Cardinal Assignment Mechanisms: Money Matters More than it Should

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Cardinal Assignment Mechanisms: Money matters more than it should.

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Abstract

Most environments where (possibly random) assignment mechanisms are used are such that participants have outside options. For instance private schools and private housing are options that participants in a public choice or public housing assignment problems may have. We postulate that cardinal mechanisms, as opposed to ordinal mechanisms, may be unfair for agents with less access to outside options. Chances *inside* the assignment process could favor agents with better outside options.

Key words: Random Assignments; Ordinal vs. Cardinal Mechanisms; Outside Options; Unequal Access

JEL codes: D47, D63

1 Introduction

Centralized matching markets are used to assign vacancies in Kindergarten, school, colleges, public housing or hospitals. The match is done through an assignment procedure because of the belief that income shall not determine access to such spots. However, the analysis of centralized mechanisms often ignores that participants in assignment problems often have outside options (mostly privately provided options) in case the obtained assignment is not good enough.

For instance, in the context of school choice under the Boston mechanism, Calsamiglia, Martínez-Mora and Miralles (2017) find that the existence of private schools that are available only for richer families will decrease the probability of median income families of entering the best schools in the public system.

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That is, the fact that the outside option differs across individuals introduces an inequality in the probability of assignment within the public system even when preferences over public schools are identical.

Consider the following example that serves to illustrate the problem at hand. We have two public schools with one slot each (schools 1 and 2), one public school with three slots (school 3,) and five students. Students 1 and 2 have valuations $v = (1.2, 1, 0)$ where entries refer to, respectively, school 1, school 2 and school 3. Students 3, 4 and 5 have valuations $v' = (1.2, 1, 0.9999)$ Consider the following random assignment: probabilities $q = (0, 1/2, 1/2)$ for both students 1 and 2 and $q' = (1/3, 0, 2/3)$ for students 3,4, and 5. The aforementioned assignment is ex-ante both efficient and envy-free, indeed coinciding with a Competitive Equilibrium from Equal Incomes assignment à la Hylland and Zeckhauser (1979).

Suppose now that school 3 is instead a private school outside of the assignment process, that one can always obtain access to if whished by paying a tuition fee. We argue that in this case this assignment may not be a convincingly fair assignment of probabilities anymore given the "outside option" nature of school 3. One could imagine that agents 3 to 5 give more value to the outside option than students 1 and 2 entirely or partly because they have more income and therefore less marginal utility of money. This fact triggers an advantage that wealthier students have in accessing school 1, the best *public* school, as compared to poorer students. Wealthier students are better-equipped to bear the risk of ending up with the outside option.[1]

The outside option is a *special* object in this sense. Just as priorities are not exactly "school preferences" even if they could be mathematically modelled as such, the outside option may be conceptually different from an object with sufficiently many copies, even if there is apparently no difference between the two approaches from a purely methodological point of view. We claim that an assignment mechanism should avoid the distortions that an outside option may have on the assignment of the other objects.

We construct a model to illustrate our case, in which agent's wealth positively affects the valuation of the outside option relatively more than it affects the valuation of the objects we allocate. Before we come up with our results, we would like to stress that the ideas here explained go beyond the presence of income differences. Consider for example a student who is talented enough for a scholarship at the school of arts. Yet she would prefer to attend the best public school in the area. Given that she has an extra outside option available, she could better bear the risks of applying for the most popular school than a not so talented student. Differences in talent jointly with selective schools may also generate disruptions in the assignment of public slots. Any other discriminatory criterion, given maybe by other socioeconomics (religion, etnicity etc.) that give some agents higher access to reservation objects, may trigger a similar normative requirement.

[1]The lack of enough supply of public slots in this example could just be offset by the existence of other public schools of very low quality that everyone would like to avoid (ghetto schools.)

We postulate the following no-regret robustness condition: that no agent prefers a different (interim) assignment other than the one assigned should she become marginally wealthier. For the reader who thinks that a robust mechanism should simply assign the same interim probabilities to any two agent's types differing only in the valuation for the outside option, we indeed show (Lemma 1) that such a normative requirement is a consequence of our robustness property. Adding Bayesian Incentive Compatibility, one can think of this simplified approach and ours as equivalent.

Also, one might think that our robustness notion is equivalent to robustness to a small noise in cardinal preferences. It is not entirely the case, for two reasons. First, because it is motivated differently, in our case due to a fairness concern (unequal access to outside options, as opposed to uncertainty about own preferences.) The second reason is methodological: we do not allow for any deviation in cardinal preferences, only for deviations in directions compatible with an increase in income. Even under such a restriction we obtain a strong implication as seen below.

