\{i\})}(w_{(1,\alpha)}) = \mathbf{0} = \Psi^{HM}_{N(N\setminus\{i\})}(w_{(1,\alpha)})$ for all $N \in \mathcal{P}_{fin}(\mathcal{U})$ for all $\alpha \in \mathbb{R}$ and for all $i \in N$.

Second, we show that $\Psi_{N(N\setminus\{i\})}(\alpha u_T) = \Psi^{HM}_{N(N\setminus\{i\})}(\alpha u_T)$ for all $T \in 2^N \setminus \{\varnothing, N\}$ for all $\alpha \in \mathbb{R}$ and for all $i \in N$ by induction on n.

For N with $|N| = 2$, the proof is identical to that of Theorem 1 since Ψ^{HM} and Ψ^{Sub} coincide for the proper subsets T of N and we did not use anti-efficiency in that part of the proof.

Consider now any N and suppose that the induction property holds for any set with fewer than n players. (a) When $i \notin T$ then i is a null player in αu_T, hence $\Psi_{N(N\setminus\{i\})}(\alpha u_T) = \alpha u_T\,|_{2^{N\setminus\{i\}}} = \Psi^{HM}_{N(N\setminus\{i\})}(\alpha u_T)$ because both Ψ and Ψ^{HM} satisfy null player out. (b) When $i \in T$ and $T \subsetneq N$, take any $j \in N \setminus T$. Player j is a null player in αu_T and, under the permanent null player property, also in $\Psi_{N(N\setminus\{i\})}(\alpha u_T)$. Therefore, (b1) if $S \subseteq N \setminus \{i, j\}$, then equation (13) holds by the same arguments as in the proof of Theorem 1. Using also the induction argument, we have $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \Psi_{(N\setminus\{j\})(N\setminus\{i,j\})}(\alpha u_T\,|_{2^{N\setminus\{j\}}})(S) = \Psi^{HM}_{(N\setminus\{j\})(N\setminus\{i,j\})}(\alpha u_T\,|_{2^{N\setminus\{j\}}})(S)$. Since j is a null player in αu_T and Ψ^{HM} satisfies null player out and path independence, we have $\Psi^{HM}_{(N\setminus\{j\})(N\setminus\{i,j\})}(\alpha u_T\,|_{2^{N\setminus\{j\}}})(S) = \Psi^{HM}_{(N\setminus\{j\})(N\setminus\{i,j\})}\big(\Psi^{HM}_{N(N\setminus\{j\})}(\alpha u_T)\big)(S) = \Psi^{HM}_{N(N\setminus\{i\})}(\alpha u_T)(S)$. (b2) If $j \in S$, then $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \Psi_{N(N\setminus\{i\})}(\alpha u_T)(S \setminus \{j\})$ because j is a null player in $\Psi_{N(N\setminus\{i\})}(\alpha u_T)$. Now we can apply equation (13) to $S \setminus \{j\}$ and, by the same argument as in (b1), $\Psi_{N(N\setminus\{i\})}(\alpha u_T)(S) = \Psi^{HM}_{N(N\setminus\{i\})}(\alpha u_T)(S\setminus\{j\}) = \Psi^{HM}_{N(N\setminus\{i\})}(\alpha u_T)(S)$, where the last equality holds because j is a null player in u_T.

Therefore, if a v-f reduction satisfies the five properties, then it is equal to Ψ^{HM}.

Finally, we notice that $\Psi^{HM}_{N\{i\}}(v)(\{i\}) = Sh_i(v|_{2^{\{i\}\cup(N\setminus\{i\})}}) = Sh_i(v)$ for all $v \in \mathcal{G}^N$ and for all $i \in N$. Therefore, Ψ^{HM} induces the Shapley value. □

Proof of Proposition 4. Let Ψ be a v-f reduction that satisfies 1-addition invariance axiom. Our aim is to verify that Ψ^* satisfies $(n-1)$-addition invariance axiom.

We first show that the dual of the game $w_{(n-1,\alpha)}$ is $w_{(1,-\alpha)}$. Indeed, $w^*_{(n-1,\alpha)}(S) = w_{(n-1,\alpha)}(N) - w_{(n-1,\alpha)}(N \setminus S) = -w_{(n-1,\alpha)}(N \setminus S) = w_{(1,-\alpha)}(S)$.

Then, for any $\alpha \in \mathbb{R}$, we have $\Psi^*_{NN'}(v + w_{(n-1,\alpha)}) = \big(\Psi_{NN'}((v + w_{(n-1,\alpha)})^*)\big)^* = \big(\Psi_{NN'}(v^* + w^*_{(n-1,\alpha)})\big)^* = \big(\Psi_{NN'}(v^* + w_{(1,-\alpha)})\big)^* = \big(\Psi_{NN'}(v^*)\big)^* = \Psi^*_{NN'}(v)$, where the first and last equalities follow from Definition 8, the second from the additivity of $v \mapsto v^*$ (see Lemma 1), the third equality follows from the fact that the dual of $w_{(n-1,\alpha)}$ is $w_{(1,-\alpha)}$, and the fourth from 1-addition invariance of Ψ. Therefore, $(n-1)$-addition invariance is dual to 1-addition invariance. □

Proof of Theorem 3. The *ONHF* v-f reduction is dual to the *HM* v-f reduction. Then, by Proposition 3, Ψ^{ONHF} satisfies additivity, null player out, permanent null player, and path independence, because they are self-dual properties and Ψ^{HM} satisfies them. Similarly, Ψ^{ONHF} satisfies $(n-1)$-invariance, which is dual to 1-addition invariance (see Proposition 4), because Ψ^{HM} satisfies 1-addition invariance.

