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School Choice: Nash Implementation of Stable Matchings through Rank-Priority Mechanisms*

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Abstract

We consider school choice problems (Abdulkadiroğlu and Sönmez, 2003) where students are assigned to public schools through a centralized assignment mechanism. We study the family of so-called rank-priority mechanisms, each of which is induced by an order of rank-priority pairs. Following the corresponding order of pairs, at each step a rank-priority mechanism considers a rank-priority pair and matches an available student to an unfilled school if the student and the school rank and prioritize each other in accordance with the rank-priority pair. The Boston or immediate acceptance mechanism is a particular rank-priority mechanism. Our first main result is a characterization of the subfamily of rank-priority mechanisms that Nash implement the set of stable matchings (Theorem 1). We show that our characterization also holds for "sub-implementation" and "sup-implementation" (Corollaries 3 and 4). Our second main result is a strong impossibility result: under incomplete information, no rank-priority mechanism implements the set of stable matchings (Theorem 2).

Keywords: school choice; rank-priority mechanisms; stability; Nash implementation.
JEL–Numbers: C78; D61; D78; I20.

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1 Introduction

An important application of mechanism design is school choice (Abdulkadiroğlu and Sönmez, 2003). This paper analyzes so-called rank-priority mechanisms for the centralized assignment of students to public schools based on the students' rankings (preferences) over schools and the schools' priorities over students. Rank-priority mechanisms were first studied in Roth (1991) in the context of the assignment of medical school graduates to consultants in the UK.

A rank-priority mechanism is determined by an order of all rank-priority pairs. Here, rank refers to the position that a student assigns to a school in his ranking, while priority refers to the position that a school assigns to a student in its priority ordering. Given a profile of rankings and priority orderings, a matching is determined step-wise by going through the order of rank-priority pairs. More specifically, at each step a rank-priority pair (r, f) is considered. If a school is ranked r^{th} by some student and if the student has priority f for the school, then the student is assigned to the school provided that the school still has vacant seats (after which the student and the seat are removed from the market). A student remains unassigned if he is not assigned to a school at any step. A well-known rank-priority mechanism is the immediate acceptance or "Boston" mechanism which is widely used in school choice.[1] The immediate acceptance mechanism is lexicographic as it first considers the student preferences and only then the school priorities.[2]

Roth (1991, Proposition 10) shows that no rank-priority mechanism is stable. Stability is a central concept in the matching literature and puts together three desiderata: individual rationality, non-wastefulness, and no justified envy.[3] Roth (1991) also illustrates that under rank-priority mechanisms agents have incentives to misrepresent their rankings.[4] Assuming complete information, Ergin and Sönmez (2006) study the strategic games induced by "monotonic" rank-priority mechanisms. Here, monotonicity refers to the requirement that for any distinct rank-priority pairs (r, f) and (r', f'), if $r \leq r'$ and $f \leq f'$, then (r, f) is considered before (r', f'). Ergin and Sönmez (2006, Proposition 4) show that monotonic rank-priority mechanisms Nash implement the set of stable matchings. Since the immediate acceptance mechanism is monotonic, it Nash implements the set of stable matchings (Ergin and Sönmez, 2006, Proposition 1). While monotonicity may seem natural, the economic appeal of a mechanism does not necessarily stem from its definition per se. Instead, the potential interest of a mechanism is probably mostly determined by the properties of the matchings that it induces. Therefore we investigate the following questions:

[1] See, for instance, Abdulkadiroğlu and Sönmez (2003), Basteck et al. (2015), Calsamiglia and Güell (2018), and Kojima and Ünver (2014). We refer to Pathak (2011) and Abdulkadiroğlu (2013) for surveys on mechanism design in school choice.

[2] It can be easily checked that the immediate acceptance mechanism is the rank-priority mechanism based on the order $(1, 1), (1, 2), \cdots, (1, n), (2, 1), \cdots, (2, n), \cdots, (m, 1), \cdots, (m, n)$, where m and n are the number of schools and students, respectively.

[3] We refer to the next section for the formal definition.

[4] It is well-known that the immediate acceptance mechanism is not an exception.

- are there non-monotonic rank-priority mechanisms that Nash implement the set of stable matchings?
- are there non-monotonic rank-priority mechanisms that Nash implement a (potentially interesting) subset[5] or superset of the set of stable matchings?
- if so, can we identify these subfamilies of rank-priority mechanisms?

The assumption of complete information and the study of Nash equilibria is far from unusual in the school choice literature.[6] However, Ergin and Sönmez (2006, Section 8) also consider an incomplete information environment where students do know the priorities and the capacities of the schools but not the realizations of the other students' types. They show that the immediate acceptance mechanism may induce Bayesian Nash equilibria with unstable matchings in their support.[7] This result prompts us to ask the following questions:

- is there another monotonic or a non-monotonic rank-priority mechanism that guarantees that all its Bayesian Nash equilibria have a support with stable matchings only?
- if so, can we identify the subfamily of such rank-priority mechanisms?

We answer all questions above. Regarding the complete information environment, we characterize the family of rank-priority mechanisms that implement the set of stable matchings. Our necessary and sufficient condition is that the order of rank-priority pairs be "quasi-monotonic." Loosely speaking, an order satisfies quasi-monotonicity if each priority appears in the sequence only after the precedent priority has appeared with sufficiently small ranks.[8] One might suspect that by demanding only "sub-implementation" or "sup-implementation" (rather than "full implementation") one would obtain a larger family of rank-priority mechanisms than the family of quasi-monotonic mechanisms. However, for any non-quasi-monotonic mechanism we exhibit a school choice problem such that the set of equilibrium outcomes is non-empty, the set of stable matchings is a singleton, and yet neither of the two sets is a subset of the other (Proposition 2). So, our result also holds for "sub-implementation" and "sup-implementation": a rank-priority mechanism sub/sup-implements the set of stable matchings if and only if it is quasi-monotonic (Corollary 3/Corollary 4).

Regarding the incomplete information environment, our second main result (Theorem 2) is a strong impossibility result: all rank-priority mechanisms exhibit the same feature as

[5]The lattice structure of the set of stable matchings allows for potentially interesting subsets.

[6]See, e.g., Pathak and Sönmez (2008), Haeringer and Klijn (2009), and more recently, Bando (2014), Dur and Morrill (2016), Dur et al. (2018,2019), and Dur (2019), among others.

[7]Ehlers (2008) also assumes incomplete information and studies a class of mechanisms that contains the class of monotonic rank-priority mechanisms. He shows that each non-truncation strategy is first-order stochastically dominated by a truncation strategy provided that students' information satisfies a symmetry property (Ehlers, 2008, Theorem 3.2).

[8]We refer to Section 3 for the formal definition and examples of quasi-monotonic rank-priority mechanisms.

the immediate acceptance mechanism. Therefore, one conclusion that can be drawn from our results is that in terms of stability of equilibrium outcomes there is no rank-priority mechanism that outperforms the immediate acceptance mechanism.

The remainder of the paper is organized as follows. In Section 2, we describe the school choice problem and rank-priority mechanisms. In Sections 3 and 4, we present our results for complete and incomplete information settings, respectively. Section 5 concludes.

2 Model

Let $\boldsymbol{I} = \{i_1, \ldots, i_n\}$ be the set of **students** and $\boldsymbol{S} = \{s_1, \ldots, s_m\}$ be the set of **schools**. We assume that $n \geq 2$ and $m \geq 1$. The sets I and S are kept fixed throughout.

Each student $i \in I$ has a complete, transitive, and strict **preference relation** $\boldsymbol{P_i}$ over the schools and "being unmatched" (e.g., attending a private school or being home-schooled), which is denoted by $\emptyset$. For each pair $s, s' \in S \cup \{\emptyset\}$, we write $s\, P_i\, s'$ if i prefers s to s', and $s\, R_i\, s'$ if i finds s as desirable as s', i.e., $s\, P_i\, s'$ or $s = s'$. A school $s \in S$ is acceptable (under P_i) if $s\, P_i\, \emptyset$. Given that only acceptable schools will be relevant, we often write a preference relation as a ranking (i.e., ordered list) of acceptable schools (and $\emptyset$ to indicate the end of the list). Preference relation P_i can also be encoded through a function $r_i : S \to \{1, \ldots, m, \infty\}$ by setting $r_i(s) \equiv k$ if s is the k^{th} highest ranked acceptable school under P_i. (So, if $r_i(s) = 1$ then s is student i's most preferred acceptable school.) Otherwise, $r_i(s) \equiv \infty$. We refer to $r_i(s)$ as the **rank** of s in P_i. We will use P_i and r_i interchangeably. Let $P \equiv (P_i)_{i \in I}$ be the preference profile. For each $i \in I$, $P_{-i} \equiv (P_j)_{j \neq i}$.

Each school $s \in S$ has a **capacity** $\boldsymbol{q_s} \geq 1$ which is the (integer) number of seats it offers. Let $q = (q_{s_1}, \ldots, q_{s_m})$ be the capacity vector. Each school $s \in S$ has a complete, transitive, and strict **priority relation** $\boldsymbol{\succ_s}$ over the students and a "vacant seat" denoted by $\emptyset$. For each pair $i, i' \in S \cup \{\emptyset\}$, we write $i \succ_s i'$ if i has a strictly higher priority than i' for s, and $s \succeq_i s'$ if i has a weakly higher priority than i' for s, i.e., $i \succ_i i'$ or $i = i'$. A student $i \in I$ is acceptable (under $\succ_s$) if $i \succ_s \emptyset$. Given that only acceptable students will be relevant, we often write a priority relation as an ordered list of acceptable students. A priority relation can also be encoded through a function $f_s : I \to \{1, \ldots, n, \infty\}$ by setting $f_s(i) \equiv k$ if i is the k^{th} highest priority acceptable student for school s, and $f_s(i) \equiv \infty$ otherwise. (So, a small value of $f_s(\cdot)$ indicates a high priority for school s. E.g., if $f_s(i) = 1$ then i has the highest priority for s.) We refer to $f_s(i)$ as the **priority** of i for s. We will use $\succ_s$ and f_s interchangeably. Let $\succ \equiv (\succ_s)_{s \in S}$ be the profile of priority relations.

A **problem** is a list $\boldsymbol{(I, S, P, \succ, q)}$ or, when no confusion is possible, $\boldsymbol{P}$ for short. Let $\mathcal{P}$ be the class of all problems. A **matching** $\boldsymbol{\mu}$ for problem $P \in \mathcal{P}$ is a function $\mu : I \cup S \to 2^I \cup S$ such that (a) each student is assigned to one school or is unassigned,[9] i.e., for each $i \in I$, $\mu(i) \in S \cup \{\emptyset\}$; (b) each school is assigned to a set of students that does not exceed its capacity, i.e., for each $s \in S$, $\mu(s) \in 2^I$ and $|\mu(s)| \leq q_s$; and (c) assignments are "consistent," i.e., for

[9]Note that $\emptyset \in 2^I$.

each $i \in I$ and $s \in S$, $\mu(i) = s$ if and only if $i \in \mu(s)$. We call $\mu(i)$ the match of student i and if $\mu(i) = s \in S$, we say that student i is assigned to school s. Let $\boldsymbol{\mathcal{M}(P)}$ denote the **set of matchings** for problem $P \in \mathcal{P}$.

Next, we describe desirable properties of matchings. First, we are interested in a voluntary participation condition. A matching μ is **individually rational** for problem P if for each $i \in I$ and $s \in S$ with $\mu(i) = s$, $s\ P_i\ \emptyset$ and $i \succ_s \emptyset$. Second, a matching is non-wasteful if no student prefers a school with some empty seat to his match and the school finds the student acceptable. Formally, a matching μ is **non-wasteful** for problem P if there is no student i and a school s such that $s\, P_i\, \mu(i)$, $|\mu(s)| < q_s$, and $i \succ_s \emptyset$. Finally, a student i is said to have justified envy if there is a school s such that i prefers s to his match, and i has higher priority at s than some student assigned to s. Formally, a student i has **justified envy** at μ for problem P if there is a school s and a student $j \in \mu(s)$ such that $s\, P_i\, \mu(i)$ and $i \succ_s j$. A matching μ is **stable** for P if it is *individually rational*, *non-wasteful*, and no student has *justified envy* for P. Let $\boldsymbol{\mathcal{S}(P)}$ denote the **set of stable matchings** for problem $P \in \mathcal{P}$. From Gale and Shapley (1962) it follows that for each $P \in \mathcal{P}$, $\mathcal{S}(P) \neq \emptyset$.

A mechanism φ is a function that selects for each problem a matching, i.e., for each $P \in \mathcal{P}$, $\varphi(P) \in \mathcal{M}(P)$. In this paper we focus on the family of rank-priority mechanisms which are defined next. Let $\pi : \{1, \ldots, m\} \times \{1, \ldots, n\} \to \{1, \ldots, m \cdot n\}$ be a bijection. Each element $\boldsymbol{(r, f)} \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ is interpreted as a **rank-priority pair**, i.e., r is a rank and f is a priority. We often equivalently denote π by its induced order of rank-priority pairs, i.e., $(r^1, f^1), (r^2, f^2), \ldots, (r^{m \cdot n}, f^{m \cdot n})$ where for all k, $\pi(r^k, f^k) = k$. Thus, we will refer to $\boldsymbol{\pi}$ as an **order of rank-priority pairs**. Then, the **rank-priority mechanism $\boldsymbol{\varphi^\pi}$** is defined as follows. Let Q be a profile of students' preferences. Set $\tilde{I} \equiv I$. For each $s \in S$, set $\tilde{q}_s \equiv q_s$. Matching $\varphi^\pi(Q)$ is obtained in $m \cdot n$ steps:

STEP $k = 1, \ldots, m \cdot n$: As long as there are $i \in \tilde{I}$ and $s \in S$ such that

(c1) s has rank r^k in Q_i,

(c2) i has priority f^k for s, and

(c3) s still has some empty seat, i.e., $\tilde{q}_s > 0$,

assign student i to school s and set $\tilde{q}_s \equiv \tilde{q}_s - 1$ and $\tilde{I} \equiv \tilde{I} \backslash \{i\}$.

After step $m \cdot n$, the students in $\tilde{I}$ remain unmatched. Let $\varphi^\pi(Q)$ denote the thus induced matching. Note that at each step of the algorithm multiple students can be assigned, but at most one to each school (because for any school only one student has a given priority). Let $\boldsymbol{\mathcal{F}}$ denote the **family of rank-priority mechanisms**.

