Government Debt Management: The Long and the Short of It

Elisa Faraglia
Albert Marcet
Rigas Oikonomou
Andrew Scott

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Government Debt Management: The Long and the Short of It*

Faraglia, Elisa † Marcet, Albert‡ Oikonomou, Rigas § Scott, Andrew ¶

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Abstract

Standard optimal Debt Management (DM) models prescribe a dominant role for long bonds and advocate against issuing short bonds. They require very large positions in order to complete markets and assume each period that governments *r*epurchase all outstanding bonds and *r*eissue (r/r) new ones. These features of DM are inconsistent with US data. We introduce incomplete markets via small transaction costs which serves to make optimal DM more closely resemble the data : r/r are negligible, short bond issuance substantial and persistent and short and long bonds positively co-vary. Intuitively long bonds help smooth taxes over states and short bonds over time. Solving incomplete market models with multiple assets is challenging so a further contribution of this paper is introducing a novel computational method to find global solutions.

JEL codes: C63, E43, E62, H63

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†University of Cambridge and CEPR.

‡Institut d'Anàlisi Econòmica-CSIC, ICREA, Barcelona GSE, UAB, MOVE and CEPR.

§Université Catholique de Louvain. Email: Rigas.Oikonomou@uclouvain.be

¶London Business School and CEPR.

1 Introduction

What type of debt should a government issue? A sizeable literature (see *inter alia* Angeletos (2002) and Barro (2003)) studies optimal debt management (DM) using the canonical framework of a Ramsey planner under full commitment, stochastic government expenditure and distortionary taxes. This literature concludes that governments should focus on issuing long bonds. Based on a negative covariance between deficits and long bond prices, issuing long bonds ensures that the market value of debt falls when there is an adverse fiscal shock allowing the government to effectively complete markets and smooth taxes.

Buera and Nicolini (2004) point out that the optimal portfolios emerging from this approach feature very large long term debt issuance, by several multiples of GDP, and governments investing in short term bonds. Faraglia, Marcet and Scott (2010) show that introducing habits and capital leads to bond positions which are even larger and more volatile and characterised by a negative correlation between issuance of short and long bonds.

The above papers assume effectively complete markets but the importance of long bonds in optimal DM can survive even when market completeness cannot be achieved. For instance, Lustig et al. (2008) and Nosbusch (2008) rule out the ability of governments to invest in private assets and find it is still optimal for governments to focus almost exclusively on long term debt.[1,2]

As we document in Section 2, US debt management is at odds with these recommendations. Firstly the US government issues substantial amounts of short term debt with an average share of debt under one year of 43%. This share is far from zero in all years, never falling below 24% in our sample. Furthermore, the portfolio share of short bonds is relatively stable and highly persistent. In addition, the government tends to increase the stock of both long and short bonds together in response to a deficit shock.

Observed debt policy deviates from the optimal DM literature in yet another dimension. All the papers mentioned so far assume that each period governments repurchase the whole stock of previously issued bonds with this repurchase financed by freshly issued bonds. This can be described as a *r*epurchase/*r*eissuance (*r/r*) operation.[3] This assumption lends simplicity to the analysis as it reduces the number of state variables. However assuming a full *r/r* is widely at odds with the data as bonds are rarely repurchased before maturity, as documented for the US in Garbade and Rutherford (2007), for OECD in Marchesi (2006) and Blommestein et al (2012) and for the UK in Ellison and Scott (2017).

From the above studies one could draw the normative recommendation that governments should issue a much larger share of long bonds and engage in *r/r* operations. However for the normative

[1]Lustig et al (2008) use a monetary model to demonstrate a further reason for long term debt: inflation is more effective in lowering the debt burden when the deficit is high if long bonds have been issued.

[2]All of these papers assume that governments are creditworthy and can fully commit to their tax plans. A few papers have moved away from these assumptions, of no default and full commitment. An older literature (Calvo (1988) and Blanchard and Missale (1994)) considers moral hazard factors that lead governments to issue short term debt. Broner, Lorenzoni and Schmulker (2013), Aguiar et al (2016) and Arellano and Ramanarayanan (2012), Acharya and Rajan (2013), explain the interaction between debt management and default. He and Xiong (2012) study the interplay between liquidity and credit risks in the corporate bond market but their findings can also be applied to government finances. Finally, Debortoli, Nunes and Yared (2016) show that modifying the Angeletos (2002) model to allow for lack of commitment leads to an increase in long term interest rates such that governments issue substantial amounts of short term debt as well as long term debt.

[3]To be precise, in a model where only 10-period bonds are issued, an *r/r* involves replacing long bonds issued last year that have now 9-years to maturity, by new 10-year bonds.

insights of a Ramsey model to be useful it is crucial that they remain robust. If optimal policy changes considerably by introducing plausible bond markets frictions then the above normative implications would be mute. With this motivation we systematically compare the data with the recommendations that emerge from optimal DM under various financial frictions. Thus our approach mixes both normative and positive analysis to study what key ingredients are needed to build a useful theory of debt management that provides insights into the trade-offs policymakers face.

The central issue we address is the following. Full buyback involves very large repurchases and reissuances (*r/r)* which could be costly if there are transaction costs of any type. These costs may outweigh the fiscal insurance benefits and render *r/r* undesirable, changing the nature of optimal DM policy. Indeed, our discussions with Debt Management Offices (DMO) reveal considerable nervousness about the possibility of operating large scale *r/r* operations, with concerns expressed over market disruption, transaction costs and fears that large scale purchases and issuances would adversely affect bond prices. Thus we consider two questions around *r/r* operations: *a)* if a DMO does not buy back debt does this have a substantive impact on optimal DM? and *b)* why might a DMO *choose* not to buy back debt each period?

In Section 2 we outline observed features of US debt management. We argue that due to a number of financial frictions DMOs may be reluctant to perform *r/r*'s. This leads us to consider an extreme "no-buyback" assumption, where bonds are never repurchased before maturity. This version of the model is new to the literature.

No buyback imposes a limitation on the bond payments: under buyback long bonds issued in period t pay something in period $t+1$ and nothing afterwards, so the timing of payments is the same for long and short bonds. The only difference is that the long bonds' payoff, namely their price in the secondary market, is stochastic. By contrast, under no buyback a long bond pays a large amount at maturity date but nothing before. Since no buybacks impose additional restrictions on bonds it is of interest to address the issue of whether markets can be effectively completed under no buyback.

We show the answer is 'yes' if enough maturities are available and the *total market value* of bonds is subject to a No-Ponzi Game condition. However, the government achieves this outcome by purchasing increasingly large amounts of private bonds whilst making large bond issuances, the quantity of each bond going to infinity. Therefore complete markets does not seem like a useful benchmark for studying the effect of forbidding *r/r* as the resulting bond positions involve ever larger transactions and ever larger transaction costs under incomplete markets.

In Section 3 we begin our analysis of *a)* by introducing two sources of market incompleteness: we assume that the number of states of nature is much larger (a continuum) than the number of bonds, and we introduce bond limits that prevent huge bond positions from arising. We then outline two extreme models - the first model has *r/r*, every period the government buys back all outstanding debt and then reissues an optimal portfolio and the second model assums the government only buys back at the prespecified maturity date. The first model is motivated by the existing optimal DM literature and the second by empirical evidence on DM practice presented in Section 2.

Analysing optimal DM under these two extremes allows us to focus on the effects of the different timing of cash flows discussed above. Through specific analytical cases in Section 3 we show how no buyback reduces the attractiveness of long bonds. Firstly under no buyback whilst long bonds still provide fiscal insurance this effect is attenuated. Secondly under no buyback issuing N period bonds introduces N cycles into taxes through lumpy rollovers of debt. By contrast issuing short tem debt helps smooth cash flows and promotes tax smoothing.

To gain a deeper and more robust understanding of the impact of no buyback on optimal DM we perform calibrated simulations. Solving such models under incomplete markets is computationally demanding. This is particularly true in our case as not only are we looking at multiple assets but also the set of available assets does not span the state space. Further under no buyback it is necessary to keep track of all outstanding bonds and for large maturities this leads to a rapidly expanding state space. A significant contribution of this paper is therefore Section 4 where we outline a computational method, based on the Parameterized Expectations Algorithmn (PEA) of den Haan and Marcet (1990) to solve for optimal portfolios globally. We confront two difficulties: *i)* the size of the state space is very large and *ii)* using a standard formulation of first order conditions the optimal portfolio choice is indeterminate. We solve the first issue by introducing the Condensed PEA and the second through the Forward States PEA. The Condensed PEA significantly reduces the size of the state space, by forming an initial solution to the model using a small size vector of core state variables, and subsequently finding a few linear combinations of remaining state variables that summarize these variables efficiently. We also use this idea to introduce relevant non-linear terms of higher order, as these are often necessary for a good approximation. Forward States circumvents indeterminacy by approximating the integrand terms inside the expectations at t with a function of $t+1$ state variables. Furthermore, Forward States promotes a good approximation by embedding features of the correct solution into the approximations. These numerical procedures are likely to be useful in many other applications involving large portfolios and large state spaces.

In Section 5 we use this solution method to examine optimal DM when the government can issue both short and long term debt. We consider four different market environments: buyback/no-buyback combined with unrestricted/non-negative bond issuance (lending/no-lending). We find that the introduction of a no buyback constraint has substantial implications for optimal debt management - now the government should issue short term debt, in some cases even more than long term debt; portfolio shares are much more stable and persistent and the stock of both short and long run bonds positively co-move. Viewed in this light, Ramsey policy does not seem to urge governments to issue much larger amounts of long debt than at present.

These findings lead us to consider our second substantive issue around DM. The full *r/r* assumption in the buyback model enables the government to fully utilise the covariance of long bonds with fiscal deficit shocks, thus it achieves fiscal insurance - why would an optimising government avoid *r/r* operations? We turn to the analysis of this issue by explicitly introducing transaction costs. In the first part of Section 6 we perform a shadow cost computation of transaction costs by valuing the utility loss from buyback and no-buyback using Lagrange Multipliers from the extreme (buyback/nobuyback) cases of Section 5. Based on a U.S calibration of transaction costs we show that the no buyback solution leads to higher levels of welfare than the canonical *r/r* assumption. We then solve for DM when repurchases are chosen optimally under transaction costs. We find that the government hardly ever chooses to buy back debt before its maturity date, it only does on rare occasions when fiscal surpluses are very large. Optimal DM involves issuing a portfolio of both short and long bonds much more consistent with the basic facts in the data displayed in Section 2.

In Section 7 we turn to the robustness of our results. A relatively unexamined feature of observed

DM is the fact that bonds pay a fixed semi annual coupon. The existence of coupons becomes more important under no buyback, as coupons effectively mimic the cash flow of short bonds and so can be thought of as a security design aimed at mitigating the no buyback restriction. We first show that fixed coupons cannot complete the markets and then apply our simulation techniques to compare the model with the data. We also explore the effects of introducing a third bond under no buyback. We finally introduce a simple form of callable bonds. In all these cases we find very little influence on our debt management implications - governments should still issue long term bonds to achieve fiscal insurance but need to issue short term bonds to smooth cash flows and short and long bond positions should be positively correlated. Intuitively long bonds help smooth taxes over states and short bonds help smooth taxes over time. Finally we look at issues of accuracy of solutions and a final section concludes.

2 US Government Debt Management

This Section documents stylised facts about the U.S Treasury's management of marketable government debt[4] held by the public over the period 1955-2015. We use these facts as guidance for the interpretation and robustness of the optimal policy recommendations arising from Ramsey models and to uncover the ingredients that are important for the analysis of DM.

The full details of our data and calculations are contained in Appendix A. We use data from the CRSP about gross government debt issued. As a reference, we classify as "short" debt payments due in less than one year and as "long" debt payments due in over a year. Most of our stylised facts are quoted based on converting bonds into a zero coupon form (e.g coupon payments are treated as a separate bond with a maturity date set to when the coupon is paid) so when we refer to "short" or "long" term bonds we are referring to redemption *and* coupon payments that are due in either before or after one year.

Figure 1 shows the share of the market value of short bonds as a proportion of the total market value of U.S government debt (S_t) and reveals :

Fact 1 Portfolio shares of long and short maturities are *both* substantial.

On average S_t is 43% and ranges between 24% and 57%.

Fact 2 Portfolio shares are *never* close to zero or negative.

Fact 3 Portfolio shares are highly stable over time, with a low standard deviation and high serial correlation.

The first order serial correlation of S_t is 0.94 and its standard deviation 0.078.

[Figure 1 About Here]

[4]Our focus is on the Treasury's DM practice, hence we leave aside the Federal Reserve balance sheet from our statistics. The QE bond purchases by the Fed during the financial crisis were a component of "non-conventional" monetary policy. They were viewed as part of the transmission of monetary policy rather than optimal debt management. As such they relate to mechanisms that lie outside the models of the DM literature surveyed in the introduction and outside the scope of this paper. As in the rest of the literature our focus is on the interaction between DM and fiscal policy only. Hence we do not take into account QE bond purchases in our empirical analysis. We could avoid the issue of how to treat QE purchases by using data only up to 2008. The DM moments in the first column of Table 4 change slightly for this subsample, but they are still in full agreement with our description of Facts 1-7.

The underlying data also shows:

Fact 4 Short debt is positively correlated with long term debt.

The correlation between S_t and the same ratio for long debt is 0.86.[5] In other words, the government issues both short and long debt in response to a deficit shock.

All of these facts are in sharp contrast to the optimal DM recommendations from available Ramsey models with effectively complete markets. These usually produce very large issuance of long bonds, large short (negative) positions on short bonds, and, in models with time-varying bond positions such as Faraglia, Marcet and Scott (2010), considerable volatility and a negative correlation between short and long issuance. Facts 1-2 are also unlike the models with non-negativity constraints on bond issuances, as in Nosbusch (2008) and Lustig et al. (2008), where $S_t = 0$ frequently.

Our focus is on models of incomplete markets and in this environment the timing of cash flows matters. This is what motivates our remaining stylised facts which focus on the timing of cash flows by the Treasury - specifically around when they buy back bonds from investors.

Figure 2 shows the total issuance of government debt (long and short) each period over the total stock of debt held and illustrates the following.[6]

[Figure 2 About Here]

Fact 5 The ratio of total (gross) bond issuance over the stock of outstanding debt is never close to 100%.

More specifically, this ratio is never larger than 60%. This is in sharp contrast to the available optimal DM literature where the assumption of full r/r each period causes the above ratio to fluctuate around 100%.[7]

To better understand government behaviour around buying back debt before maturity we examine the dates at which the US Treasury has bought back bonds over our sample period. Consider first the case of non-callable bonds. As shown in Table 1, for the whole sample 99.8% of all long maturity government debt is redeemed either at maturity or within a year of its stated maturity date (and 98.86% one quarter before maturity). As mentioned in the introduction, this practice is not confined to the US but is standard practice across the OECD. This leads us to state

[Table 1 About Here]

Fact 6 Non-callable long bonds are effectively redeemed only at their maturity date and not before.

We have found that the extent to which governments rarely buy back debt before maturity is not widely known. This seems to be based on an awareness of the buybacks that occurred in the 1920s and between 2000-2001 as well as with the previously widespread use of callable bonds. However our dataset does extend as far back as the 1920s so this period is included in our analysis. Further whilst

[5] In calculating this correlation we divide by GDP to remove non-stationarity. The high correlation remains under alternative detrending techniques. For example, using a linear trend we obtain a correlation of 0.84 and with a linear quadratic trend the correlation is 0.79. We conclude that the high correlation of fluctuations in long and short debt is robust to detrending.

[6] See the Appendix A for details on how the series displayed in Figure 2 was constructed.

[7] Assuming that on average the primary deficit is in balance and that there is no nominal GDP growth.

the government did repurchase outstanding long bonds before maturity this was due to large and persistent budget surpluses and was done to avoid running down issues of short maturity bills[8]. In other words, this is an example of a true repurchase and not a *r/r*.

Closer examination of the case of callable bonds is also not supportive of the buyback assumption. Callable bonds are long maturity government debt which embed an option for the government to redeem the principal (at par) prior to maturity. The period prior to maturity containing the dates when the bond can be redeemed (or recalled) is dubbed the "call window". As shown in Figure 3 in 1955 around 50% of long bonds outstanding were callable although this declined to around 10% by the early 70s before rising once again. The last column of Table 2 shows the fraction of every issuance of callable bonds which has been redeemed prior to the maturity date. Aside from a few cases in the late 50s/early 60s it is typical for all callable debt to be bought back before maturity, i.e. for the government to exercise the option to redeem it before the bond matures.

[Figure 3 About Here]

[Table 2 About Here]

Given the magnitude of callable bonds in the first half of our sample and that they were nearly always redeemed before maturity it may seem that the "buyback" assumption may be relevant after all. However closer inspection shows this not to be the case.

Firstly, in nearly every case the callable bonds were bought back within the call window and often at the first date in the call window. Table 3 counts the call windows for all callable bonds issued by the US during this time period. The first row shows that there were three five-year callable bonds issued and they all had a call window which started two years from their maturity date, i.e. once the bonds had been outstanding for three years. It shows that all ten-year callable bonds could be recalled only two years prior to maturity at the earliest, and so on. Furthermore, Figure 4 shows the year callable bonds were redeemed within the call window, for different call windows. We see that around 80% of debt is repurchased at the first opportunity in the call window across maturities, and the remaining debt is repurchased within a year of the stated original redemption date.

[Table 3 About Here]

[Figure 4 About Here]

Whilst debt managers exercise call options, they do not buyback callable bonds before the callable window starts, and since call windows are close to maturity it means that most callable long bonds are recalled close to their maturity date. Further, an important feature of the buyback of callable bonds is that it is made at par and not the prevailing market price for bonds which is an important distinction from the usual r/r assumption.

Therefore we have

Fact 7 Most callable bonds in the US are redeemed at their first call date. For long bonds the first call date is most commonly close to the redemption date.

[8] Garbade and Rutherford (2007) document the details of the 2000-2001 buyback.

Facts 5-7 suggest that bonds issued by the Treasury tend to stay in private hands until maturity. Only callable bonds, which are a declining fraction of long debt, were redeemed before maturity but even in this case they were redeemed close to maturity and at par, not market value.

In a section aimed at summarising the practical operation of debt management it is worth considering what market features might explain the near nonexistence of r/r's in practice. In seeking an answer to this question we have held various informal discussions with debt management officers. In short, their answers are that DMO's are mainly worried about issuing cheaply, and for this they need to promote bond market stability. This response is broadly consistent with models such as Greenwood and Vayanos (2010), Gorton, Levellen and Metrick (2012), Guibaud et al (2013), Greenwood et al (2015) and Quinn and Roberds (2017) which emphasise investors being attracted to liquid and safe assets where bonds function as money or investors having a strong preference for particular habitats. In this environment large repurchases or sales are costly to manage and may disrupt the market[9]. Only in a debt reduction environment might buybacks be needed in order to maintain a desirable mix of maturities. However even this is not strictly speaking an r/r operation but a "pure repurchase" without reissuance.

In general we can think of three different reasons why r/r may be costly and all three we capture under the term "transaction costs". The first are simply the resources required to run the government's debt management office e.g buildings, personnel, equipment. It seems these are fixed costs the Treasury would have to pay anyway to run its issuance operations and so are unlikely to influence whether buyback or no buyback is optimal. The second category is due to the existence of bid-ask spreads (as documented by Amihud and Mendelson (1991) and Engle et al (2012)). A bid-ask spread will make full scale r/r every period more expensive by creating a wedge between the buying and selling price. The third category of transaction costs arise from price pressures and the belief that the supply and demand curves for government bonds are not perfectly elastic so issuing or purchasing more of specific bonds will influence the market price[10]. Lou, Yan and Zhang (2013), Breedon and Turner (2016) and Song and Zhu (2016) all derive estimates of how bond prices are influenced by large scale purchases or issues of government debt. As with bid-ask spreads, the existence of these "auction effects" will add additional costs to r/r which may make buyback suboptimal. In Section 6 we calibrate these costs and see if they can explain why governments do not perform r/r operations. For now we simply advance that DMOs tend to offer these facts as a reason for the absence of buybacks documented by Facts 5-7.