The main result (Theorem 1) is the prescription of ordinality: in the universal domain of vNM preferences, without loss of generality[2] we can restrict attention to (interim) **ordinal mechanisms**, where only ordinal preferences are taken into account. Moreover, **Ordinal Bayesian Incentive Compatibility** is another consequence of our robustness requirement.

When we assess the relevance of such a recommendation, one has to pay attention to plausible restrictions in the domain of preferences over lotteries. In an extreme case in which all objects are acceptable (i.e. better than the outside option) for every agent and there is sufficient supply of (copies of) objects, our robustness condition is innocuous and there is full scope for cardinal mechanisms. In environments in which three or more objects are acceptable for every agent, there is still scope for taking cardinal preferences into account.

There has been some literature stressing the fact that outside options shall be studied in depth in assignment problems: for instance Kesten and Kurino (2016) and Pycia and Unver (2017.) According to this new strand, outside options are more than an object with infinite capacity. However, we are not aware of other papers considering this fact as the source for a robustness concept in cardinal mechanisms.

A significant array of papers (Hylland and Zeckhauser, 1979; Miralles, 2008; Abdulkadiroglu, Che and Yasuda, 2011 and 2015; Ashlagi and Shi, 2016; Miralles and Pycia, 2014; He, Miralles, Pycia and Yan 2015; Pycia 20014; Featherstone and Niederle, 2016; or Kim, 2017 for voting schemes) have stressed the importance of taking cardinal utilities into account. Empirically, and particularly the school choice case, the seminal paper by Black (1999) evidences that parents have cardinal preferences for the schools that can be expressed in monetary terms as willingness to pay through the residential market.

However, the preference for mechanisms eliciting cardinal preferences is not

[2] Throughout this paper, "without loss of generality" means that there is an interim payoff-equivalent feasible assignment accomplishing with the desired property.

so clear-cut. A recent paper by Carroll (2018) is most related to ours. Carroll elaborates on the literature of robust mechanism design (see for instance the seminal paper by Bergemann and Morris, 2005) applied to Social Choice Correspondences. He summarizes the debate and postulates that, in environments with uncertainty about the own cardinal preferences, simple mechanisms eliciting information on ordinal preferences only shall be preferable. In contrast with Carroll's approach, we allow cardinal preferences to vary in a way that alter ordinal preferences, even though these cardinal preferences cannot be shaken in any direction (only in those compatible with an increase in the valuation of the outside option.) Moreover, our conclusions do not depend upon a social planner wishing to implement an ordinal social choice correspondence (e.g. ex-post Pareto-optimality, ordinal efficiency, etc.)

In a similar trend of literature, Hylland (1980) and more recently Dutta, Peters and Sen (2007) have shown that the only strategy-proof cardinal decision scheme satisfying a weak unanimity property is the random dictatorship, hence eliminating cardinality. Quite recently, Ehlers, Majumdar, Mishra and Sen (2014) establish that under some continuity criteria, incentive-compatible cardinal mechanisms are ordinal. Our approach is appealing since our model starts from a pure cardinal approach with no bias in favor of ordinality. As argued in Carroll (2018), Ehlers et al. gives some initial advantage to ordinal mechanisms since continuity criteria are not required around indifferences over sure objects.

To sum up, the present paper constitutes a warning. If one takes outside options seriously, there will be insurmountable trade-offs between cardinal efficiency and the robustness of the mechanism with respect to outside options. We hope that this contribution will foster further research on this important matter.

Section 2 introduces the model and robustness concepts. Section 3 extracts properties of the mechanisms as direct implications. Section 4 concludes.

2 Notation and definitions

There is a finite set of agents I, each one to be assigned to exactly one of a set of objects $S \cup \{o\}$ where o is the outside option. Notation for agents include i, i'... whereas notation for objects include s, s'... We use $-i$ $(-s)$ for all the agents (objects) that are not i (s). Each object s has a positive number of copies η_s, and we assume $\eta_o = \infty$. The supply vector is denoted with $\eta = (\eta_s)_{s \in S \cup \{o\}}$. Agents i have vNM *valuations* for the objects $v_i = (v_i^s)_{s \in S \cup \{o\}} \in \mathcal{V}_i \subset \mathbb{R}^{|S|+1}$. We allow valuations to depend on agent's income: $v_i = \tilde{v}_i(m_i)$. We use the following assumption:

Axiom 1 *Income influences the valuation for the outside option relatively more. For every agent i and for every pair of incomes $m' > m$ we have*
1) $\tilde{v}_i^s(m') - \tilde{v}_i^s(m) = \tilde{v}_i^{s'}(m') - \tilde{v}_i^{s'}(m) \geq 0$ *for all pairs of objects* $s, s' \neq o$
2) $\tilde{v}_i^o(m') - \tilde{v}_i^o(m) > \tilde{v}_i^s(m') - \tilde{v}_i^s(m)$ *for all* $s \neq o$

This assumption says that income increases improve the relative valuation of the outside option, while preferences for other objects are vis-a-vis unchanged. Considering that preferences over lotteries are invariant to affine transformations of vNM valuations, we assume that $\mathcal{V}_i$ contains all affine transformations of itself: [3]

$$\mathcal{V}_i = \{v \in \mathbb{R}^{|S|+1} : \exists \alpha > 0, \beta = \beta_i \in \mathbb{R} | \alpha v + \beta 1_{|S|+1} \in \mathcal{V}_i\}$$

$\mathcal{V} = \prod_{i \in I} \mathcal{V}_i$ is the overall preference domain. We say that the *domain (of cardinal preferences) is universal* if $\mathcal{V} = (\mathbb{R}^{|S|+1})^{|I|}$.

Each agent $i \in I$ knows the distribution of valuations μ with support over the preference domain. She also learns her own continuous function $\tilde{v}_i(m_i)$ and her income type m_i, assumed independent from the other agents' income types. [4]

A (direct random) assignment mechanism q is a function $q = (q_i)_{i \in I} : \mathcal{V} \to \Delta(S \cup \{o\})^{|I|}$ satisfying the feasibility condition $\sum_{i \in I} q_i(v) \leq \eta$.[5] For every $i \in I$ we denote with $Q_i(v_i) = E_{v_{-i}}(q_i(v_i, v_{-i})|\mu)$ the *interim mechanism* for agent i. We restrict attention to mechanisms that are invariant to affine transformations expressing the same preferences over lotteries, that is $Q_i(\alpha v_i + \beta 1_{|S|+1}) = Q_i(v_i)$ for every $\alpha > 0$ and $\beta \in \mathbb{R}$, where again $1_{|S|+1}$ is a vector of ones with dimension $|S| + 1$. We also assume that Q_i is invariant to the valuation for objects that are (weakly) less-preferred than the outside option, for which zero probability is assigned. An assignment mechanism is ordinal if it only responds to the ordinal component of agents' preferences.

Definition 1 *$\tilde{Q}_i(v_i; m)$ is an* adaptation *of $Q_i(v_i)$ to m if for all $s \in S$ we have $\tilde{Q}_i^s(v_i; m) = Q_i^s(v_i)$ if $v_i^s > \max\{\tilde{v}_i^o(m), v_i^o\}$ and $\tilde{Q}_i^s(v_i; m) = 0$ otherwise, whereas $\tilde{Q}_i^o(v_i; m) = \sum_{s \in S \cup \{o\} : \max\{\tilde{v}_i^o(m), v_i^o\} \geq v_i^s} Q_i^s(v_i)$.*

An adaptation is simply the acknowledgement that no agent may obtain positive chances at an object that is less-preferred than the outside option, when the valuation of the latter increases. If that were the case, the agent would reject the assigned object in favor of the outside option. No mechanism may force an agent to take a unacceptable object.

Definition 2 *For a vector $\varepsilon = (\varepsilon_i)_{i \in I} >> 0$, a mechanism is $\varepsilon-$ **Income-Robust** (or just $\varepsilon-$ **Robust**) if for every $i \in I$, every pair $v_i, v_i' \in \mathcal{V}_i$ and every $m \in [m_i, m_i + \varepsilon_i]$ we have $\tilde{Q}_i(v_i; m) \cdot \tilde{v}_i(m) \geq \tilde{Q}_i(v_i'; m) \cdot \tilde{v}_i(m)$.*

[3] In the formula below, $1_{|S|+1}$ is a vector of ones with dimension $|S| + 1$.

[4] A slightly modified version of our model would also work under correlated types. In such a model the valuation for the outside option would depend on two components, a (possibly correlated) signal and a private, independent component (including income.) Our proofs are based upon variations in the valuation for the outside option. By restricting attention to variations in its private component, beliefs are not modified by such variations, and the arguments below would follow.