Example 1. [A rank-priority mechanism]

Consider the school choice problem with $I = \{i_1, \ldots, i_6\}$, $S = \{s_1, \ldots, s_7\}$, $q = (1, \ldots, 1)$, and preferences P and priorities $\succ$ as given in Table 1. In each student's column, higher placed schools are more preferred. For instance, $r_{i_4}(s_1) = 5$. In each school's column, higher placed students have higher priority. For instance, $f_{s_1}(i_4) = 2$. Since all students and all

Students						Schools						
P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$	$\succ_{s_6}$	$\succ_{s_7}$
$\boldsymbol{s_1}$	$\boldsymbol{s_2}$	$\boldsymbol{s_3}$	$\boxed{\boldsymbol{s_4}}$	s_7	$\boxed{\boldsymbol{s_7}}$	$\boxed{i_3}$	$\boxed{i_1}$	i_1	i_2	i_4	$\boldsymbol{i_5}$	$\boxed{\boldsymbol{i_6}}$
s_7	s_7	s_2	s_5	$\boldsymbol{s_6}$	s_1	i_4	i_3	$\boldsymbol{i_3}$	i_1	i_2	i_3	i_2
s_6	s_6	s_4	s_6	s_1	s_2	i_5	i_4	$\boxed{i_2}$	$\boxed{\boldsymbol{i_4}}$	i_1	i_2	i_5
s_5	s_5	s_5	s_7	s_2	s_3	i_2	i_5	i_4	i_3	i_3	i_1	i_1
s_4	s_4	s_6	s_1	s_3	s_4	$\boldsymbol{i_1}$	$\boldsymbol{i_2}$	i_5	i_5	$\boxed{i_5}$	i_4	i_3
s_3	$\boxed{s_3}$	s_7	s_2	s_4	s_5	i_6	i_6	i_6	i_6	i_6	i_6	i_4
$\boxed{s_2}$	s_1	$\boxed{s_1}$	s_3	$\boxed{s_5}$	s_6							

Table 1: Preferences P and priorities $\succ$ in Example 1.

schools are mutually acceptable, we have omitted $\emptyset$ from the preference relations and priority relations. Consider any order of rank-priority pairs π such that

$$\begin{aligned} \pi \quad : \quad & (7,1),(6,2),(6,3),(6,4),(5,2),(5,4),(7,5),(6,5),(5,3),(1,3),(4,3),(3,2), \\ & (2,1),(4,5),(1,4),(1,1),(2,3),(2,4),(3,5),(5,5),(1,6),\ldots. \end{aligned} \tag{1}$$

To illustrate the algorithm above, we compute $\varphi^\pi(P)$, going through the steps defined by the sequence π:

- At step $1 = \pi(7,1)$, rank-priority pair $(7,1)$ is considered, i.e., rank 7 in each student's preference relation together with priority 1 in each school's priority relation. School s_2 has rank 7 in the preference relation of student i_1. In addition, student i_1 has priority 1 for school s_2. Moreover, school s_2 has still an empty seat. Hence, conditions (c1), (c2), and (c3) are satisfied for i_1 and s_2, and student i_1 is assigned to school s_2. Similarly, student i_3 is assigned to school s_1.
- At step $2 = \pi(6,2)$, no student-school pair satisfies conditions (c1), (c2), and (c3), and hence no student is assigned.
- At step $3 = \pi(6,3)$, student i_2 is assigned to school s_3.
- At steps $4 = \pi(6,4)$, $5 = \pi(5,2)$, and $6 = \pi(5,4)$, no student-school pair satisfies conditions (c1), (c2), and (c3), and hence no student is assigned.
- At step $7 = \pi(7,5)$, student i_5 is assigned to school s_5.
- At steps $8 = \pi(6,5)$ and $9 = \pi(5,3)$, no student-school pair satisfies conditions (c1), (c2), and (c3), and hence no student is assigned.
- At step $10 = \pi(1,3)$, student i_4 is assigned to school s_4.
- At steps $11,\ldots,15$, no student-school pair satisfies conditions (c1), (c2), and (c3), and hence no student is assigned.
- Finally, at step $16 = \pi(1,1)$, student i_6 is assigned to school s_7.

Hence, at problem P, the mechanism φ^π yields the "boxed" matching in Table 1:

$$\varphi^\pi(P) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & \emptyset \\ s_2 & s_3 & s_1 & s_4 & s_5 & s_7 & s_6 \end{pmatrix},$$

which is not stable, since the unique stable matching is

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & \emptyset \\ s_1 & s_2 & s_3 & s_4 & s_6 & s_7 & s_5 \end{pmatrix}, \tag{2}$$

i.e., the boldfaced matching in Table 1. ◇

The fact that the rank-priority mechanism in Example 1 yields an unstable matching is not surprising: Roth (1991, Proposition 10) shows that for each rank-priority mechanism, there is some problem such that the mechanism yields an unstable matching.

We assume that priorities are determined by laws and that capacities are commonly known by the students.[10] Hence, students are the only strategic agents. A **strategy** is a preference relation. For each $i \in I$, let $\mathcal{P}_i$ denote the set of strategies. With slight abuse of notation, let $\mathcal{P} \equiv \prod_{i\in I} \mathcal{P}_i$. Given a rank-priority mechanism φ^π, a **game** is a quadruple $\Gamma = (I, (\mathcal{P}_i)_{i\in I}, \varphi^\pi, P)$, or $\boldsymbol{\Gamma = (\varphi^\pi, P)}$ for short, where I is the set of players, $\mathcal{P}_i$ is the set strategies of player $i \in I$, φ^π is the outcome function, and the outcome is evaluated through the (true) preference relations P of the students.

Example 2. [A rank-priority mechanism, Example 1 cont'd]
Consider again the school choice problem and the order of rank-priority pairs π (see (1)) from Example 1. At $\varphi^\pi(P)$, student i_1 is assigned to his least preferred school s_2. Can student i_1 obtain a more preferred school by submitting a different strategy? Since student i_3 is assigned to school s_1 at step 1, for any ranking $\tilde{P}_{i_1}$ for student i_1,

$$\varphi^\pi_{i_1}(\tilde{P}_{i_1}, P_{-i_1}) \neq s_1.$$

So, student i_1 cannot obtain his most preferred school s_1. However, he can obtain another school that is also preferred to s_2. For instance, student i_1 can obtain school s_5 by submitting the strategy $P'_{i_1} \equiv *, *, *, *, *, s_5, \emptyset$, where $*, *, *, *, *$ are five different schools in $S\backslash\{s_5\}$. It can be easily verified that at profile (P'_{i_1}, P_{-i_1}), student i_1 is assigned to school s_5 at step $3 = \pi(6, 3)$. Note that student i_1 does *not* obtain the seat at school s_5 by submitting the strategy $P''_{i_1} \equiv s_5, \emptyset$, because at profile (P''_{i_1}, P_{-i_1}), student i_5 is assigned to school s_5 at step $\pi(7, 5) < \pi(1, 3)$. ◇

Roth (1991, Proposition 5) already detects and formalizes the problem that Example 2 exhibits: rank-priority mechanisms are vulnerable to manipulation. For this reason, we will

[10] In many school choice applications, students are prioritized at each school using some exogenous criteria, e.g., neighborhood or walk-zone priority (see Pathak, 2011 and Abdulkadiroğlu, 2013). Capacities are also often determined by laws. In particular, capacities cannot be manipulated (cf. Sönmez, 1997).

study the Nash equilibria of the games induced by rank-priority mechanisms. A strategy-profile $Q \in \mathcal{P}$ is a (Nash) equilibrium of the game (φ^π, P) if for each student i and for each $Q'_i \in \mathcal{P}_i$, $\varphi^\pi_i(Q_i, Q_{-i})\, R_i\, \varphi^\pi_i(Q'_i, Q_{-i})$. Let $\boldsymbol{\mathcal{E}(\varphi^\pi, P)}$ denote the **set of equilibria**. Let $\boldsymbol{\mathcal{O}(\varphi^\pi, P)}$ denote the **set of equilibrium outcomes**, i.e.,

$$\mathcal{O}(\varphi^\pi, P) = \{\mu \in \mathcal{M}(P) : \mu = \varphi^\pi(Q) \text{ and } Q \in \mathcal{E}(\varphi^\pi, P)\}.$$

Mechanism φ^π **(Nash) implements the set of stable matchings** if for each problem $P \in \mathcal{P}$, $\mathcal{O}(\varphi^\pi, P) = \mathcal{S}(P)$. Ergin and Sönmez (2006, Theorem 4) show that if φ^π is *monotonic*, then it implements the set of stable matchings. A rank-priority mechanism $\varphi^\pi \in \mathcal{F}$ is **monotonic** (Ergin and Sönmez, 2006) if

$$[(r, f) \neq (r', f'),\ r \leq r',\ \text{and}\ f \leq f'] \implies \pi(r, f) < \pi(r', f'). \tag{3}$$

Since (3) is in fact a condition on π, we will interchangeably refer to the *monotonicity* of π and φ^π. Let $\boldsymbol{\mathcal{F}^m}$ denote the **family of monotonic rank-priority mechanisms**. The **immediate acceptance** or **Boston mechanism** $\varphi^{\pi^{ia}}$ (Abdulkadiroğlu and Sönmez, 2003) is a particular rank-priority mechanism where π^{ia} lexicographically orders pairs (r, f): $\pi^{ia} : (1,1), (1,2), \cdots, (1,n), (2,1), \cdots, (2,n), \cdots, (m,1), \cdots, (m,n)$. Note that the immediate acceptance mechanism is *monotonic*, i.e., $\varphi^{\pi^{ia}} \in \mathcal{F}^m$. In the next section we will see that *monotonicity* is not necessary for the implementation of the set of stable matchings.

3 Characterization

In this section, we introduce a weaker monotonicity property and prove that it characterizes the subfamily of rank-priority mechanisms that implement the set of stable matchings.

Let π be an order of rank-priority pairs. For any priority $f \in \{1, \ldots, n\}$, we let $\pi(f)$ denote the first position in the order where priority f appears, i.e.,

$$\boldsymbol{\pi(f)} \equiv \min\{\, \pi(r, f) : r \in \{1, \ldots m\}\}.$$

Rank-priority mechanism φ^π is **quasi-monotonic** if $\pi(1) = 1$ and for each priority $f \in \{2, \ldots, n-1\}$ and for each priority $f' \in \{1, \ldots, n-2\}$, $f' < f$, there is a rank $r_{f'} \in \{1, \ldots, m\}$ such that

$$(i)\ \pi(r_{f'}, f) < \pi(f+1) \qquad \text{and} \qquad (ii)\ r' < r_{f'} \implies \pi(r', f') > \pi(r_{f'}, f). \tag{4}$$

We provide some examples of rank-priority mechanisms to illustrate quasi-monotonicity in Example 3. Since (4) is a condition on π, we will interchangeably refer to the *quasi-monotonicity* of π and φ^π. Note that *quasi-monotonicity* in fact only imposes restrictions on the rank-priority pairs that appear in π before position $\pi(n)$, i.e., the position in which priority n appears for the first time. Let $\boldsymbol{\mathcal{F}^q}$ denote the **family of quasi-monotonic rank-priority mechanisms**. The following lemma shows that *monotonicity* implies *quasi-monotonicity*.

Lemma 1. Monotonic *rank-priority mechanisms are* quasi-monotonic, *i.e.,* $\mathcal{F}^m \subseteq \mathcal{F}^q$.

Proof. Let φ^π be a *monotonic* rank-priority mechanism. First, $\pi(1) = \pi(1,1) = 1$. Second, let $f \in \{2, \ldots, n-1\}$ and $f' < f$. Let $r_{f'} = 1$. Then, by *monotonicity*,

$$\pi(r_{f'}, f) = \pi(1, f) < \pi(1, f+1) = \pi(f+1),$$

which proves (i) in (4). Since there is no rank $r' < 1 = r_{f'}$, (ii) in (4) is vacuously satisfied. Hence, φ^π is *quasi-monotonic.* $\square$

We say that π (or equivalently, φ^π) satisfies **unit increments of priority (UIP)** if

$$\pi(1) < \pi(2) < \cdots < \pi(n).$$

Let $\boldsymbol{\mathcal{F}^u}$ denote the **family of rank-priority mechanisms that satisfy UIP**. The following result is immediate.

Lemma 2. Quasi-monotonicity *implies* unit increments of priority, *i.e.,* $\mathcal{F}^q \subseteq \mathcal{F}^u$.

Proof. Follows immediately from $\pi(1) = 1$ and condition (i) in (4). $\square$

Before we state and prove our main result, we provide some examples of rank-priority mechanisms to illustrate quasi-monotonicity.

Example 3. [Rank-priority mechanisms]
Consider the following orders of rank-priority pairs. A priority is in boldface whenever it appears for the first time in the order.

$$\begin{aligned}
\pi^{ia} \equiv \pi^1 \ &: \ (1,\mathbf{1}),(1,\mathbf{2}),\cdots,(1,\boldsymbol{n}),(2,1),\cdots,(2,n),\cdots,(m,1),\cdots,(m,n).\\
\pi^2 \ &: \ (1,\mathbf{1}),(2,1),\cdots,(m,1),(1,\mathbf{2}),\cdots,(m,2),\cdots,(1,\boldsymbol{n}),\cdots,(m,n).\\
\pi^3 \ &: \ \pi^3(r,f) < \pi^3(r',f') \iff r\cdot f < r'\cdot f' \text{ or } [r\cdot f = r'\cdot f' \text{ and } r < r'].\\
\pi^4 \ &: \ \pi^4(r,f) < \pi^4(r',f') \iff r\cdot f < r'\cdot f' \text{ or } [r\cdot f = r'\cdot f' \text{ and } f < f'].\\
\pi^5 \ &: \ (2,\mathbf{1}),(3,1),(3,\mathbf{2}),(2,2),(1,1),(1,\mathbf{3}),(2,\mathbf{4}),\cdots \text{ where } n=4.\\
\pi^6 \ &: \ (3,\mathbf{1}),(2,\mathbf{2}),(3,\mathbf{3}),(1,1),(2,3),(2,1),(2,\mathbf{4}),\cdots \text{ where } n=4.\\
\pi^7 \ &: \ (2,\mathbf{1}),(3,1),(3,\mathbf{2}),(2,2),(1,2),(2,\mathbf{3}),(1,3),(1,1),(2,\mathbf{4}),\cdots \text{ where } n=4.\\
\pi^8 \ &: \ (3,\mathbf{1}),(3,\mathbf{2}),(3,\mathbf{3}),(3,\mathbf{4}),\cdots \text{ where } n=4.\\
\pi^9 \ &: \ (4,\mathbf{1}),(3,1),(3,\mathbf{2}),(2,2),(1,1),(1,\mathbf{4}),\cdots.\\
\pi^{10} \ &: \ (2,\mathbf{1}),(3,1),(3,\mathbf{2}),(1,1),(2,2),(1,\mathbf{3}),\cdots.
\end{aligned}$$

For $k = 1, \ldots, 4$, mechanism φ^{π^k} is *monotonic.* It is not difficult to check that for $k = 5, \ldots, 8$, mechanism φ^{π^k} is *quasi-monotonic*, but not *monotonic.*[11] In the case of $k = 5$, condition

[11] So, $\mathcal{F}^q \not\subseteq \mathcal{F}^m$.

(4) is satisfied for priorities $f = 2$ and $f' = 1$ (with rank $r_{f'} = 2$ but not $r_{f'} = 3$), and for priorities $f = 3$ and $f' \in \{1, 2\}$ (with $r_{f'} = 1$). In particular, in the case of $k = 5$ for a given f, we can take the same $r_{f'}$ for all $f' < f$.

In the case of $k = 6$, condition (4) is satisfied for priorities $f = 2$ and $f' = 1$ (with rank $r_{f'} = 2$), for priorities $f = 3$ and $f' = 1$ (with rank $r_{f'} = 3$ but not $r_{f'} = 2$), and for priorities $f = 3$ and $f' = 2$ (with rank $r_{f'} = 2$ but not $r_{f'} = 3$).