[9] In a related paper, Faraglia, Marcet, Oikonomou and Scott (2017) outline a three period model where the government has superior information than investors around future public finances. This informational asymmetry, in a similar spirit to Myers and Majluf (1984), generates the result that the government chooses not to buy back previously issued debt before maturity. Doing so triggers a bond market shut down as investors believe that the government is trying to reschedule its debt ahead of poor public finances.

[10] A particular concern of debt managers are "auction failures". Given the size of government debt there are frequent issues of new debt and an orderly market requires these to be sold at or near prevailing market prices. Large scale issues which go unsold or create market volatility are perceived as very damaging. This fear naturally produces a conservatism in issuance and a reluctance to do r/r as that would increase the number and scale of auctions and increase the probability of an auction failure.

3 The Model

In this section we compare the extreme assumption of full r/r with the opposite extreme where the government lets bonds stay in the hands of private investors until their redemption date. In Section 6.3 we will examine a model with transaction costs where each period the government can choose how much to repurchase. If we found that the case of buyback produces similar portfolio recommendations as no buyback then pursuing the modelling of transaction costs and the complexity of partial buybacks would be an unnecessary distraction. The extreme cases discussed in this section are useful for simplicity and to provide useful intuition for the transaction cost case.

For both the case of buyback and no buyback we examine the Ramsey policy equilibrium with perfect commitment and two bonds.[11] Essentially it can be seen as adding a long bond to the model of Aiyagari et al. (2002). We also follow the literature and consider the existence of a non-negativity constraint on bond issuance.

We assume a single representative household whose preferences over consumption, c_t, and leisure, x_t, are given by $E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(x_t)\right]$, where u and v are strictly increasing and strictly concave functions, they satisfy Inada conditions to avoid corners for c, x, and $0 < \beta < 1$ is the discount factor.

The economy produces a single good that cannot be stored. The household is endowed with T units of time that it allocates between leisure and labour. Technology for every period t is given by:

$$c_t + g_t = T - x_t \tag{1}$$

where g_t represents government expenditure which is assumed to be stochastic and exogenous and is the only source of uncertainty in the model. The representative firm maximizes profits. Both the household and the firm take prices and taxes as given.

The government engages in the following activities to finance spending: First, it levies distortionary taxes τ_t on labor income and second, it issues debt. Bond issuance of the government at period t is a vector $\mathbf{b}_t = (b_t^S, b_t^N)$ where N denotes the long and S the short bond. Both are real, zero-coupon, riskless bonds: the short (long) bond promises to pay one unit of consumption in S (N) periods with certainty, we take the integer $S \geq 1$ to be much lower than the integer N. Let p_t^i be the price of a bond of maturity $i = 1, \ldots, N$ with $p_t^0 = 1$.

In the standard case with *buyback* the period-t government budget constraint can be written as:

$$\sum_{i=\{S,N\}} b_t^i p_t^i = \sum_{i=\{S,N\}} b_{t-1}^i p_t^{i-1} + g_t - \tau_t(T - x_t) \tag{2}$$

The left side of this equation is the value of the bond portfolio issued this period. The first term on the right side is the market value of debt outstanding and $g_t - \tau_t(T - x_t)$ is the primary deficit. Notice that with this constraint the government is assumed to perform a full repurchase/reissuance operation (r/r) every period.

In the case where government debt is held to maturity i.e *no buyback*, then the government's

[11] Our benchmark model is purposefully simplistic allowing the government to issue debt in short and long bonds and in Section 7 we will also consider a third asset. Though government portfolios are indeed more complex in reality, the work of Piazzesi and Schneider (2010) shows that parsimonious portfolios, with a few zero coupon bonds, can span payoffs of more complex portfolios.

budget constraint becomes:

$$\sum_{i\in\{S,N\}} b_t^i p_t^i = \sum_{i\in\{S,N\}} b_{t-i}^i + g_t - \tau_t(T - x_t). \tag{3}$$

The left hand side of (3) once more corresponds to the market value of new debt issued in period t but the first term on the right hand side now measures not the total value of debt outstanding but instead the total value of debt maturing that period. Even though the government issues only ever issues two kinds of bonds, at any point in time there are N kinds of bonds outstanding, namely $(b_t^1 + b_{t-N+1}^N, b_{t-N+2}^N ..., b_t^N)$. Even though non-maturing bonds do not show up in the government's or consumer's budget constraint at t they may nonetheless affect the actions of agents since they influence the income that will be available in the future. The maturity of previously issued bonds declines each period - long bonds issued today will eventually become short bonds as they approach their redemption date. This provides a mechanical channel whereby for a given proportion of long/short bonds issued, the overall portfolio shows a greater reliance on short bonds than under the case of full buyback.

Under market incompleteness it is standard to allow for ad-hoc limits that constrain b_t^i (see Aiyagari et al. (2002))[12]

$$\frac{\underline{M}_i}{\beta^i} \leq b_t^i \leq \frac{\overline{M}_i}{\beta^i} \tag{4}$$

for some $|\underline{M}_i|, |\overline{M}_i| < \infty$. In the case where $\underline{M}_i < 0$ the government can purchase private bonds, this is ruled out when $\underline{M}_i = 0$ (as in Lustig et al. (2008) and Nosbusch (2008)). We shall refer to the latter as the "No Lending" case. The upper bound $\overline{M}_i$ plays several roles. It is a simple way of introducing transaction costs for large issuances, with an infinite cost of issuing more than $\overline{M}_i/\beta^i$ and zero cost of issuing below this bound in a given period. Under this interpretation it is reasonable to calibrate $\overline{M}_i$ so that it gives a level of average debt roughly as in the data. In section 6 we will introduce a more involved transaction costs function that matches better actual costs of issuing bonds, and we will allow for r/r. Another practical reason for using this bound is that it helps to stabilize the simulations, as noted by Maliar and Maliar (2003). Notice that in (4) we scale both the upper and lower bounds of maturity i by the steady state price of debt for that maturity $p^i = \beta^i$ (see formula for prices in section 3.1). We will keep this convention across all economies considered with different types of bonds. This facilitates the interpretation of the M's as they are in units of the (steady state) market value of debt issued each period for each type of bonds available.

In the case of no buyback we have to modify these constraints as the amount of debt outstanding per period is no longer given by debt issued that period. Then, given the issuances between t and $t - N + 1$, the market value of debt in N bonds still outstanding using steady state prices is: $\sum_{j=1}^{N} \beta^j b_{t-N+j}^N$. Therefore we normalize the debt constraints for $i = \{S, N\}$ as

$$\frac{\underline{M}_i}{\sum_{j=1}^{i} \beta^j} \leq b_t^i \leq \frac{\overline{M}_i}{\sum_{j=1}^{i} \beta^j}. \tag{5}$$

[12] As in Aiyagari et al. (2002) we assume for simplicity that the debt limits that the government faces are tighter than the consumer's debt limits, thus the consumer is always at the margin.

Note that, for the same M's, this puts the same limits as for the case of buyback on the steady state value of debt in each bond and, therefore, the same limits for the total market value of debt.

3.1 Ramsey Problem

As is standard in the Ramsey policy literature we assume the government chooses tax and bond policies knowing the implied equilibrium quantities and seeking to maximize household utility. We first summarize the competitive equilibrium in a few equations.

The consumer budget constraint is analogous to (2) or (3). In a standard manner we derive from the maximization problem of the consumer the equilibrium bond prices condition $p_t^i = \beta^i E_t(\frac{u_{c,t+i}}{u_{c,t}})$, where $u_{c,t} \equiv u'(c_t)$ and equilibrium condition for taxes $\tau_t = 1 - \frac{v_{x,t}}{u_{c,t}}$. Substituting these conditions and using the budget constraint under buyback we obtain the implementability conditions

$$\text{(6)} \qquad \sum_{i\in\{S,N\}} b_t^i E_t\left(\beta^i u_{c,t+i}\right) = \sum_{i\in\{S,N\}} b_{t-1}^i E_t\left(\beta^{i-1} u_{c,t+i-1}\right) + g_t u_{c,t} - (u_{c,t} - v_{x,t})(g_t + c_t)$$

for all t a.s. As argued in Aiyagari et al. (2002) it is not possible to simplify further under incomplete markets, (6) has to be imposed for all t. For the no buyback case we get

$$\text{(7)} \qquad \sum_{i\in\{S,N\}} b_t^i E_t\left(\beta^i u_{c,t+i}\right) = \sum_{i\in\{S,N\}} b_{t-i}^i u_{c,t} + g_t u_{c,t} - (u_{c,t} - v_{x,t})(g_t + c_t).$$

Using standard arguments we have that $\left\{c_t, b_t^S, b_t^N\right\}$ is a competitive equilibrium if and only if it satisfies the implementability constraint (6) (or (7)) and debt limits (4) (or (5)) almost surely for all t. The Ramsey equilibrium solves a planner's problem choosing sequences $\left\{c_t, b_t^S, b_t^N\right\}$ to maximize the household's utility subject to (6) (or (7)) and (4) (or (5)) a.s. for all t. The Lagrangean for the planner's problem under buyback is :

$$\text{(8)} \quad \mathcal{L} = E_0 \sum_t \beta^t \Bigg[u(c_t) + v(T - c_t - g_t) + \lambda_t [\sum_{i\in\{S,N\}} b_t^i \beta^i u_{c,t+i} - \sum_{i\in\{S,N\}} b_{t-1}^i \beta^{i-1} u_{c,t+i-1}$$
$$- g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t)] + \sum_{i\in\{S,N\}} \xi_{U,t}^i (\frac{\overline{M}_i}{\beta_i} - b_t^i) + \sum_{i\in\{S,N\}} \xi_{L,t}^i (b_t^i - \frac{\underline{M}_i}{\beta_i}) \Bigg].$$

Here $\xi_{L,t}^i$ and $\xi_{U,t}^i$ denote the multipliers on the lower and upper bounds respectively and λ_t is the multiplier of (6).

Under no buyback the corresponding Lagrangean is

$$\text{(9)} \quad \mathcal{L} = E_0 \sum_t \beta^t \Bigg[u(c_t) + v(T - c_t - g_t) + \lambda_t [\sum_{i\in\{S,N\}} b_t^i \beta^i u_{c,t+i} - \sum_{i\in\{S,N\}} b_{t-i}^i u_{c,t}$$
$$- g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t)] + \sum_{i\in\{S,N\}} \xi_{U,t}^i (\frac{\overline{M}_i}{\sum_{j=1}^i \beta^j} - b_t^i) + \sum_{i\in\{S,N\}} \xi_{L,t}^i (b_t^i - \frac{\underline{M}_i}{\sum_{j=1}^i \beta^j}) \Bigg]$$

3.2 Optimality Conditions under Buyback

In the case of buyback the first order conditions are:

$$u_{c,t} - v_{x,t} + \lambda_t \left[u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t}\right] + \tag{10}$$
$$+u_{cc,t} \sum_{i\in\{S,N\}} (\lambda_{t-i} - \lambda_{t-i+1}) b^i_{t-i} = 0$$

$$\beta^i E_t \left(u_{c,t+i}\lambda_t - u_{c,t+i}\lambda_{t+1}\right) + \xi^i_{L,t} - \xi^i_{U,t} = 0 \qquad \text{for } i = S, N. \tag{11}$$

Equation (10) represents the first order optimaliy condition of consumption and (11) of b^i_t. For the case of loose debt limits we have $\xi^i_{L,t} = \xi^i_{U,t} = 0$. Then, using the arguments of Aiyagari et al. (2002) we see that (11) states that $\lambda_t = E_t u_{c,t+i}\lambda_{t+1}/E_t(u_{c,t+i})$ which evolves as a risk-adjusted random walk with two risk measures, namely $u_{c,t+i}/E_t(u_{c,t+i})$ for $i = S, N$.

Extending the argument in Marcet and Marimon (2012) the optimal solution has a recursive formulation where the optimal tax schedule may be written as:

$$\tau_t = \tau(\mathbf{X}_t) \text{ for} \tag{12}$$

$$\mathbf{X}_t = (g_t, \lambda_{t-1}, \lambda_{t-2}, \ldots, \lambda_{t-N}, b^S_{t-1}, \ldots, b^S_{t-S}, b^N_{t-1}, \ldots, b^N_{t-N}) \tag{13}$$

for a time-invariant function $\tau(\cdot)$ as long as we constrain $\lambda_{-1} = \ldots = \lambda_{-N} = 0$.

The reason that N lags of λ and b are needed as state variables is because they contain the relevant information from the past needed to implement the full commitment solution. Faraglia et al. (2016) explain in detail the nature of the promises made under full commitment, they show optimal policy involves "interest rate twisting": if g_t is high the government can partly offset this adverse shock by promising lower taxes at $t + N$, as this increases c_{t+N}, lowering the cost of long bonds currently issued p^N_t and it lowers the increase in τ_t needed to maintain solvency. This promise is implemented mechanically in a recursive form by the term $\sum_{i\in\{S,N\}}(\lambda_{t+N-i} - \lambda_{t+N-i+1})b^i_{t+N-i}$, which appears in the optimality condition (10) at $t+N$ to implement the tax cut that was promised when the high g_t was observed. Once we realise that $\lambda_{t-S}, \lambda_{t-S+1}, \lambda_{t-N}, \lambda_{t-N+1}$ enter (10) at t it is clear that all the λ's and b^N's between $t-1$ and $t-N$ matter for the decision at t, since all the tax cuts previously promised influence the total discounted tax revenue that the government can raise between t and $t+N$. Initial λ's are set to zero because in full commitment Ramsey solutions no promises from the past that are relevant at $t = 0$.

This is why the state space for this solution is so large, the dimension of $\mathbf{X}_t$ is $2N + S + 1$. In the very simple model below, when we take $S = 1$ and $N = 10$, this amounts to 22 state variables. Solving this model is computationally demanding because of the magnitude of the state space and in the next section we outline a new computational method which offers an efficient solution procedure.

3.3 Optimality Conditions under No Buyback

The model in section 3.2 under buyback is analogous to others in the literature whereas the focus of the paper is on the implications of assuming no buyback. In this case the first order optimality conditions for consumption and bonds are

$$
\begin{aligned}
& u_{c,t} - v_{x,t} + \lambda_t \left[u_{cc,t} c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t} \right] + \\
(14) \qquad & + u_{cc,t} \sum_{i \in \{S,N\}} (\lambda_{t-i} - \lambda_t) b^i_{t-i} \; = \; 0 \\
(15) \qquad & \beta^i E_t(u_{c,t+i}\lambda_t - u_{c,t+i}\lambda_{t+i}) + \xi^i_{L,t} - \xi^i_{U,t} \; = \; 0 \qquad \text{for} \qquad i = S, N
\end{aligned}
$$

Now we have that off corners $\lambda_t = E_t u_{c,t+S}\lambda_{t+S}/E_t(u_{c,t+S})$ and $\lambda_t = E_t u_{c,t+N}\lambda_{t+N}/E_t(u_{c,t+N})$. The only difference between the martingale result here and the one under buyback is that we now have λ_{t+i} instead of λ_{t+1}. Therefore under no buyback λ_t shows more complex dynamics displaying (risk-adjusted) cycles of periodicity S and N. As we discuss in detail below this is because under no buyback a shock to the budget constraint today impacts on the budget constraint S and N periods ahead, as the bonds issued to absorb today's shock will only affect budget constraints in those terms. Long bonds under no buyback provide less possibilities for fiscal insurance because the available bonds do not allow smoothing of shocks across all periods, but in cycles of S and N periods.

Although the first order conditions are different the state vector is the same as under buyback so the optimal allocation satisfies (12)-(13).

3.4 Effectively Complete vs. "Truly" Incomplete Markets

We now study a case where, in our model with riskless bonds, the complete market allocation is reached. This serves to highlight available results in the literature on DM, to show how they are related to our formulation and to set up the notation that will be used throughout the paper.

Throughout the paper we use the following notation. Let $\mathbf{g}^t = (g_0, g_1, ..., g_t)$. Denote the primary government surplus as $s_t = \tau_t(T - x_t) - g_t$ for equilibrium taxes $\tau_t = 1 - \frac{v_{x,t}}{u_{c,t}}$ and feasible allocations. For a given allocation $\{c_t\}$ denote expected present discounted value of surpluses as

$$
(16) \qquad z_t(\mathbf{g}^{t-1}, g_t) \equiv E_t \sum_{i=0}^{\infty} \beta^i \frac{u_{c,t+i}}{u_{c,t}} s_{t+i}.
$$

3.4.1 Buyback

As noted by Angeletos (2002) and Buera and Nicolini (2004) (ABN hereafter), optimal DM in the framework of section 3.2 implements the complete market allocations if the bounds M are sufficiently large in absolute value and the government issues only long bonds. Let us now prove the ABN result using the optimality conditions derived above. For more details on the derivations in this subsection, see Online Appendix A.1.

Assume g_t is a Markov process taking only *two possible values* (g^H, g^L), one-period bonds $S = 1$ and (without loss of generality) $g_0 = g^H$. Denote by $\left\{c_t^{CM}\right\}$ the complete market allocation. As is well known, $c_t^{CM} = c^H, c^L$ contingent on g_t, for values c^H, c^L constant through time for $t \geq N$.

We guess and verify that constant $\lambda_t = \Delta$ and $\xi^i_{j,t} = 0$ satisfy all the first order conditions. First, it is obvious that $\lambda_t = \Delta$ and $\xi^i_{j,t} = 0$ satisfies the martingale condition (11). As pointed out by Aiyagari et al. (2002), in this case consumption solving (10) is the same as $\left\{c_t^{CM}\right\}$. If we can find bond holdings $\left\{b_t^1, b_t^N\right\}$ satisfying bond limits and budget constraints for this consumption allocation and the corresponding equilibrium prices and taxes, we will have proved markets are effectively complete.

To find such $\left\{b_t^1, b_t^N\right\}$, substitute equilibrium conditions in the budget constraint to give

$$b_{t-1}^1 + p_t^{N-1} b_{t-1}^N = z_t \quad \forall t, \text{ a.s.} \tag{17}$$

The variables z and p^{N-1} that correspond to $\left\{c_t^{CM}\right\}$ can only take two values (z^H, z^L) and (p_H^{N-1}, p_L^{N-1}) for $t \geq N$. Therefore (17) defines two equations for $i = H, L$ that can be used to show that the portfolio

$$b_t^1 \;=\; B_N^{BB} \equiv \frac{z^H - z^L}{p_H^{N-1} - p_L^{N-1}} \tag{18}$$

$$b_t^N \;=\; B_1^{BB} \equiv z^H - p_H^{N-1} B_N^{BB} \text{ for all } t \geq N \tag{19}$$

satisfies (17) for all periods. Hence this portfolio supports $\left\{c_t^{CM}\right\}$, markets are effectively complete in the optimum. As noted by ABN, the key properties of the solution are: *i)* the issuance of each security is constant over time and, *ii)* since $z^H < z^L$ and $p_H^{N-1} < p_L^{N-1}$ it is clear from the above equations that $B_N^{BB} > 0$ and $B_N^{BB} < 0$ i.e. issue long and invest short.[13]

A constant issuance effectively completes markets because long bond prices and the primary deficit are perfectly negatively correlated: a high g brings about a high deficit and a lower price of long bonds, hence $b_t^N > 0$ means that debt servicing goes down with a higher deficit, and in this way the left side of (17) can equal the right side a.s. Roughly speaking, the level of b's are chosen to match total debt, while the difference $B_N^{BB} - B_1^{BB}$ generates a variability of the value of debt matching the variability in z.

The intuition about why λ_t is constant is straightforward: under complete markets higher (discounted) income at $t > 0$ provides the same marginal increase in utility as a higher income in period $t = 0$, since all that matters for the solution is total discounted wealth at $t = 0$. Therefore under complete markets λ_t is non-stochastic. As long as the corresponding allocations are also feasible under incomplete markets (as shown by the above derivations) then λ_t is constant under incomplete markets as well.

3.4.2 No buyback

As mentioned previously, no buyback has been mostly ignored in the academic literature on optimal policy in macro and bond portfolios in finance. There is a common presumption that no buyback is an unnecessary complication that only serves to multiply the number of state variables and is presumably irrelevant under complete markets. In this section we first show that in the *no buyback* model of Section 3.3 markets can be effectively complete, supporting the presumption of nobuyback is irrelevant. But then we show that the portfolios needed for this purpose would have ever larger transaction costs. Some details are left to Online Appendix A.2.