For the other direction, consider a v-f reduction Ψ satisfying all the stated axioms. Then, the dual Ψ^* of Ψ satisfies all the axioms stated in Theorem 2, which implies $\Psi^* = \Psi^{HM}$. Hence, the dual v-f reductions of Ψ^* and Ψ^{HM}, i.e., Ψ and Ψ^{ONHF}, must coincide, as we wanted to prove.

Finally, Corollary 1 implies that Ψ^{ONHF} induces the Shapley value since it is a self-dual value. □

Proof of Theorem 4. First, we verify that Ψ^{PW} satisfies all the stated properties. It is linear and hence additive, because it is the composition of linear functions.

To show path independence, linearity ensures that it suffices to verify that unanimity games satisfy the property. Consider any $T \in 2^N \setminus \{\varnothing\}$, then $\Psi^{PW}_{NN'}(u_T)(S) = u_T(S) - \sum_{i\in S} Sh_i(u_T|_{2^{N'}}) + \sum_{i\in S} Sh_i(u_T) = u_T|_{2^{N'}}(S) - \sum_{i\in S} Sh_i(u_T|_{2^{N'}}) + \frac{|T\cap S|}{t}$. Notice that $u_T|_{2^{N'}} = 0$ if $T \not\subseteq N'$ and $\sum_{i\in S} Sh_i(u_T|_{2^{N'}}) = \frac{|T\cap S|}{t}$ if $T \subseteq N'$. Thus we have, for all $S \subseteq N'$,

$$\Psi^{PW}_{NN'}(u_T)(S) = \begin{cases} u_T|_{2^{N'}}(S) & \text{if } T \subseteq N'; \\ \frac{|T\cap S|}{t} & \text{if } T \not\subseteq N'. \end{cases}$$

The previous expression implies that $\Psi^{PW}_{NN'}(u_T)$ is equal to $\Psi^{Sub}_{NN'}(u_T)$ if $T \subseteq N'$. Otherwise, each player in $N' \setminus T$ is a null player and the rest of the players have a constant marginal contribution $\frac{1}{t}$ to any coalition in $\Psi^{PW}_{NN'}(u_T)$ and its subgames.

Now we verify that Ψ^{PW} is path-independent. Take $N_3 \subseteq N_2 \subseteq N_1$. First, if $T \subseteq N_3$, then $\Psi^{PW}_{N_2N_3}(\Psi^{PW}_{N_1N_2}(u_T)) = \Psi^{PW}_{N_1N_3}(u_T) = u_T|_{2^{N_3}}$ by path independence of Ψ^{Sub}. Second, if $T \not\subseteq N_3$, then $\Psi^{PW}_{N_1N_3}(u_T) = \frac{|T\cap S|}{t}$. There are two possibilities: (a) if $T \subseteq N_2$, it is immediate that $\Psi^{PW}_{N_2N_3}(\Psi^{PW}_{N_1N_2}(u_T)) = \frac{|T\cap S|}{t}$; (b) if $T \nsubseteq N_2$, then for $S \subseteq N_3$, it happens that $\Psi^{PW}_{N_2N_3}(\Psi^{PW}_{N_1N_2}(u_T))(S) = \Psi^{PW}_{N_1N_2}(u_T)(S) - \sum_{i\in S} Sh_i(\Psi^{PW}_{N_1N_2}(u_T)\,|_{2^{N_3}}) + \sum_{i\in S} Sh_i(\Psi^{PW}_{N_1N_2}(u_T)) = \frac{|T\cap S|}{t} - \frac{|T\cap S|}{t} + \frac{|T\cap S|}{t} = \frac{|T\cap S|}{t}$, where the first equality follows from equation (7), and the terms in the second equality follow from (i) the expression of the game $\Psi^{PW}_{N_1N_2}(u_T)(S) = \frac{|T\cap S|}{t}$ and its subgames, (ii) each player $i \in T \cap N_2$ has a constant marginal contribution $\frac{1}{t}$, and (iii) the rest of the players are null players. Therefore, the PW v-f reduction is path-independent.

We verify the null player out property, i.e., $\Psi^{PW}_{N(N\setminus\{i\})}(v) = v|_{2^{N\setminus\{i\}}}$ for all $v \in \mathcal{G}^N$ such that $i \in N$ is a null player in v. We notice that for all $S \subseteq N \setminus \{i\}$, $\Psi^{PW}_{N(N\setminus\{i\})}(v)(S) = v(S) - \sum_{j\in S} Sh_j(v\,|_{2^{N\setminus\{i\}}}) + \sum_{j\in S} Sh_j(v) = v(S)$, where the first equality follows (7) and the second holds because $Sh_j(v\,|_{2^{N\setminus\{i\}}}) = Sh_j(v)$ if i is a null player in v. Therefore Ψ^{PW} satisfies null player out.