[5] We do not impose anonymity. In particular, priority structures might apply. The latter inequality guarantees implementability as a lottery over feasible deterministic assignments by virtue of the Birkhoff-von Neumann theorem.

Lemma 1 *$\varepsilon-$ Robustness as defined above implies Bayesian Incentive Compatibility. $\varepsilon-$Robustness also implies that, without loss of generality, we can restrict attention to mechanisms in which $Q_i(v_i)$ is invariant with respect to v_i^o, as long as the variation of v_i^o does not alter ordinal preferences.*

Proof. The first statement is an obvious consequence of $m = m_i$ being an admissible value. To see the second statement, and for a type v_i with strict associated ordinal preferences, consider a sufficiently small increment of m_i to $m < m_i + \varepsilon_i$, such that $\tilde{v}_i(m_i + \varepsilon_i)$ keeps ordinal preferences unchanged (and so does $\tilde{v}_i(m)$.) Notice that since ordinal preferences remain unvaried, $\tilde{Q}_i(v_i; m) = Q_i(v_i)$. Take $v_i' = \tilde{v}_i(m)$. (Trivially, $\tilde{Q}_i(\tilde{v}_i(m); m) = Q_i(\tilde{v}_i(m))$.) $\varepsilon-$Robustness implies $Q_i(v_i) \cdot \tilde{v}_i(m) \geq Q_i(\tilde{v}_i(m)) \cdot \tilde{v}_i(m)$ and at the same time Bayesian Incentive Compatibility implies $Q_i(v_i) \cdot v_i' \leq Q_i(v_i') \cdot v_i'$. Recalling $v_i' = \tilde{v}_i(m)$ we obtain $Q_i(v_i) \cdot \tilde{v}_i(m) = Q_i(\tilde{v}_i(m)) \cdot \tilde{v}_i(m)$.

We now have to show that also $Q_i(v_i) \cdot v_i = Q_i(\tilde{v}_i(m)) \cdot v_i$. By Bayesian Incentive Compatibility it is clear that $Q_i(v_i) \cdot v_i \geq Q_i(\tilde{v}_i(m)) \cdot v_i$. Suppose now that $Q_i(v_i) \cdot v_i > Q_i(\tilde{v}_i(m)) \cdot v_i$.

Knowing $Q_i(v_i) \cdot \tilde{v}_i(m) = Q_i(\tilde{v}_i(m)) \cdot \tilde{v}_i(m)$ and the fact that $Q_i(\tilde{v}_i(m))$ is invariant to affine transformations of $\tilde{v}_i(m)$, and noticing that

$$\begin{aligned} & \tilde{v}_i(m) - (\tilde{v}_i^1(m) - \tilde{v}_i^1(m_i), ..., \tilde{v}_i^1(m) - \tilde{v}_i^1(m_i)) \\ = \quad & (v_i^{-o}, v_i^o + \tilde{v}_i^o(m) - \tilde{v}_i^o(m_i) - (\tilde{v}_i^1(m) - \tilde{v}_i^1(m_i)) > v_i \end{aligned}$$

(the inequality comes from Axiom 1), and finally using the notation $v_i^{o\prime} = v_i^o + \tilde{v}_i^o(m) - \tilde{v}_i^o(m_i) - (\tilde{v}_i^1(m) - \tilde{v}_i^1(m_i))$, we see that

$$Q_i(v_i) \cdot (v_i^{-o}, v_i^{o\prime}) = Q_i(\tilde{v}_i(m)) \cdot (v_i^{-o}, v_i^{o\prime})$$

Since we were supposing $Q_i(v_i) \cdot (v_i^{-o}, v_i^o) > Q_i(\tilde{v}_i(m)) \cdot (v_i^{-o}, v_i^o)$ and provided $v_i^{o\prime} > v_i^o$, we must conclude that $Q_i^o(v_i) < Q_i^o(\tilde{v}_i(m))$. Yet that cannot happen, as we show next.