In the case of $k = 7$, condition (4) is satisfied for priorities $f = 2$ and $f' = 1$ (with rank $r_{f'} \in \{1, 2\}$ but not $r_{f'} = 3$), for priorities $f = 3$ and $f' = 1$ (with rank $r_{f'} \in \{1, 2\}$), and for priorities $f = 3$ and $f' = 2$ (with rank $r_{f'} = 1$ but not $r_{f'} = 2$).

Finally, mechanisms φ^{π^9} and $\varphi^{\pi^{10}}$ are not *quasi-monotonic*. For φ^{π^9}, condition (i) in (4) is not satisfied for $f = 3$ since $\pi(3) > \pi(4)$. For $\varphi^{\pi^{10}}$, condition (ii) in (4) is not satisfied for $f = 2$ and $f' = 1$.[12] To see this, note first that the only candidates for $r_{f'}$ are $r_1 = 2$ and $r_1 = 3$. However, if $r_1 = 2$ then $(r', f') = (1, 1)$ violates (ii), and if $r_1 = 3$ then $(r', f') = (2, 1)$ violates (ii). $\diamond$

Our main result shows that *quasi-monotonicity* is a necessary and sufficient condition for the Nash implementation of the set of stable matchings.

Theorem 1. [Implementation: characterization]
A rank-priority mechanism $\varphi^\pi \in \mathcal{F}$ Nash implements the set of stable matchings if and only if it is quasi-monotonic*, i.e., $\varphi^\pi \in \mathcal{F}^q$.*

Theorem 1 immediately follows from Propositions 1 and 2, which are stated and proved below. We first provide an example that gives insights into how a *quasi-monotonic* rank-priority mechanism can implement the set of stable matchings (which is formalized in Proposition 1).

Example 4. [A rank-priority mechanism, Example 1 cont'd]
Consider again the school choice problem from Example 1. We show that the order of rank-priority pairs π in (1) is *quasi-monotonic*. More specifically, for each $f \in \{2, 3, 4, 5\}$ and each $f' \in \{1, 2, 3, 4\}$ with $f' < f$, we specify in Table 2 all ranks $r_{f'}$ such that condition (4) is satisfied.

From Theorem 1 (or more specifically, Proposition 1), it follows that there exists an equilibrium of the game induced by φ^π that yields the (unique) stable matching μ (given in (2)). A slightly naive approach to find such an equilibrium is to consider the strategy-profile $\tilde{Q}$ where each student i only lists the school $\mu(i)$, see Table 3. Obviously, $\varphi(\tilde{Q}) = \mu$. But $\tilde{Q}$ is not an equilibrium: $\tilde{Q}'_{i_5} \equiv s_6, s_1, s_7, \emptyset$ is a profitable deviation for student i_5 because $\varphi_{i_5}(\tilde{Q}'_{i_5}, \tilde{Q}_{-i_5}) = s_7 \, P_{i_5} \, s_6 = \varphi_{i_5}(\tilde{Q})$.

[12] It is easy to complete π^{10} so that $\varphi^{\pi^{10}}$ satisfies UIP. So, $\mathcal{F}^u \not\subseteq \mathcal{F}^q$.

$f =$	$f' =$	$r_{f'} \in$
2	1	{6}
3	1	{6}
	2	{6}
4	1	{5,6}
	2	{5,6}
	3	{5,6}
5	1	{6,7}
	2	{3}
	3	{6}
	4	{4}

Table 2: Verification of *quasi-monotonicity* of π in (1).

$\tilde{Q}_{i_1}$	$\tilde{Q}_{i_2}$	$\tilde{Q}_{i_3}$	$\tilde{Q}_{i_4}$	$\tilde{Q}_{i_5}$	$\tilde{Q}_{i_6}$
s_1	s_2	s_3	s_4	s_6	s_7
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 3: Strategy-profile $\tilde{Q}$ in Example 4.

To find an equilibrium, we instead construct a strategy-profile Q^* such that for each $i \in I$,

(q1) ranking Q_i^* lists school $\mu(i)$ in the last rank, say r_i^*;
(q2) $\pi(r_i^*, f_{\mu(i)}(i)) < \pi(f_{\mu(i)}(i) + 1)$;
(q3) student i is *not* assigned to a school *until step* $\pi(r_i^*, f_{\mu(i)}(i))$, independently of Q_{-j}^*.

Since μ is a (feasible) matching, it follows from (q1) and (q3) that for each $i \in I$, at step $\pi(r_i^*, f_{\mu(i)}(i))$ student i is assigned to school $\mu(i)$, i.e., $\varphi^\pi(Q^*) = \mu$. Condition (q2) guarantees that Q^* is a Nash equilibrium.[13] Lemma 3, which is stated after the example, shows that it is always possible to find a rank r_i^* and a ranking Q_i^* that satisfy (q1), (q2), and (q3). Remark 3, which appears at the end of the Appendix, discusses a construction of Q_i^* using computationally efficient algorithms from graph theory. Below, we take a slightly different, direct approach to illustrate some of the aspects of the construction.

Consider student i_1 and school $\mu(i_1) = s_1$. Since $f_{\mu(i_1)}(i_1) = f_{s_1}(i_1) = 5$, the possible candidates for $r_{i_1}^*$ that satisfy (q2) are $3, 4, 5, 6, 7$. Does $r_{i_1}^* = 3$ allow some $Q_{i_1}^*$ to satisfy (q3)? To answer this question, we have to check whether it is possible to put some school $s \neq s_1$ at rank 1 and some other school $s' \neq s, s_1$ at rank 2 such that student i_1 is *not* assigned to either of schools s and s' until step $\pi(r_{i_1}^*, f_{\mu(i_1)}(i_1)) = \pi(3, 5)$, independently of the other students' rankings. One easily verifies that at rank 1,

[13] This can be seen in the proof of Proposition 1.

- we cannot put $s \in \{s_2, s_3\}$ because $\pi(1,1) < \pi(3,5)$ (otherwise i_1 could potentially be assigned to s at step $\pi(1,1)$);
- we cannot put $s = s_5$ because $\pi(1,3) < \pi(3,5)$;
- we cannot put $s \in \{s_6, s_7\}$ because $\pi(1,4) < \pi(3,5)$.

Therefore, the only possible school that we can put at rank 1 is school s_4. But at rank 2,

- we cannot put $s' \in \{s_2, s_3\}$ because $\pi(2,1) < \pi(3,5)$;
- we cannot put $s' = s_5$ because $\pi(2,3) < \pi(3,5)$;
- we cannot put $s' \in \{s_6, s_7\}$ because $\pi(2,4) < \pi(3,5)$.

In other words, the only possible school that we can put at rank 2 is again school s_4. But since each school can be put at (no more than) one rank, there is no $Q^*_{i_1}$ that satisfies (q3). So, $r^*_{i_1} \neq 3$.[14] Next, we consider the next candidate: does $r^*_{i_1} = 4$ allow some $Q^*_{i_1}$ to satisfy (q3)? Proceeding in a similar way as above, one easily verifies that

- at rank 1 we can put any of s_2, s_3, s_4, s_6, s_7 (but not s_5 because $\pi(1,3) < \pi(4,5)$);
- at rank 2 we can put any of s_4, s_5, s_6, s_7 (but not s_2 nor s_3 because $\pi(2,1) < \pi(4,5)$);
- at rank 3 we can put any of s_2, s_3, s_5, s_6, s_7 (but not s_4 because $\pi(3,2) < \pi(4,5)$).

Therefore, $r^*_{i_1} \equiv 4$ and $Q^*_{i_1} \equiv s_4, s_5, s_6, s_1, \emptyset$ satisfy conditions (q1), (q2), and (q3).

Next, consider student i_2 and school $\mu(i_2) = s_2$. Since $f_{\mu(i_2)}(i_2) = f_{s_2}(i_2) = 5$, the possible candidates for $r^*_{i_2}$ that satisfy (q2) are $3, 4, 5, 6, 7$ (as in the case of $r^*_{i_1}$). However, it can be easily verified that this time rank 3 can be used, i.e., $r^*_{i_2} \equiv 3$ together with (for instance) $Q^*_{i_1} \equiv s_5, s_7, s_2, \emptyset$ satisfies conditions (q1), (q2), and (q3). Continuing this procedure for the other students as well, we obtain a strategy-profile that satisfies (q1), (q2), and (q3). One such profile is Q^* depicted in Table 4.

$Q^*_{i_1}$	$Q^*_{i_2}$	$Q^*_{i_3}$	$Q^*_{i_4}$	$Q^*_{i_5}$	$Q^*_{i_6}$
s_4	s_5	s_1	s_1	s_1	s_1
s_2	s_7	s_2	s_2	s_2	s_2
s_5	s_2	s_4	s_3	s_3	s_3
s_1	$\emptyset$	s_5	s_5	s_4	s_4
$\emptyset$		s_6	s_6	s_5	s_5
		s_3	s_4	s_7	s_6
		$\emptyset$	$\emptyset$	s_6	s_7
				$\emptyset$	$\emptyset$

Table 4: Equilibrium Q^* in Example 4.

[14] In particular, this shows that even if r^*_i satisfies (q2) it need not allow for a Q^*_i that satisfies (q3).

As mentioned earlier, due to (q1) and (q3), $\varphi^\pi(Q^*) = \mu$. It is easy to see that Q^* is an equilibrium. Suppose it is not an equilibrium. Since only student i_5 does not get his most preferred school at μ, there is a strategy Q'_{i_5} such that

$$\varphi_{i_5}(Q'_{i_5}, Q^*_{-i_5})\, P_{i_5}\, \varphi_{i_5}(Q^*) = s_6.$$

Then, $\varphi_{i_5}(Q'_{i_5}, Q^*_{i_5}) = s_7$. However, student i_6 is assigned to school s_7 at step $1 = \pi(7,1)$, independently of the other students' strategies. Hence, $\varphi_{i_5}(Q'_{i_5}, Q^*_{i_5}) \neq s_7$. So, Q^* is an equilibrium. $\diamond$

The following lemma formalizes the observations from Example 4 and will be key in the proof of Proposition 1. We use the convention that $\pi(n+1) \equiv \infty$.

Lemma 3. *Let φ^π be quasi-monotonic. Let i be a student and s be a school. Suppose $f \equiv f_s(i) < \infty$. Then there is a rank $r^* \equiv r^*(i,s)$ with $\pi(r^*, f) < \pi(f+1)$ and a strategy*

$$Q^*_i \equiv \cdots, \underbrace{s}_{\text{at rank } r^*}, \emptyset$$

such that if the rank-priority algorithm of φ^π is applied to $Q = (Q^*_i, Q_{-i})$, where Q_{-i} is any strategy-profile of the other students, then student i remains unassigned until the end of step $\pi(r^*, f) - 1$ (and hence is assigned to school s at step $\pi(r^*, f)$ if at that point the school still has an empty seat).

The proof of Lemma 3 is relegated to the Appendix.

Remark 1. Lemma 3 is key in the proof of Nash implementation (Proposition 1) for two reasons. First, for any stable matching it allows us to find an equilibrium that induces this matching. Second, for any profile that yields an unstable matching it allows us to find a profitable deviation. Even though the statement of the lemma only deals with the existence of a particular strategy, in Remark 3 (in the Appendix) we briefly discuss how such a strategy can be constructed in polynomial time. $\diamond$

Remark 2. Without the requirement $\pi(r^*, f) < \pi(f+1)$, Lemma 3 would be trivial (take $r^* = 1$). For the same reason we could dispense with the lemma if we were to prove Nash implementation for *monotonic* rank-priority mechanisms.[15] Put differently, the strategies that are needed and used in Proposition 1 (see Remark 1) for *monotonic* rank-priority mechanisms can simply consist of a single school. $\diamond$

We can now state and prove the propositions that imply Theorem 1.

Proposition 1. [Quasi-monotonic mechanisms: implementation]
If a rank-priority mechanism is quasi-monotonic*, then it Nash implements the set of stable matchings.*

[15] This follows from the fact that for any monotonic π we have that for each priority $f \in \{1, \ldots, n-1\}$, $\pi(1, f) < \pi(f+1)$.

Proof. Let φ^π be *quasi-monotonic*. We show that φ^π implements the set of stable matchings, i.e., for each problem $P \in \mathcal{P}$, $\mathcal{O}(\varphi^\pi, P) = \mathcal{S}(P)$. Let $P \in \mathcal{P}$.

We first prove the inclusion $\mathcal{S}(P) \subseteq \mathcal{O}(\varphi^\pi, P)$. Let $\mu \in \mathcal{S}(P)$. For each $i \in I$ with $\mu(i) \neq \emptyset$, let $f(i) \equiv f_{\mu(i)}(i) < \infty$. For each $i \in I$, consider a strategy

$$Q_i^* \equiv \begin{cases} \emptyset & \text{if } \mu(i) = \emptyset; \\ \cdots, \underbrace{\mu(i)}_{\text{at rank } r^*(i,\mu(i))}, \emptyset & \text{if } \mu(i) \neq \emptyset, \end{cases}$$

as defined in Lemma 3.

Obviously, for each $i \in I$ with $\mu(i) = \emptyset$, $\varphi_i^\pi(Q^*) = \emptyset = \mu(i)$. Now let $i \in I$ with $\mu(i) \neq \emptyset$. Since for each $s \in S$, $|\mu(s)| \leq q_s$, it follows from Lemma 3 that each student $i \in I$ with $\mu(i) \neq \emptyset$ is assigned to $\mu(i)$ at step $\pi(r^*(i, \mu(i)), f(i))$. Hence, $\varphi^\pi(Q^*) = \mu$.

Next, we show that Q^* is an equilibrium. Suppose that some student $i \in I$ has a deviation Q_i' such that $\varphi_i^\pi(Q') = s \; P_i \; \mu(i)$ where $Q' \equiv (Q_i', Q_{-i}^*)$ and $s \in S$ (otherwise, μ would not be individually rational, contradicting $\mu \in \mathcal{S}(P)$). Then, $f_s(i) \leq n$ and under Q', student i is assigned to school $s \in S$ at a step $\pi(r, f_s(i)) \geq \pi(f_s(i))$ for some $r \in \{1, \ldots, m\}$.

Since $\mu \in \mathcal{S}(P)$ and $s \; P_i \; \mu(i)$, (a) $|\mu(s)| = q_s$ and (b) for each $j \in \mu(s)$, $f(j) = f_s(j) < f_s(i)$ (so, in particular, $f(j) \neq n$). In view of (a), let $j \in \mu(s)$ such that $\varphi_j^\pi(Q') \neq s$. Since $f(j) \neq n$, it follows from the definition of $r^*(j, \mu(j))$ that $\pi(r^*(j, \mu(j)), f(j)) < \pi(f(j)+1)$. From (b), $f(j) + 1 \leq f_s(i)$. Hence from UIP, $\pi(f(j)+1) \leq \pi(f_s(i))$. Hence, $\pi(r^*(j, \mu(j)), f(j)) < \pi(f_s(i))$. So, $\pi(r^*(r, \mu(j)), f(j)) < \pi(r, f_s(i))$. Then, since under Q' student i is assigned to school s at step $\pi(r, f_s(i))$, there is still an empty seat at s at step $\pi(r^*(j, \mu(j)), f(j))$. Since $Q_j' = Q_j^*$, it follows from Lemma 3 that under Q' student j is assigned to s, which contradicts $\varphi_j^\pi(Q') \neq s$. Hence, there is no profitable deviation for any student. So, $Q \in \mathcal{E}(\varphi^\pi, P)$. Hence, $\mu \in \mathcal{O}(\varphi^\pi, P)$. So, $\mathcal{S}(P) \subseteq \mathcal{O}(\varphi^\pi, P)$.