As for permanent null player, let $i \in N'$ be a null player in v. Then, for all $S \subseteq N' \setminus \{i\}$, $\Psi^{PW}_{NN'}(v)(S \cup \{i\}) - \Psi^{PW}_{NN'}(v)(S) = v(S \cup \{i\}) - \sum_{j\in S\cup\{i\}} Sh_j(v\,|_{2^{N'}}) + \sum_{j\in S\cup\{i\}} Sh_j(v) - \big(v(S) - \sum_{j\in S} Sh_j(v\,|_{2^{N'}}) + \sum_{j\in S} Sh_j(v)\big) = (v(S\cup\{i\}) - v(S)) - Sh_i(v\,|_{2^{N'}}) + Sh_i(v) = 0$, where the third equality holds because i is a null player in v and its subgames and from the null player property of the Shapley value. Therefore Ψ^{PW} satisfies permanent null player.

We check the proportional addition invariance property. By additivity, it suffices to show that $\Psi^{PW}_{NN'}(\sum_{k=1}^{n-1} w_{(k,k\alpha)})(S) = 0$. Indeed, $\Psi^{PW}_{NN'}(\sum_{k=1}^{n-1} w_{(k,k\alpha)})(S) = s\alpha - \sum_{i\in S} Sh_i(\sum_{k=1}^{n'} w_{(k,k\alpha)}\,|_{2^{N'}}) + \sum_{i\in S} Sh_i(\sum_{k=1}^{n-1} w_{(k,k\alpha)}) = s\alpha - \sum_{i\in S}\sum_{k=1}^{n'} Sh_i(w_{(k,k\alpha)}\,|_{2^{N'}}) + \sum_{i\in S}\sum_{k=1}^{n-1} Sh_i(w_{(k,k\alpha)}) = s\alpha - \sum_{i\in S} Sh_i(w_{(n',n'\alpha)}\,|_{2^{N'}}) = s\alpha - \sum_{i\in S} \frac{n'\alpha}{n'} = 0$, where the first equality follows from (7), the second from additivity, and the third and fourth equalities hold because the Shapley value of each player in a symmetric game is equal to an equal share of the worth of the grand coalition.

To prove the reverse implication we need a previous lemma:

Lemma 3. *For all $N \in \mathcal{P}_{fin}(\mathcal{U})$ such that $|N| > 2$, the set $\{u_T \mid T \subsetneq N, T \neq \varnothing\} \cup \{\sum_{k=1}^{n-1} w_{(k,k)}\}$ forms a basis of $\mathcal{G}^N$.*

Proof of Lemma 3. Take any $N \in \mathcal{P}_{fin}(\mathcal{U})$. To prove Lemma 3, we first note that:

$$nu_N = \big(\sum_{i\in N} u_{\{i\}}\big) - \big(\sum_{k=1}^{n-1} w_{(k,k)}\big), \tag{21}$$

which is easily seen by recognizing that $\sum_{i\in N} u_{\{i\}}(S) = s$ for all $S \in 2^N \setminus \{\varnothing\}$.

Given that equation (21) holds and $\{u_T \mid T \subseteq N, T \neq \varnothing\}$ forms a basis of $\mathcal{G}^N$, then the set resulting from replacing u_N with $\sum_{k=1}^{n-1} w_{(k,k)}$ on this basis spans $\mathcal{G}^N$, which proves Lemma 3. $\square$

The proof that $\Psi = \Psi^{PW}$ if the v-f reduction Ψ satisfies the five properties is very similar to the proof of Theorem 2. The only difference is in the proof that $\Psi_{N(N\setminus\{i\})}(v) = \Psi^{PW}_{N(N\setminus\{i\})}(v)$ for all $i \in N$, when $v = \sum_{k=1}^{n-1} w_{(k,k\alpha)}$. In this case, additivity and proportional addition invariance imply that $\Psi_{N(N\setminus\{i\})}(\sum_{k=1}^{n-1} w_{(k,k\alpha)}) = \mathbf{0} = \Psi^{PW}_{N(N\setminus\{i\})}(\sum_{k=1}^{n-1} w_{(k,k\alpha)})$ for all $N \in \mathcal{P}_{fin}(\mathcal{U})$, for all $\alpha \in \mathbb{R}$ and for all $i \in N$.

Therefore, if a v-f reduction satisfies the five properties, then it is equal to Ψ^{PW}.

Finally, regarding the value induced by Ψ^{PW}, we notice that, for all $v \in \mathcal{G}^N$, for $i \in N$, then $\Psi^{PW}_{N\{i\}}(v)(\{i\}) = v(\{i\}) - Sh_i(v|_{2^{\{i\}}}) + Sh_i(v) = Sh_i(v)$. Therefore, Ψ^{PW} induces the Shapley value. $\square$

Proof of Proposition 5. We prove that if Ψ satisfies proportional addition invariance then Ψ^* satisfies reverse-proportional addition invariance.

We first show that the dual of the game $w_{(n-k,k\alpha)}$ is $w_{(k,-k\alpha)}$, for $k = 1, 2, \ldots, n-1$. Indeed, $w^*_{(n-k,k\alpha)}(S) = w_{(n-k,k\alpha)}(N) - w_{(n-k,k\alpha)}(N \setminus S) = -w_{(n-k,k\alpha)}(N \setminus S) = w_{(k,-k\alpha)}(S)$.