Assume g and S as in Section 3.4.1. As with buyback, constant λ_t and $\xi_{j,t}^i = 0$ satisfy the martingale condition (15) and give rise to complete markets allocations. We now check if it is possible to find bonds $\left\{b_t^1, b_t^N\right\}$ satisfying bond limits and budget constraints for this allocation.

[13] Strictly speaking some more assumptions are needed, see Online Appendix A.1.

Now there are N kinds of bonds outstanding at the beginning of period t, each bond with $i = 0, ..., N-1$ periods left to maturity and held in the amounts $(b_{t-1}^1 + b_{t-N}^N, b_{t-N+1}^N, ..., b_{t-1}^N,)$. Adding and subtracting the value of outstanding bonds held by the government $\sum_{i=1}^{N-1} p_t^{N-i} b_{t-i}^N$ to both sides of (3) and using that in equilibrium $p_t^i = E_t(p_{t+j}^{i-j} p_t^j)$ we substitute forward to see that, analogous to (17), the equilibrium value of all bonds outstanding has to equal the discounted sum of surpluses:

$$\sum_{j=1}^{N} p_t^{N-j} b_{t-j}^N + b_{t-1}^1 = z_t \quad \forall t \text{ a.s.} \tag{20}$$

Steady State To find the *steady state* of bond issuances compatible with complete markets, set $b_t^N = B_N^{ss,NBB}$ and $b_t^1 = B_1^{ss,NBB}$ for all t and all $\mathbf{g}^t$, and $z_t = z^H, z^L$ in (20) to find

$$B_N^{ss,NBB} = \frac{z^H - z^L}{\sum_{j=1}^{N-1}\left(p_H^j - p_L^j\right)} \tag{21}$$

$$B_1^{ss,NBB} = z^H - \sum_{j=0}^{N-1} p_H^j B_N^{ss,NBB}. \tag{22}$$

As with buyback, $B_N^{ss,NBB} > 0$, $B_1^{ss,NBB} < 0$ generically.

Since $\left|\sum_{i=0}^{N-1} p^{i,H} - p^{i,L}\right| > \left|p^{N-1,H} - p^{N-1,L}\right|$ the amount of bonds issued each period is lower under no buyback. Clearly, a given amount total debt involves issuing less each period if bonds are not repurchased. Therefore transaction costs (if there are any) of issuing bonds will be lower with no buyback.

Stability of the steady state Clearly, convergence to steady state was very quick in the model under buyback: in section 3.4.1 bonds go from their initial condition b_{-1}^1, b_{-1}^N to the steady state B_1^{BB}, B_N^{BB} in N periods. This is because first order conditions are time-invariant for $t \geq N$ (see Online Appendix A.1).

Let us now examine convergence to a steady state under no buyback. Equations (18), (20) at g^H and g^L and simple algebra show that if $\left\{c_t^{CM}\right\}$ is implemented the "true" dynamics of b^N satisfy

$$b_t^N = B_N^{BB} + \sum_{i=1}^{N-2} \frac{p_H^i - p_L^i}{p_L^{N-1} - p_H^{N-1}} b_{t-i}^N \tag{23}$$

for all $t \geq N$, given initial bonds b_{-j}^N, $j = 1, ..., N-2$.

Therefore, to effectively complete markets b_t^N should satisfy a deterministic linear difference equation of order $N-2$. It is trivial to check that convergence to steady state will generically fail. For example, in the case $N = 3$, and using the approximation $p_j^i \simeq \beta^i K_j$ for all i and $j = H, L$, (23) gives the approximation

$$b_t^N \simeq B_N^{BB} - \frac{1}{\beta} b_{t-1}^N. \tag{24}$$

Therefore, except for very special initial conditions, b_t^N does not converge to $B_N^{ss,NBB}$. To effectively complete markets b_t^N has to oscillate between positive and negative values and $\left|b_t^N\right| \to \infty$

geometrically at a rate $\beta^{-1} > 1$. Therefore we have the following

Result *Optimal DM achieves the complete markets solution under no buyback if constraints (5) are replaced by* ***bounds on total value of debt:*** $\underline{M} \leq \sum_{j=1}^{N} p_i^{N-j} b_{t-j}^N + b_{t-1}^1 \leq \overline{M}$, *for* $\overline{M}, -\underline{M}$ *sufficiently large. Under bond limits such as (5) and* $N > 2$ *markets cannot be generically completed.*

Obviously, gross issuance satisfying (24) would be infeasible even with minor transaction costs of the type considered in Section 6. Therefore, under no buyback we obtain a result analogous to ABN, but the implementation of the optimal solution would be even extremely costly in the presence of transaction costs.

3.4.3 "Truly" Incomplete markets

In the case where g takes a continuum of possible values the complete markets allocation $\left\{c_t^{CM}\right\}$ cannot be implemented. To see this, for a given $\mathbf{g}^{t-1}$, denote by $f(\cdot) \equiv z_t(\mathbf{g}^{t-1}, \cdot)$ the discounted sum corresponding to $\left\{c_t^{CM}\right\}$ in the right side of (17). Thus, f is a pre-specified function of a continuum of values g_t. Similarly, bond prices corresponding to $\left\{c_t^{CM}\right\}$ $h(\cdot) \equiv p_t^{N-1}(\mathbf{g}^{t-1}, \cdot)$ give a *different* pre-specified function h of g_t. It would be a coincidence if there were constants K_1, K_2 such that $K_1 + K_2 h = f$ for all values of g. Therefore one cannot find bond values $\left(b_{t-1}^1, b_{t-1}^N\right)$ for which (17) holds for all values of g_t, $\left\{c_t^{CM}\right\}$ cannot be implemented and markets are *truly* incomplete.[14]

Deficits and long bond prices are still conditionally perfectly negatively correlated, as they are both a function of g_t only, but the dependence of the value of bonds on g is not sufficient to offset the variability in z_t that is needed to effectively complete markets. As emphasized in Aiyagari et al. (2002) and Angeletos (2002) (20) becomes an active constraint, restricting allocations that can be chosen today as a function of past bonds.

Therefore, markets can not be effectively complete with two bonds and a continuous g, both with buyback and no buyback. A constant multiplier λ_t and $\xi = 0$ may not arise, therefore the martingale conditions stated after (11) or (15) have to hold stochastically. Intuitively,[15] what happens is that the marginal utility from an additional unit of income depends on the wealth at t namely $\left(b_{t-1}^1, b_{t-1}^N\right)$ and under incomplete markets wealth is uncertain as it depends on the whole realisation $\mathbf{g}^{t-1}$, causing λ_t to be random.

The presence of bond limits (4) and (5) introduces a further reason that prevents complete markets from arising with these limits playing the role of a transaction cost function as explained below equation (4). Under buyback the bond limit is likely to be binding for calibrated bounds M, but under no buyback it is binding for *any* value of M as stated in the Result at the end of section 3.4.2. Hence, the bond limits are more constraining under no buyback, and the optimal allocation is likely to be even further away from complete markets.

An additional effect comes from the fact that now there are bonds of many maturities outstanding. As pointed out by ABN, issuing long bonds helps with tax smoothing, since both long bond prices and z go down with a high realisation of g. But because of no buyback all outstanding bonds issued over the past N periods now influence the dynamics of the value of total debt outstanding, and

[14] This explanation echoes the argument in Aiyagari et al. (2002), although their one-bond case is simpler to demonstrate.

[15] See the last paragraph of section 3.4.1 about the intuition for a constant λ under complete markets.

fluctuations in z cannot be absorbed as they were with buyback due to previously issued outstanding long bonds. In effect, under no buyback, long bonds become short bonds reducing their ability to provide fiscal insurance.

3.4.4 N cycles

Long bonds under no buyback introduce an additional problem - they add N cycles to taxes and so contribute to tax volatility rather than tax smoothing. To illustrate this point consider the following special case :

i) no short bonds can be issued or purchased, that is $\underline{M}_S = \overline{M}_S = 0$;
ii) g_t is deterministic and higher in the first period: $g_0 > g_1 = g_2 = \ldots$;
iii) debt limits on the long bond are not binding
iv) $b^N_{-i} = \overline{b}$ for $i = 1, \ldots, N$;
Equation (15) for b^N_t gives:

$$\lambda_t = \lambda_{t+N} \qquad \text{for} \qquad \text{all} \qquad t = 0, 1, \ldots \quad . \tag{25}$$

Furthermore $\lambda_1 = \lambda_2 = \ldots = \lambda_{N-1}$ [16].
Putting all this together implies that λ has a simple $N-period$ cycle.

$$\begin{aligned} \lambda_{tN} &= \lambda_0 \quad \text{for} \quad t = 1, 2, \ldots \\ \lambda_{tN+i} &= \overline{\lambda} \quad \text{for} \quad i = 1, \ldots, N-1, \quad \text{and} \quad t = 0, 1, 2, \ldots \quad . \end{aligned}$$

Equation (14) implies that this cycle also arises for consumption and taxes.

The evolution of taxes in this example is shown in Figure 5 assuming $b^N_{-i} = 0$ for $i = 1, \ldots, N$. The dashed line represents taxes when the government issues just a three year bond ($N = 3$), the crossed line for a ten year bond ($N = 10$) and the solid line for when just a one year bond is issued. Clearly taxes are more volatile under no buyback if only a long bond is issued. The higher tax needed in period $t = 0$ because of high g_0 reverberates every N periods, even if there are no further high values of g. This in turn causes a large increase in taxes in future periods at intervals of N periods ahead, while the high g_0 has no effect on taxes in the intervening periods. Obviously the longer the maturity of long bonds the greater the volatility in taxes at longer frequencies.

To understand the reason for this result notice that through forward iteration on the budget constraint (3) we can express bonds today as follows

$$\sum_{j=0,N,2N,\ldots} \beta^j \frac{u_{c,t+j}}{u_{c,t}} (\tau_{t+j}(T - x_{t+j}) - g_{t+j}) = b^N_{t-N} \qquad \text{for all } t. \tag{26}$$

To emphasize, notice the summation index is over $j = 0, N, 2N, \ldots$ This shows that if only one long bond is issued taxes can only be compensated at $t + N, t + 2N, \ldots$ and intervening periods become disconnected. Given the large value of g_0 in assumption *ii*) above there is a rise in taxes and issuance of debt in $t = 0$. But the new debt issued at $t = 0$ has to be redeemed in N years at which point taxes have to increase to pay the accumulated interest and further debt has to be issued. It is pointless to

[16] To prove this notice that (25) and (14) imply $c_i = c_{i+Nt}$ for $i = 1, 2, \ldots, N-1$ and all $t = 1, 2, \ldots$ Together with (26) this implies $u_{c,i}(\tau_i(T - x_i) - \overline{g}) = u_{c,i}\overline{b}(1 - \beta^N)$, therefore $c_i = \overline{c}$ and $\lambda_i = \overline{\lambda}$, for $i = 1, 2, \ldots, N-1$.

increase taxes in intervening periods when spending is back at the steady state value. More precisely, it is not possible to reduce τ_0 by increasing, for example, τ_1 or τ_2. The additional tax revenue at $t = 2$ cannot be utilized to reduce the debt accumulated at $t = 0$ if only long bonds are issued under no buy back and short bonds are not available or at a limit. An increase in τ_2 would only reduce τ_{2+N} and therefore induce even more volatility in the intervening periods.

[Figure 5 About Here]

This example is chosen to illustrate a stark result but the finding is robust. This is an implication of the fact that λ_{t+1} is replaced by λ_{t+i} in going from (11) to (15), generating an N-period cycle in λ. One key result of this paper is that debt management can offset this N-period lumpiness through issuing short term debt. Short debt helps offset tax volatility by distributing debt payments in between these $N-$period cycles. Short bonds under no buyback have additional smoothing properties over long bonds in general.

The above discussion suggests that assuming no buyback will influence optimal DM as it makes long bonds less effective at providing fiscal insurance, induces N cycles in taxes and provides a tax smoothing role for short bonds. What isn't of course clear from this section is whether these channels are quantitatively significant and for that we need to turn to simulations. However, as mentioned above, solving Ramsey models under incomplete markets with multiple bonds is a computationally challenging task and even more so under the assumption of no buyback. In the next section we introduce two new computational methods that help significantly to produce numerical solutions of this model.

4 The Solution Method

In solving our model we apply the widely used Parameterized Expectations Algorithm (PEA) of den Haan and Marcet (1990). Solving our model requires introducing two modifications to PEA. The first modification is necessary because the state vector X_t in our model may be very large requiring a method to reduce the state space.[17] The second modification is required because using PEA in the standard way yields a system of equations that is indeterminate. We refer to the first modification as Condensed PEA and to the second as the Forward States PEA. Whilst our focus is on a problem of government debt management these computational methods have much wider applicability.[18] In order to set up the notation and to clarify the discussion we first give a description of PEA.

[17] Sometimes, in order to reduce the dimensionality of the state space, the literature has assumed bonds consist of geometrically decaying coupons. One justification for this simplification is that the decay may capture a given portfolio with decaying weights on higher bond maturities outstanding. Since the objective of this paper is to aim precisely at explaining portfolio choices taking as fixed the weights of the bond portfolio seems self defeating. Furthermore, Faraglia, Marcet, Oikonomou and Scott (2016) show, in the context of models of optimal fiscal policy, that this approach is at best a weak approximation whilst Hilscher, Raviv, and Reis (2014) show that actual portfolio of bonds outstanding is not geometrically decaying but hyperbolic.

[18] To conserve space we mention here only the principles of these methods. In the Online Appendix B we describe the technical aspects of their implementation. Faraglia, Marcet, Oikonomou and Scott (2014) provide a detailed description of how to solve many optimal fiscal policy problems with this extended PEA.

4.1 The Conventional PEA Approach

We first describe how a standard application of PEA would proceed in this model. For the sake of simplicity we focus on solving the model under buyback described in Section 3.2 when debt limits are not binding. Given the vector $\mathbf{X}_t$ our aim is to solve the system of equations (6), (10) and (11) to obtain the current value of consumption c_t, the bond quantities $b_t^i,\ i = S, N$, and the multiplier λ_t. Parameterized expectations requires approximating the terms $E_t(u_{c,t+i})$ and $E_t(u_{c,t+i}\lambda_{t+i})$ with functions of the state vector $\mathbf{X}_t$, in other words :

$$(27) \qquad E_t(u_{c,t+i}) = \Phi^i(\mathbf{X}_t, \boldsymbol{\gamma}^i) \qquad \text{and} \qquad E_t(\lambda_{t+1}u_{c,t+i}) = \Psi^i(\mathbf{X}_t, \boldsymbol{\delta}^i) \quad i = S, N$$

where Φ^i and Ψ^i belong to a class of functions such that $\Phi^i(\cdot, \boldsymbol{\gamma}^i)$ and $\Psi^i(\cdot, \boldsymbol{\delta}^i)$ can approximate the conditional expectations arbitrarily well. We will take Φ^i and Ψ^i to be polynomials of a given order so γ^i and δ^i will be coefficients on the variables in $\mathbf{X}_t$ as well as their squares, cubes, cross-products and so on, depending on the order of the approximating polynomial of Φ^i and Ψ^i that is used.[19]

The system (6), (10) and (11) has four equations that we hope will give a solution for the four variables $(c_t, b_t^S, b_t^N, \lambda_t)$ given the parameterized expectations. In Section 4.3 we discuss how to set up this system so that $(c_t, b_t^S, b_t^N, \lambda_t)$ can be conveniently solved for.

PEA then iterates to find parameter values $\boldsymbol{\gamma}^{i,f}$ and $\boldsymbol{\delta}^{i,f}$ that satisfy the following fixed point property: the series for $\{c_t, b_t^S, b_t^N, \lambda_t\}$ generated by $(\boldsymbol{\gamma}^{i,f}, \boldsymbol{\delta}^{i,f})$ is such that $\Phi^i(\mathbf{X}_t, \boldsymbol{\gamma}^{i,f})$ and $\Psi^i(\mathbf{X}_t, \boldsymbol{\delta}^{i,f})$ are the best predictors of the objects inside the conditional expectations (27) among any other $\boldsymbol{\gamma}^i, \boldsymbol{\delta}^i$.

4.2 The Condensed PEA

Despite the simplicity of our model the state vector is very large. For the case where the government issues one- and ten-year bonds (i.e $S = 1$, $N = 10$) the state vector $\mathbf{X}_t$ has 22 elements. Allowing the government to issue all maturities between 1 and 10 increases the length of $\mathbf{X}_t$ to 67, as every maturity m adds m lags of bond quantities to the state vector. Since debt limits play a role in our model perturbation methods are not appropriate as they cannot approximate well the solution both near and away from the debt limits, so we strive to approximate the non-linear solution globally. In this situation a state vector of such dimension is difficult to handle even for our relatively basic model.

However, there are reasons to believe that, for most models, the dimensions of the state vector can be effectively reduced. With so many state variables our numerical methodology has a tendency towards close collinearity in the elements of $\mathbf{X}_t$. Furthermore, in models with incomplete markets both λ and b have near unit roots.[20] This means that the regressions used to compute parameters $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are nearly undefined and in plain PEA this often leads the algorithm to either circle indefinitely or even diverge.

However this multicollinearity is in a way encouraging: it suggests that in the optimal solution many elements of $\mathbf{X}_t$ influence the conditional expectation only slightly. Therefore it is likely that

[19] For the sake of clarity we represent the approximating functions using ordinary polynomials though it should be noted that the technique may be applied to orthogonal polynomials (such as Chebyshev, Hermite and Legendre families). We utilize polynomials that are additively separable in the state variables as this allows us to calculate the coefficients with linear methods.

[20] See Aiyagari et al. (2002).

the "relevant" information in $\mathbf{X}_t$ can be condensed in a few state variables to obtain a good approximation. After all, we know some models where this is exactly right. For example, under complete markets all past bond issuances can be "condensed" in total wealth, which is the only relevant state variable. In our case we may expect this to be approximately true.

Furthermore, in PEA we only need that the variable "condensing" $\mathbf{X}_t$ have good properties in predicting the objects *inside* the conditional expectation of the equilibrium conditions. In the solutions we compute each of the elements of $\mathbf{X}_t$ determine the simulation through their role in the budget constraints and other optimality conditions - only the role of $\mathbf{X}_t$ *in the conditional expectations* is condensed.

Intuitively, the dimensionality of $\mathbf{X}_t$ can be reduced for two reasons: first because many elements of $\mathbf{X}_t$ may be perfectly correlated with the rest of the states, and second because they may be (nearly) irrelevant in predicting the objects they should predict along the optimal solution.[21]

More specifically, in solving the buyback model we approximate the expectation

$$E_t(u_{c,t+i}) \tag{28}$$

which appears in the implementability constraint (6) and the first order condition (11).[22] We partition $\mathbf{X}_t$ into two parts: a subset of n state variables $\{\mathbf{X}_t^{core}\} \subset \{\mathbf{X}_t\}$, where n is small and an omitted subset of state variables $\{\mathbf{X}_t^{out}\} = \{\mathbf{X}_t\} - \{\mathbf{X}_t^{core}\}$. Although in our later application the approximating function $\Phi^i(\cdot, \boldsymbol{\gamma}^i)$ includes higher order terms in the solution we study below for the sake of the exposition we illustrate assuming $\Phi^i(\cdot, \boldsymbol{\gamma}^i)$ is linear.

The idea is to first solve the model including only $\mathbf{X}_t^{core}$ as state variables and find a fixed point $\boldsymbol{\gamma}^{i,f,core}$ when only $\mathbf{X}_t^{core}$ is included in Φ^i. We subsequently define the prediction error:

$$\phi_{t+i} \equiv u_{c,t+i} - \Phi^i(\mathbf{X}_t^{core}, \boldsymbol{\gamma}^{i,f,core}). \tag{29}$$

If this is a good approximation then $E_t(u_{c,t+i}) \simeq \Phi^i(\mathbf{X}_t^{core}, \boldsymbol{\gamma}^{i,f,core})$ and the error ϕ_{t+i} would be linearly unpredictable with $\mathbf{X}_t^{out}$. In this case we would claim the solution with core variables is a good approximation. But if $\mathbf{X}_t^{out}$ is correlated with ϕ_{t+i} it means that some elements of $\mathbf{X}_t^{out}$ help predict $u_{c,t+i}$ above the prediction provided by $\mathbf{X}_t^{core}$. We then find the linear combination $\mathbf{X}_t^{out}$ that has the highest predictive power for ϕ_{t+i}, say $\mathbf{X}_t^{out\prime} \cdot \boldsymbol{\alpha}$, we add this linear combination (only one more variable) to the set of state variables in Φ^i, solve the model again with $(\mathbf{X}_t^{core}, \mathbf{X}_t^{out\prime} \cdot \boldsymbol{\alpha})$ as state variables, find a new fixed point $\boldsymbol{\gamma}^{i,f,1}$ with one more element, check if $\mathbf{X}_t^{out}$ can predict the new error ϕ_{t+i} and possibly add new linear combinations of $\mathbf{X}_t^{out}$. Once we find that $\mathbf{X}_t^{out}$ does not have any linear predictive power for the prediction errors we claim that we have found a sufficient summary of the whole state vector $\mathbf{X}_t$.