Finally, we prove the inclusion $\mathcal{O}(\varphi^\pi, P) \subseteq \mathcal{S}(P)$. Let $Q \in \mathcal{E}(\varphi^\pi, P)$ and $\mu = \varphi^\pi(Q)$. Suppose $\mu \notin \mathcal{S}(P)$. Since $\mu = \varphi^\pi(Q)$, for each $i \in I$ and $s \in S$ with $\mu(i) = s$, $i \succ_s \emptyset$. Therefore, we can distinguish between the following two cases.

Case 1: There is $i^* \in I$ with $\emptyset \, P_{i^*} \, \mu(i^*)$.
Let student i^* report $Q_{i^*}' = \emptyset$. Then, for $Q' \equiv (Q_{i^*}', Q_{-i^*})$, $\varphi_{i^*}^\pi(Q') = \emptyset$. Hence, Q_{i^*}' is a profitable deviation.

Case 2a: There are $i^* \in I$ and $s^* \in S$ with $s^* \, P_{i^*} \, \mu(i^*)$, $|\mu(s^*)| < q_{s^*}$, and $i^* \succ_{s^*} \emptyset$.
(Note that $i^* \succ_{s^*} \emptyset$ is equivalent to $f_{s^*}(i^*) < \infty$.)
Case 2b: There are $i^*, j^* \in I$ and $s^* \in S$ with $s^* \, P_{i^*} \, \mu(i^*)$, $j^* \in \mu(s^*)$, and $i^* \succ_{s^*} j^*$. (Note that $i^* \succ_{s^*} j^*$ is equivalent to $f_{s^*}(i^*) < f_{s^*}(j^*)$. Moreover, $j \succ_{s^*} \emptyset$, or, equivalently, $f_{s^*}(j^*) < \infty$.)
Let $f^* \equiv f_{s^*}(i^*) < \infty$. Take $r^* \equiv r^*(i^*, s^*)$ as defined in Lemma 3. Let $k^* \equiv \pi(r^*, f^*)$ and

$$Q_{i^*}^* \equiv \cdots, \underbrace{s^*}_{\text{at rank } r^*}, \emptyset,$$

as defined in Lemma 3. Using the following claim we will prove that $Q^*_{i^*}$ is a profitable deviation for student i^*.

Claim. *Consider the rank-priority algorithm of φ^π for Q and $Q^* \equiv (Q^*_{i^*}, Q_{-i^*})$. Then, at the* beginning *of each step k, $1 \leq k \leq k^*$,*
(1.) *For each student i with $i \neq i^*$, if i is already assigned under Q, then he is also already assigned under Q^*.*
(2.) *For each school s, there are at least as many unassigned seats under Q^* as under Q.*

Proof of Claim. We prove the Claim by induction. Since the rank-priority algorithm starts with each student unassigned, the Claim holds for $k = 1$. Suppose the Claim holds for some step k, $1 \leq k < k^*$. Let $(r, f) \equiv \pi^{-1}(k)$. We will show that it also holds for step $k + 1$.
(1.) Let i, $i \neq i^*$, be a student who is already assigned at the beginning of step $k + 1$ under Q. If i got assigned to a school at some step l with $l < k$ under Q, then, by part 1 of the induction assumption, he is already assigned at step $l < k + 1$ under Q^*.

Now assume that i got assigned to a school, say $\bar{s}$, at step k under Q. Hence, student i has priority f for school $\bar{s}$ and student i's strategy Q_i lists $\bar{s}$ at rank r. We will prove that i is assigned to a school by the end of step k under Q^*. School $\bar{s}$ has at least one empty seat at the beginning of step k under Q. From part 2 of the induction assumption it follows that school $\bar{s}$ has at least one empty seat at the beginning of step k under Q^* as well. Suppose i is still unassigned at the beginning of step k under Q^*. Note that the rank-priority algorithm for Q^* considers at step k (student, school)–pairs such that the student has priority f for the school and the student lists the school at rank r in Q^*. Since $i \neq i^*$, $Q^*_i = Q_i$, and hence student i is assigned to $\bar{s}$ at step k under Q^*. Hence, i is assigned to a school by the end of step k under Q^*.

(2.) Let $s \in S$. From part 2 of the induction assumption it follows that it is sufficient to show that if at step k under Q^* a student gets assigned to s, then at step k under Q the student also gets assigned to s or there are no seats left at s.

Let i be a student who gets assigned to s at step k under Q^*. Hence, student i has priority f for school s and student i's strategy Q^*_i lists s at rank r. Recall $k < k^* = \pi(r^*, f^*)$. From Lemma 3 it follows that under Q^* student i^* is not assigned until step $\pi(r^*, f^*)$. Hence, $i \neq i^*$. By assumption, i is still unassigned at the beginning of step k under Q^*. Then, from part 1 of the induction assumption it follows that i is still unassigned at the beginning of step k under Q as well. Then, since $i \neq i^*$, $Q_i = Q^*_i$, and hence student i is assigned to s at step k under Q if at that point s still has an empty seat. □

We complete the proof by showing that $Q^*_{i^*}$ is a profitable deviation in both Case 2a and Case 2b. We first show that in both cases school s^* has at least one empty seat at the beginning of step $\pi(r^*, f^*)$ under Q.

In Case 2a, school s^* has at least one empty seat after applying the rank-priority algorithm to Q. Hence, school s^* has at least one empty seat at the beginning of step $\pi(r^*, f^*)$ under Q.

In Case 2b, student j^* is assigned to school s^* at a step $\pi(r', f_{s^*}(j^*))$ (where $r' \in \{1, \ldots, m\}$) under Q. Hence, school s^* has at least one empty seat at the beginning of step $\pi(r', f_{s^*}(j^*))$ under Q. From UIP and $f^* = f_{s^*}(i^*) < f_{s^*}(j^*)$, $\pi(f^* + 1) \leq \pi(f_{s^*}(j^*))$. From the definition of $r^* \equiv r^*(i^*, s^*)$ (Lemma 3), $\pi(r^*, f^*) < \pi(f^* + 1)$. Hence, $\pi(r^*, f^*) < \pi(f_{s^*}(j^*))$. In particular, $\pi(r^*, f^*) < \pi(r', f_{s^*}(j^*))$. Hence, school s^* has at least one empty seat at the beginning of step $\pi(r^*, f^*)$ under Q.

By part 2 of the Claim, school s^* has at least one empty seat at the beginning of step $k^* = \pi(r^*, f^*)$ under Q^* as well. Hence, by Lemma 3, i^* is assigned to s^* at step $\pi(r^*, f^*)$ under Q^*. So, $\varphi^{\pi}_{i^*}(Q^*) = s^*$. Hence, $Q^*_{i^*}$ is a profitable deviation, which contradicts $Q \in \mathcal{E}(\varphi^{\pi}, P)$. Hence, $\mu \in \mathcal{S}(P)$. So, $\mathcal{O}(\varphi^{\pi}, P) \subseteq \mathcal{S}(P)$. □

Since any *monotonic* rank-priority mechanism is *quasi-monotonic* (Lemma 1) and the immediate acceptance mechanism is a *monotonic* rank-priority mechanism, we immediately obtain the following two corollaries to Proposition 1.

Corollary 1. [Ergin and Sönmez, 2006, Theorem 4]
Each monotonic *rank-priority mechanism Nash implements the set of stable matchings.*

Corollary 2. [Ergin and Sönmez, 2006, Theorem 1]
The immediate acceptance mechanism Nash implements the set of stable matchings.

In Section 5 we also show that Proposition 1 and its proof imply Theorem 5.2 in Dur et al. (2018): any mechanism in the class considered by Dur et al. (2018) Nash implements the set of stable matchings.

Next, we show that non-quasi-monotonic mechanisms do not Nash implement the set of stable matchings. In fact, we prove a stronger result: for any non-quasi-monotonic mechanism we construct a school choice problem for which (a) the unique stable matching cannot be obtained as an equilibrium outcome *and* (b) some equilibrium outcome is not stable. Propositions 1 and 2 prove Theorem 1.

Proposition 2. [Non-quasi-monotonic mechanisms: no implementation]
Let π violate quasi-monotonicity. *Then, there is a problem P with $\mathcal{O}(\varphi^{\pi}, P) \neq \emptyset$, $|\mathcal{S}(P)| = 1$, $\mathcal{S}(P) \nsubseteq \mathcal{O}(\varphi^{\pi}, P)$, and $\mathcal{O}(\varphi^{\pi}, P) \nsubseteq \mathcal{S}(P)$. In particular, φ^{π} does not Nash implement the set of stable matchings.*

Proof. It is convenient to first introduce some more notation. For any priority $f \in \{1, \ldots, n\}$, we let $r(f)$ denote the rank such that $(r(f), f)$ is the first pair in π in which priority f appears. In other words, $r(f) \in \{1, \ldots, m\}$ is such that for each $r \in \{1, \ldots, m\}$, $\pi(f) = \pi(r(f), f) \leq \pi(r, f)$.

In view of Lemma 2 it is sufficient to distinguish between the following two cases.

Case 1: φ^{π} violates UIP, i.e., $\varphi^{\pi} \notin \mathcal{F}^u$.

Then, there is a *smallest* priority $f \in \{1, \ldots, n-1\}$ such that for some priority $\bar{f}$ with $\bar{f} > f$, $\pi(\bar{f}) < \pi(f)$. Thus, π takes the following form:

$$\pi: \quad \underbrace{\cdots, \cdots, \cdots, \cdots, \cdots}_{\substack{\text{only priorities } 1,2,\ldots,f-1 \\ \text{(respecting UIP here)}}}, \quad \cdots, \quad \underbrace{(r(\bar{f}), \bar{f})}_{\substack{\text{first time} \\ \text{priority } \bar{f} \text{ appears}}}, \quad \cdots, \quad \underbrace{(r(f), f)}_{\substack{\text{first time} \\ \text{priority } f \text{ appears}}}, \cdots. \tag{5}$$

Consider the school choice problem $(P, \succ, q)$ where preferences over schools P and priorities over students $\succ$ are given by the columns[16] in Table 5. Each school $s \in S$ has capacity $q_s = f$. One easily verifies that $\mathcal{S}(P) = \{\mu\}$ where the unique stable matching μ is such that for each $k = 1, \ldots, f$, $\mu(i_k) = s_1$ and for each $k = f+1, \ldots, n$, $\mu(i_k) = \emptyset$.

Students' preferences		Schools' priorities
$P_{\{i_1, i_2, \ldots, i_f, i_{\bar{f}}\}}$	$P_{\{i_{f+1}, i_{f+2}, \ldots, i_n\} \setminus \{i_{\bar{f}}\}}$	$\succ_S$
s_1	$\emptyset$	i_1
$\emptyset$		i_2
		$\vdots$
		i_n

Table 5: School choice problem in Case 1.

We first show $\mathcal{S}(P) \not\subseteq \mathcal{O}(\varphi^\pi, P)$. Suppose there is an equilibrium $Q \in \mathcal{E}(\varphi^\pi, P)$ such that $\varphi^\pi(Q) = \mu$. Since $\mu(i_f) = s_1$ and student i_f has priority f for school s_1, it follows from (5) that student i_f is assigned to school s_1 after step $\pi(r(\bar{f}), \bar{f})$ under Q. Hence, school s_1 has at least 1 empty seat at the beginning of step $\pi(r(\bar{f}), \bar{f})$ under Q.

Consider any strategy of the form

$$Q'_{i_{\bar{f}}} \equiv \cdots, \underbrace{s_1}_{\text{at rank } r(\bar{f})}, \emptyset,$$

for student $i_{\bar{f}}$. Let $Q' \equiv (Q'_{i_{\bar{f}}}, Q_{-i_{\bar{f}}})$. Since student $i_{\bar{f}}$ has priority $\bar{f}$ for all schools, $i_{\bar{f}}$ is not assigned to any school before step $\pi(r(\bar{f}), \bar{f})$ under Q'. Since school s_1 has at least 1 empty seat at the beginning of step $\pi(r(\bar{f}), \bar{f})$ under Q, it follows that school s_1 has at least 1 empty seat at the beginning of step $\pi(r(\bar{f}), \bar{f})$ under Q' as well. Hence, student $i_{\bar{f}}$ is assigned to school s_1 at step $\pi(r(\bar{f}), \bar{f})$ under Q'. But then, since $\varphi^\pi_{i_{\bar{f}}}(Q) = \mu(i_{\bar{f}}) = \emptyset$, $Q'_{i_{\bar{f}}}$ is a profitable deviation for student $i_{\bar{f}}$ at Q, contradicting $Q \in \mathcal{E}(\varphi^\pi, P)$. Hence, $\mu \in \mathcal{S}(P) \setminus \mathcal{O}(\varphi^\pi, P)$. So, $\mathcal{S}(P) \not\subseteq \mathcal{O}(\varphi^\pi, P)$.

Next, we show $\mathcal{O}(\varphi^\pi, P) \not\subseteq \mathcal{S}(P)$. Consider strategy-profile Q in Table 6. Each student i_k with $k \in \{1, 2, \ldots, f-1, \bar{f}\}$ submits a list where school s_1 appears at rank $r(k)$. All other students submit the empty list.

[16]So, each student in $\{i_1, i_2, \ldots, i_f, i_{\bar{f}}\}$ only finds s_1 acceptable, all other students find all schools unacceptable, and all schools $s \in S$ have the same priority relation $i_1 \succ_s i_2 \succ_s \ldots \succ_s i_n$.

	Students' strategies	
	$Q_{i_k}, k \in \{1,2,\ldots,f-1,\bar{f}\}$	$Q_{i_k}, k \in \{f,f+1,\ldots,n\}\backslash\{\bar{f}\}$
	$\vdots$	$\emptyset$
	$\vdots$	
$r(k) \rightarrow$	s_1	
	$\vdots$	

Table 6: Strategy-profile in Case 1.

Let $\mu' \equiv \varphi^\pi(Q)$. From (5) and the fact that each student i_k with $k \in \{1,2,\ldots,f-1,\bar{f}\}$ has priority k for all schools, it follows that for each $k \in \{1,2,\ldots,f-1,\bar{f}\}$, student i_k is assigned a seat at school s_1 at step $\pi(r(k),k)$ under Q. Thus, for each $k = 1,2,\ldots,f-1,\bar{f}$, $\mu'(i_k) = s_1$ and for each $k = f, f+1, \ldots, n$ with $k \neq \bar{f}$, $\mu'(i_k) = \emptyset$. Since $\mu' \neq \mu$, μ' is not stable, i.e., $\mu' \notin \mathcal{S}(P)$.

We show that $Q \in \mathcal{E}(\varphi^\pi, P)$. First, no student i_k with $k \in \{1,2,\ldots,n\}\backslash\{f\}$ has a profitable deviation (since each of them gets his most preferred match). Second, student i_f cannot obtain a seat at his only acceptable school s_1 by means of some deviation Q'_{i_f}. To see this, let $Q' \equiv (Q'_{i_f}, Q_{-i_f})$. Since student i_f has priority f for all schools, i_f is not assigned to any school before step $\pi(r(f),f)$ under Q'. Since $\pi(r(f),f) > \pi(r(\bar{f}),\bar{f})$ and school s_1 has no more empty seats after step $\pi(r(\bar{f}),\bar{f})$ under Q, school s_1 has no more empty seats after step $\pi(r(\bar{f}),\bar{f})$ under Q' either. So, student i_f does not obtain a seat at school s_1 under Q'. Hence, $Q \in \mathcal{E}(\varphi^\pi, P)$. Hence, $\mu' \in \mathcal{O}(\varphi^\pi, P)\backslash\mathcal{S}(P)$. So, $\mathcal{O}(\varphi^\pi, P) \not\subseteq \mathcal{S}(P)$.