Then, for any $\alpha \in \mathbb{R}$, $\Psi^*_{NN'}(v+\sum_{k=1}^{n-1} w_{(k,(n-k)\alpha)}) = \big(\Psi_{NN'}((v+\sum_{k=1}^{n-1} w_{(k,(n-k)\alpha)})^*)\big)^* = \big(\Psi_{NN'}(v^*+\sum_{k=1}^{n-1} w^*_{(k,(n-k)\alpha)})\big)^* = \big(\Psi_{NN'}(v^*+\sum_{k=1}^{n-1} w_{(k,-k\alpha)})\big)^* = \big(\Psi_{NN'}(v^*)\big)^* = \Psi^*_{NN'}(v)$, where the first and last equalities follow from Definition 8, the second from the additivity of $v \mapsto v^*$ in Lemma 1, the third follows from the property that the dual of $w_{(n-k,k\alpha)}$ is $w_{(k,-k\alpha)}$, and the fourth from the proportional addition invariance of Ψ. Therefore, reverse-proportional addition invariance is dual to proportional addition invariance. $\square$

Proof of the expression in Example 5. We prove that the expression for Ψ^{PW^*} corresponds to that provided in Example 5:

$$\begin{aligned}
\Psi^{PW^*}_{NN'}(v)(S) =& \big(\Psi^{PW}_{NN'}(v^*)\big)^*(S) = \Psi^{PW}_{NN'}(v^*)(N') - \Psi^{PW}_{NN'}(v^*)(N' \setminus S) \\
=& [v^*(N') - \sum_{i \in N'} Sh_i(v^*|_{2^{N'}}) + \sum_{i \in N'} Sh_i(v^*)] \\
& - [v^*(N' \setminus S) - \sum_{i \in N' \setminus S} Sh_i(v^*|_{2^{N'}}) + \sum_{i \in N' \setminus S} Sh_i(v^*)] \\
=& v^*(N') - v^*(N' \setminus S) - \sum_{i \in S} Sh_i(v^*|_{2^{N'}}) + \sum_{i \in S} Sh_i(v^*)
\end{aligned}$$

$$=[v(N) - v(N \setminus N')] - [v(N) - v(N \setminus (N' \setminus S))] - \sum_{i \in S} Sh_i(v^* |_{2^{N'}}) + \sum_{i \in S} Sh_i(v)$$
$$=v(S \cup (N \setminus N')) - v(N \setminus N') - \sum_{i \in S} Sh_i\big((v^* |_{2^{N'}})^*\big) + \sum_{i \in S} Sh_i(v),$$

where the first equality follows from Definition 8, the second from the defining equation (3) of dual game, the third from (7), the fifth from (3) and the self-duality of the Shapley value, which also leads to the sixth equality.

Finally, we show that $(v^* |_{2^{N'}})^* = v^{N \setminus N'}$. Consider $T \subseteq N'$. By repeated application of the definition of dual game, for each $T \subseteq N'$, $(v^* |_{2^{N'}})^*(T) = v^* |_{2^{N'}} (N') - v^* |_{2^{N'}} (N' \setminus T) = v^*(N') - v^*(N' \setminus T) = [v(N) - v(N \setminus N')] - [v(N) - v(N \setminus (N' \setminus T))] = v(T \cup (N \setminus N')) - v(N \setminus N') = v^{N \setminus N'}(T)$. □

Proof of Theorem 5. The proof of this theorem is identical to that of Theorem 3. □

Proof of Theorem 6. First, we verify that Ψ^{Ban} satisfies the properties. It satisfies linearity and hence additivity because it is the composition of linear functions.

To verify the null player out property, let $i \in N$ be a null player in v. Then, for all $S \subseteq N \setminus \{i\}$, $\Psi^{Ban}_{N(N \setminus \{i\})}(v)(S) = \sum_{T \subseteq \{i\}} \frac{1}{2}[v(S \cup T) - v(T)] = \frac{1}{2}v(S) + \frac{1}{2}[v(S \cup \{i\}) - v(\{i\})] = v(S)$, where the first equality follows from the defining equation (10) and the second holds because i is a null player. Therefore Ψ^{Ban} satisfies null player out.

To verify that Ψ^{Ban} satisfies permanent null player, suppose that $i \in N'$ is a null player in $v \in \mathcal{G}^N$. Then, for all $S \subseteq N' \setminus \{i\}$, $\Psi^{Ban}_{NN'}(v)(S \cup \{i\}) = \sum_{T \subseteq N \setminus N'} \frac{1}{2^{n-n'}}[v((S \cup \{i\}) \cup T) - v(T)] = \sum_{T \subseteq N \setminus N'} \frac{1}{2^{n-n'}}[v(S \cup T) - v(T)] = \Psi^{Ban}_{NN'}(v)(S)$, where the first and last equalities follows from (10) and the second from the property that i is a null player in v. Therefore Ψ^{Ban} satisfies permanent null player.

To verify maximum ignorance, we substitute $N' = N \setminus \{i\}$ and $W = N$ in equation (10), then $\Psi^{Ban}_{N(N \setminus \{i\})}(\alpha u_N)(S) = \sum_{T \subseteq \{i\}} \frac{1}{2}[\alpha u_N(S \cup T) - \alpha u_N(T)] = \frac{1}{2}[\alpha u_N(S) - \alpha u_N(\varnothing)] + \frac{1}{2}[\alpha u_N(S \cup \{i\}) - \alpha u_N(\{i\})] = \frac{\alpha}{2} u_N(S \cup \{i\})$. Therefore Ψ^{Ban} satisfies maximum ignorance.