We now provide a more formal definition of Condensed PEA. Given a selection for core variables

Step 1 Parameterize the expectation as

$$E_t(u_{c,t+i}) = \Phi^i(\mathbf{X}_t^{core}, \boldsymbol{\gamma}^{i,core}). \tag{30}$$

[21] Reiter (2009) addresses a related issue in solving dynamic models with heterogeneous agents. He applies techniques used in control theory to reduce the dimensionality of the agents' distribution of wealth.

[22] Of course, the remaining conditional expectations that appear in the equilibrium conditions must be handled with this procedure as well.

Since we consider for now linear Φ^i (with a constant term) we have $\boldsymbol{\gamma}^{i,core} \in R^{n+1}$. Find $\boldsymbol{\gamma}^{i,f,core}$ that satisfies the usual PEA fixed point i.e where the series generated by $\Phi^i(\mathbf{X}_t^{core}, \boldsymbol{\gamma}^{i,f,core})$ predicts $u_{c,t+i}$ better than with any other $\boldsymbol{\gamma}^{i,core}$.

The next step orthogonalizes the information in $\mathbf{X}_t^{out}$. This will be helpful to give good initial conditions for the next iteration and to arrive at a well conditioned fixed point problem in Step 4.

Step 2 Using a simulation of T periods, for a large[23] T , run a regression of each element of $\mathbf{X}_t^{out}$ on the core variables. That is, letting $X_{j,t}^{out}$ be the $j-$th element, we now run the regression

$$X_{j,t}^{out} = (1, \mathbf{X}_t^{core\prime}) \cdot \boldsymbol{\omega_j}^1 + v_{j,t}^1$$

$\boldsymbol{\omega}_j^1 \in R^{n+1}$ for $j = 1, 2, \ldots, 2N + S + 1 - n$ and calculate the residuals

$$X_{j,t}^{res,1} = X_{j,t}^{out} - (1, \mathbf{X}_t^{core\prime}) \cdot \boldsymbol{\omega}_j^1. \tag{31}$$

It is clear that $\mathbf{X}^{res,1}$ adds the same information to $\mathbf{X}^{core}$ as $\mathbf{X}^{out}$ does, but $\mathbf{X}^{res,1}$ has the advantage of being orthogonal to $\mathbf{X}^{core}$.

Step 3 Find the first linear combination $\boldsymbol{\alpha}^1 \in R^{2N+S+1-n}$ through the following OLS regression:

$$\boldsymbol{\alpha}^1 = \arg\min_{\boldsymbol{\alpha}} \sum_{t=1}^{T} \left(u_{c,t+i} - (1, \mathbf{X}_t^{core\prime}) \cdot \boldsymbol{\gamma}^{i,f,core} - \mathbf{X}_t^{res,1\prime} \cdot \boldsymbol{\alpha}\right)^2. \tag{32}$$

If introducing $\mathbf{X}_t^{res,1\prime} \cdot \boldsymbol{\alpha}$ does not reduce significantly the sum of squared residuals relative to the solution with only $\mathbf{X}^{core}$ we claim the core solution is sufficiently accurate and stop. Otherwise there is evidence that more state variables should be added to the solution and we go to the next step.

Step 4 Apply PEA adding $\mathbf{X}_t^{res,1\prime} \cdot \boldsymbol{\alpha}^1$ as a state variable, i.e. parameterizing the conditional expectation as

$$E_t(u_{c,t+i}) = \Phi^i(\mathbf{X}_t^{core}, \mathbf{X}_t^{res,1\prime}\boldsymbol{\alpha}^1, \overline{\boldsymbol{\gamma}}^i)$$

where $\overline{\boldsymbol{\gamma}}^i \in R^{n+2}$. Find a fixed point $\overline{\boldsymbol{\gamma}}^{i,f}$ for this parameterized expectation. Because $\boldsymbol{\gamma}^{i,f,core}$ is a fixed point and since $\mathbf{X}_t^{core}$ and $\mathbf{X}_t^{res,1}$ are orthogonal and the linear combination α^1 has high predictive power, in order to find the fixed point $\overline{\boldsymbol{\gamma}}^{i,f}$ it makes sense to start iterations with the initial conditions

$$\underset{(n+2)\times 1}{\overline{\boldsymbol{\gamma}}^i} = \begin{pmatrix} \boldsymbol{\gamma}^{i,f,core} \\ 1 \end{pmatrix}.$$

Go to Step 2 with $\left(\mathbf{X}_t^{core}, \mathbf{X}_t^{res,1\prime}\boldsymbol{\alpha}^1\right)$ in the place of $\mathbf{X}_t^{core}$, check if a new linear combination reduces squared residuals, etc.

A couple of remarks are in order. First, note that the Condensed PEA proposed in this section is designed to deal with a very large number of state variables. Our focus is on debt management

[23] This definition assumes we are interested in the steady state distribution. This step could be modified in the usual way (i.e. running the model with many short samples) to take into account transitions. See, for example, Faraglia, Marcet, Oikonomou and Scott (2014) for a detailed description.

and more broadly portfolio models but the method should be useful in many other applications with high-dimensional states including models with many sectors or heterogeneous agents. Second, note that in the presence of many state variables the literature has often solved dynamic economic models by adding state variables one by one in some "order" until the next variable does not materially influence the solution. For example, if many lags are needed the typical approach is to add the first lag, then the second lag, and so on. If at some step the solution changes very little it is claimed that the solution is sufficiently accurate. But it is easy to find reasons why this argument may fail. For instance, maybe the variables further down the list are more relevant, as is the case in our model since a simple inspection of (10) suggests that the N-th lags of both b^N and λ play a special role in the solution. Also, it can be that a linear combination of the remaining variables makes a difference but these variables do not make a difference one by one. The condensed PEA gives a chance to all state variables to make a difference in the solution in only one step and it will pick up the relevance of combinations of state variables (for example, capturing that under complete markets only total wealth matters).

4.3 Forward States PEA

As we discussed above a key step in PEA relies on solving for equilibrium variables using given functions $\Psi^N(\cdot, \boldsymbol{\delta}^N)$ and $\Phi^N(\cdot, \boldsymbol{\gamma^N})$. In particular, off corners the system of four equations (6), (10) and (11) should deliver a solution for the four variables $(c_t, b_t^S, b_t^N, \lambda_t)$. However, because of the multiplicity of assets the system (6), (10) and (11) is not well determined. Note that the two Euler equations imply

$$\lambda_t = \frac{\Psi^S(\mathbf{X}_t, \boldsymbol{\delta}^S)}{\Phi^S(\mathbf{X}_t, \boldsymbol{\gamma}^S)} \qquad \text{and} \qquad \lambda_t = \frac{\Psi^N(\mathbf{X}_t, \boldsymbol{\delta}^N)}{\Phi^N(\mathbf{X}_t, \boldsymbol{\gamma}^N)}. \tag{33}$$

Since the vector $\mathbf{X}_t$ contains only predetermined variables, (33) gives us two equations to solve for the variable λ_t, so this multiplier is overdetermined while the values for bond holdings are indeterminate.[24] Note that this is not a fundamental indeterminacy in the model, it is only an indeterminacy of the particular way PEA solves this problem. We overcome this through the following modification.

4.3.1 Solution through Forward States (FS)

Our proposal is to formulate conditional expectations as functions of current values of state variables. We accomplish this using the following two steps. First, instead of approximating (27) we approximate

$$E_t(u_{c,t+i-1}) = \Phi^i(\mathbf{X}_t, \boldsymbol{\gamma}^i) \qquad \text{and} \qquad E_t(\lambda_t u_{c,t+i-1}) = \Psi^i(\mathbf{X}_t, \boldsymbol{\delta}^i) \quad i = S, N,$$

[24] Marcet and Singleton (1999) and den Haan (1995) (MSDH) already identified this problem in related models. Applying their procedure to the current model is done as follows: replace $E_t(u_{c,t+1})$ by $E_t(u_{c,t+1}H(\boldsymbol{\zeta}, b_t^1))/H(\boldsymbol{\zeta}, b_t^1)$ where H is some function with fixed parameters $\boldsymbol{\zeta}$, invertible in b_t^i, and $H > 0$. Then parameterize $E_t(u_{c,t+i}H(\boldsymbol{\zeta}, b_t^i)) = \overline{\Phi}^S(\mathbf{X}_t, \boldsymbol{\gamma}^S)$. The multiplier can be recovered with the second equation in (33) and bond holdings from $H(\boldsymbol{\zeta}, b_t^1) = \frac{\Phi^S(\mathbf{X}_t, \boldsymbol{\gamma}^1)}{\Psi^S(\mathbf{X}_t, \boldsymbol{\delta}^1)}\lambda_t$. When we used this approach for the current model the algorithm diverges or circles indefinitely. We discuss in the last footnote of this section the reason why FS may work better.

where the only difference with the parameterization in (27) is that we have substracted -1 from the subindeces of the variables inside the conditional expectation. Second, we invoke the law of iterated expectations to write, for example, $E_t(u_{c,t+i}) = E_t(\Phi^i(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^i))$ in order to approximate (11). Similarly, $E_t(\Psi^i(\mathbf{X}_{t+1}, \boldsymbol{\delta}^i))$ approximates $E_t(u_{c,t+i}\lambda_{t+1})$. Substituting these expressions in the system of first order conditions we get that (33) becomes

$$\lambda_t = \frac{E_t(\Psi^i(\mathbf{X}_{t+1}, \boldsymbol{\delta}^i))}{E_t(\Phi^i(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^i))} \quad \text{for} \quad i = S, N \tag{34}$$

$$\sum_{i\in\{S,N\}} b_t^i \beta^i E_t(\Phi^i(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^i)) = \sum_{i\in\{S,N\}} b_{t-1}^i \beta^{i-1} \Phi^i(\mathbf{X}_t, \boldsymbol{\gamma}^i) + g_t u_{c,t} - (u_{c,t} - v_{x,t})(g_t + c_t) \tag{35}$$

Now current b's do enter the right side of (34), therefore these equations plus (10) determine $\left(c_t, b_t^S, b_t^N, \lambda_t\right)$ given $\Psi^i(\cdot, \boldsymbol{\delta}^i)$ and $\Phi^i(\cdot, \boldsymbol{\gamma}^i)$ and the first order conditions with respect to b_t^i hold.[25] In the Online Appendix B we describe further the details of applying this procedure in the model at hand.[26]

5 Optimal Debt Management

Having outlined our solution method we now turn to examine numerically optimal DM under four different market scenarios: full buyback and no buyback, each combined with loose lending constraints (i.e. large $|\underline{M}_i|, |\overline{M}_i|$) and no-lending ($\underline{M}_i = 0$). The no lending constraint $\underline{M}_i = 0$ follows a number of DM papers (e.g Lustig et al. (2008) and Nosbusch (2008)) and is clearly consistent with the stylised facts of Section 2. A number of candidate possible explanations spring to mind such as the uninsurable risk involved in holding private assets and controversies over exactly which private assets the government should buy.[27]

To calibrate all four models we follow Marcet and Scott (2009). We choose β=0.95, set utility $u(c_t) + v(x_t) = \log(c_t) + \eta\frac{(x_t)^{1-\gamma}}{1-\gamma}$ and use a time endowment T=100. We choose a value of $\gamma = 2$ and target a value for η so that on average the household's leisure is 30% of the time endowment; with taxes that balance the budget at the deterministic steady state, this gives η =12.857. Finally, our

[25] In particular we compute

$$E_t\Phi^i(X_{t+1}, \gamma^i) = \int \Phi^i(g', \lambda_t, \ldots, b_{t-N+1}^N, \gamma^i) f_{g'|g_t} dg'$$

analytically, we give the formula in the Online Appendix B.1. This expression shows how this term depends on b_t^N, b_t^S in addition to predetermined variables.

[26] In a previous footnote we mentioned that the approach of MSDH did not work in practice for the current paper, while FS works in the very many versions of the model that we have tried. We do not have a theorem that FS works better than MSDH in general, but we can offer two reasons why it may behave better: first the function H in MSDH is arbitrary, it has to be such that its realized value correlates significantly with marginal utility, it has to be well-approximated by PEA. It is difficult to know beforehand which function H has such properties. FS avoids such an arbitrary choice. Second, our approximation to $E_t(u_{c,t+i})$ under FS depends at most on lags $t - N + 1$, it does not depend on lags dated $t - N$. Obviously, the true solution also has this property. We contend that in this way FS imposes more closely features of the true solution in the numerical approximation and this lends more stability to the algorithm.

[27] Of course the Fed and Treasury have purchased private assets during the financial crisis. This seems not to do with debt management but more tackling financial market disruptions that we, as does the rest of the literature, abstract from. Introducing self-fulfilling debt crises, as in Conesa and Kehoe (2017), in our model would justify that in periods when a debt crisis may occur the government purchases its own bonds while in other periods bond issuance is governed by the mechanisms we describe in the paper.

parameterization of the stochastic process for spending shocks is :

$$g_t = (1 - \rho_g)\overline{g} + \rho_g \, g_{t-1} + \epsilon_t. \tag{36}$$

We set ρ_g=0.95, σ_ε^2 =1.44. We further truncate the value of spending so that it always lies within an interval of 15% to 35% of steady state GDP. $\overline{g}$ is chosen so that the ratio of spending to output is 25% in the deterministic steady state.

We consider two maturities $S = 1$ and $N = 10$, in other words we focus on the case where the government issues a one and a ten year bond. As discussed previously, the bounds $\underline{M}_i$ and $\overline{M}_i$ are in units of steady state market value of debt and we set the upper bounds $\overline{M}_i$ equal to 100% of GDP. This implies that the government can issue debt equal to a maximum of 200% GDP if both bonds are at their upper bound.[28]

In Table 4 we show the key moments to summarize Facts 1 to 4, namely, the serial correlation, standard deviation and the mean of the share of short bonds over total debt (denoted $\mathcal{S}_t$) and the correlation of the market value of short debt with long debt normalized by output. We show results for each of the four cases considered and for comparison purposes we also show U.S data in the first line.

5.1 Buyback

Consider first the case of buyback with lending i.e. loose borrowing constraints. We summarize the output of this model in Figure 6 and the second row of Table 4. In Figure 6 we show a simulated part of the optimal portfolio plotting the market value of short and long bonds for a typical realization of the shocks in g_t.[29]

[Figure 6 About Here]

[Table 4 About Here]

The simulations show clearly that several of the predictions for DM of the complete market models of ABN carry over to the case of incomplete markets. In periods when long bonds are away from the upper bound the government "issues long and saves short", so that on average the market value of short bonds in the sample equals -22.9 and the value of long debt is 45.8.

As is standard under incomplete markets, *total* debt varies through time as it performs the role of a buffer smoothing out shocks.[30] This causes total debt to have very high serial correlation and

[28]In our simulations the government never hits this upper bound for total debt. In our simulations the overall debt level is rarely as high as 120-130% of GDP. Under buyback and when lending is permitted, long term bonds may hit their upper bound constraint and short bonds their lower bound.

[29]The moments displayed in Table 4 are constructed from simulating the model 1000 times over 60 years. For each sample we feed to the model the initial conditions for the debt to GDP ratio and the share of short maturity debt we find in the data (see Online Appendix B.2 for further details). The sample shown in Figure 6 starts from zero total debt to make it comparable to the benchmark of ABN. We show 350 periods to make the figure readable.

[30]See for example Aiyagari et al (2002). When markets are incomplete the government in the long run accumulates savings for precautionary reasons, so that it can fund future adverse shocks without having to raise taxes. Aiyagari et

a positive comovement with the primary deficit. Marcet and Scott (2009) emphasised that the data shows clearly this behavior, making incomplete markets much more empirically plausible than complete markets. All the models considered in this paper confirm this finding and in the interest of space we do not discuss this empirical implication any further.

In contrast to the complete markets case the optimal portfolio is time varying,[31] as is clear from Figure 6 and from the large variance of the share of short bonds (see Table 4). This is due to total debt varying over time and the decision of which bond to issue depending on the level of total debt.

Perhaps more surprising is the fact that, as shown in Figure 6, there are many periods where bond positions b_t^N and b_t^S move in opposite directions. One may have expected a positive correlation because, as explained in Section 3.4.1, the hedging properties of the optimal portfolio are created by the *difference* between the amounts of long- and short-term debt. To the extent that the volatility of government spending is the same in high- or low- debt periods we would expect the difference between short and long term debt to not change much over time, implying a positive correlation of both bond maturities.

The intuition in the last paragraph certainly works for partial equilibrium models, when the volatility of the deficit and interest rates is exogenous to the total debt. But in our model higher debt means higher taxes which alters equilibrium allocations for each g and, therefore, it alters interest rates. The variability of deficits and interest rates change in different ways if taxes go up causing the difference between long and short bonds to vary over time.

More precisely, consider the complete market model of Section 3.4.1. The numerator and the denominator that determine B_N^{BB} in (18) will change in different proportions in response to a higher τ depending on the wage elasticity of leisure. It turns out that for the calibrated model optimal long bonds B_N^{BB} increase more than proportionally across economies with different initial debt b_{-1}^g. Formally $\frac{\partial B_N^{BB}}{\partial b_{-1}^g} > 1$. Now, if $\frac{\partial B_N^{BB}}{\partial b_{-1}^g} > 1$ it must be the case that short bonds decrease with initial debt in order for the budget constraint to hold, therefore $\frac{\partial B_1^{BB}}{\partial b_{-1}^g} < 0$. Putting all this together we see that, given our calibration, across complete market economies with different initial debt, B_1^{BB} and B_N^{BB} move in opposite directions.[32] We describe this issue more carefully in Appendix B.

To the extent that the government tries to replicate complete markets allocations and total debt fluctuates over time we expect b_t^N to increase, and b_t^S to decrease with higher total debt, providing a force for a negative correlation of b_t^N and b_t^S.

In our simulations of Figure 6 we see that in periods when bond limits are loose (and therefore the government has more freedom to behave close to complete markets) b_t^N, b_t^S tend to move in oppposite directions as described in the previous paragraph. On the other hand, in periods when bond limits are tight there is no room to implement these portfolios and b_t^N, b_t^S tend to move in the same direction.

al (2002) show one example where it can be proved analytically that government savings go to a very large number, so that the government can implement the first best in the long run. It is a common feature of optimal dynamic contracts that if the planner can implement the first best in the long run a martingale convergence theorem leads the economy to this first best as time goes by. Albanesi and Armenter (2012) give a set of sufficient conditions for this to arise. In our model the government cannot implement the first best even in the long run because of the lower bound on debt, but savings tend to be large amount, the average market value of total debt to GDP ratio in the sample that we use to solve the model is roughly -50 percent.

[31] In ABN the *value* of the shares of short bonds are not exactly constant but their variance is near zero since the position is constant and the price has very high serial correlation.

[32] A similar phenomenon was found in the economy with endogenous capital and complete markets of Faraglia, Marcet and Scott (2010). Figure 2 in that paper shows that a negative correlation of long and short position emerges.

These two forces turn out to cancel out in the long run and this is why we see a correlation of b_t^N, b_t^S over time close to zero as reported in Table 4.

In summary: the general insights of standard Ramsey models carry over to incomplete markets to some extent - on average $b_t^N > b_t^S$, debt positions are large, away from debt limits b_t^N, b_t^S have a negative correlation but there is now an additional variability of total debt and DM positions over time due to behavior around the bounds.

Now consider *no-lending*, where $\underline{M}_i = 0$. A typical simulation is displayed in Figure 7 based on the same sequence of spending shocks as Figure 6. The solid line represents one year debt and the dashed line ten year debt.