Case 2: φ^π violates *quasi-monotonicity* but satisfies UIP, i.e., $\varphi^\pi \in \mathcal{F}^u\backslash\mathcal{F}^q$.
Since φ^π satisfies UIP, it follows that

$$\text{for each } \bar{f} \in \{1,\ldots,n-1\},\ \pi(r(\bar{f}),\bar{f}) < \pi(r(\bar{f}+1),\bar{f}+1). \tag{6}$$

But then in view of (4), let $f \in \{2,\ldots,n-1\}$ and $f' \in \{1,\ldots,n-2\}$, $f' < f$, be two priorities such that for each rank $\tilde{r} \in \{1,\ldots,m\}$ with $\pi(\tilde{r},f) < \pi(f+1)$,

$$\text{there is a rank } \tilde{r}' < \tilde{r} \text{ with } \pi(\tilde{r}',f') < \pi(\tilde{r},f). \tag{7}$$

It follows from (6) that there exists a rank $r \in \{1,\ldots,m\}$ with $\pi(r,f) < \pi(f+1)$. From (7), there is a rank $r' < r$. The existence of two different ranks implies that there are at least two different schools, i.e., $m \geq 2$.

Consider a school choice problem $(P,\succ,q)$ where preferences over schools P and priorities over students $\succ$ are given by the columns in Table 7. All students are acceptable for all schools. School s_1 has capacity $q_{s_1} = 1$. Each school $s \neq s_1$ has capacity $q_s = n$. One easily verifies that $\mathcal{S}(P) = \{\mu\}$ where the unique stable matching μ is such that $\mu(i_1) = s_1$ and for each $i \neq i_1$, $\mu(i) = \emptyset$.

Students' preferences				Schools' priorities	
P_{i_1}	P_{i_2}	$P_{I\setminus\{i_1,i_2\}}$		$\succ_{s_1}$	$\succ_{S\setminus\{s_1\}}$
s_1	s_1	$\emptyset$		$\vdots$	$\vdots$
$\emptyset$	$\emptyset$		$f' \rightarrow$	$\vdots$	i_1
				$\vdots$	$\vdots$
			$f \rightarrow$	i_1	$\vdots$
			$f+1 \rightarrow$	i_2	i_2
				$\vdots$	$\vdots$

Table 7: School choice problem in CASE 2.

We first show $\mathcal{S}(P) \nsubseteq \mathcal{O}(\varphi^\pi, P)$. Suppose there is an equilibrium $Q \in \mathcal{E}(\varphi^\pi, P)$ such that $\varphi^\pi(Q) = \mu$. Then, under Q, student i_1 is assigned to school s_1 at some step $\pi(\bar{r}, f)$ where $\bar{r} \in \{1, \ldots, m\}$. Then, Q_{i_1} lists school s_1 at rank $\bar{r}$. Moreover, $\pi(\bar{r}, f) < \pi(r(f+1), f+1)$. To see this, we can use arguments similar to those in CASE 1. We include the arguments for the sake of completeness. Suppose that $\pi(\bar{r}, f) > \pi(r(f+1), f+1)$. Then, from $\varphi^\pi(Q) = \mu$ it follows that no student is assigned to s_1 before or at step $\pi(r(f+1), f+1)$ under Q. Consider any strategy of the form

$$Q'_{i_2} \equiv \cdots, \underbrace{s_1}_{\text{at rank } r(f+1)}, \emptyset,$$

for student i_2. Let $Q' \equiv (Q'_{i_2}, Q_{-i_2})$. Since student i_2 has priority $f+1$ for all schools, i_2 is not assigned to any school before step $\pi(r(f+1), f+1)$ under Q'. Since no student is assigned to s_1 before or at step $\pi(r(f+1), f+1)$ under Q, it follows that no student is assigned to s_1 before or at step $\pi(r(f+1), f+1)$ under Q' either. Hence, student i_2 is assigned to school s_1 at step $\pi(r(f+1), f+1)$ under Q'. Hence, Q'_{i_2} is a profitable deviation for student i_2, contradicting $Q \in \mathcal{E}(\varphi^\pi, P)$. So, $\pi(\bar{r}, f) < \pi(r(f+1), f+1) = \pi(f+1)$.

Since $\pi(\bar{r}, f) < \pi(f+1)$, (7) implies that there is a rank $\bar{r}' < \bar{r}$ such that $\pi(\bar{r}', f') < \pi(\bar{r}, f)$. Since Q_{i_1} lists at least $\bar{r}$ schools, it lists some school, say $\bar{s}'$, at rank $\bar{r}'$. Obviously, $\bar{s}' \neq s_1$. Since i_1 has priority f' for school $\bar{s}'$ and $q_{\bar{s}'} = n$, it follows that student i_1 is assigned to some school before or at step $\pi(\bar{r}', f')$ under Q. Since $\pi(\bar{r}', f') < \pi(\bar{r}, f)$, this contradicts the fact that student i_1 is assigned to school s_1 at step $\pi(\bar{r}, f)$ under Q. So, $Q \notin \mathcal{E}(\varphi^\pi, P)$. Hence, $\mu \in \mathcal{S}(P) \setminus \mathcal{O}(\varphi^\pi, P)$. So, $\mathcal{S}(P) \nsubseteq \mathcal{O}(\varphi^\pi, P)$.

Next, we show $\mathcal{O}(\varphi^\pi, P) \nsubseteq \mathcal{S}(P)$. Consider strategy-profile Q in Table 8. Student i_2 submits a list where school s_1 appears at rank $r(f+1)$. All other students submit the empty list.

Let $\mu' \equiv \varphi^\pi(Q)$. Obviously, for each $i \neq i_2$, $\mu'(i) = \emptyset$ and $\mu'(i_2) = s_1$. Since $\mu' \neq \mu$, μ' is not stable, i.e., $\mu' \notin \mathcal{S}(P)$. We show that $Q \in \mathcal{E}(\varphi^\pi, P)$. First, none of the students $i_2, \ldots, i_n$

	Students' strategies		
	Q_{i_1}	Q_{i_2}	$Q_{I\setminus\{i_1,i_2\}}$
	$\emptyset$	$\vdots$	$\emptyset$
		$\vdots$	
$r(f+1) \rightarrow$		s_1	
		$\vdots$	

Table 8: Strategy-profile in CASE 2.

has a profitable deviation (since each of them gets his most preferred match). Second, consider student i_1. The only possible improvement would be to get the seat at school s_1. Suppose Q'_{i_1} is such that $\varphi^\pi_{i_1}(Q') = s_1$ where $Q' \equiv (Q'_{i_1}, Q_{-i_1})$. Then, i_1 is assigned to s_1 before step $\pi(r(f+1), f+1)$ under Q'. (Otherwise i_2 would again grab the unique seat at s_1.) Since i_1 has priority f for s_1, i_1 is assigned to s_1 at some step $\pi(\bar{r}, f) < \pi(r(f+1), f+1)$ where $\bar{r} \in \{1, \ldots, m\}$. In particular, the list Q'_{i_1} consists of at least $\bar{r}$ schools and school s_1 appears at rank $\bar{r}$. It follows from (7) that there exists a rank $\bar{r}' \in \{1, \ldots, m\}$ with $\bar{r}' < \bar{r}$ such that $\pi(\bar{r}', f') < \pi(\bar{r}, f)$. Since Q'_{i_1} lists a school (different from s_1), say s', at rank $\bar{r}' < \bar{r}$, and since student i_1 has priority f' for s' and $q_{s'} = n$, it follows that before or at step $\pi(\bar{r}', f')$ student i_1 is assigned to a school, which contradicts the fact that student i_1 is assigned to a school at step $\pi(\bar{r}, f)$. Hence, i_1 does not have a profitable deviation. Hence, $Q \in \mathcal{E}(\varphi^\pi, P)$. Hence, $\mu' \in \mathcal{O}(\varphi^\pi, P)\setminus\mathcal{S}(P)$. So, $\mathcal{O}(\varphi^\pi, P) \nsubseteq \mathcal{S}(P)$. □

Mechanism φ^π **(Nash) sub-implements the set of stable matchings** if for each problem $P \in \mathcal{P}$, $\mathcal{O}(\varphi^\pi, P) \subseteq \mathcal{S}(P)$. Similarly, mechanism φ^π **(Nash) sup-implements the set of stable matchings** if for each problem $P \in \mathcal{P}$, $\mathcal{S}(P) \subseteq \mathcal{O}(\varphi^\pi, P)$. Clearly, a mechanism implements the set of stable matchings if and only if it both sub-implements and sup-implements the set of stable matchings. As corollaries to Propositions 1 and 2 we obtain the following two results.

Corollary 3. [Sub-implementation: characterization]
A rank-priority mechanism $\varphi^\pi \in \mathcal{F}$ sub-implements the set of stable matchings if and only if it is quasi-monotonic, *i.e., $\varphi^\pi \in \mathcal{F}^q$.*

Corollary 4. [Sup-implementation: characterization]
A rank-priority mechanism $\varphi^\pi \in \mathcal{F}$ sup-implements the set of stable matchings if and only if it is quasi-monotonic, *i.e., $\varphi^\pi \in \mathcal{F}^q$.*

4 Incomplete information

In the analysis of Section 3 we rely on the concept of Nash equilibrium. In particular, we assume complete information about preferences. A natural question is whether our result still holds when this assumption is relaxed. Ergin and Sönmez (2006, Section 8) consider an incomplete information environment where students do know the priorities and the capacities of the schools but not the realizations of the other students' types. They show that the immediate acceptance mechanism may induce a Bayesian Nash equilibrium with unstable matchings in its support. In this section, we prove a strong impossibility result: *all* rank-priority mechanisms exhibit the same feature.

As before, let $I = \{i_1, \ldots, i_n\}$ and $S = \{s_1, \ldots, s_m\}$ denote the fixed set of students and schools, respectively. Furthermore, let $\succ = (\succ_s)_{s \in S}$ be the profile of priority relations and $q = (q_{s_1}, \ldots, q_{s_m})$ be the capacity vector. Each student $i \in I$ is now endowed with a **von Neumann-Morgenstern utility function** (or type) $\boldsymbol{u_i} : S \cup \{\emptyset\} \to \mathbb{R}$. We assume that for all $s, s' \in S \cup \{\emptyset\}$ with $s \neq s'$, $u_i(s) \neq u_i(s')$. Let $\mathcal{U}_i$ be the set of possible utility functions for student i. (For $i \neq j$, it is possible that $\mathcal{U}_i \neq \mathcal{U}_j$.) In our incomplete information setting, all students know the **probability distribution** $\mathbb{P}_{\boldsymbol{i}}$ over $\mathcal{U}_i$ where, without loss of generality, for each $u_i \in \mathcal{U}_i$, $\mathbb{P}_i(u_i) > 0$ and $\sum_{u_i \in \mathcal{U}_i} \mathbb{P}_i(u_i) = 1$, but only student i knows its realization. Let $\tilde{u}_i$ denote the random variable that determines student i's utility function. We assume that the collection $(\tilde{u}_i)_{i \in I}$ is independent. A **problem of incomplete information** is a list $\boldsymbol{(I, S, (\mathcal{U}_i)_{i \in I}, (\mathbb{P}_i)_{i \in I}, \succ, q)}$.

As before, we assume that students are the only strategic agents. For each $i \in I$, let $\mathcal{P}_i$ be the set of all complete, transitive, and strict preference relations over $S \cup \{\emptyset\}$. A **strategy** of student i is a function $\sigma_i : \mathcal{U}_i \to \mathcal{P}_i$. Let Σ_i denote the set of student i's strategies and let $\Sigma \equiv \prod_{i \in I} \Sigma_i$. Given a rank-priority mechanism φ^π, $\boldsymbol{\Gamma = (I, (\mathcal{U}_i)_{i \in I}, (\mathbb{P}_i)_{i \in I}, (\Sigma_i)_{i \in I}, \varphi^\pi)}$ is a **Bayesian game**.

A strategy-profile $\sigma = (\sigma_{i_1}, ..., \sigma_{i_n}) \in \Sigma$ is a **(Bayesian Nash) equilibrium** of Γ if for each student $i \in I$, σ_i assigns an optimal action to each $u_i \in \mathcal{U}_i$, i.e., maximizes student i's expected payoff given the other students' strategies. Formally, for each $i \in I$, each $u_i \in \mathcal{U}_i$, and each $P'_i \in \mathcal{P}_i$,

$$\mathbb{E}\left[u_i[\ \varphi^\pi_i(\sigma_i(u_i), (\sigma_j(\tilde{u}_j))_{j \neq i})\ ] \right] \geq \mathbb{E}\left[u_i[\ \varphi^\pi_i(P'_i, (\sigma_j(\tilde{u}_j))_{j \neq i})\ ] \right], \tag{8}$$

where the expected payoff is computed with respect to the vector of random variables $(\sigma_j(\tilde{u}_j))_{j \neq i}$. Let $\boldsymbol{\mathcal{E}(\Gamma)}$ denote the **set of equilibria**. The **support** of a strategy-profile $\sigma \in \Sigma$ is the set of matchings that can be obtained with strictly positive probability, i.e.,

$$\left\{ \mu \ :\ \text{ there is } (u_i)_{i \in I} \in \prod_{i \in I} \mathcal{U}_i \ \text{ s.t. } \varphi^\pi\big(\sigma_{i_1}(u_{i_1}), \ldots, \sigma_{i_n}(u_{i_n})\big) = \mu \right\}.$$

Next, we show that for each rank-priority mechanism, there is a problem of incomplete information with a Bayesian Nash equilibrium such that its support contains an unstable matching.

Theorem 2. [Incomplete information: impossibility of "stable support"]
Let $m \geq 3$ and $n \geq 4$. For each rank-priority mechanism φ^π, there is a problem of incomplete

information with a Bayesian Nash equilibrium such that its support contains an unstable matching, i.e., for each $\varphi^\pi \in \mathcal{F}$*, there is* $\sigma \in \mathcal{E}(\Gamma)$ *such that for some* $(u_i)_{i\in I} \in \prod_{i\in I} \mathcal{U}_i$*,*

$$\varphi^\pi\big(\sigma_{i_1}(u_{i_1}), \ldots, \sigma_{i_n}(u_{i_n})\big) \notin \mathcal{S}(P_{u_{i_1}}, \ldots, P_{u_{i_n}}),$$

where for each $i \in I$*,* P_{u_i} *is the preference relation over* $S \cup \{\emptyset\}$ *such that for all* $s, s' \in S \cup \{\emptyset\}$*,* $s \, P_{u_i} \, s'$ *if* $u_i(s) > u_i(s')$*.*

Proof. Let φ^π violate *quasi-monotonicity*. Then, the statement follows immediately from Proposition 2. Let φ^π be *quasi-monotonic*. Assume there are $n = 4$ students and $m = 3$ schools.[17] Let $(I, S, (\mathcal{U}_i)_{i\in I}, (\mathbb{P}_i)_{i\in I}, \succ, q)$ be any school choice problem[18] of incomplete information with $I = \{1,2,3,4\}$, $S = \{a,b,c\}$, $\mathcal{U}_1 = \{u_1\}$, $\mathcal{U}_2 = \{u_2\}$, $\mathcal{U}_3 = \{u_3\}$, $\mathcal{U}_4 = \{u_4^\emptyset, u_4^a\}$, $\mathbb{P}_1(u_1) = 1$, $\mathbb{P}_2(u_2) = 1$, $\mathbb{P}_3(u_3) = 1$, and $\mathbb{P}_4(u_4^\emptyset) = \mathbb{P}_4(u_4^a) = \frac{1}{2}$. The utility functions u_1, u_2, u_3, u_4^a, and $u_4^\emptyset$ are given by the columns in Table 9. The only condition (apart from the partial specification of u_1) that we impose on the utility functions is that the induced preferences are those described by the corresponding columns[19] in Table 10. The profile of priority relations $\succ$ is also described in Table 10. Finally, school a has capacity 2 and schools b and c each have capacity 1.