To verify path independence, we use the following claim:

Claim 1. *For all $N, N', W, S \in \mathcal{P}_{fin}(\mathcal{U})$, such that $N', W \in 2^N \setminus \{\varnothing\}$ and $S \subseteq N'$,*

$$\Psi^{Ban}_{NN'}(u_W)(S) = \begin{cases} 2^{-|W \setminus N'|} u_W(S \cup (W \setminus N')) & \text{if } S \cap W \neq \varnothing \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

To verify the claim, we have,

$$
\begin{aligned}
\Psi^{Ban}_{NN'}(v)(u_W)(S) =& \sum_{T\subseteq N\setminus N'} \frac{1}{2^{n-n'}}[u_W(S\cup T)-u_W(T)]\\
=& \sum_{T\subseteq N\setminus N': T\supseteq W\setminus N'} \frac{1}{2^{n-n'}}[u_W(S\cup T)-u_W(T)]\\
=& \sum_{T'\subseteq (N\setminus N')\setminus W} \frac{1}{2^{n-n'}}[u_W(S\cup (W\setminus N')\cup T')-u_W(W\setminus N')]\\
=& \sum_{T'\subseteq (N\setminus N')\setminus (W\setminus N')} \frac{1}{2^{n-n'}}[u_W(S\cup (W\setminus N'))-u_W(W\setminus N')]\\
=& \sum_{T'\subseteq (N\setminus N')\setminus (W\setminus N')} \frac{2^{n-n'-|W\setminus N'|}}{2^{n-n'}}[u_W(S\cup (W\setminus N'))-u_W(W\setminus N')]\\
=& 2^{-|W\setminus N'|}[u_W(S\cup (W\setminus N'))-u_W(W\setminus N')]\\
=& \begin{cases} 2^{-|W\setminus N'|}[u_W(W\setminus N')-u_W(W\setminus N')]=0 & \text{if } S\cap W=\varnothing\\ 2^{-|W\setminus N'|}u_W(S\cup (W\setminus N')) & \text{if } S\cap W\neq\varnothing \end{cases}
\end{aligned}
$$

where the first equality follows from (10), the fourth from $u_W(S\cup(W\setminus N')\cup T') = u_W(S\cup(W\setminus N'))$ if $T'\cap W=\varnothing$ and the first case of the seventh from the same reasoning, the second case of the seventh from $u_W(W\setminus N')=0$ if $S\cap W\neq\varnothing$, which implies $N'\cap W\neq\varnothing$.

Then, to check path independence, first notice that equation (22) is equivalent to

$$
\Psi^{Ban}_{NN'}(u_W) = \begin{cases} 2^{-|W\setminus N'|}u_{W\cap N'}|_{2^{N'}} & \text{if } W\cap N'\neq\varnothing\\ \mathbf{0} & \text{otherwise.}\end{cases} \tag{23}
$$

Let $N_1, N_2, N_3, T \in \mathcal{P}_{fin}(N)$ such that $N_3\subseteq N_2\subseteq N_1$ and $T\subseteq N_1$. To compute $\Psi^{Ban}_{N_2N_3}(\Psi^{Ban}_{N_1N_2}(u_T))$, there are three different possibilities to consider: (i) If $T\subseteq N_3$, then $\Psi^{Ban}_{N_2N_3}(\Psi^{Ban}_{N_1N_2}(u_T)) = \Psi^{Ban}_{N_2N_3}(2^{-|T\setminus N_2|}u_{T\cap N_2}|_{2^{N_2}}) = 2^{-|T\setminus N_2|}\Psi^{Ban}_{N_2N_3}(u_{T\cap N_2}|_{2^{N_2}}) = 2^{-|T\setminus N_2|}\cdot 2^{-|(T\cap N_2)\setminus N_3|}u_{(T\cap N_2)\cap N_3}|_{2^{N_2}}|_{2^{N_3}} = 2^{-|T\setminus N_3|}u_{T\cap N_3}|_{2^{N_3}} = \Psi^{Ban}_{N_1N_3}(u_T)$, where the first, the third and the last equalities follow from equation (23) and the second from linearity. (ii) If $T\nsubseteq N_3$ and $T\subseteq N_2$, then $T\cap N_2\nsubseteq N_3$. We have $\Psi^{Ban}_{N_2N_3}(\Psi^{Ban}_{N_1N_2}(u_T)) = \Psi^{Ban}_{N_2N_3}(2^{-|T\setminus N_2|}u_{T\cap N_2}|_{2^{N_2}}) = 2^{-|T\setminus N_2|}\Psi^{Ban}_{N_2N_3}(u_{T\cap N_2}|_{2^{N_2}}) = 2^{-|T\setminus N_2|}\cdot\mathbf{0} = \mathbf{0} = \Psi^{Ban}_{N_1N_3}(u_T)$. (iii) If $T\nsubseteq N_2$, then $\Psi^{Ban}_{N_2N_3}(\Psi^{Ban}_{N_1N_2}(u_T)) = \Psi^{Ban}_{N_2N_3}(\mathbf{0}) = \mathbf{0} = \Psi^{Ban}_{N_1N_3}(u_T)$. Therefore, Ψ^{Ban} satisfies the path independence.

We now prove the reverse implication of the theorem by showing that if the v-f reduction Ψ satisfies the five properties, then $\Psi = \Psi^{Ban}$. By path independence and additivity, it suffices to show the equality restricted to one-player operators $\left(\Psi_{N(N\setminus\{i\})}\right)$, for any $N \in \mathcal{P}_{fin}(\mathcal{U})$ and $i \in N$, restricted to a set of of all scalar multiples of elements in a basis of $\mathcal{G}^N$. We choose the set $\left(\alpha u_T\right)_{T\in 2^N\setminus\{\varnothing\},\alpha\in\mathbb{R}}$.