[Figure 7 About Here]

Figure 7 shows that it is still optimal to issue mostly long term debt under buyback and no lending. Short bonds are often close to their lower limit and indeed in some periods we have $b_t^1 = 0$. The third row of Table 4 shows the key moments for the model. The average share of short bonds is at a low 12%, with $b_t^1 = 0$ in 13.1% of the periods and it is less than 10% more than half of the time. Recall that in the data the lowest share short bonds was 24%.

The intuition for this result is that since it is now impossible to build the "issue long, save short" portfolio that provides fiscal insurance, the government gets as close as possible to this portfolio by setting short bonds close to zero.[33] For high levels of total debt we see the government issuing both short and long debt so that overall there is a weakly positive correlation between them. There is no longer scope for increasing long bonds by more than one-to-one when debt increases. This is because there is no possibility of compensating this higher long bond issuance with negative short bond issuance. Therefore the force for a negative correlation of long and short bonds is suppressed and the correlation is now positive.

Our conclusion is that in this case fiscal insurance concerns still dominate, leading the government to prefer to issue mostly long bonds. The average share of short debt is relatively minor and concentrated in times when total government debt is high. The selected DM moments of this model in the third line of Table 4 are still far from the data.

5.2 No Buyback

As we mentioned in the introduction our paper has both normative and positive aspects: we systematically compare our models with the data and to the extent that large differences arise one may conclude that actual policy should change. As the optimal buyback no-lending policy is at odds with the data (i.e the first and third lines in Table 4 are very different) several normative recommendations emerge: governments should issue a much larger proportion of long bonds to achieve fiscal insurance; short bonds should be issued only when debt is already very large; the government should repurchase previously issued bonds.

But the buyback assumption was introduced in the literature for convenience not because it describes actual bond markets. Motivated by empirical observations of Facts 5-7 it is of interest to introduce no buyback in the model. If we should find that introducing no buyback helps match

[33] This intuition was already mentioned in Nosbusch (2008) and Lustig et al. (2008).

the first line of Table 4 and if we can justify that no buyback is closer to the data then the above normative recommendations would be mute.

We now describe the optimal portfolio under no buyback. Later we discuss why no buyback may arise. Notice that in this model at any point in time there are bonds of all maturities between 1 and 10 outstanding in the market, so that ten-year bonds issued nine years ago are now short bonds. We take this properly into account in the statistics we compute below.

The typical realizations with loose debt limits are displayed in Figure 8 and with no lending in Figure 9. Under *lending* the most striking difference with buyback is the strong positive correlation between issued short and long term debt. The usual incomplete market result that governments should accumulate assets still holds in this case. However, under no buyback the government funds a larger deficit by issuing both short and long run debt at the same time. As explained in Section 3.4.4, the reason is that under no buyback issuing only long term debt is less effective at providing fiscal insurance and adds volatility in taxes through N cycles. The implication is that short bonds are a valuable asset in mitigating these cycles. The result is a much stronger co-movement of short and long bond issuance.

[Figures 8 and 9 About Here]

When we add to no buyback the No-Lending constraint the tendency for short and long debt to co-move is strengthened further but now short term debt plays a much more substantial role (with an average portfolio share of 48%). Under no lending the share of short term debt is also less volatile and more persistent than under buyback.

The fourth and fifth rows in Table 4 confirm that under no buyback the moments of the data are matched quite closely. Since we are engaged in comparing statistics in the model and the data it is of interest to make this comparison systematic by using standard inference. Table 5 shows t-statistics for the null hypothesis that the various statistics reported from our simulations are equal to those for US data in our sample period. The t-statistics are simply the difference of the model and data moment divided by the standard deviation of the moment.[34] The standard deviations of the moments are shown in the first row of Table 5 . The t-statistics are shown in rows 2 to 7 of the Table.

Our aim in calculating t-statistics is not to see if our model can explain the data. Our modelling of both the economy and the bond market are extremely simple and easy to falsify. Instead the purpose of Table 5 is to provide some form of statistical gauge to our previous observation that buyback is very far from the data while no buyback is much closer. Table 5 suggests that once allowance is made for no buyback and no lending the discrepancy between Ramsey recommendations and observed practice is very small, in fact the government should issue more short term debt. Therefore, if no buyback is an optimal strategy for debt managers then the policy recommendations arising from buyback (namely, as we mentioned earlier, to issue much more long bonds, to repurchase often previously issued bonds) are, indeed, mute.

[34] We estimate the standard deviation of each moment considered in Table 4 from the data. For this purpose we use the fact that, for a mean zero stationary and ergodic process x_t we have $\frac{\sum^T x_t}{\sqrt{T}} \to N(0, S_w)$ in distribution. We estimate $S_w = \sum_{j=-\infty,\infty} E(x_t x_{t-j})$ using the Newey-West statistic. The asymptotic standard deviation of functions of moments (such as correlations) are found with the delta method.

5.2.1 In-sample model fit

An approach that is often used to visualize the goodness of fit of a dynamic model is to check if a variable determined by the model compares well with the data. For this purpose one plugs in the observed values for the state variables into the model's law of motion and then compares the resulting value of the endogenous variable with the data. In our case we would like to compare the short share $\mathcal{S}_t$ determined by the model with the data. For this purpose we should solve out the short bond issuance using the law of motion

$$\mathcal{S}_t = \mathcal{S}(\mathbf{X}_t)$$

where $\mathcal{S}(\cdot)$ is the time-invariant policy function determining the share implied by the model solution.

One difficulty in our case is that $\mathbf{X}_t$ contains non observables, namely, the lagged λ's. Therefore we replace $\mathcal{S}(\cdot)$ by an approximate law of motion that is a function only of observables. These observables are selected with an eye to using variables that are stationary, so that their data counterpart is a reasonable value to be plugged into the model's law of motion. For this purpose we use model simulations to regress $\mathcal{S}_t$ on its first and second lags, two lags of the market value of total debt over GDP, the current value and first lag of the spending over GDP series. We also include higher order terms of these variables to produce a 'good fit' to the model's policy function. This was done separately for the buyback and no buyback models, since the DM policies differ across these two models. The regression is performed with all the periods and realizations used in constructing the moments reported in Table 4.[35] Second, we feed the 'data state variables' to each model's approximate policy function to obtain the graphs shown in Figure 10. The data line for $\mathcal{S}_t$ is the same as in Figure 1, but because of the inclusion of two lags in the regressions the sample starts in 1957 in Figure 10.

As can be seen the fit of the model under no buyback tracks the short share quite well both in terms of its level as well as its movements in response to government spending shocks.[36] Given the extreme simplicity of the model economy the fit of the no buyback model is surprisingly good. The fit of the model under buyback is much poorer. The correlation between the series produced by the buyback model and the data series equals 0.4. The analogous correlation between the no buyback model and the data equals 0.88.

[Figures 10 About Here]

6 Optimal Bond Repurchases and Transaction Costs

The previous sections considered optimal DM when the government was constrained to either repurchasing as much as possible (buyback) or not to repurchase at all (no-buyback). Both are extreme restrictions on government behaviour. As we have shown optimal DM differs substantially between

[35] To capture better the part of the state space that is active for the CRSP sample, we only use in the regression observations with market value to GDP ratio between 20 percent and 70 percent.

[36] The fit is not perfect - there is an obvious 'lag' in the data relative to the model. We attribute this to the fact that spending does not respond contemporaneously to higher output in the US (see for example the considerable literature on identifying spending shocks in SVARs).

them and no buyback fares much better in reproducing observed stylized DM observations. The question then is: which set of assumptions is most plausible for the study of DM?

Viewed from the perspective of the buyback model cancelling the access to r/r's may appear self-limiting. Indeed, for the calibrated model of the previous section the consumer utility is slightly higher in the optimal buyback allocation than no-buyback, so one could claim that the no-buyback model is in fact incompatible with an optimizing government.[37] Furthermore, although rare, some bond repurchases are observed in the data, as discussed after Fact 6 in Section 2. Perhaps governments should engage in the repurchases more often, as they do in the buyback model of Section 5.1.

However, as discussed at the end of Section 2, many features of actual government bond markets make r/r costly. The standard debt management literature assumes governments can issue and repurchase debt at the same market price but the existence of transaction costs means that in practice this is not the case. So an alternative view is that no buyback as in section 5.2 is the relevant exercise, as transaction costs make the huge r/r in the buyback model too costly.

A way to settle this issue is to model transaction costs as in the data and to allow the government to choose optimally how much to repurchase. We have motivated the upper bound M as an extreme transaction costs function, where bond issues above $\overline{M}$ become suddenly very costly. In that setup the government never actually pays a transaction cost. However the empirical literature on transaction costs of bond issuances suggests a smoother transaction cost function than these bond limits imply and so we can imagine the government choosing to pay transaction costs sometimes as part of optimal DM."

A priori it is not obvious what optimal DM may look like assuming smaller and smoother transaction costs. In practice such costs are very small expressed as a percentage of the total bond issue. This suggests transaction costs would have little effect however it is also the case that the welfare gains from r/r are also quite small (see previous footnote) so that even small transaction costs might change DM substantially. The intuition for why the gains from r/r are small gain can be found from Aiyagari et al (2002, page 1248) who report small gains between the full complete market outcome and the optimal incomplete market policy under buyback. Given we have shown that under no buyback the government can use short bonds to get close to the complete market outcome as well then we are comparing two incomplete market economies which are close to complete markets and so we should expect relatively small welfare differences between them.

Of course depending on the level of transaction costs the optimal behaviour may be an intermediate position with the government sometimes buying back some of the debt. In that case, the observed pattern that repurchases occur in periods of total debt reduction (see discussion after Fact 6) provide another challenge for the model.

It is to these issues that we now turn. In this section we perform three exercises. The first is to use various studies of the US government debt market to calibrate a reduced form function capturing the various transaction costs discussed at the end of Section 2. The second is to use these estimates of transaction costs and perform a shadow cost analysis of the two extreme cases of full buyback and

[37] More precisely, the consumption-equivalent increase in utility ω satisfying

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[u\left((1+\omega)\, c_t^{NBB}\right) + v\left(x_t^{NBB}\right)\right] = E_0 \sum_{t=0}^{\infty} \beta^t \left[u\left(c_t^{BB}\right) + v\left(x_t^{BB}\right)\right]$$

turns out to be $\omega = 0.00416$.

no-buyback and see whether the fiscal insurance benefits of buyback outweigh the transaction costs involved in r/r. Finally we move away from the two extreme cases and allow for the government to choose each period how much debt to repurchase or leave outstanding.

6.1 Calibrating Transaction Costs

At the end of Section 2 we outlined a number of factors that will lead r/r operations to be costly. These factors will be reflected in bid-ask spreads and auction effects, whereby the more of a specific bond the government wishes to issue or purchase the more the price shifts against the government. It is beyond the scope of this paper to introduce explicit microfoundations for these effects. Instead our interest is in establishing whether concern over these costs should substantially influence optimal policy. To do so we extend our model to allow for a reduced form transactions cost function. We denote issuance costs by $\mathcal{T}^i(b_t^i)$ and the cost of repurchasing a bond that was issued at $t-1$ by $\mathcal{T}^R(R_t)$. Assuming these costs are ad valorem e.g the bid-ask spreads are a percentage of price, gives total transaction costs at t as

$$\mathcal{T}ot_t = \sum_{i\in\{S,N\}} p_t^i \, b_t^i \, \mathcal{T}^i(b_t^i) + p_t^{N-1} \, R_t \, \mathcal{T}^R(R_t).$$

Based on our discussion in Section 2 we specify the following functions for transaction costs: $\mathcal{T}^i(b_t^i) = \alpha_0^i + \alpha_1^i b_t^i$ for $i = S, N$ and $\mathcal{T}^R(R_t) = \alpha_0^R + \alpha_1^R R_t$.[38] The bid-ask spread margin is independent of the scale of purchases and so will be reflected in the intercept terms α_0 whilst the auction effects pin down the slope effects. The fact that the costs are linear in issuance/repurchases means that the term $\mathcal{T}^i(b_t^i)$ which appears in the total costs is linear quadratic. Assuming a linear quadratic function is a standard specification in the literature on transaction costs and captures the notion, common in our conversations with debt managers, that price pressures increase the larger the transaction.[39]

As usual in the transaction cost literature we assume that these costs are in terms of hours worked so that feasibility now requires[40]

$$c_t + g_t + \mathcal{T}ot_t = T - x_t$$

Amihud and Mendelson (1991) calculate that bid aks spreads and brokerage fees amount to 0.0381 percent of the price for bonds and 0.0099 percent for Treasury bills. This gives us estimates of $\alpha_0^1 = 0.000099$ and $\alpha_0^N = \alpha_0^R = 0.000381$ (given the face value of a bond is 1 in our model and bid-ask spreads are symmetrical on buyers and sellers). To calibrate the slope terms α_1 we use the estimates of Lou, Yan and Zhang (2013) such that yields are affected by 3 basis points on average due to auction effects on issuance/repurchases.[41] Their estimate is common across all maturities.

The fact the impact is calibrated in terms of yearly yields and the costs $\mathcal{T}$ above are paid only at issuance means that we need to translate the 3 bps estimate into issuance costs $\mathcal{T}$. The effect

[38] If b_t^i would be allowed to be negative this function would present a kink at $b_t^i = 0$, giving rise to non-smooth solutions that would be hard to compute. Since we impose $b_t^i \geq 0$ in this subsection there is no kink, solutions are smooth, and all costs considered are indeed positive if we take $\mathcal{T}^i \geq 0$.

[39] Support for this is also to be found in Breedon and Turner (2016) Table A2.1.

[40] Not all transaction costs require resources to be deducted from the resource constraint. Our findings in this section remain if no such deduction is made.

[41] Lou et al actually give a range of 2-3 bps. These estimates are broadly similar to Breedon and Turner (2016) Table 2 and substantially less than many estimates of the impact of QE on yields.

of issuance/purchase is larger on longer maturity bonds (the impact on bond prices is proportional to the impact on yield multiplied by the duration of the bond). This calibration means that the steady state yields of bonds, after taking into account auction effects, is $1/\beta + 0.0003 - 1$, i.e auction effects increase the cost of issuing bonds across all maturities by 3 basis points on average. The annualized yield plus auction costs implied by the above transaction cost function is $\left(\frac{1}{p^i(1-\alpha_1^i b^i)}\right)^{1/i} - 1$, where p^i and b^i are averages. Equating these expressions and rearranging gives an estimate of $\alpha_1^i b^i = 1 - (1 + 0.0003\beta)^{-i}$. For $i = 10, 9, 1$ this gives average auction costs ($\alpha_1^i b^i$) of 0.0028, 0.0026 and 0.000284 respectively. To calibrate the slope terms we need to divide these average auction effects through by the average issuance in our simulations. Since the no-buyback model of Section 5.2 matches the data reasonably well we use average issuances in that model to arrive at the following estimates for the slope coefficients : $\alpha_1^1 = 0.000021$, $\alpha_1^{10} = 0.001$ and $\alpha_1^R = 0.000926$.[42]

6.2 A Shadow Cost Calculation

As mentioned before, for the calibration used here, the utility of the optimal allocation is indeed higher under buyback. But since the buyback strategy involves many more purchases and sales it is not clear if the benefit of fiscal insurance under buyback will dominate if transaction costs exist. We now make an approximate "shadow" calculation of the loss in utility due to transaction costs.[43] This calculation has the virtue of being independent of the precise way that we model repurchases. Since the transaction costs are small and they indicate a larger government expenditure, the lagrange multipliers of each solution translate transaction costs into an approximate utility loss.

Let superindex ^{BB} denote the solution with buyback of Section 5.1 and ^{NBB} the solution under no buyback in Section 5.2.

A transaction cost plays the same role as higher g_t both in the government budget constraint and in the utility term $\beta^t v(T - c_t - g_t)$. The total marginal utility loss of higher g_t in t is

$$\widetilde{\lambda}_t^{BB} + \beta^t v'(T - c_t^{BB} - g_t^{BB} - \mathcal{T}ot_t^{BB})$$

where $\widetilde{\lambda}_t^{BB}$ is the "plain" lagrange multiplier of the government budget constraint. Since we have normalized the plain lagrange multipliers in the usual way,[44] we have that the lagrange multipliers of Section 5.1 are given by

$$\widetilde{\lambda}_t^{BB} = \beta^t \lambda_t^{BB} u_{c,t}^{BB}.$$

[42] We calibrate repurchases assuming that the auction effects are symmetric in buying and selling. We therefore use estimates for $i = 9$ to calibrate the repurchase auction effects one year after a ten year bond has been issued. Point estimates in Breedon and Turner (2016) suggest that repurchase auction effects may actually be larger than issuance effects. However given the importance of no buyback in our model we make the more conservative assumption that the two effects are symmetric and our results are clearly robust to attributing higher effects to repurchases.

[43] We thank Dimitri Vayanos for suggesting this calculation.

[44] The "plain" lagrange multiplier would be the one that would come out of a standard Lagrangean, namely

$$\begin{aligned}\mathcal{L} \;=\;& E_0 \sum_{t=0}^{\infty} \beta^t (u(c_t) + v(T - c_t - g_t)) + \\ & E_0 \sum_{t=0}^{\infty} \widetilde{\lambda}_t \left(\sum_{i \in \{S,N\}} b_t^i E_t \left(\beta^i \frac{u_{c,t+i}}{u_{c,t}} \right) - \sum_{i \in \{S,N\}} b_{t-1}^i E_t \left(\beta^{i-1} \frac{u_{c,t+i-1}}{u_{c,t}} \right) + g_t - (1 - \frac{v_{x,t}}{u_{c,t}})(g_t + c_t) \right).\end{aligned}$$

Therefore the total shadow transaction costs of buyback, in term of utility, is

$$\mathcal{T}otal^{BB} = E_0 \sum_{t=0}^{\infty} \beta^t \left(\lambda_t^{BB} u_{c,t}^{BB} + v_{x,t}^{BB}\right) \mathcal{T}ot_t^{BB} .$$

The total costs of the no buyback strategy $\mathcal{T}otal^{NBB}$ are found similarly, using the lagrange multipliers of the no-buyback solution and $R_t = 0$.

Denoting utility of the optimal allocation under each environment as $U^i = E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t^i) + v(x_t^i)\right]$ for $i = BB, NBB$, we define *total utility net of transaction costs* for each policy as $U^i - \mathcal{T}otal^i$. Then, we find the scaling factor χ such that if the transaction cost functions would be $\chi\left(\mathcal{T}^R, \mathcal{T}^S, \mathcal{T}^N\right)$ the government is indifferent between the two strategies. This factor is given by

$$\chi = \frac{U^{BB} - U^{NBB}}{\mathcal{T}otal^{BB} - \mathcal{T}otal^{NBB}}.$$

If it turns out that $\chi < 1$ it means that, given the approximate utility calculation, the transaction costs of buyback outweigh their benefit.[45]

We find $\chi = 0.07$. In other words, once we take into account transaction costs the buyback strategy is more costly than no buyback. This would be so even if transaction costs would only be one tenth lower.

6.3 A Model of Optimal Bond Repurchases

The previous shadow cost calculation supports our intuition that if one considers the two extremes, buyback or no-buyback, the former involves such large bond transaction that it ceases to be optimal once transaction costs are taken into account. That result is largely robust to our transaction cost calibration, as it would still hold if transaction costs were one tenth of the calibrated value.

We now examine the same issue by studying a model where repurchases of any amount are allowed and the optimal level of repurchases is decided each period. The budget constraint of the government is as follows

$$(37) \quad \sum_{i\in\{S,N\}} p_t^i b_t^i (1 - \mathcal{T}^i(b_t^i)) \;=\; b_{t-S}^S + b_{t-N}^N - R_{t-N+1} + p_t^{N-1} R_t (1 + \mathcal{T}^R(R_t)) + g_t - \tau_t(T - x_t)$$

$$(38) \quad 0 \;\leq\; b_t^i \leq \frac{\overline{M}_i}{\sum_{j=1}^{i} \beta^j}, \qquad 0 \leq R_t \leq b_{t-1}^N$$

There are *three* differences relative to the models of Sections 3.2 and 3.3: *i)* we introduce transaction costs $\mathcal{T}^i$, *ii)* repurchases R_t appear as a cost in period t and *iii)* repurchases R_{t-N+1} appear as an income at $t-N$, the amount of long bonds that mature at t is now $\left(b_{t-N}^N - R_{t-N+1}\right)$. The buyback model of section 3.2 imposes $R_t = b_{t-1}^N$, the no-buyback model of Section 3.3 imposes $R_t = 0$ whilst the current model allows any value in between.