	Students				
	u_1	u_2	u_3	$u_4^\emptyset$	u_4^a
a	3	$*$	$*$	$*$	$*$
b	2	$*$	$*$	$*$	$*$
c	$*$	$*$	$*$	$*$	$*$
$\emptyset$	0	$*$	$*$	$*$	$*$

Table 9: The utility functions in Theorem 2. Each $*$ can be arbitrarily chosen provided that each column induces the preferences in the corresponding column of Table 10.

Students					Schools		
P_1	P_2	P_3	$P_4^\emptyset$	P_4^a	$\succ_a$	$\succ_b$	$\succ_c$
a	a	a	$\emptyset$	a	2	1	2
b	$\emptyset$	b		$\emptyset$	4	3	4
$\emptyset$		c			1	2	1
		$\emptyset$			3	4	3

Table 10: Induced preferences and the priorities in Theorem 2.

Let $(r(f), f)$ be the first pair in π in which priority f appears. Let $(r^2(4), 4)$ and $(r^3(4), 4)$ be the pair in π in which priority 4 appears for the second and third time, respectively. Note that since $m = 3$, $\{r(4), r^2(4), r^3(4)\} = \{1,2,3\}$. Also observe that since φ^π satisfies UIP, before step $\pi(r(2), 2)$ only pairs with priority 1 are considered. In particular, $\pi(r(1), 1) = 1$. We will use these facts in the remainder of the proof.

Consider any strategy-profile $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ such that

[17] A proof for the case with $m > 3$ or $n > 4$ can easily be obtained by introducing unacceptable schools.

[18] For the sake of clarity, we let integers and letters denote students and schools, respectively.

[19] Note that we simplify notation by writing P_1,P_2,P_3,$P_4^\emptyset$, and P_4^a instead of P_{u_1},P_{u_2},P_{u_3},$P_{u_4^\emptyset}$, and $P_{u_4^a}$.

	1	2	3	4
$\mu^{\emptyset}$	b	a	a	$\emptyset$
μ^{a}	b	a	c	a

Table 11: The support of σ in Theorem 2.

- at $\sigma_1(u_1)$, b has rank $r(1)$,
- at $\sigma_2(u_2)$, a has rank $r(1)$,
- at $\sigma_3(u_3)$,

  case I: if $r(2) \neq r(4), r^2(4)$, then b has rank $r(2)$, a has rank $r(4)$, and c has rank $r^2(4)$,
  case II: if $r(2) = r(4)$, then b has rank $r(2)$, a has rank $r^2(4)$, and c has rank $r^3(4)$,
  case III: if $r(2) = r^2(4)$, then b has rank $r(2)$, a has rank $r(4)$, and c has rank $r^3(4)$, and
- $\sigma_4(u_4^{\emptyset}) = P_4^{\emptyset}$ and at $\sigma_4(u_4^a)$, a has rank $r(2)$.

We compute the support of σ. Since $\pi(r(1), 1) = 1$, at step 1, student 1 is assigned to school b and student 2 is assigned to school a (independently of the realization of student 4's utility function ($u_4^{\emptyset}$ or u_4^a)). Students 3 and 4 are not assigned to a school at step 1, because students 3 and 4 do not have priority 1 for any school. To determine the assignment of the latter two students, we consider the two possible realizations of student 4's utility function separately.

First, consider realization $u_4^{\emptyset}$. In this case, student 4 obviously remains unassigned. As a consequence, student 3 is assigned to school a at step $\pi(r(4), 4)$ or $\pi(r^2(4), 4)$. To see this, note that after step 1, only schools a and c have an empty seat. Moreover, student 3 has priority 4 for both schools a and c. In case I, since $\pi(r(4), 4) < \pi(r^2(4), 4)$, student 3 is assigned to school a at step $\pi(r(4), 4)$. In case II, since $\pi(r^2(4), 4) < \pi(r^3(4), 4)$, student 3 is assigned to school a at step $\pi(r^2(4), 4)$. In case III, since $\pi(r(4), 4) < \pi(r^3(4), 4)$, student 3 is assigned to school a at step $\pi(r(4), 4)$. So, under realization $u_4^{\emptyset}$, the resulting outcome is matching $\mu^{\emptyset}$ as depicted in Table 11.

Second, consider realization u_4^a. Recall that before step $\pi(r(2), 2)$ only pairs with priority 1 are considered. Since student 2 is the only student with priority 1 for school a and since he has been assigned a seat at school a at step 1, at the beginning of step $\pi(r(2), 2)$ school a has still one empty seat. Moreover, since student 4 does not have priority 1 for any school, student 4 is not assigned to any school before step $\pi(r(2), 2)$. But then (by definition of $\sigma_4(u_4^a)$) student 4 is assigned to school a at step $\pi(r(2), 2)$. Consequently, student 3 is assigned to school c at step $\pi(r^2(4), 4)$ or $\pi(r^3(4), 4)$. So, under realization u_4^a, the resulting outcome is matching μ^a as depicted in Table 11.

Next, we show that σ is an equilibrium by checking that none of the four students has a profitable deviation, i.e., inequality (8) is satisfied for each student $i \in I$.

Since student 2 gets his most preferred match, student 2 does not have a profitable deviation. Since student 4 gets his most preferred match, either being unassigned or school a under each realization of his utility function, student 4 does not have a profitable deviation.

Consider student $i = 1$. Since $\mathcal{U}_1 = \{u_1\}$, we only have to check inequality (8) for $u_i = u_1$. It follows immediately from Table 11 that

$$\mathbb{E}\Big[u_1[\ \varphi_1^\pi(\sigma_1(u_1), (\sigma_j(\tilde{u}_j))_{j\neq 1})\ ]\Big] = 2. \tag{9}$$

Suppose there is $P_1' \in \mathcal{P}_1$ such that

$$\mathbb{E}\Big[u_1[\ \varphi_1^\pi(P_1', (\sigma_j(\tilde{u}_j))_{j\neq 1})\ ]\Big] > \mathbb{E}\Big[u_1[\ \varphi_1^\pi(\sigma_1(u_1), (\sigma_j(\tilde{u}_j))_{j\neq 1})\ ]\Big]. \tag{10}$$

Since $\mathbb{P}_4(u_4^\emptyset) = \mathbb{P}_4(u_4^a) = \frac{1}{2}$,

$$\begin{aligned}\mathbb{E}\Big[u_1[\ \varphi_1^\pi(P_1', (\sigma_j(\tilde{u}_j))_{j\neq 1})\ ]\Big] = \ & \frac{1}{2}u_1\big(\varphi_1^\pi(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^\emptyset))\big) + \\ & \frac{1}{2}u_1\big(\varphi_1^\pi(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^a))\big).\end{aligned} \tag{11}$$

Consider the rank-priority algorithm for π at $(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^a))$. Since student 1 has priority 3 for school a and since before step $\pi(r(2), 2)$ only pairs with priority 1 are considered, student 1 cannot be assigned to school a before or at step $\pi(r(2), 2)$. However, by the end of step $\pi(r(2), 2)$, school a does no longer have empty seats: student 2 is assigned to a at step $\pi(r(1), 1)$ and student 4 is assigned to a at step $\pi(r(2), 2)$. Hence, $\varphi_1^\pi(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^a))) \neq a$.

From $\varphi_1^\pi(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^a))) \neq a$, (9), (10), (11), and the specification of u_1 in Table 9 it follows that P_1' is a ranking that includes a as an acceptable school and

$$\varphi_1^\pi(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^\emptyset)) = a. \tag{12}$$

Then, together with the fact that before step $\pi(r(2), 2)$ only pairs with priority 1 are considered, we have that for each $r \in \{1, 2, 3\}$ and each $f \in \{1, 2, 3, 4\}$ with $\pi(r, f) < \pi(r(2), 2)$, $f = 1$ and at P_1' school b does not have rank r (otherwise, student 1 would be assigned to school b at $(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^\emptyset))$, which contradicts (12)).[20]

Turning back to the rank-priority algorithm for π at $(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^a))$, we already found that student 1 does not obtain a seat at school a (and that by the end of step $\pi(r(2), 2)$ the two seats at school a have been taken by students 2 and 4). But now we can also conclude that student 1 is not assigned a seat at school b. To see this, recall first that for each $r \in \{1, 2, 3\}$ and each $f \in \{1, 2, 3, 4\}$ with $\pi(r, f) < \pi(r(2), 2)$, $f = 1$ and at P_1' school b does not have rank r. So, student 1 is not assigned to school b before step $\pi(r(2), 2)$. And at step $\pi(r(2), 2)$, student 3 is assigned to the unique seat at school b. So, after step $\pi(r(2), 2)$ schools a and b do no longer have empty seats. Hence, $\varphi_1^\pi(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^a)) \in \{\emptyset, c\}$. Hence, from the specification of u_1 in Table 9,

$$u_1\big(\varphi_1^\pi(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^a))\big) \leq 0. \tag{13}$$

[20] In fact, at P_1', school b may not even be acceptable.

Substituting (12) and (13) in (11) yields a contradiction with (9) and (10). We conclude that there is no $P_1' \in \mathcal{P}_1$ that satisfies (10), i.e., student 1 does not have a profitable deviation.

Finally, consider student $i = 3$. Since $\mathcal{U}_3 = \{u_3\}$, we only have to check inequality (8) for $u_i = u_3$. Since $\mathbb{P}_4(u_4^{\emptyset}) = \mathbb{P}_4(u_4^a) = \frac{1}{2}$, it follows immediately from Table 11 that

$$\mathbb{E}\left[\, u_3[\ \varphi_3^{\pi}(\sigma_3(u_3), (\sigma_j(\tilde{u}_j))_{j\neq 3})\ ]\right] = \frac{1}{2}u_3(a) + \frac{1}{2}u_3(c). \tag{14}$$

Suppose there is $P_3' \in \mathcal{P}_3$ such that

$$\mathbb{E}\left[\, u_3[\ \varphi_3^{\pi}(P_3', (\sigma_j(\tilde{u}_j))_{j\neq 3})\ ]\right] > \mathbb{E}\left[\, u_3[\ \varphi_3^{\pi}(\sigma_3(u_3), (\sigma_j(\tilde{u}_j))_{j\neq 3})\ ]\right]. \tag{15}$$

Note that

$$\begin{aligned}\mathbb{E}\Big[u_3[\ \varphi_3^{\pi}(P_3', (\sigma_j(\tilde{u}_j))_{j\neq 3})\,]\Big] = \ & \frac{1}{2}u_3\big(\varphi_3^{\pi}(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^{\emptyset}))\big) + \\ & \frac{1}{2}u_3\big(\varphi_3^{\pi}(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^{a}))\big)\end{aligned} \tag{16}$$

Equations (14), (15), and (16) yield

$$\begin{aligned} u_3\big(\varphi_3^{\pi}(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^{\emptyset}))\big) \ + \ u_3\big(\varphi_3^{\pi}(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^{a}))\big) \ > \\ u_3(a) \ + \ u_3(c). \end{aligned} \tag{17}$$

Since at both $(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^{\emptyset}))$ and $(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^{a}))$, student 1 is assigned to the unique seat at b at step $1 = \pi(r(1), 1)$, student 3 cannot be assigned to school b, i.e.,

$$\varphi_3^{\pi}(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^{\emptyset})) \neq b \quad \text{and} \tag{18}$$

$$\varphi_3^{\pi}(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^{a})) \neq b. \tag{19}$$

Then, from (17), (18), (19), and the conditions imposed on u_3 by Table 10, it follows that

$$\varphi_3^{\pi}(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^{\emptyset})) = \varphi_3^{\pi}(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^{a})) = a. \tag{20}$$

Now consider the rank-priority algorithm for π at $(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^{a}))$. Recall that before step $\pi(r(2), 2)$ only pairs with priority 1 are considered. At step $\pi(r(1), 1)$, student 1 is assigned to b and student 2 is assigned to a; no further assignments take place until step $\pi(r(2), 2)$; and at step $\pi(r(2), 2)$, student 4 is assigned to school a. Hence, $\varphi_3^{\pi}(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^{a})) \neq a$, which contradicts (20). We conclude that there is no $P_3' \in \mathcal{P}_3$ that satisfies (15), i.e., student 3 does not have a profitable deviation. Hence, σ is an equilibrium.

Finally, to complete the proof, notice that the support of σ contains the unstable matching $\mu^{\emptyset}$ (see Table 11): at $\mu^{\emptyset}$, student 1 has justified envy with respect to student 3 and school a. □

Notice that to tackle any rank-priority mechanism $\varphi^\pi \in \mathcal{F}^q$ in the proof of Theorem 2, we only use the fact that $\varphi^\pi \in \mathcal{F}^u$. Moreover, for all rank-priority mechanisms in $\mathcal{F}^q$ we use the same problem of incomplete information and the same strategy-profile to prove the statement. Therefore we obtain the following corollary.

Corollary 5.
Let $m \geq 3$ and $n \geq 4$. There is a problem of incomplete information and a strategy-profile σ such that for each rank-priority mechanism $\varphi^\pi \in \mathcal{F}^u$, σ is a Bayesian Nash equilibrium with an unstable matching in its support.

5 Concluding remarks

Our analysis shows that in terms of the induced equilibrium outcomes the family of *quasi-monotonic* rank-priority mechanisms is equivalent to the immediate acceptance mechanism in the complete information framework (Theorem 1). Any mechanism that violates *quasi-monotonicity* may induce unstable equilibrium outcomes (Proposition 2). In the incomplete information framework that we studied, all rank-priority mechanisms suffer from the same problem as the immediate acceptance mechanism: there are equilibria whose support contains unstable matchings (Theorem 2). Therefore, one conclusion that can be drawn from our study is that in terms of the stability of equilibrium outcomes there is no rank-priority mechanism that outperforms the immediate acceptance mechanism.

Our "negative" results (Proposition 2 and Theorem 2) do not hinge on the fact that a student can be unacceptable for some school: one easily verifies that in the problems exhibited in the proofs of Proposition 2 and Theorem 2, all students are acceptable for all schools. It is an open question what happens in an environment where all schools are acceptable for all students and students have to submit strategies that contain all schools.