We show that $\Psi_{N(N\setminus\{i\})}(\alpha u_T) = \Psi^{Ban}_{N(N\setminus\{i\})}(\alpha u_T)$ for all $T \in 2^N \setminus \{\varnothing\}$, for all $\alpha \in \mathbb{R}$ and for all $i \in N$ by induction on the number of players n. We notice that maximum ignorance implies that $\Psi_{N(N\setminus\{i\})}(\alpha u_N) = \frac{\alpha}{2} u_{N\setminus\{i\}}$. Thus, we only need to check the remaining elements in the set, that is, the games $\left(\alpha u_T\right)_{T\in 2^N\setminus\{\varnothing,N\},\alpha\in\mathbb{R}}$. The proof of this part is identical to the corresponding part of the proof of Theorem 1.

Therefore, a v-f reduction that satisfies the five properties coincides with Ψ^{Ban}.

Finally, we show that Ψ^{Ban} induces the Banzhaf value: $\varphi_i^{\Psi^{Ban}}(v) = \Psi_{N\{i\}}(v)(\{i\}) = \sum_{T\subseteq N\setminus\{i\}} \frac{1}{2^{n-1}}[v(T\cup\{i\}) - v(T)] = Ban_i(v)$, where the second and the third equality follows from the defining equation (10). Therefore, Ψ induces Ban. $\square$

Proof of Proposition 8. It is easy to see that Ψ^X is additive as a result of the linearity of the Shapley value. Moreover, $\Psi^X = \Psi^{PW}$ if $X = \varnothing$ and $\Psi^X = \Psi^{HM}$ if $X = \mathcal{U}$. Equivalently, $\Psi^X_{NN'} = \Psi^{PW}_{NN'}$ if $(N\setminus N')\cap X = \varnothing$ and $\Psi^X_{NN'} = \Psi^{HM}_{NN'}$ if $(N\setminus N')\cap X = N\setminus N'$. Therefore, the reduction of a game from N to $N\setminus\{j\}$ is different depending on whether the player j belongs to X or not. Hence, Ψ^X does not satisfy anonymity if $X \neq \varnothing$ and $X \neq \mathcal{U}$. Finally, Ψ^X satisfies null player out and permanent null player if Ψ^X satisfies path independence, which we show next.

For ease of notation, for each $T \subseteq N$, let us define $e_T \in \mathcal{G}^N$ as $e_T(S) \equiv \frac{|T\cap S|}{t}$ for all $S \subseteq N$. It is easy to see that

$$\Psi^X_{NN'}(e_T)(S) = e_T\,|_{2^{N'}} \; . \tag{24}$$

By linearity of Ψ^X, it suffices to verify the path independence of Ψ^X operating on a basis $\left(u_T\right)_{T\in 2^N\setminus\{\varnothing\}}$. We need to consider three different cases of T: (i) If $T \subseteq S\cup((N\setminus N')\cap X)$, then $\Psi^X_{NN'}(u_T)(S) = \sum_{i\in S} Sh_i(u_T|_{2^{S\cup((N\setminus N')\cap X)}}) - \sum_{i\in S} Sh_i(u_T|_{2^{N'\cup((N\setminus N')\cap X)}}) + \sum_{i\in S} Sh_i(u_T) = \sum_{i\in S} Sh_i(u_T|_{2^{S\cup((N\setminus N')\cap X)}}) - \sum_{i\in S} Sh_i(u_T|_{2^{S\cup((N\setminus N')\cap X)}}) + \sum_{i\in S} Sh_i(u_T) = \frac{|T\cap S|}{t}$. (ii) If $T \nsubseteq S\cup((N\setminus N')\cap X)$ and $T \subseteq N'\cup((N\setminus N')\cap X)$, then $\Psi^X_{NN'}(u_T)(S) = \sum_{i\in S} Sh_i(u_T|_{2^{S\cup((N\setminus N')\cap X)}}) - \sum_{i\in S} Sh_i(u_T|_{2^{N'\cup((N\setminus N')\cap X)}}) + \sum_{i\in S} Sh_i(u_T) = -\sum_{i\in S} Sh_i(u_T|_{2^{N'\cup((N\setminus N')\cap X)}}) + \sum_{i\in S} Sh_i(u_T) = -\sum_{i\in S} Sh_i(u_T|_{2^{N'\cup((N\setminus N')\cap X)}}) + \sum_{i\in S} Sh_i(u_T\,|_{2^{N'\cup((N\setminus N')\cap X)}}) = 0$. Finally, (iii) if $T \nsubseteq N'\cup((N\setminus N')\cap X)$, then

$\Psi^X_{NN'}(u_T)(S) = \sum_{i\in S} Sh_i(u_T|_{2^{S\cup((N\setminus N')\cap X)}}) - \sum_{i\in S} Sh_i(u_T|_{2^{N'\cup((N\setminus N')\cap X)}}) + \sum_{i\in S} Sh_i(u_T) = \sum_{i\in S} Sh_i(u_T) = \frac{|T\cap S|}{t}$.