We assume only the government pays all transaction costs, hence the consumer/investor budget constraint is unchanged relative to previous sections. This simplifies the model. Since we look at optimal policy the government will take into account the presence of all transaction costs anyway,

[45] Details on the approximations can be found in the Online Appendix B.4.

hence results should be very similar even if consumers pay part of the transaction costs. Another assumption in the above model is that we only consider repurchases of long bonds issued in the previous period. This keeps as close as possible to the standard optimal DM literature and considerably simplifies the analysis.[46]

The Lagrangean is straightforward to write and is shown in Online Appendix A.3. Endogenous repurchases complicate the simulations in various dimensions: obviously now we have an additional decision variable R_t. Furthermore, since R_{t-N+1} enters the budget constraint at t it might seem that we now have to add $N-1$ lags of R to the state variables X_t so we end up with $S+3N$ state variables. However, after some manipulations one can show that a sufficient set of state variables is

$$X_t = \left[g_t, \left(B_{t-i}, B\lambda_{t-i} \right)_{i=1}^{S}, \left(B_{t-i}^{net}, B\lambda_{t-i}^{net} \right)_{i=1}^{N-S+1}, \lambda_{t-N}, b_{t-N}^{N} \right]$$

where

$$\begin{aligned} B_t^{net} &\equiv b_{t-1}^N - R_t \\ B_t &\equiv b_t^S + B_{t-N+1+S}^{net} \\ B\lambda_t^{net} &\equiv \lambda_{t-1}(1-\mathcal{T}^N) b_{t-1}^N + \lambda_t (1+\mathcal{T}^R) R_t \\ B\lambda_t &\equiv \lambda_t (1-\mathcal{T}^S) b_t^S + B\lambda_{t-N+1+S}^{net} \end{aligned}$$

see Online Appendix A.3. Therefore we have "only" $1+2N$ state variables.

6.3.1 Simulation Results

Solving our model with transaction costs calibrated as above but allowing the government to choose how much debt to repurchase each period effectively generates the no buy back situation. The government chooses to repurchase only very rarely and even then only during periods of strong debt reduction due to large government surpluses - similar to the US in the 1920s and 2000-1. In Figure 11 we plot the ratio of the market value of debt to GDP, and the absolute level of repurchases using the same sample for g as in Figures 6 to 9. Notice that repurchases are a tiny fraction of GDP and that indeed the government repurchases debt only in periods where debt falls sharply. The moments are as reported in the final row of Table 4 and they match closely the analogous objects under no buyback and no lending [47] and those of the US data very well. The t-statistics reported in Table 5 confirm that the model is close to the data.

[Figure 11 About Here]

This confirms the main point of the paper, namely, that once we take into account small transaction costs resembling those found in the data a portfolio with a substantial share of short bonds and

[46] Since we will find that buyback of any amount is very rare there will be bonds outstanding of many maturities in this model. Hence we could entertain a model where bonds of any maturity could be repurchased. This would complicate the analysis as there would be $N-2$ more decision variables. By considering only immediate repurchases we are maximizing the possible fiscal insurance benefits of long bonds, since the price of one-year-old long bonds is the one that provides the largest amount of fiscal insurance as it is the longest bond outstanding.

[47] For brevity we do not show the portfolio of long and short bonds in a separate figure. However, as the results in Table 4 show the behavior of the portfolio in the optimal repurchase model is very close to the no buyback model, e.g. Figure 9

no repurchases achieves higher utility than the portfolios under buyback that resemble the recommendations arising from effectively complete market models. Therefore the basic Facts 1-7 described in Section 2 can be matched by a model where debt management is decided optimally. More critically allowing for transaction costs means no buyback is a preferred strategy and the assumption that optimal DM requires a reliance on long bonds is no longer accurate with short bonds playing a significant smoothing role.

7 Robustness and Accuracy

In this section we explore the robustness of our main results to the introduction of various relevant features. First of all, as US government bonds tend to pay a fixed semi-annual coupon we introduce coupons in the model to see if this alters our findings. Second, we introduce a third bond in the analysis enabling us to talk about short, medium and long issuance. Third, given that callable bonds have been used in the sample period we study a model consistent with Fact 7 where bonds are recalled before, but close to, maturity. Finally we explore the accuracy of our solutions.

7.1 Coupons

Although ignored in most academic papers, it is a fact that individual long term bonds in the US pay constant semi-annual coupons. The effect of coupons on DM is non-trivial for two reasons. Firstly, because coupons add another element of flexibility in bond payments opening up another channel with which to complete the markets. Secondly, the existence of coupons means that a bond's duration (the measure of how long it takes to recoup the price paid for a bond in terms of its cash flow) is distinct from its maturity. This distinction doesn't exist for the zero coupon bonds but as we have established in the presence of no buyback the duration of bonds matters for DM. Therefore the effects of coupons are non-trivial. Let us discuss the two issues we have raised.

Completing Markets with Coupons

We show that if coupons are fixed at the time of issuance and kept constant for the duration of the bond markets can not be completed. This occurs even if the fixed coupons are contingent on the information at issuance. Markets can be effectively completed only if the government promises to pay coupons *contingent* on future shocks complete the markets.

The outline of the argument is as follows. Consider the process for g and the notation in section 3.4. Assume the government could issue consol bonds b_t^{∞} in period t that paid coupons κ_t^j in period $t+j$ contingent on the realisation of $\mathbf{g}^{t+j}$ for all periods $t+j$, all $j>0$. As in sections 3.4.1 and 3.4.2, in order to complete the markets the following analog of (17) should hold for these consol bonds[48]

$$\sum_{j=1}^{t+1} b_{t-j}^{\infty}(p_t^{j,\infty} + \kappa_{t-j}^{j}) = z_t \tag{39}$$

where $p_t^{j,\infty}$ is the price in the secondary market at t of a consol issued at $t-j$.

[48]To simplify things we consider the case where the government can not repurchase previously issued console bonds.

We guess and verify that the following DM policy achieves this equality for the complete market allocations: issue nothing after $t=0$, ie. $b_t^\infty = 0$ for $t>0$, normalise $b_0^\infty = 1$; contingent coupons for the issued bond are given by $\kappa_0^j = \kappa^H, \kappa^L$ contingent on $g_j = g^H, g^L$ for all j.

The values κ^H, κ^L are determined as follows. If complete markets are implemented bond prices in the secondary market at t satisfy $p_t^{0,\infty} = E_t\left(\sum_{k=1}^\infty \beta^k \frac{u'(c_{t+k}^{CM})}{u'(c_t^{CM})}\kappa_t^k\right)$. Given that $c_{t+j}^{CM} = c^H, c^L$ as in section 3.4.1 we have

$$
\begin{aligned}
p_t^{0,\infty} &= K_i^H\kappa^H + K_i^L\kappa^L \text{ if } g_t = g^i \text{ for } i = H, L \text{ where} \\
K_i^k &= \sum_{j=1}^\infty \beta^j \mu_i^{k,j}\frac{u'(c^k)}{u'(c^i)} \text{ for } k, i = H, L,
\end{aligned}
$$

where $\mu_i^{k,j} = Prob(g_{t+j} = g^k \mid g_t = g^i)$ is given by g's Markov chain. Combining this with (39) we find values κ^H, κ^L have to satisfy

$$
K_i^H\kappa^H + K_i^L\kappa^L + \kappa^i = z^i \text{ for } i = H, L \tag{40}
$$

where z^H, z^L are as in section 3.4. This gives two equations to determine κ^H, κ^L. Clearly (39) holds by construction since $p_t^{0,\infty} + \kappa_0^t = z_t$ for all periods a.s. Therefore issuing one bond at $t=0$ that pays the contingent coupons defined by (40) effectively completes markets.

Consider now the case when coupons are restricted to be fixed for the whole life of the bond, namely $\kappa_t = \kappa_t^j$ for all j. Most bonds actually issued pay (nominally) fixed coupons. In this case we can not effectively complete markets. To see this, notice if we implemented complete markets we would have

$$
p_t^{j,\infty} = \mathcal{D}_t\kappa_{t-j} \text{ for } \mathcal{D}_t \equiv E_t\sum_{k=1}^\infty \beta^k \frac{u'(c_{t+k}^{CM})}{u'(c_t^{CM})}
$$

so that for (39) to hold we would need

$$
(\mathcal{D}_t + 1)\sum_{j=1}^{t+1} b_{t-j}^\infty \kappa_{t-j} = z_t. \tag{41}
$$

But since $\mathcal{D}_t$ and z_t are both different functions of g_t, and since the term $\sum_{j=1}^{t+1} b_{t-j}^\infty \kappa_{t-j}$ is not contingent on g_t, equation (41) will not hold a.s.[49]

Therefore, for a consol to complete the markets the government needs to issue debt with contingent coupons. This is akin to issuing contingent Arrow securities with the added difficulty that the coupon amounts κ^H, κ^L need to satisfy equations (40) requiring a very detailed knowledge of the economy. Obviously, if g would have a continuum of realisations the coupons paid would need to take a continuum of values satisfying even more complicated equations.

Fixed coupons as a mixture of long and short bonds

From now one we consider a case where long bonds issued in t pay a (possibly time dependent) constant coupon κ_t from periods $t+1$ to $t+N$ and in addition it pays the principal (normalized to unity) at $t+N$. Coupons can then be thought of as a means of lessening or even overcoming the

[49] Notice this reasoning is analogous to the one we gave at the end of section 3.4.3 to conclude that markets are truly incomplete with zero coupon long bonds.

problem of no buyback that has been our focus. If the government attaches to each bond a sequence of fixed coupon payments this makes long bonds closer to short bonds (reduces their duration) and therefore the consequences of no buyback are less severe. In this section we examine to what extent coupon payments overcome the N-period cycles of Section 3.4.4 and attenuation of fiscal insurance. In other words, are coupons a way of using security design to make long bonds more attractive?

It is easy to show that the competitive equilibrium price of this bond is

$$q_t^N = \kappa_t \sum_{j=1}^{N} \beta^j E_t\left(\frac{u_{c,t+j}}{u_{c,t}}\right) + \beta^N E_t\left(\frac{u_{c,t+N}}{u_{c,t}}\right)$$

i.e. the price is now the sum of prices of zero coupon bonds of maturity $j < N$ ($p_t^j = \beta^j E_t(\frac{u_{c,t+j}}{u_{c,t}})$) weighted by the coupon payments plus the value of the bond repayment. Obviously, in the steady state (with constant coupons and consumption) the bond price becomes equal to $\kappa \sum_{j=1}^{N} \beta^j + \beta^N$.

According to the CRSP data, bonds issued by the US government trade close to par when they are issued, namely $q_t^N \approx 1$. To choose coupons that are consistent with this observation we set $\kappa_t = \kappa = \frac{1-\beta}{\beta}$ for all t. It turns out that in this case bonds trade close to par, our simulations yield q_t^N close to 1 in all periods.

7.1.1 The Ramsey Program with Coupons

We now find the optimal policy assuming that long bonds pay a yearly coupon κ and assuming no buyback. Debt limits are:

$$b_t^N \in \left[\frac{\underline{M}_N}{\sum_{j=1}^{N} \beta^j + \kappa \sum_{j=1}^{N} \sum_{i=1}^{j} \beta^i} \quad , \quad \frac{\overline{M}_N}{\sum_{j=1}^{N} \beta^j + \kappa \sum_{j=1}^{N} \sum_{k=1}^{j} \beta^k}\right] \equiv [\widetilde{\underline{M}}_N, \ \widetilde{\overline{M}}_N]$$

and $[\widetilde{\underline{M}}_1, \ \widetilde{\overline{M}}_1] \equiv [\frac{\underline{M}_1}{\beta}, \frac{\overline{M}_1}{\beta}]$ for short bonds.[50] The planning problem and the FOC with respect to consumption are given in the Online Appendix A.4. Off corners the first order conditions for b_t^1 and b_t^N are:

$$\lambda_t E_t(u_{c,t+1}) = E_t(\lambda_{t+1} u_{c,t+1}) \tag{42}$$

$$\lambda_t E_t(\kappa \sum_{j=1}^{N} \beta^j u_{c,t+j} + \beta^N u_{c,t+N}) = E_t(\kappa \sum_{j=1}^{N} \beta^j u_{c,t+j} \lambda_{t+j} + \beta^N u_{c,t+N} \lambda_{t+N}). \tag{43}$$

Equation (43) is the analogue of equation (15) (when the debt constraints are loose). It reveals that the multiplier follows a complicated pattern which equates it with the sum of all expected future terms $u_{c,t+j}\lambda_{t+j}$ for $j = 1, 2, \ldots, N$ weighted by the payments that the bond promises. Intuitively a zero-coupon N maturity bond produced a N cycle in λ_t, if we think of coupons as themselves zero coupon bonds then we are adding additional smaller cycles in λ_t at 1,2,...,$N-1$

In Faraglia et al (2016) we study the properties of the single bond coupon model analyzing the effects of fiscal shocks. We show that the optimal policy is characterized by an important N cycle component and the main features of optimal fiscal policy under no buyback persist, since basically

[50] In the US coupon payments are usually six monthly. However, since our model's horizon is one year we model one year debt as zero coupon.

under coupons the bulk of interest rate payments is still concentrated every N periods. For brevity we refer the reader to Faraglia et al. (2016) for details.

In the fifth row of Table 4 we show the simulation results of the no buyback, no lending, coupon-paying model. The results show relatively minor changes from the zero coupon case - a small increase in the portfolio share of short bonds, a less volatile, less persistent share and a slightly higher correlation between short and long bond issuance. Introducing coupons to long bonds does not alter our conclusions.

7.2 Three Bonds

A relevant robustness exercise is to consider a government that issues a third bond. In particular, we assume that in addition to 1- and 10- year bonds now the government can issue a 5-year bond. This also serves as a test for the ability of the algorithm to deal with larger models, as the state vector now has five more variables and an additional decision variable.

We keep the zero-coupon assumption for comparability and we only study the no buyback case. We do not write the model and optimality conditions as they are similar to those in Section 3.3.

As can be seen from the corresponding row in Table 4, now we find that the average share of short debt is lower than NBB with two bonds and lower than the data. There is a similar phenomenon with the serial correlation of $\mathcal{S}_t$. Other statistics remain the same.

The intuition for these results is clear: the five-year bond attenuates the $N-$period cycles that we have discussed. By issuing 5-year bonds the height of the spikes of taxes described in Section 3.4.4 is half of the spike with a 10-year bond, hence the government is less reluctant to issuing long bonds and reduces issuance of 1-year bonds. This example demonstrates one feature of the model at hand: small changes in the model may have considerable impact on $E(\mathcal{S}_t)$ hence this value is not closely determined by the theory. But there are many features of optimal DM that *are* well determined and robust in the models we have considered, namely: $\mathcal{S}_t$ is never close to zero, it is stable and short bonds are highly correlated with long bonds. The main focus of our paper is to establish a role for short term bonds in optimal debt management which is clearly preserved in this case.

7.3 Callable Bonds

In our section on the stylised facts of US government debt we described how in the first half of our sample period considerable use was made of callable bonds. Just as coupons can be thought of as shortening the life of a bond so too can a callable bond. By preannouncing a future date at which the government can buy back a bond at par the government lessens the duration. If, as is the case in practice, governments tend to repurchase at the first call date, then cash flow is affected and bonds are not bought back at maturity. This raises another way in which security design may lessen the problems with long bonds and reduce the need for short bonds. We investigate this (results in the Online Appendix A.5) by considering the case where the government issues bonds of maturity N but repurchases them after m years where $m < N$ e.g we assume that governments always repurchases at the call date. We find that our main result still holds, if callable bonds are redeemed after m years they offer less fiscal insurance and introduce greater tax volatility at lower frequencies opening up a positive role for issuing short term debt to support optimal tax smoothing. If governments cannot repurchase debt each and every period then long bonds provide additional volatility and short bonds can help tax smoothing.

7.4 Accuracy

We have performed thorough accuracy tests of all the models for which we report numerical simulations. To check the accuracy of the solution we run Euler Equation Error (EEE) tests (see for example Aruoba et al (2006) for an exhaustive description of the methodology). Essentially this methodology checks that first order conditions hold with an acceptable degree of precision at many points in the state vector. For example, in the model with buyback one needs to check that Euler equations (11) hold approximately. To interpret the size of the error, the approximation is expressed in terms of the change in consumption units that would be required for the Euler equation to hold exactly.

The tests require to numerically calculate each of the conditional expectations in the Euler equations. Ours is not a routine application of this accuracy test because we have expectations involving up to N leads, so that exact integration would be too costly. For this reason we use Monte-Carlo integration to compute these expectations, using many simulations starting at each point considered of the state space. Details on how we perform the test are in the Online Appendix B.3.

Since we have two Euler equations (or three in the case of the optimal repurchases model) we check separately each of them by calculating the value of the multiplier and for consumption in period $\bar{t}$ implied by the expectations Ξ_t generated by our approximation, given the portfolio $b_{1,\bar{t}}, b_{N,\bar{t}}$. Table 6 summarises the test for the main models presented in the paper. [51] As in Aruoba et al. (2006) we report the absolute errors using base 10 logarithms to make our findings comparable with the rest of the literature. A value of -3 means a 1\$ mistake per 1000\$, a value of -4 a mistake of \$1 per \$10000 and so on.

Table 6 shows that the average of the errors are between -3 and -4 and that the maximum errors are not large. Moreover, we found that it is quite unlikely that the region of the state space where the maximum error occurs is visited in simulations. We have calculated also the percentage of positive and negative errors. A good approximation should deliver evenly distributed errors between the two signs. In the Online Appendix B.3 we show that the distribution is fairly even in all the models presented with 41% to 58% of the errors being of positive sign. These results are well within the range accepted by other authors (e.g. Aruoba et al (2006)) suggesting that the model solutions are accurate.

8 Conclusion

We have studied optimal debt management under incomplete markets. The literature has to date isolated a powerful influence of fiscal insurance whereby governments can exploit the negative covariance between long bond prices and fiscal shocks in order to stabilise debt and minimise tax volatility. The implications of that channel for debt management under incomplete markets leads to a policy

[51] An additional table in the Online Appendix B.3 provides all the details for each model and each Euler equation in this work.

relying heavily on issuing long term debt. The recommendations from this canonical Ramsey approach are in stark contrast to the actual behaviour of observed US debt management. One might infer from this literature that governments should issue more long bonds, pursue very active portfolio management and actively repurchase previously issued long bonds.

Our approach has been to consider whether there are other market frictions that introduce additional considerations into debt management. If the implications of fiscal insurance are robust across a range of market frictions then the existing findings in the literature increase in their relevance. If however the implications from the standard Ramsey model are non-robust then it becomes important to isolate the exact market frictions that need to be considered in order to better understand the implications for actual debt management.

Once we introduce incomplete markets, non-negativity constraints on bonds issued and small transaction costs, the picture changes considerably. Optimal policy now involves repurchasing very rarely and a sizable and stable proportion of short bonds is key in achieving tax smoothing. Short bonds play a more important role in supporting optimal debt management than in previous papers, as short bonds provide a flexibility and optionality that helps reduce tax volatility in the face of fiscal shocks. Portfolio shares should be more stable and persistent and governments should issue positive quantities of both short and long debt in response to a shock to deficit. All of these conclusions are diametrically opposite to the standard recommendations from the canonical Ramsey model mentioned earlier, making observed US debt management look much closer to the recommendations of optimal debt management.

These main features of optimal debt management under no buyback are very robust, they stand the introduction of coupons, more bonds, and a callable bond. Clearly the recommendations around optimal debt management for governments can benefit from a deeper understanding of the reasons for market incompleteness and their practical relevance. It is of course entirely feasible that other microfounded features of market incompleteness e.g the clientele effects, asymetric information, liquidity provision, reduced rollover risk, etc. restore a preference for long bonds. Critically however they would do so for reasons other than fiscal insurance and merely reiterate the importance under incomplete markets of specifying the reasons for market incompleteness.