Take $m_i + \varepsilon_i > m'' > m' > m$ (recall that $\tilde{v}_i(m'')$ still keeps ordinal preferences unchanged, and so does $\tilde{v}_i(m')$.) By a similar argument as above, and defining $v_i^{o\prime\prime} = v_i^o + \tilde{v}_i^o(m') - \tilde{v}_i^o(m_i) - (\tilde{v}_i^1(m') - \tilde{v}_i^1(m_i))$ and $v_i^{o\prime\prime\prime} = v_i^o + \tilde{v}_i^o(m'') - \tilde{v}_i^o(m_i) - (\tilde{v}_i^1(m'') - \tilde{v}_i^1(m_i))$, we have these four equations

$$\begin{aligned} Q_i(v_i) \cdot (v_i^{-o}, v_i^{o\prime\prime}) &= Q_i(\tilde{v}_i(m')) \cdot (v_i^{-o}, v_i^{o\prime\prime}) \\ Q_i(\tilde{v}_i(m)) \cdot (v_i^{-o}, v_i^{o\prime\prime}) &= Q_i(\tilde{v}_i(m')) \cdot (v_i^{-o}, v_i^{o\prime\prime}) \\ Q_i(v_i) \cdot (v_i^{-o}, v_i^{o\prime\prime\prime}) &= Q_i(\tilde{v}_i(m'')) \cdot (v_i^{-o}, v_i^{o\prime\prime\prime}) \\ Q_i(\tilde{v}_i(m)) \cdot (v_i^{-o}, v_i^{o\prime\prime\prime}) &= Q_i(\tilde{v}_i(m'')) \cdot (v_i^{-o}, v_i^{o\prime\prime\prime}) \end{aligned}$$

From here we obtain

$$\begin{aligned} Q_i(v_i) \cdot (v_i^{-o}, v_i^{o\prime\prime}) &= Q_i(\tilde{v}_i(m)) \cdot (v_i^{-o}, v_i^{o\prime\prime}) \\ Q_i(v_i) \cdot (v_i^{-o}, v_i^{o\prime\prime\prime}) &= Q_i(\tilde{v}_i(m)) \cdot (v_i^{-o}, v_i^{o\prime\prime\prime}) \end{aligned}$$

Subtracting, we get:

$$\begin{aligned} 0 &= [Q_i(\tilde{v}_i(m)) - Q_i(v_i)] \cdot [(v_i^{-o}, v_i^{o'''}) - (v_i^{-o}, v_i^{o''})] \\ &= [Q_i^o(\tilde{v}_i(m)) - Q_i^o(v_i)](v_i^{o'''} - v_i^{o''}) \end{aligned}$$

This implies $Q_i^o(\tilde{v}_i(m)) = Q_i^o(v_i)$ in contradiction with $Q_i^o(v_i) < Q_i^o(\tilde{v}_i(m))$. We conclude that $Q_i(v_i) \cdot v_i = Q_i(\tilde{v}_i(m)) \cdot v_i$. Together with $Q_i(v_i) \cdot \tilde{v}_i(m) = Q_i(\tilde{v}_i(m)) \cdot \tilde{v}_i(m)$, there is a standard "ironing" argument by which there is another mechanism averaging out the assignments of all such types while keeping interim payoffs constant. ■

3 Robustness and ordinality

We are ready to state the main result of this paper, that our robustness notion implies ordinality without loss of generality. Moreover, any two agent's types sharing identical ordinal preferences over objects up to some position in the ranking obtain the same probabilities for those objects. Finally we also find the necessity of Ordinal (FOSD) Bayesian Incentive Compatibility, which is defined as $\sum_{v_i^{s'} \geq v_i^s} Q_i^{s'}(v_i) \geq \sum_{v_i^{s'} \geq v_i^s} \tilde{Q}_i^{s'}(v_i'; m_i)$ for every $i \in I$, $s \in S \cup \{o\}$ and $v_i, v_i' \in \mathcal{V}_i$.

Theorem 1 *Fix I, S, η and $\mathcal{V}$ being the universal domain of preferences, with beliefs μ. For a fixed $\varepsilon = (\varepsilon_i)_{i \in I} >> 0$, if a mechanism q is an $\varepsilon-$ Robust then it is ordinal without loss of generality. Moreover, any two agent's types sharing identical ordinal preferences over objects up to some position in the ranking obtain the same probabilities for those objects. Finally, Q_i satisfies Ordinal (FOSD) Bayesian Incentive Compatibility for every $i \in I$.*

Proof. We focus on some generic agent i. It is enough if we study the set of vNM types that (strictly) prefer object 1 to object 2 to object 3 and so on. Their ordinal preferences differ only in the position the outside option o occupies in the agents' ranking. We prove the following **induction argument**: if for the set of valuations in which o occupies the $n-$th position in the agents' ranking, all of them obtain same interim probabilities namely $Q_i^1, ..., Q_i^{n-1}$ for the objects ranked above o, then for the set of types for which o occupies the $(n+1)-$th position we have that all of them obtain the same probabilities $Q_i^1, ..., Q_i^{n-1}, Q_i^n$ for the objects ranked above o, without loss of generality.