To sum up, if $T \subseteq N' \cup ((N\setminus N')\cap X)$, for $S \subseteq N'$,

$$\Psi^X_{NN'}(u_T)(S) = \begin{cases} 0 & \text{if } T \nsubseteq S \cup ((N\setminus N')\cap X); \\ \frac{|T\cap S|}{t} & \text{if } T \subseteq S \cup ((N\setminus N')\cap X). \end{cases}$$

The previous expression means that, if $T \subseteq N' \cup ((N\setminus N')\cap X)$,

$$\Psi^X_{NN'}(u_T) = \frac{|T\cap N'|}{t} u_{T\cap N'}\,|_{2^{N'}}\,. \tag{25}$$

whereas if $T \nsubseteq N' \cup ((N\setminus N')\cap X)$,

$$\Psi^X_{NN'}(u_T) = e_T\,|_{2^{N'}}\,. \tag{26}$$

Now we can verify that $\Psi_{N_2N_3}(\Psi_{N_1N_2}(u_T)) = \Psi_{N_1N_3}(u_T)$ for all $N_1, N_2, N_3, S \in \mathcal{P}_{fin}(\mathcal{U})$ such that $S \subseteq N_3 \subseteq N_2 \subseteq N_1$, for all $T \subseteq N_1$. We have three possibilities: (c1) $T \subseteq N_2 \cup ((N_1\setminus N_2)\cap X)$ and $T\cap N_2 \subseteq N_3\cup((N_2\setminus N_3)\cap X)$; (c2) $T \subseteq N_2\cup((N_1\setminus N_2)\cap X)$ and $T\cap N_2 \nsubseteq N_3\cup((N_2\setminus N_3)\cap X)$; (c3) $T \nsubseteq N_2\cup((N_1\setminus N_2)\cap X)$ and $T \subseteq N_3\cup((N_2\setminus N_3)\cap X)$.

For (c1), $\Psi^X_{N_2N_3}(\Psi^X_{N_1N_2}(u_T)) = \Psi^X_{N_2N_3}(\frac{|T\cap N_2|}{t}u_{T\cap N_2}\,|_{2^{N_2}}) = \frac{|T\cap N_2|}{t}\Psi^X_{N_2N_3}(u_{T\cap N_2}\,|_{2^{N_2}}) = \frac{|T\cap N_2|}{t}\frac{|T\cap N_2\cap N_3|}{|T\cap N_2|}u_{T\cap N_2\cap N_3}\,|_{2^{N_2}}|_{2^{N_3}} = \frac{|T\cap N_3|}{t}u_{T\cap N_3}\,|_{2^{N_3}} = \Psi^X_{N_1N_3}(u_T)$, where the first and the third equality follow from equation (25), the second from linearity of Ψ^X, and the last from the fact that (c1) implies that $T \subseteq N_3 \cup ((N_1\setminus N_3)\cap X)$.

For (c2), $\Psi^X_{N_2N_3}(\Psi^X_{N_1N_2}(u_T)) = \Psi^X_{N_2N_3}(\frac{|T\cap N_2|}{t}u_{T\cap N_2}\,|_{2^{N_2}}) = \frac{|T\cap N_2|}{t}\Psi^X_{N_2N_3}(u_{T\cap N_2}\,|_{2^{N_2}}) = \frac{|T\cap N_2|}{t}e_{T\cap N_2}\,|_{2^{N_3}} = e_T\,|_{2^{N_3}} = \Psi^X_{N_1N_3}(u_T)$, where the first equality follows from equation (25), the second from linearity of Ψ^X, the third from equation (26), the fifth from the fact that (c2) implies that $T \nsubseteq N_3 \cup ((N_1\setminus N_3)\cap X)$.

For (c3), $\Psi^X_{N_2N_3}(\Psi^X_{N_1N_2}(u_T)) = \Psi^X_{N_2N_3}(e_T\,|_{2^{N_2}}) = e_T\,|_{2^{N_2}}|_{2^{N_3}} = e_T\,|_{2^{N_3}} = \Psi^X_{N_1N_3}(u_T)$, where the first and second equality follow from equation (26), the fourth from the fact that (c3) implies that $T \nsubseteq N_3 \cup ((N_1\setminus N_3)\cap X)$.

Therefore, Ψ^X is path-independent. $\square$

Example of a v-f reduction that does not satisfy linearity. We construct a v-f reduction that satisfies additivity, null player out, permanent null player and path independence but not linearity.

We can invoke path independence to define $\Psi_{NN'}$, for any $N' \subseteq N$ once we will determine the functions taking the form $\Psi_{N(N\setminus\{k\})}$ such that $k \in N$. Moreover, it suffices to construct a non-linear function $\Psi_{\{i,j\}\{i\}} : \mathcal{G}^{\{i,j\}} \to \mathcal{G}^{\{i\}}$ that satisfies null player out, permanent null player and additivity. For concreteness, we let the rest of functions, i.e., $\Psi_{N(N\setminus\{k\})}$ such that $k \in N$ and $|N| > 2$ coincide with the subgame operator.