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Appendix

A Database and Construction of Figures

Data on US Treasuries was obtained from the CRSP US Treasury Database and is comprised of all types of marketable treasury securities (bills, notes, bonds and inflation protected securities (TIPS)). Observations are available at monthly frequency. As discussed in text to compute key moments such as the share of short maturity debt or the ratio of issuances over total debt, we focus on data from the years 1955-2015. We extended this sample to include observations from the 1920s 30s and 40s when we report moments on the frequency of repurchases or the timing of buybacks of callable bonds (e.g. Tables 2 and 3).

As discussed, not all bonds are recorded by the CRSP in the 1920s-1940s. The missing amounts outstanding are as high as 60 percent in the mid 30s and 40 percent in the early 40s. This constrains our empirical analysis to include observations from the 1950s onwards when all bond records on marketable US government debt are included in our dataset. We report the properties of share of short maturity debt over total debt in the CRSP from 1955 onwards, so that our estimates exclude the build up of public debt during the Korean war. Our results however would not be affected if we included the early 1950s in our sample.

Date variables of particular interest for this study include the quote date, the date of the first coupon and the maturity date. Amounts outstanding of the bonds are usually available in the CRSP although missing for certain observations. Gaps in amounts outstanding were filled with preceding observations for the same bond when possible or future observations if no preceding data points existed. Coupons are typically paid every six months from the date the bond is issued and until maturity.

To construct the share of short term debt (e.g. in Figure 1) we stripped the coupons. The strips, were given distinct maturity dates, face values and market values. For a ten year bond paying coupons every six months the first coupon is counted as six month maturity debt (at the issuance date), the second coupon as one year debt and so on. [52] Market values for the strips were imputed using the yield-maturity data.

Our sample includes both nominal and inflation protected debt (TIPS). TIPS typically represent long maturity claims (five, ten or thirty year debt) and therefore contribute towards reducing the average value of the share of short term debt. The first inflation protected security in the US government bond market was issued in 1997 (before then there was no indexed government debt).

Finally, to construct the time series of the issuances (Figure 2) we adopt the following approach. Because our definition of short term debt in the data includes maturities which are less than one year (one month to six months in the CRSP data) we define the total issuance in short maturity debt as : $I_{S,t} = \frac{I_{m,t}}{12} + \frac{I_{3m,t}}{4} + \frac{I_{6m,t}}{2} + \frac{I_{12m,t}}{1}$ where $I_{xm,t}$ denote claims of maturity x months in the data [53] and the quantities $I_{xm,t}$ are sums of issuances over year t. Note that differently from the case of the

[52] Our approach to strip the coupons and assign an appropriate maturity to each strip is compatible with the notion that the benefits from fiscal insurance are proportional to the amount of long term government debt outstanding. If a long term bond is issued paying coupons every six months then this bond provides less hedging to the governments budget than a zero coupon long term bond, as is claimed in text. When we construct the model counterparts for the series plotted in Figure 1 we follow essentially the same approach.

[53] This definition includes the first coupons of long term notes and bonds.

stocks displayed in Figure 1 the issuances share is formed by a flow variable in the numerator and a stock variable in the denominator. This approach enables us therefore to not bias the share upwards by counting very short term issuances (monthly debt) many times over the year.

B Response of long bonds to higher debt under Buyback

We examine whether issuance of long bonds increases more than proportionally with initial debt in the complete market case of section 3.4.1, ie. $\frac{\partial B_N^{BB}}{\partial b_{-1}^g} > 1$. This plays a role in the intuition we give for the behavior of DM in Figure 6, section 5.1.

Taking derivatives in the equation determining long bonds (18) we have

$$\frac{\partial B_N^{BB}}{\partial b_{-1}^g} = B_N^{BB}\left[\frac{\Delta' z}{\Delta z} - \frac{\Delta' p}{\Delta p}\right]\frac{\partial \tau}{\partial b_{-1}^g} \tag{44}$$

where in this appendix we denote $\Delta z \equiv z^H - z^L$, $\Delta' z \equiv \frac{\partial \Delta z}{\partial \tau}$, similarly $\Delta p \equiv p_H^{N-1} - p_L^{N-1}$ and for all other variables. The values of z and p are evaluated at the complete markets solution.

Note that in this complete market economy Δx gives the variability of variable x across time. Therefore $\frac{\partial B_N^{BB}}{\partial b_{-1}^g}$ depends not only on the bond level itself B_N^{BB}, but also on the tax elasticity of the *variability of the yield curve* $(\Delta' p/\Delta p)$ and of the *variability of surpluses* $(\Delta' z/\Delta z)$.

It is clear that $\frac{\partial \tau}{\partial b_{-1}^g} > 0$ and $B_N^{BB} > 0$.[54] In a partial equilibrium model we would have $\Delta' p = 0$, but in our general equilibrium model the variability of interest rates depends on consumption and, therefore, it varies with income taxes so that $\Delta' p/\Delta p$ is not zero. We study this dependence in detail below.

We first find $\frac{\partial B_N^{BB}}{\partial b_{-1}^g}$ by calculating B_N^{BB} for different initial debt given our calibration in section 5. We fit a two-value, symmetric g process that roughly reproduces the mean, variance and serial correlation of (36) and we use $g^H = 18.94$, $g^L = 16.06$, $\mu = \Pr ob(g_{t+1} = g^i \mid g_t = g^i) = 0.95$. Figure 12 shows $\frac{\partial B_N^{BB}}{\partial b_{-1}^g}$ in the vertical axis and initial debt b_{-1}^g in the horizontal axis. Clearly $\frac{\partial B_N^{BB}}{\partial b_{-1}^g} > 1$ for b_{-1}^g within the debt limits. This explains why in some periods b_t^N and b_t^1 move in opposite directions in the lending model of section 5.1.

[Figure 12 About Here]

Uncertain signs of the derivative

After showing numerically that $\frac{\partial B_N^{BB}}{\partial b_{-1}^g} > 1$ for a specific calibration we now explore the robustness of this property. We find that this is not a robust feature of the model: the various elasticities of consumption can combine to produce different $\frac{\partial B_N^{BB}}{\partial b_{-1}^g}$, one can even find calibrations where $\frac{\partial B_N^{BB}}{\partial b_{-1}^g} < 0$.

To see this we now study the terms in (44) relying on various approximations. We skip large amounts of algebra - details are available from the authors. We show in bold the sentences where the signs of the elements in the formula are determined.

Recall that **from ABN we know** $\Delta z < 0$.

For a symmetric g denote $\mu_j = \Pr ob(g_{t+j} = g^i \mid g_t = g^i)$ for $i = H, L$. Standard calculations give $\mu_j = \frac{1+(2\mu-1)^j}{2}$.

[54] The first inequality follows from efficient taxes being in the increasing side of the Laffer curve, the second inequality has been shown in section 3.4.1.

As shown in Online Appendix A.1, log utility implies

$$\Delta p = \beta^{N-1}(1-\mu_{N-1})\left[\frac{c^H}{c^L} - \frac{c^L}{c^H}\right]$$

therefore $\Delta p < 0$.

Using various Taylor approximations and $\Delta c \simeq c'_g \Delta g$, where c'_g is the derivative in equilibrium consumption with respect to g, we have

$$\Delta p \simeq \beta^{N-1}(1-\mu_{N-1})\ 2c'_g \Delta g/\widetilde{c}$$

where $\widetilde{c}$ is average consumption.

Taking derivatives with respect to τ and using Taylor approximations

$$\frac{\Delta' p}{\Delta p} \simeq \frac{c''_{g,\tau}}{c'_g} - \frac{c'_\tau}{\widetilde{c}} \tag{45}$$

where $c''_{g,\tau}$ is the cross derivative of equilibrium consumption with respect to g, τ.

This shows that the term $\frac{\Delta' p}{\Delta p}$ depends on various elasticities of consumption and that in general equilibrium $\frac{\Delta' p}{\Delta p}$ is not zero.

To characterize further this expression note that equilibrium taxes $\tau_t = 1 - \frac{v_{x,t}}{u_{c,t}}$ for the utility in section 5 imply $\eta c\,(1-g-c)^{-\gamma} = 1 - \tau_t$. Taking total derivatives in this equality we find

$$\begin{aligned} c'_g &= \frac{-\gamma}{x/c+\gamma} < 0 \\ c'_\tau &= \frac{-1}{(1-\tau)\left[1/c+\gamma/x\right]} < 0 \\ c''_{\tau,g} &= -c'_\tau \frac{\gamma(x+c)}{(x+\gamma c)^2} > 0 \end{aligned}$$

Plugging this in (45) we have $\frac{\Delta' p}{\Delta p} > \mathbf{0}$.

Now we turn to the term $\frac{\Delta' z}{\Delta z}$. For the utility function at hand,

$$\begin{aligned} b^g_{-1} &= z_H = \sum_{t=0}^{\infty}\beta^t\left[\mu_t s^H + \frac{c^H}{c^L}(1-\mu_t)s^L\right] \\ z_L &= \sum_{t=0}^{\infty}\beta^t\left[\mu_t s^L + \frac{c^L}{c^H}(1-\mu_t)s^H\right] \end{aligned}$$

using the approximation $\left(\frac{c^H}{c^L}s^L - \frac{c^L}{c^H}s^H\right) \simeq \frac{\widetilde{s}}{\widetilde{c}}2(c^H - c^L) - (s^H - s^L)$ we have

$$\Delta z \simeq K_1(s^H - s^L) + K_2\frac{\widetilde{s}}{\widetilde{c}}2\left(c^H - c^L\right)$$

for constants

$$K_1 = \sum_{t=0}^{\infty} \beta^t (2\mu_t - 1) = \frac{1}{1-\beta(2\mu-1)}$$
$$K_2 = \sum_{t=0}^{\infty} \beta^t (1-\mu_t) = \frac{1}{2}\left[\frac{1}{1-\beta} - K_1\right]$$

Since optimal policy involves tax smoothing we take the appoximation $\tau^H \simeq \tau^L$ to have $s^H - s^L \simeq \tau(\Delta c + \Delta g) - \Delta g$, then

$$\Delta z \simeq K_1 \left[\tau(c'_g + 1)\Delta g - \Delta g\right] + K_2 \frac{\widetilde{s}}{\widetilde{c}} 2 c'_g \Delta g$$

(46) $$\Delta' z \simeq K_1 \left[(c'_g + 1)\Delta g + \tau c''_{g,\tau} \Delta g\right] + K_2 \Delta g \frac{\widetilde{s}' c'_g 2}{\widetilde{c}}$$

where we use the approximation $\widetilde{s} \simeq 0$, which is approximately right if initial debt is close to zero. However, note $\widetilde{s}' = (1-x) + \tau c'_\tau$ is non-zero.

The sign of $\Delta' z$ is uncertain. Since the numerical calculations of Figure 12 shows that $\frac{\partial B_N^{BB}}{\partial b_{-1}^g}$ is positive and as we know $B_N^{BB} > 0$ the bracket of (44) has to be positive for the calibration above. As we have already shown that $\frac{\Delta' p}{\Delta p} > 0$ and $\Delta z < 0$ this implies that **for our calibration $\Delta' z < 0$**.

However, it is easy to see that under the above approximations the term in brackets in (44) could be zero or even negative. Consider, for example, the case when $\mu \simeq 1$, that is when g is nearly a constant. In this case $K_2 \simeq 0$ so that, since $c'_g > -1$ **equation (46) means $\Delta' z > 0$. Therefore for very high μ we have $\frac{\Delta' z}{\Delta z} - \frac{\Delta' p}{\Delta p} < 0$ and $\frac{\partial B_N^{BB}}{\partial b_{-1}^g} < 0$.**

Remaining Term (in quarters)	**Normalized Count**
0	98.86%
1	0.70%
2-4	0.24%
5-9	0.05%
10-14	0.02%
≥15	0.13%

Notes: The table provides information on the redemption profiles of non callable bonds. The data are all non callable debt issued by the US Treasury since the 1920s. 'Remaining Term' counts the number of quarters remaining until maturity when debt is bought back. When 0 this signifies that debt is bought at maturity. The data are extracted from the CRSP.

Table 1: **Remaining Term at Time of Buyback**

Year	**Amount Issued (in millions)**	**Share Called**
1931	755	100%
1934	491	100%
1935	2611	100%
1936	5616	100%
1938	3588	100%
1939	1689	100%
1940	1404	100%
1941	9326	55.35%
1942	14061	38.06%
1943	16763	29.46%
1944	26986	14.16%
1945	28172	11.71%
1952	921	100%
1953	1606	0%
1960	470	0%
1962	365	0%
1963	550	54.55%
1973	1618	100%
1974	587	100%
1975	3616	100%
1976	1574	100%
1977	2638	100%
1978	4516	100%
1979	4523	100%
1980	7794	100%
1981	4626	100%
1982	3163	100%
1983	4921	100%
1984	16142	100%

Notes: The table lists (by year of issuance) the total amounts of callable bonds which have been called prior to maturity. The data are extracted from the CRSP.

Table 2: **Share of Redeemed Callable Treasuries by Year of Issuance**

Bond Term (in years)	Call Window* (in years)					
	2	**3**	**4**	**5**	**10**	**15**
5	3	0	0	0	0	0
7	1	0	0	0	0	0
9	1	0	0	0	0	0
10	8	0	0	0	0	0
11	1	0	0	0	0	0
12	3	0	0	0	0	0
13	1	0	0	0	0	0
14	3	1	1	0	0	0
15	1	2	0	0	0	0
16	1	1	0	0	0	0
17	1	2	1	0	0	0
18	0	3	0	0	0	0
20	0	0	1	1	0	0
23	0	1	0	0	0	0
24	0	0	1	0	0	0
25	0	0	0	8	1	2
26	0	0	0	2	0	0
27	0	0	0	6	0	0
29	0	0	0	0	0	2
30	0	0	0	13	2	2

Notes: The table shows the call windows (maturity minus first possible call date) for callable bonds in the US. The sample used is the same as the sample used for Table 2.

Table 3: **Bond Terms and Call Windows**

		$\overline{\mathcal{S}}_t$ (%)	$\sigma_{\mathcal{S}_t}$ (%)	$\rho_{(\mathcal{S}_{1,t},\mathcal{S}_{1,t-1})}$	$\rho_{\left(\tilde{b}^S_t,\tilde{b}^N_t\right)}$	$\%_{\mathcal{S}_t}=0$	$\%_{\mathcal{S}_t}\leq 0.1$
US DATA		43%	7.8%	0.94	0.86	0	0
BuyBack							
	'Lend.'	$4{\cdot}10^3$%	$3{\cdot}10^5$%	0.47	-0.01	-	-
	'No Lend.'	12%	13.0%	0.86	0.46	13.1%	56.6%
No Buy back							
	'Lend.'	76%	$3{\cdot}10^3$%	0.42	0.87	-	-
	'No Lend.'	48%	8.1%	0.92	0.92	0.01%	0.02%
	'No Lend.+Coupons'	51%	4.9%	0.90	0.94	0.01%	0.02%
	'3 Bonds'	31%	5.5%	0.81	0.93	0.11%	0.64%
Repurchases+ $\mathcal{T}$ Costs							
	'No Lend.'	45%	9.0%	0.92	0.93	0.01%	0.01%

Notes: $\mathcal{S}_t$ denotes the share of debt of maturity less than or equal to one year over the total (market value) of debt. $\overline{\mathcal{S}}_t$ represents the average share and $\sigma_{\mathcal{S}_t}$ denotes the standard deviation. The statistic $\rho_{(\tilde{b}^S_t,\tilde{b}^N_t)}$ is the correlation between the market value of short debt and the value of long debt both divided by GDP. The exact definition of the market values, varies depending on the model specification. For example under buyback it holds that $\rho_{(\tilde{b}^S_t,\tilde{b}^N_t)} \equiv \rho(\frac{p^S_t b^S_t}{GDP_t}, \frac{p^N_t b^N_t}{GDP_t})$. Under no buyback and no coupons $\tilde{b}^N_t \equiv \frac{\sum_{i=S+1}^{N} p^i_t b_{t-N+i}}{GDP_t}$. Therefore, when $S=1$ the value of long debt outstanding is the value of all debt which in t is of maturity greater than one year and $\tilde{b}^S_t$ is the market value of all outstanding debt less than one year maturity, divided by GDP. The data counterpart is constructed applying this logic (see text).
$\%_{\mathcal{S}_t}\leq x$ denotes the percentage of times that $\mathcal{S}_t$ is less than or equal to x.
Finally $\mathcal{T}$ denotes the transaction cost function specified in Section 6. See text for details.

Table 4: **Moments: Data and Models**

		$\overline{\mathcal{S}}_t$	$\sigma_{\mathcal{S}_t}$	$\rho_{(\mathcal{S}_t,\mathcal{S}_{t-1})}$	$\rho_{\left(\tilde{b}^S_t,\tilde{b}^N_t\right)}$
std. of sample moment		0.0314	0.0139	0.0209	0.0354
Buyback					
	'Lend.'	-1355	247540	22.44	24.65
	'No Lend.'	9.87	-3.76	3.79	11.39
No Buyback					
	'Lend.'	-10.52	-2744	24.83	-0.17
	'No Lend.'	-1.60	-0.24	0.92	-1.59
	'No Lend.+Coupons'	-2.55	2.05	1.88	-2.15
	'3 Bonds'	3.81	1.61	6.18	-1.87
Repurchases					
	'No Lend.'	-0.65	-0.89	0.92	-1.87

Notes: The Table presents t statistics testing the hypothesis that the data moments are equal to the model generated moments summarized in Table 4.

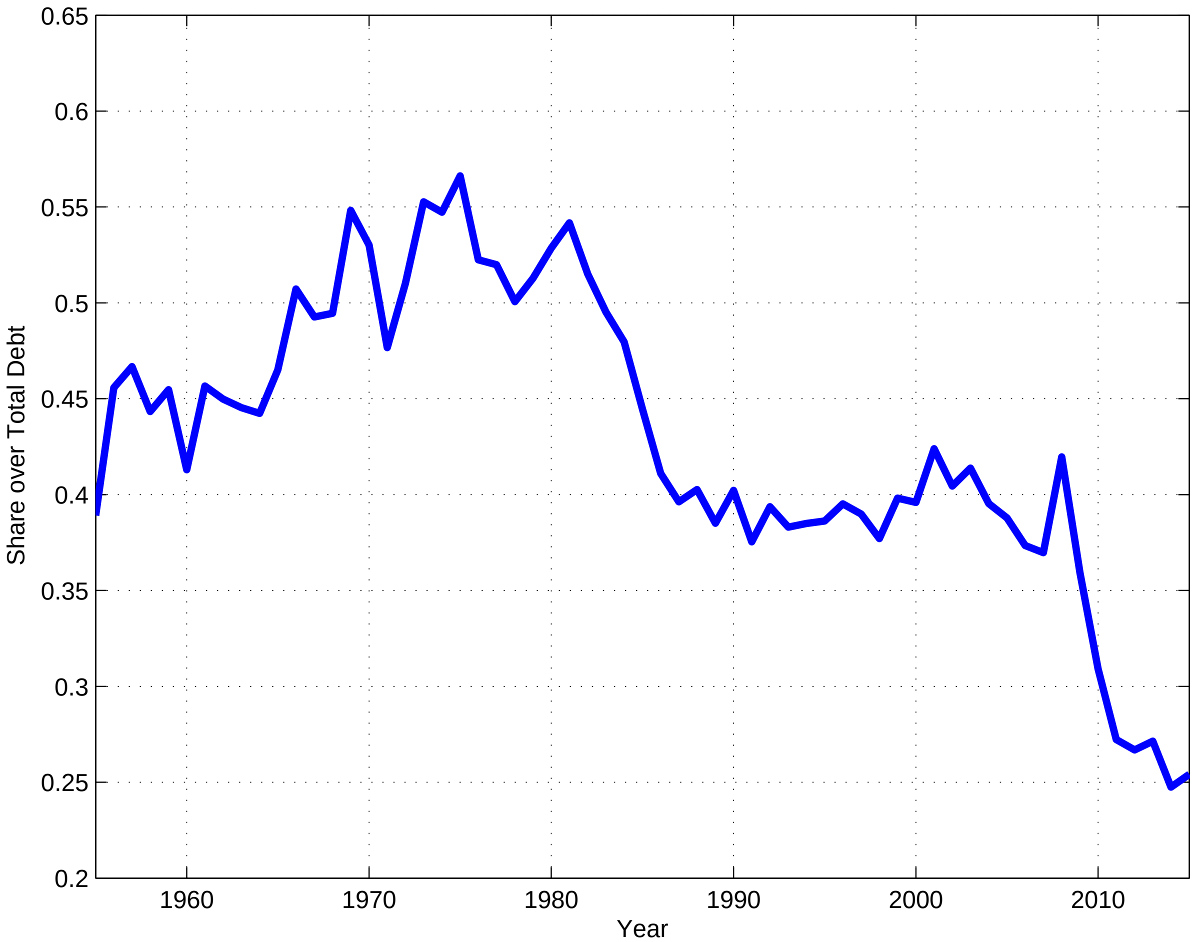
Table 5: **t stats: Data and Model Moments**

		BB		NBB		
		lending	no lending	lending	no lending	opt. repurch.
EEE^1	ave	-3.97	-3.72	-3.84	-3.86	-3.84
	max	-2.30	-2.28	-2.50	-2.64	-2.79
EEE^N	ave	-3.18	-3.06	-3.53	-3.51	-3.18
	max	-1.81	-1.93	-2.55	-2.47	-2.12
EEE^{N-1}	ave					-3.23
	max					-1.94

Notes: The Table reports average and maximum Euler equation errors (EEE) for the benchmark models (buyback/ no buyback, lending / no lending, and optimal repurchases). Additional moments and errors for other models considered in this paper can be found in the Online Appendix B.3.