Even though we have restricted our attention to the family of rank-priority mechanisms, we can easily obtain Nash implementation of the set of stable matchings for a class of mechanisms that are not rank-priority mechanisms. More specifically, let φ be a mechanism that consists of the following two phases. In the first phase, students are matched to schools according to some *quasi-monotonic* mechanism φ^π, but only considering the rank-priority pairs that appear in π up to and including step $\pi(n)$. At the end of step $\pi(n)$, the second phase starts: unmatched students are matched to vacant seats in any way that guarantees individual rationality and non-wastefulness. Then, since *quasi-monotonicity* only imposes restrictions on the rank-priority pairs that appear in π before position $\pi(n)$, it is easy to see that the proof of Proposition 1 yields the Nash implementation of the set of stable matchings for φ. In particular, we obtain Nash implementation for the immediate acceptance with skips (or adaptive Boston) mechanism (Alcalde, 1996; Miralles, 2008; Dur, 2019; Harless, 2019).

More generally, Dur et al. (2018, Definition 3.4) consider the class of so-called first-choice mechanisms, which are the mechanisms that (1) maximize the number of students matched to their reported first choices and (2) yield a matching in which no student forms a blocking pair with his first choice. They show that the set of students that receive their first choice under each of these mechanisms always coincides with the set of students that receive their first choice under the immediate acceptance mechanism (Dur et al., 2018, Lemma 4.3). Hence, any mechanism in their class can be expressed as a "two-phase" mechanism (as described in the previous paragraph) where the first phase consists of the first $n = \pi^{ia}(n)$ rank-priority steps of the immediate acceptance mechanism. Thus, we obtain Theorem 5.2 in Dur et al. (2018): any mechanism in the class considered by Dur et al. (2018) Nash implements the set of stable matchings.

Corollary 6. *All first-choice mechanisms (Dur et al., 2018, Definition 3.4) Nash implement the set of stable matchings. In particular, the immediate acceptance with skips mechanism Nash implements the set of stable matchings.*

A Proof of Lemma 3

To keep the notation of the variables relatively simple in its proof we first state Lemma 3 with a slightly different notation. Also recall that we use the convention that $\pi(n+1) \equiv \infty$.

Lemma 3. *Let φ^π be quasi-monotonic. Let i^* be a student and s^* be a school. Suppose $f^* \equiv f_{s^*}(i^*) < \infty$. Then there is a rank $r^* \equiv r^*(i^*, s^*)$ with $\pi(r^*, f^*) < \pi(f^* + 1)$ and a strategy*

$$Q^*_{i^*} \equiv \cdots, \underbrace{s^*}_{\text{at rank } r^*}, \emptyset$$

such that if the rank-priority algorithm of φ^π is applied to $Q = (Q^*_i, Q_{-i})$, where Q_{-i} is any strategy-profile of the other students, then student i remains unassigned until the end of step $\pi(r^*, f^*) - 1$ (and hence is assigned to school s^* at step $\pi(r^*, f^*)$ if at that point the school still has an empty seat).

Proof. If $f^* = n$, then by taking $r^* = 1$ the statement follows. In the remainder of the proof we will suppose that $f^* < n$. We first introduce some useful sets to reformulate the statement.

Let R be the set of ranks that accompany f^* before $f^* + 1$ appears for the first time, i.e.,

$$R \equiv \{r \in \{1, \ldots, m\} \mid \pi(r, f^*) < \pi(f^* + 1)\}.$$

From Lemma 2 it follows that $R \neq \emptyset$.

For each candidate rank $\bar{r} \in R$, we define the set of "dangerous" schools for each rank $r < \bar{r}$. These are the schools that i^* cannot report at rank r when he reports school s^* at rank $\bar{r}$, because i^* might be assigned to one of these schools instead of s^*. Formally, for each $\bar{r} \in R$ and each $r < \bar{r}$, the set of dangerous schools is[21]

$$D(\bar{r}, r) \equiv \{s \in S \backslash \{s^*\} \mid \text{there is } f \leq f^* \text{ with (i) } f_s(i^*) = f \text{ and (ii) } \pi(r, f) < \pi(\bar{r}, f^*)\}.$$

[21] Since $r \neq \bar{r}$, (ii) is equivalent to $\pi(r, f) \leq \pi(\bar{r}, f^*)$.

Note that if for a school $s \in S \backslash \{s^*\}$, $f_s(i^*) \geq f^* + 1$, then s is not dangerous because either $f_s(i^*) = \infty$ or $f_s(i^*) < \infty$ and $\pi(f_s(i^*)) \geq \pi(f^* + 1) > \pi(\bar{r}, f^*)$, where the first inequality follows from UIP and the second inequality from $\bar{r} \in R$.

For each candidate rank $\bar{r} \in R$, the set of "safe" schools for each rank $r < \bar{r}$ is $S(\bar{r}, r) \equiv S \backslash (D(\bar{r}, r) \cup \{s^*\})$, i.e.,[22]

$$S(\bar{r}, r) = \{s \in S \backslash \{s^*\} \mid \text{for each } f \leq f^*, \ \text{(i) } f_s(i^*) \neq f \text{ or (ii) } \pi(r, f) \geq \pi(\bar{r}, f^*)\}.$$

In the remainder of the proof we will show that there exists $r^* \in R$ for which we can find for each rank $r \in \{1, \ldots, r^* - 1\}$ some school $s_r \in S(r^*, r)$ such that for all distinct $r, r' \in \{1, \ldots, r^* - 1\}$, $s_r \neq s_{r'}$. Then, the existence of the required r^* and strategy as in (A) follows. It follows from Hall's (1935) marriage theorem that the existence problem above is equivalent to the existence of $r^* \in R$ that satisfies the following $r^* - 1$ associated "feasibility conditions (fc)":

(fc.1) For each $r_1 < r^*$, $|S(r^*, r_1)| \geq 1$.
(fc.2) For all distinct $r_1, r_2 < r^*$, $|S(r^*, r_1) \cup S(r^*, r_2)| \geq 2$.
(fc.3) For all distinct $r_1, r_2, r_3 < r^*$, $|S(r^*, r_1) \cup S(r^*, r_2) \cup S(r^*, r_3)| \geq 3$.
$\vdots$
(fc.r^*-1) $|S(r^*, 1) \cup S(r^*, 2) \cup \cdots S(r^*, r^* - 2) \cup S(r^*, r^* - 1)| \geq r^* - 1$.

Define

$$F \equiv \{f \in \{1, \ldots, f^*\} \mid \ \text{there is } s \in S \backslash \{s^*\} \text{ such that } f_s(i^*) = f\}.$$

We can assume that $F \neq \emptyset$. (If $F = \emptyset$, let $r^* \in R$. Then, for each $r < r^*$, $|S(r^*, r)| = |S \backslash \{s^*\}| = m - 1$. Hence, all $r^* - 1$ feasibility conditions are satisfied.)

To prove that there is $r^* \in R$ such that r^* satisfies all $r^* - 1$ feasibility conditions, we focus on a subset of R and prove that there is a rank r^* in that subset that satisfies all $r^* - 1$ feasibility conditions. For each $\bar{r} \in R$, let

$$F(\bar{r}) \equiv \{f \in F \mid \ \text{for each } r < \bar{r}, \ \pi(r, f) > \pi(\bar{r}, f^*)\}.$$

Let

$$\hat{R} \equiv \{\hat{r} \in R \mid \ \text{for each } r \in R \text{ with } r < \hat{r}, \ F(\hat{r}) \backslash F(r) \neq \emptyset\}$$

so that we can define, for each $f \in F$,

$$\hat{R}(f) \equiv \{\hat{r} \in \hat{R} \mid f \in F(\hat{r})\} = \{\hat{r} \in \hat{R} \mid \ \text{for each } r < \hat{r}, \ \pi(r, f) > \pi(\hat{r}, f^*)\}.$$

Next, we show that for each $f \in F$, $\hat{R}(f) \neq \emptyset$. Let $f \in F$. From Condition (4)(ii)[23] it follows that there exists a *smallest* $\hat{r} \in R$ such that for each $r < \hat{r}$, $\pi(r, f) > \pi(\hat{r}, f^*)$. Suppose $\hat{r} \notin \hat{R}$. Then, there is $r \in R$ with $r < \hat{r}$ such that $F(\hat{r}) \backslash F(r) = \emptyset$. But then, since $f \in F(\hat{r})$, $f \in F(r)$, which, together with $r < \hat{r}$, contradicts the definition of $\hat{r}$. Hence,

$$\text{for each } f \in F, \ \hat{R}(f) \neq \emptyset. \tag{21}$$

[22] Since $r \neq \bar{r}$, (ii) is equivalent to $\pi(r, f) > \pi(\bar{r}, f^*)$.
[23] In case $f = f^*$, take $\bar{r} \in R$ with $\pi(\bar{r}, f) = \pi(f)$.

Claim 1. *For any* $\underline{r}, \bar{r} \in \bigcup_{f \in F} \hat{R}(f)$ *with* $\underline{r} < \bar{r}$,

(i) $\pi(\bar{r}, f^*) < \pi(\underline{r}, f^*)$ *and*

(ii) for each $r < \underline{r}$, $S(\underline{r}, r) \subseteq S(\bar{r}, r)$.

Proof. We first prove statement (i). Since $\bar{r} \in \bigcup_{f \in F} \hat{R}(f) \subseteq \hat{R}$ and $\underline{r} \in \bigcup_{f \in F} \hat{R}(f) \subseteq \hat{R} \subseteq R$, there exists $f' \in F(\bar{r})$ with $f' \notin F(\underline{r})$. So, there is $f' \in F \backslash F(\underline{r})$ with $\bar{r} \in \hat{R}(f')$.

Since $f' \in F \backslash F(\underline{r})$ and $\underline{r} \in \hat{R}$, $\underline{r} \notin \hat{R}(f')$. Then, there is $r_1 < \underline{r}$ such that $\pi(r_1, f') < \pi(\underline{r}, f^*)$. Since $\bar{r} \in \hat{R}(f')$, we have that for each $r < \bar{r}$, $\pi(\bar{r}, f^*) < \pi(r, f')$. In particular (since $r_1 < \underline{r} < \bar{r}$), $\pi(\bar{r}, f^*) < \pi(r_1, f')$. Then, $\pi(\bar{r}, f^*) < \pi(\underline{r}, f^*)$.

To prove statement (ii), let $r < \underline{r}$. Let $s \in S(\underline{r}, r)$. Suppose $f_s(i^*) \in F$. Then, $\pi(r, f_s(i^*)) > \pi(\underline{r}, f^*)$, which together with statement (i) gives $\pi(r, f_s(i^*)) > \pi(\bar{r}, f^*)$. Hence, $s \in S(\bar{r}, r)$. Now suppose $f_s(i^*) \notin F$. If $f_s(i^*) = \infty$, then obviously $s \in S(\bar{r}, r)$. So suppose $f_s(i^*) < \infty$. Since $\bar{r} \in R$, $\pi(\bar{r}, f^*) < \pi(f^* + 1)$. Since $f_s(i^*) \notin F$, $f_s(i^*) > f^*$. Then, from UIP, $\pi(f^* + 1) \leq \pi(f_s(i^*))$. Hence, $\pi(\bar{r}, f^*) < \pi(r, f_s(i^*))$. So, $s \in S(\bar{r}, r)$. Hence, $S(\underline{r}, r) \subseteq S(\bar{r}, r)$. □

We will prove by contradiction that some $r \in \bigcup_{f \in F} \hat{R}(f)$ satisfies all $r - 1$ associated feasibility conditions.[24] So,

$$\text{Assume that each } r \in \bigcup_{f \in F} \hat{R}(f) \text{ violates at least one of its } r-1 \text{ feasibility conditions.} \tag{22}$$

Let

$$\bigcup_{f \in F} \hat{R}(f) = \{r_1, \ldots, r_L\} \text{ where } r_1 < r_2 < \cdots < r_L \leq m.$$

For each $\underline{r} \in \bigcup_{f \in F} \hat{R}(f)$, let

$$S(\underline{r}) = \{s \in S \backslash \{s^*\} \mid [f_s(i^*) \leq f^* \text{ and } \underline{r} \in \hat{R}(f_s(i^*))] \text{ or } f_s(i^*) > f^*\}.$$

Claim 2. *For each* $l \in \{1, \ldots, L\}$, $|\bigcup_{\underline{r} \in \{r_1, \ldots, r_l\}} S(\underline{r})| \leq r_l - 2$.

Proof. Note that for each $l \in \{1, \ldots, L\}$, the set $\bigcup_{\underline{r} \in \{r_1, \ldots, r_l\}} S(\underline{r})$ can be written as the union of mutually exclusive sets,

$$S(r_l) \cup [S(r_{l-1}) \backslash S(r_l)] \cup [S(r_{l-2}) \backslash (S(r_l) \cup S(r_{l-1}))] \cup \ldots \cup [S(r_1) \backslash (S(r_l) \cup \ldots \cup S(r_2))].$$

We will show by induction that for each $l \in \{1, \ldots, L\}$,

$$|S(r_l)| + |S(r_{l-1}) \backslash S(r_l)| + |S(r_{l-2}) \backslash (S(r_l) \cup S(r_{l-1}))| + \cdots + |S(r_1) \backslash (S(r_l) \cup \ldots \cup S(r_2))| \leq r_l - 2. \tag{23}$$

We first prove (23) for $l = 1$, i.e., $|S(r_1)| \leq r_1 - 2$. Suppose $|S(r_1)| > r_1 - 2$. Let $s \in S(r_1)$. Case 1: $f_s(i^*) = \infty$. Then, obviously, for each $r < r_1$, $s \in S(r_1, r)$. Case 2.a:

[24] From $F \neq \emptyset$ and (21) it follows that $\bigcup_{f \in F} \hat{R}(f) \neq \emptyset$.

[$f_s(i^*) \leq f^*$ and $r_1 \in \hat{R}(f_s(i^*))$]. Then, for each $r < r_1$, $\pi(r, f_s(i^*)) > \pi(r_1, f^*)$. Case 2.b: $f^* < f_s(i^*) < \infty$. Then, for each $r < r_1$, $\pi(r, f_s(i^*)) \geq \pi(f^*+1) > \pi(r_1, f^*)$, where the first inequality follows from UIP and the second inequality from $r_1 \in R$. Hence, for each $s \in S(r_1)$ in Cases 2.a and 2.b and each $r < r_1$, $\pi(r, f_s(i^*)) > \pi(r_1, f^*)$, and thus $s \in S(r_1, r)$. Putting together Cases 1, 2.a, and 2.b, we have that for each $r < r_1$, $S(r_1) \subseteq S(r_1, r)$. Then, since $|S(r_1)| > r_1 - 2$, we have the following inequalities:

(fc.1) For each $r_1' < r_1$, we have $|S(r_1, r_1')| > r_1 - 2 \geq 1$.
(fc.2) For all distinct $r_1', r_2' < r_1$, we have $|S(r_1, r_1') \cup S(r_1, r_2')| > r_1 - 2 \geq 2$.
$\vdots$
(fc.$r_1 - 1$) $|S(r_1, 1) \cup S(r_1, 2) \cup \ldots \cup S(r_1, r_1 - 1)| > r_1 - 2 \geq r_1 - 1$.

In other words, $r_1 \in \bigcup_{f \in F} \hat{R}(f)$ satisfies all $r_1 - 1$ associated feasibility conditions, which contradicts assumption (22). Hence, $|S(r_1)| \leq r_1 - 2$.