Proof of the induction argument. Take a valuation vector v_i where o occupies the $(n+1)-$th position, and her assigned interim probabilities $Q_i^1(v_i), ..., Q_i^n(v_i), Q_i^o(v_i)$. By Lemma 1 and without loss of generality, a valuation vector v_i' defined as $v_i'^s = v_i^s$, if $s \neq o$, and $v_i'^o = v_i^n - \gamma$, where γ is an arbitrarily small positive number, obtains the same assignment of interim probabilities. Notice that the transformation keeps ordinal preferences unaltered.

Take the valuation vector v_i'' defined as $v_i''^s = v_i'^s - v_i'^o$, an affine transformation of v_i', hence receiving the same interim probabilities as v_i does as well.

Since γ (and thus $v_i''^n$) can be arbitrarily small, Bayesian Incentive Compatibility imposes that $Q_i^1 v_i''^1 + ... + Q_i^{n-1} v_i''^{n-1} \leq Q_i^1(v_i) v_i''^1 + ... + Q_i^{n-1}(v_i) v_i''^{n-1}$.

Now, for *all* valuations such that o occupies the $n-$th position, the assignment interim probabilities $Q_i^1, ..., Q_i^{n-1}, Q_i^o$ must be preferable to $Q_i^1(v_i), ..., Q_i^n(v_i), Q_i^o(v_i)$, implying the FOSD condition $\sum_{s=1}^{\tilde{n}} Q_i^s \geq \sum_{s=1}^{\tilde{n}} Q_i^s(v_i)$, $\tilde{n} = 1, ..., n-1$. Together with the former equality we must have $(Q_i^1, ..., Q_i^{n-1}) = (Q_i^1(v_i), ..., Q_i^{n-1}(v_i))$. Note that this is true for every valuation vector v_i such that o occupies the $(n+1)-$th position.

It is easy to conclude (by Bayesian Incentive Compatibility) that we must also have $Q_i^n(v_i) = Q_i^n$ for all v_i such that o occupies the $(n+1)-$th position, since

$$Q_i^n(v_i) + Q_i^o(v_i) = 1 - Q_i^1 - ... - Q_i^{n-1}$$

which is invariant in v_i. The induction argument is closed by noticing that the initial condition trivially holds for $n = 1$: all valuation vectors such that the outside option is right below object 1 must obtain the same interim probability for object 1.

It only remains to check that the random assignment mechanism satisfies Ordinal Bayesian Incentive Compatibility. This is immediate by Bayesian Incentive Compatibility since the assignment of interim probabilities is ordinal. ■

Comments

If we also wished to impose ordinal efficiency and anonymity as a desideratum, by a well-known result provided by Liu and Pycia (2016) we would have:

Corollary 1 *Fix a vector $\varepsilon = (\varepsilon_i)_{i \in I} >> 0$. The set of anonymous $\varepsilon-$Robust mechanisms that are ordinally efficient allocation mechanisms (and regular in the Liu and Pycia sense) in a universal-domain growing economy converges to the Random Serial Dictatorship.*

So is there a way to scape from the "ordinal trap"? We might imagine plausible constraints in the domain of ordinal preferences. If there is a linear ordering of elements of S such that the outside option cannot be ranked ahead of the first *three* elements in the ordering, then for such an ordering $\varepsilon-$Robustness admits cardinality.[6] Environments in which ordinal preferences are highly correlated among agents might justify such assumption. Also, one might account for possible gaps in the domain of vNM preferences that break the argument in the previous proof. Gaps give some room to cardinality.

[6] Notice that two elements ahead of o are not enough. In such a case, since $Q_i(v_i)$ is invariant to v_i^o as long as ordinal preferences are kept and no matter how low the valuation of objects worse than o is, by varying only v_i^o we can generate all the preferences over lotteries that are consistent with that ordinal preference. Consequently, cardinality is not taken into account.

4 Conclusions

We have proposed a desirable property an assignment mechanism should accomplish in the presence of outside options with unequal access due to income (or talent, or socioeconomic) differences. The deduced recipe for universal domains of preferences is clear: use ordinal assignment rules that ignore the cardinality of agents' preferences. This suggestion should mainly be taken into account in environments in which the outside option (e.g. private schooling and alternatives) account for an important share of the market we analyze. Unequal access to outside options constitutes a relevant issue in paradigmatic examples of assignment problems such as school choice.

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