Let $\mathbb{Q} \subseteq \mathbb{R}$ be the set of all rational numbers. To define a non-linear additive function, we use the concept of $\mathbb{R}$ as a vector space over $\mathbb{Q}$. A linear basis of this vector space is called a Hamel basis. Let $\mathcal{H}$ be a Hamel basis. Then for each $\gamma \in \mathbb{R}$, we can find a **unique** finite set of elements $\{x_1, \ldots, x_k\} \subseteq \mathcal{H}$ such that $\gamma = \sum_{j=1}^{k} c_j x_j$ where $c_1, \ldots, c_k \in \mathbb{Q} \setminus \{0\}$. Choose an arbitrary element $y \in \mathcal{H}$. Then for each $\gamma \in \mathbb{R}$, we can determine its corresponding coefficient (which is possibly zero) in the expression of γ, coefficient that we denote $c(\gamma)$. Thus we have a function $c : \mathbb{R} \to \mathbb{Q}$ defined by the projection $\gamma \mapsto c(\gamma)$. This function is additive but not linear and it satisfies that $c(0) = 0$.

Before defining $\Psi_{\{i,j\}\{i\}}$, recall that for each $v \in \mathcal{G}^{\{i,j\}}$, v can be expressed by $\alpha u_{\{i\}} + \beta u_{\{j\}} + \gamma u_{\{i,j\}}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. Now we define $\Psi_{\{i,j\}\{i\}}(v)$ as follows:

$$\Psi_{\{i,j\}\{i\}}(v)(\{i\}) \equiv \alpha + c(\gamma), \tag{27}$$

where $\alpha, \gamma \in \mathbb{R}$ are such that $v = \alpha u_{\{i\}} + \beta u_{\{j\}} + \gamma u_{\{i,j\}}$ for some $\beta \in \mathbb{R}$.

Notice that if i is a null player in v then v must take the form of $\beta u_{\{j\}}$ and that if j is a null player in v then v must take the form of $\alpha u_{\{i\}}$. Therefore, $\Psi_{\{i,j\}\{i\}}$ satisfies null player out and permanent null player. Moreover, it is additive but not linear because the function c is additive but not linear. $\square$

References

[1] Banzhaf III, John F. "Weighted voting doesn't work: A mathematical analysis." Rutgers L. Rev. 19 (1964): 317.

[2] Béal, Sylvain, Eric Rémila, and Philippe Solal. "Axioms of invariance for TU-games." International Journal of Game Theory 44.4 (2015): 891-902.

[3] Chang, Chih, and Cheng-Cheng Hu. "Reduced game and converse consistency." Games and Economic Behavior 59.2 (2007): 260-278.

[4] Davis, Morton, and Michael Maschler. "The kernel of a cooperative game." Naval Research Logistics Quarterly 12.3 (1965): 223-259.

[5] Derks, Jean JM, and Hans H. Haller. "Null players out? Linear values for games with variable supports." International Game Theory Review 1.03n04 (1999): 301-314.

[6] Dragan, Irinel. "New mathematical properties of the Banzhaf value." European Journal of Operational Research 95.2 (1996): 451-463.

[7] Driessen, Theo. "A survey of consistency properties in cooperative game theory." SIAM review 33.1 (1991): 43-59.

[8] Driessen, Theo, and Tadeusz Radzik. "Extensions of Hart and Mas-Colell's consistency to efficient, linear, and symmetric values for TU-games." ICM Millennium Lectures on Games. Springer, Berlin, Heidelberg, 2003. 147-165.

[9] Hart, Sergiu, and Andreu Mas-Colell. "Potential, value, and consistency." Econometrica: Journal of the Econometric Society (1989): 589-614.

[10] Jaynes, Edwin T. "Discrete prior probabilities – The entropy principle." Probability Theory: The Logic of Science. Cambridge university press, 2003.

[11] Maschler, Michael. "Consistency." in "Game Theory and Applications" (T. Ichiishi, A. Neyman and Y. Tauman. Eds.)." (1990): 183-186.

[12] Moulin, Hervé. "The separability axiom and equal-sharing methods." Journal of Economic Theory 36.1 (1985): 120-148.

[13] Oishi, Takayuki, Mikio Nakayama, Toru Hokari, and Yukihiko Funaki. "Duality and anti-duality in TU games applied to solutions, axioms, and axiomatizations." Journal of Mathematical Economics 63 (2016): 44-53.

[14] Owen, Guillermo. "Multilinear extensions and the Banzhaf value." Naval Research Logistics (NRL) 22.4 (1975): 741-750.

[15] Peleg, Bezalel. "An axiomatization of the core of cooperative games without side payments." Journal of Mathematical Economics 14.2 (1985): 203-214.

[16] Pérez-Castrillo, David, and David Wettstein. "Bidding for the surplus: a non-cooperative approach to the Shapley value." Journal of Economic Theory 100.2 (2001): 274-294.

[17] Peters, Hans, and Eric Van Damme. "Characterizing the Nash and Raiffa bargaining solutions by disagreement point axioms." Mathematics of Operations Research 16.3 (1991): 447-461.

[18] Shapley, Lloyd S. "A value for n-person games." Contributions to the Theory of Games 2.28 (1953): 307-317.

[19] Sobolev, Andrej I. "The characterization of optimality principles in cooperative games by functional equations." Mathematical Methods in the Social Sciences 6 (1975): 94-151.

[20] Thomson, William. "Consistency and its converse: an introduction." Review of Economic Design 15.4 (2011): 257-291.

[21] Yanovskaya, Elena. "Consistent and covariant solutions for TU games." International Journal of Game Theory 32.4 (2004): 485-500.

[22] Yanovskaya, Elena, and Theo Driessen. "On linear consistency of anonymous values for TU games." International Journal of Game Theory 30.4 (2002): 601-609.

[23] Young, H. Peyton. "Monotonic solutions of cooperative games." International Journal of Game Theory 14.2 (1985): 65-72.