Now let $j \in \{2, \ldots, L-1\}$. Suppose that for each $l \in \{1, \ldots, j-1\}$, (23) holds. We will complete the proof of the claim by showing in three steps that (23) also holds for $l = j$. Suppose that

$$|S(r_j)| + |S(r_{j-1}) \setminus S(r_j)| + |S(r_{j-2}) \setminus (S(r_j) \cup S(r_{j-1}))| + \cdots + |S(r_1) \setminus (S(r_j) \cup \ldots \cup S(r_2))| > r_j - 2. \tag{24}$$

Step 1. We show that the following relations hold:

($R.r_j$) for each $r \in \{1, \ldots, r_j - 1\}$, $S(r_j) \subseteq S(r_j, r)$,
($R.r_{j-1}$) for each $r \in \{1, \ldots, r_{j-1} - 1\}$, $S(r_j) \cup [S(r_{j-1}) \setminus S(r_j)] \subseteq S(r_j, r)$,
($R.r_{j-2}$) for each $r \in \{1, \ldots, r_{j-2} - 1\}$, $S(r_j) \cup [S(r_{j-1}) \setminus S(r_j)] \cup [S(r_{j-2}) \setminus (S(r_j) \cup S(r_{j-1}))] \subseteq S(r_j, r)$,
$\vdots$
($R.r_1$) for each $r \in \{1, \ldots, r_1 - 1\}$, $S(r_j) \cup [S(r_{j-1}) \setminus S(r_j)] \cup [S(r_{j-2}) \setminus (S(r_j) \cup S(r_{j-1}))] \cup \ldots \cup [S(r_1) \setminus (S(r_j) \cup S(r_{j-1}) \cup \ldots \cup S(r_2))] \subseteq S(r_j, r)$.

To prove relations ($R.r_j$), let $s \in S(r_j)$. Case 1: $f_s(i^*) = \infty$. Then, obviously for each $r < r_j$, $s \in S(r_j, r)$. Case 2.a: [$f_s(i^*) \leq f^*$ and $r_j \in \hat{R}(f_s(i^*))$]. Then, for each $r < r_j$, $\pi(r, f_s(i^*)) > \pi(r_j, f^*)$. Case 2.b: $f^* < f_s(i^*) < \infty$. Then, for each $r < r_j$, $\pi(r, f_s(i^*)) \geq \pi(f^*+1) > \pi(r_j, f^*)$, where the first inequality follows from UIP and the second inequality from $r_j \in R$. Hence, for each $s \in S(r_j)$ in Cases 2.a and 2.b, we have that for each $r < r_j$, $\pi(r, f_s(i^*)) > \pi(r_j, f^*)$, and thus $s \in S(r_j, r)$. This proves relations ($R.r_j$).

Given relations ($R.r_j$), to prove relations ($R.r_{j-1}$), it is sufficient to prove that for each $r \in \{1, \ldots, r_{j-1} - 1\}$, $S(r_{j-1}) \setminus S(r_j) \subseteq S(r_j, r)$. Let $r \in \{1, \ldots, r_{j-1} - 1\}$ and $s \in S(r_{j-1}) \setminus S(r_j)$. By arguments similar to the ones in the previous paragraph, $s \in S(r_{j-1}, r)$. From Claim 1(ii) it follows that $S(r_{j-1}, r) \subseteq S(r_j, r)$. Hence, $s \in S(r_j, r)$. This proves relations ($R.r_{j-1}$).

Given relations ($R.r_j$) and ($R.r_{j-1}$), to prove relations ($R.r_{j-2}$), it is sufficient to prove that for each $r \in \{1, \ldots, r_{j-2} - 1\}$, $S(r_{j-2}) \setminus (S(r_j) \cup S(r_{j-1}) \subseteq S(r_j, r)$. Let $r \in \{1, \ldots, r_{j-2} -$

$1\}$ and $s \in S(r_{j-2})\backslash(S(r_j) \cup S(r_{j-1})$. By arguments similar to the ones in the previous paragraphs, $s \in S(r_{j-2}, r)$. From Claim 1(ii) it follows that $S(r_{j-2}, r) \subseteq S(r_j, r)$. Hence, $s \in S(r_j, r)$. This proves relations $(R.r_{j-2})$. Using the same arguments, we continue the proof until relations $(R.r_1)$. This completes the proof of relations $(R.r_j), (R.r_{j-1}), \ldots, (R.r_1)$.

Step 2. We show that the following inequalities hold:

$(I.r_1)$ for each $r \in \{1, \ldots, r_1 - 1\}$, $|S(r_j, r)| \geq r_j - 1$,
$(I.r_2)$ for each $r \in \{r_1, \ldots, r_2 - 1\}$, $|S(r_j, r)| \geq r_j - r_1$,
$(I.r_3)$ for each $r \in \{r_2, \ldots, r_3 - 1\}$, $|S(r_j, r)| \geq r_j - r_2$,
$\vdots$
$(I.r_j)$ for each $r \in \{r_{j-1}, \ldots, r_j - 1\}$, $|S(r_j, r)| \geq r_j - r_{j-1}$.

To prove inequalities $(I.r_1)$, let $r \in \{1, \ldots, r_1 - 1\}$. From relations $(R.r_1)$ and (24), $|S(r_j, r)| \geq |S(r_j)| + |S(r_{j-1})\backslash S(r_j)| + |S(r_{j-2})\backslash(S(r_j) \cup S(r_{j-1}))| + \cdots + |S(r_1)\backslash(S(r_j) \cup S(r_{j-1}) \cup \ldots \cup S(r_2))| > r_j - 2$, which proves inequalities $(I.r_1)$.

To prove inequalities $(I.r_2)$, let $r \in \{r_1, \ldots, r_2 - 1\}$. Then,

$$\begin{aligned}
|S(r_j, r)| &\geq |S(r_j)| + |S(r_{j-1})\backslash S(r_j)| + |S(r_{j-2})\backslash(S(r_j) \cup S(r_{j-1}))| + \cdots && (25)\\
&\quad \cdots + |S(r_2)\backslash(S(r_j) \cup S(r_{j-1}) \cup \ldots \cup S(r_3))| \\
&> r_j - 2 - |S(r_1)\backslash(S(r_j) \cup S(r_{j-1}) \cup \ldots \cup S(r_2))| && (26)\\
&\geq r_j - 2 - |S(r_1)| \\
&\geq r_j - 2 - (r_1 - 2) = r_j - r_1, && (27)
\end{aligned}$$

where (25) follows from relations $(R.r_2)$, (26) from (24), and (27) from (23) for the base case $l = 1$. This proves inequalities $(I.r_2)$.

To prove inequalities $(I.r_3)$, let $r \in \{r_2, \ldots, r_3 - 1\}$. Then,

$$\begin{aligned}
|S(r_j, r)| &\geq |S(r_j)| + |S(r_{j-1})\backslash S(r_j)| + |S(r_{j-2})\backslash(S(r_j) \cup S(r_{j-1}))| + \cdots && (28)\\
&\quad \cdots + |S(r_3)\backslash(S(r_j) \cup S(r_{j-1}) \cup \ldots \cup S(r_4))| \\
&> r_j - 2 - |S(r_1)\backslash(S(r_j) \cup S(r_{j-1}) \cup \ldots \cup S(r_2))| && (29)\\
&\qquad - |S(r_2)\backslash(S(r_j) \cup S(r_{j-1}) \cup \ldots \cup S(r_3))| \\
&\geq r_j - 2 - |S(r_1)\backslash S(r_2)| - |S(r_2)| \\
&\geq r_j - 2 - (r_2 - 2) = r_j - r_2, && (30)
\end{aligned}$$

where (28) follows from relations $(R.r_3)$, (29) from (24), and (30) from (23) for $l = 2 \leq j-1$. This proves inequalities $(I.r_3)$.

Similar arguments can be used to prove inequalities $(I.r_4), \ldots, (I.r_j)$. This completes the proof of inequalities $(I.r_1), (I.r_2), \ldots, (I.r_j)$.

Step 3. Finally, we prove that r_j satisfies all feasibility conditions (which contradicts (22), and hence shows that (24) is false, thus completing the proof of Claim 2).

We consider the first $r_j - 1$ feasibility conditions for r_j. First, we consider the kth feasibility conditions with $k \in \{r_j - r_1 + 1, \ldots, r_j - 1\}$ and distinct $r^1, \ldots, r^k \in \{1, \ldots, r_j - 1\}$. Since $k \in \{r_j - r_1 + 1, \ldots, r_j - 1\}$, there is at least one rank $r \in \{r^1, \ldots, r^k\}$ with $r \in \{1, \ldots, r_1 - 1\}$. By inequalities $(I.r_1)$, for each $r \in \{1, \ldots, r_1 - 1\}$, $|S(r_j, r)| \geq r_j - 1$. Hence, $|S(r_j, r^1) \cup S(r_j, r^2) \cup \ldots \cup S(r_j, r^k)| \geq r_j - 1 \geq k$. Therefore, for each $k \in \{r_j - r_1 + 1, \ldots, r_j - 1\}$, the kth feasibility condition is satisfied.

Second, we consider the kth feasibility conditions with $k \in \{r_j - r_2 + 1, \ldots, r_j - r_1\}$ and distinct $r^1, \ldots, r^k \in \{1, \ldots, r_j - 1\}$. Since $k \in \{r_j - r_2 + 1, \ldots, r_j - r_1\}$, there is at least one rank $r \in \{r^1, \ldots, r^k\}$ with $r \in \{1, \ldots, r_2 - 1\}$. If there is $r \in \{r^1, \ldots, r^k\}$ with $r \in \{1, \ldots, r_1 - 1\}$, then by inequalities $(I.r_1)$ for such r, $|S(r_j, r)| \geq r_j - 1$ and hence, $|S(r_j, r^1) \cup S(r_j, r^2) \cup \ldots \cup S(r_j, r^k)| \geq r_j - 1 \geq k$.

If there is $r \in \{r^1, \ldots, r^k\}$ with $r \in \{r_1, \ldots, r_2 - 1\}$, then by inequalities $(I.r_2)$ for such r, $|S(r_j, r)| \geq r_j - r_1$ and hence, $|S(r_j, r^1) \cup S(r_j, r^2) \cup \ldots \cup S(r_j, r^k)| \geq r_j - r_1 \geq k$. Therefore, for each $k \in \{r_j - r_2 + 1, \ldots, r_j - r_1\}$, the kth feasibility condition is satisfied.

Third, we consider the kth feasibility conditions with $k \in \{r_j - r_3 + 1, \ldots, r_j - r_2\}$ and distinct $r^1, \ldots, r^k \in \{1, \ldots, r_j - 1\}$. Since $k \in \{r_j - r_3 + 1, \ldots, r_j - r_2\}$, there is at least one rank $r \in \{r^1, \ldots, r^k\}$ with $r \in \{1, \ldots, r_3 - 1\}$. If there is $r \in \{r^1, \ldots, r^k\}$ with $r \in \{1, \ldots, r_1 - 1\}$, then by inequalities $(I.r_1)$ for such r, $|S(r_j, r))| \geq r_j - 1$ and hence, $|S(r_j, r^1) \cup S(r_j, r^2) \cup \ldots \cup S(r_j, r^k)| \geq r_j - 1 \geq k$.

If there is $r \in \{r^1, \ldots, r^k\}$ with $r \in \{r_1, \ldots, r_2 - 1\}$, then by inequalities $(I.r_2)$ for such r, $|S(r_j, r)| \geq r_j - r_1$ and hence, $|S(r_j, r^1) \cup S(r_j, r^2) \cup \ldots \cup S(r_j, r^k)| \geq r_j - r_1 \geq k$.

If there is $r \in \{r^1, \ldots, r^k\}$ such that $r \in \{r_2, \ldots, r_3 - 1\}$, then by inequalities $(I.r_3)$ for such r, $|S(r_j, r)| \geq r_j - r_2$ and hence, $|S(r_j, r^1) \cup S(r_j, r^2) \cup \ldots \cup S(r_j, r^k)| \geq r_j - r_2 \geq k$. Therefore, for each $k \in \{r_j - r_3 + 1, \ldots, r_j - r_2\}$, the kth feasibility condition is satisfied.

Similar arguments can be used up to the kth feasibility condition where $k \in \{r_j - r_j + 1, \ldots, r_j - r_{j-1}\} = \{1, \ldots, r_j - r_{j-1}\}$. Hence, r_j satisfies the kth feasibility condition for each

$$k \in \{1, \ldots r_j - r_{j-1}\} \cup \ldots \cup \{r_j - r_2 + 1, \ldots, r_j - r_1\} \cup \{r_j - r_1 + 1, \ldots, r_j - 1\} = \{1, \ldots, r_j - 1\}.$$

Therefore, r_j satisfies all $r_j - 1$ associated feasibility conditions, which contradicts (22), and hence shows that (24) is false, thus completing the proof of Claim 2. □

From Claim 2, $|\bigcup_{\underline{r} \in \{r_1, \ldots, r_L\}} S(\underline{r})| \leq r_L - 2$. Then, since $\bigcup_{\underline{r} \in \{r_1, \ldots, r_L\}} S(\underline{r}) \subseteq S \backslash \{s^*\}$ and $|S \backslash \{s^*\}| = m - 1 \geq r_L - 1$, let $s \in S \backslash \{s^*\}$ with $s \notin \bigcup_{\underline{r} \in \{r_1, \ldots, r_L\}} S(\underline{r})$. By definition of $S(\underline{r})$, for each $\underline{r} \in \{r_1, \ldots, r_L\}$, $f_s(i^*) \leq f^*$ (which implies that $f_s(i^*) \in F$) and $\underline{r} \notin \hat{R}(f_s(i^*))$. But then, since

$$\hat{R}(f_s(i^*)) \subseteq \bigcup_{f \in F} \hat{R}(f) = \{r_1, \ldots, r_L\},$$

$\hat{R}(f_s(i^*)) = \emptyset$, which is a contradiction to (21). Therefore, there is a rank $r \in \bigcup_{f \in F} \hat{R}(f)$ that satisfies all its $r - 1$ associated feasibility conditions, which completes the proof of the lemma. □

Remark 3. Even though Lemma 3 only states the existence of a particular strategy, it can be constructed in polynomial time as follows.

In view of the first part of the proof of Lemma 3, a rank r^* with $\pi(r^*, f^*) < \pi(f^*+1)$ is suitable if there are $r^* - 1$ different schools, say $s_1, \ldots, s_{r^*-1}$, such that for each $r = 1, \ldots, r^* - 1$, school s_r is safe for rank r, i.e., $s \in S(r^*, r)$. It is easy to see that the sets of safe schools $S(r^*, r)$ can be computed in polynomial time. Then, in graph-theoretic terms, the suitability of r^* boils down to the existence of a matching of size $r^* - 1$ in the bipartite graph $G(r^*) = (V, E)$ where

- $V = \{1, \ldots, r^* - 1\} \cup \bigcup_{r=1,\ldots,r^*-1} S(r^*, r)$ and
- $E = \{(r, s) : s \in S(r^*, r)\}$.

Hence, we only have to find some r^* with $\pi(r^*, f^*) < \pi(f^*+1)$ such that its graph $G(r^*)$ has a (maximum cardinality) matching of size $r^* - 1$, which gives us the required strategy. The Hopcroft-Karp (1973) / Karzanov (1973) algorithm can be used to find a maximum cardinality matching in polynomial time. $\diamond$

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