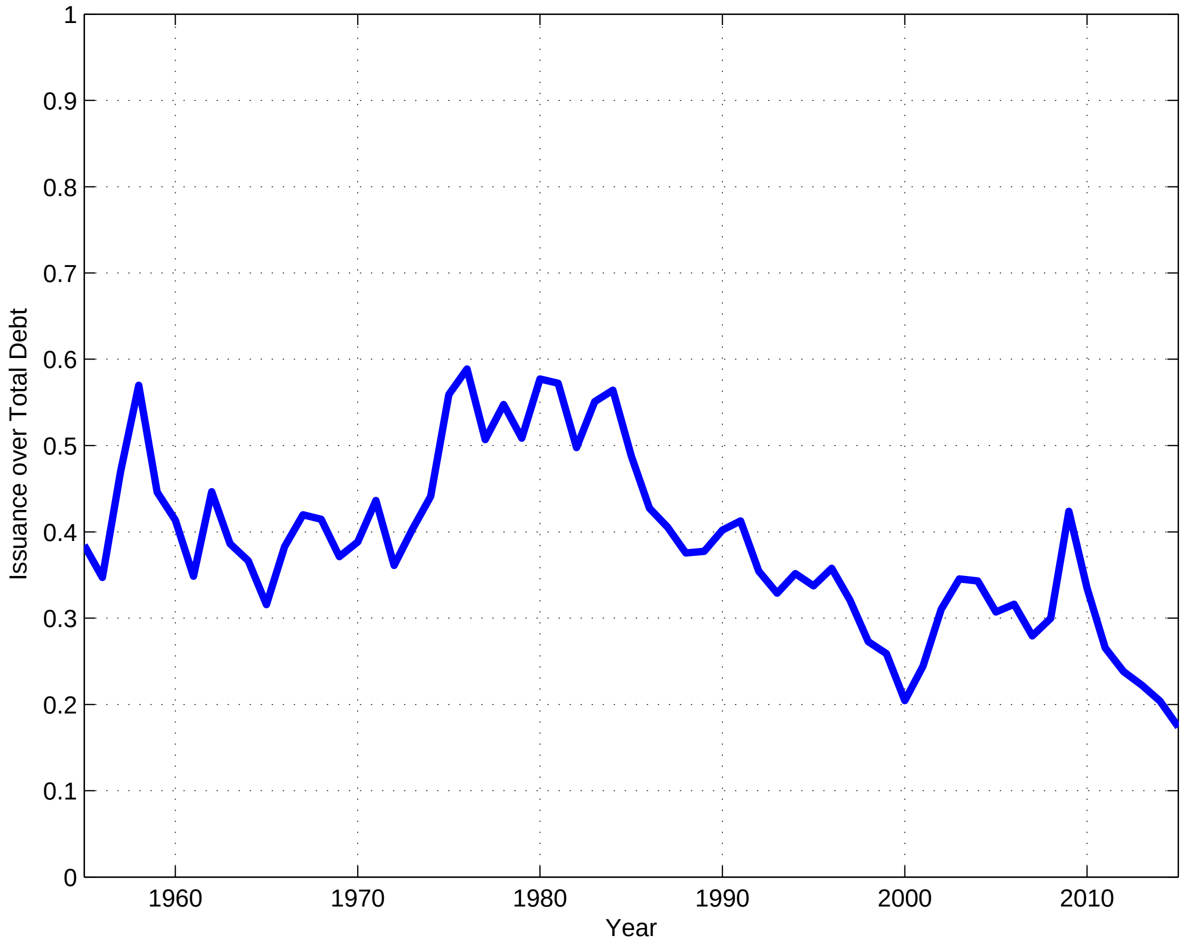
Table 6: **Accuracy Tests**

Figure 1: **Share of Short Term Debt in the US**



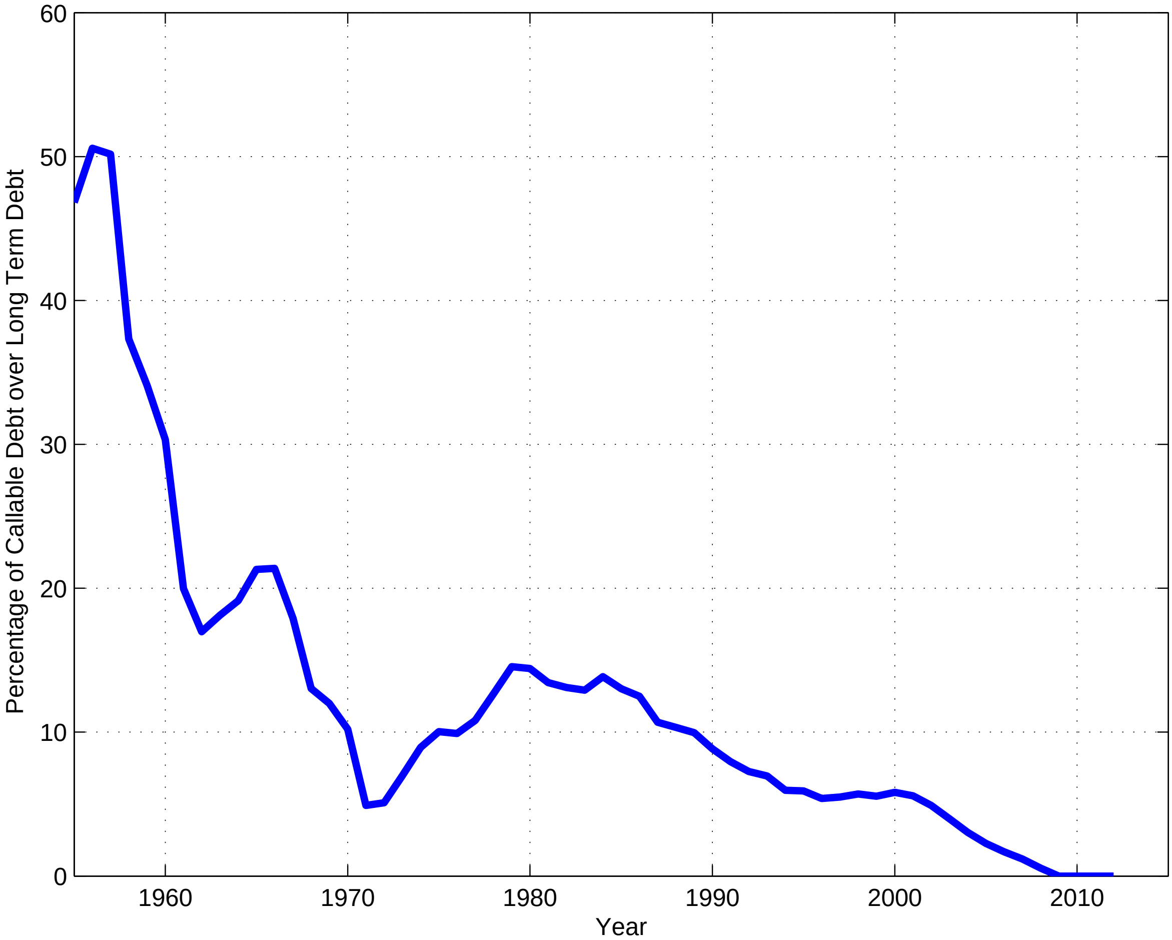
Notes: The Figure plots the share of short maturity government debt (less than or equal to one year) in the US over the period 1955-2015. The data are annual observations (time aggregated from monthly data extracted from the CRSP). Details on the data construction are contained in the Appendix A.

Figure 2: **Total Issuance as a Fraction of the Market Value of Outstanding Debt**



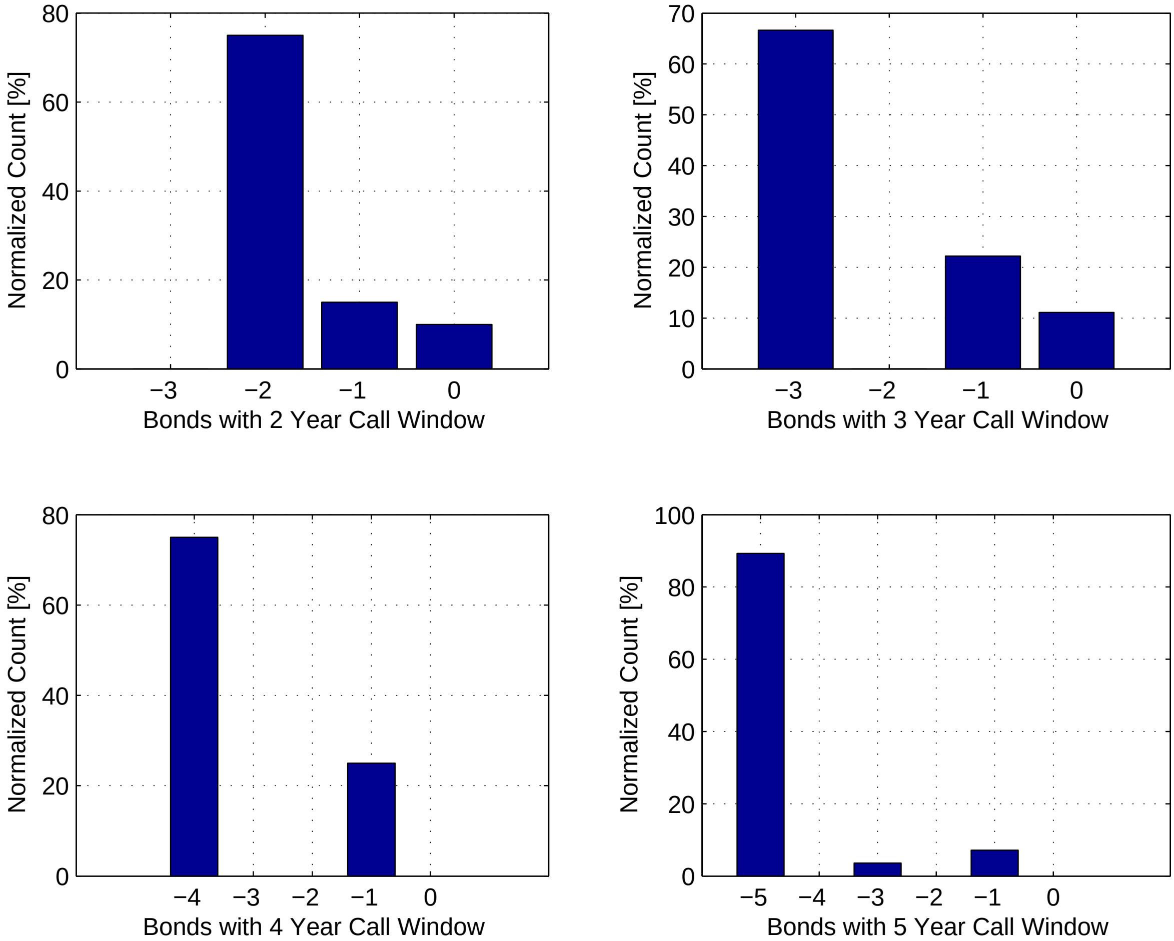
Notes: The Figure plots the issuance of new government debt by year and in market value, as a fraction of the total market value of debt outstanding in the United States. The data are from the CRSP and refer to the period 1955-2015.

Figure 3: **Callable Bonds over Long Bonds in the US data**



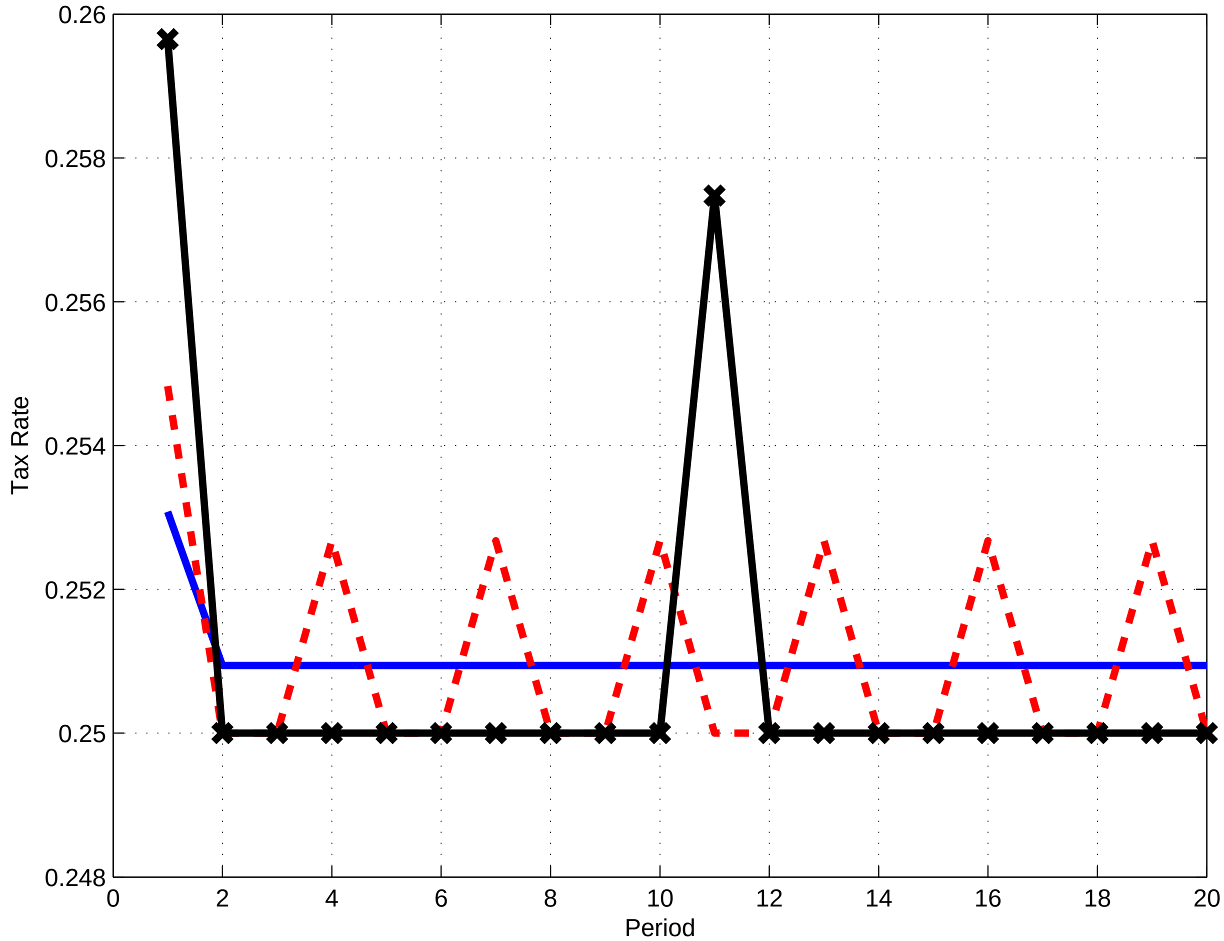
Notes: The Figure plots the fraction of long term callable debt over total long term debt outstanding in the CRSP sample.

Figure 4: **Callable Bonds: Timing of Buybacks and Call Windows**



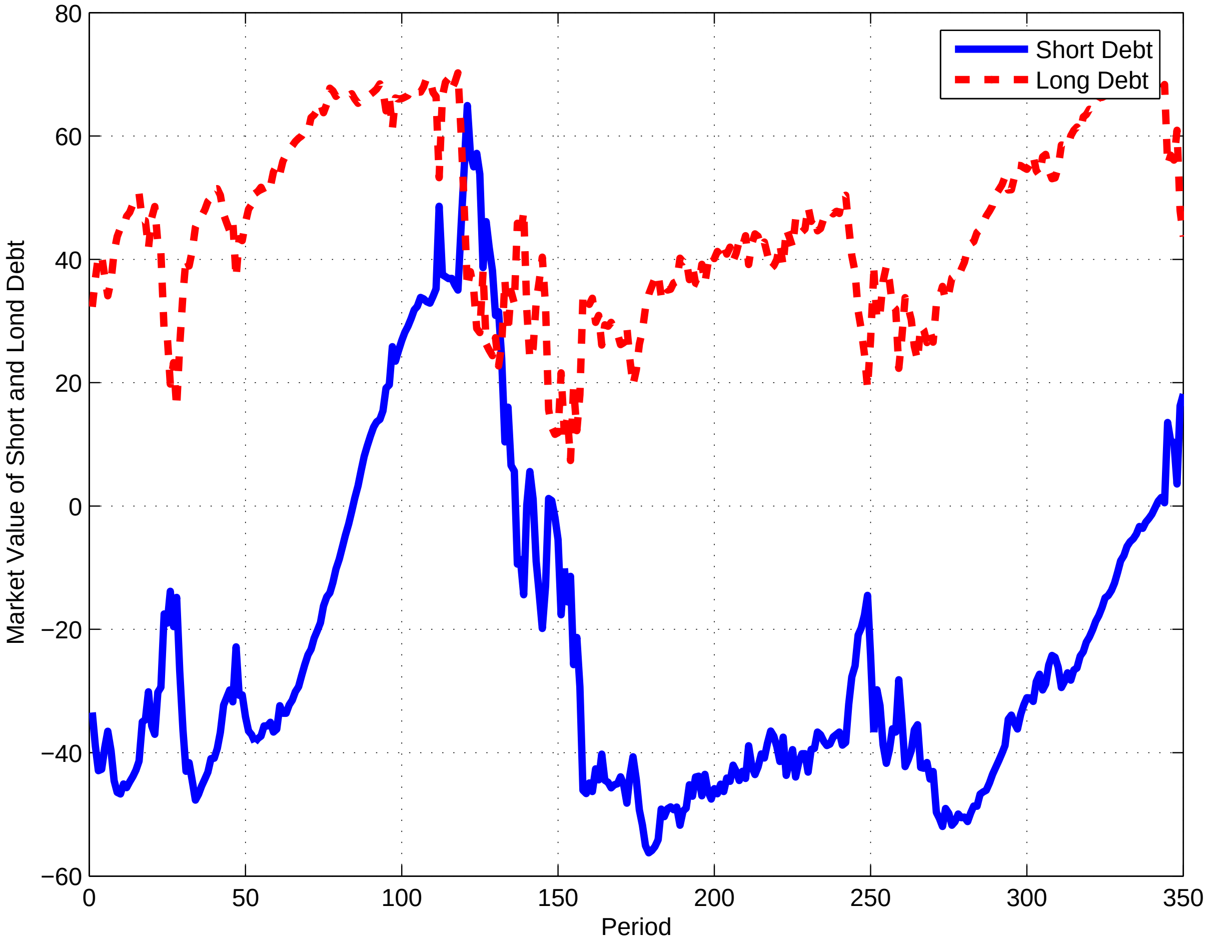
Notes: The plots shows the timing of redemptions of callable debt in the US. The top left shows this timing for bonds whose first call date is 2 years before maturity. The y-axis is in percentage points. Therefore, roughly 75 percent of the bonds are redeemed 2 years before maturity, 15 percent 1 year and 10 percent at the maturity date. The top right panel represents bonds with 3 year call windows, and the bottom panels 4 and 5 years, left and right respectively.

Figure 5: **Response of the Tax Schedule - No Buyback Model**



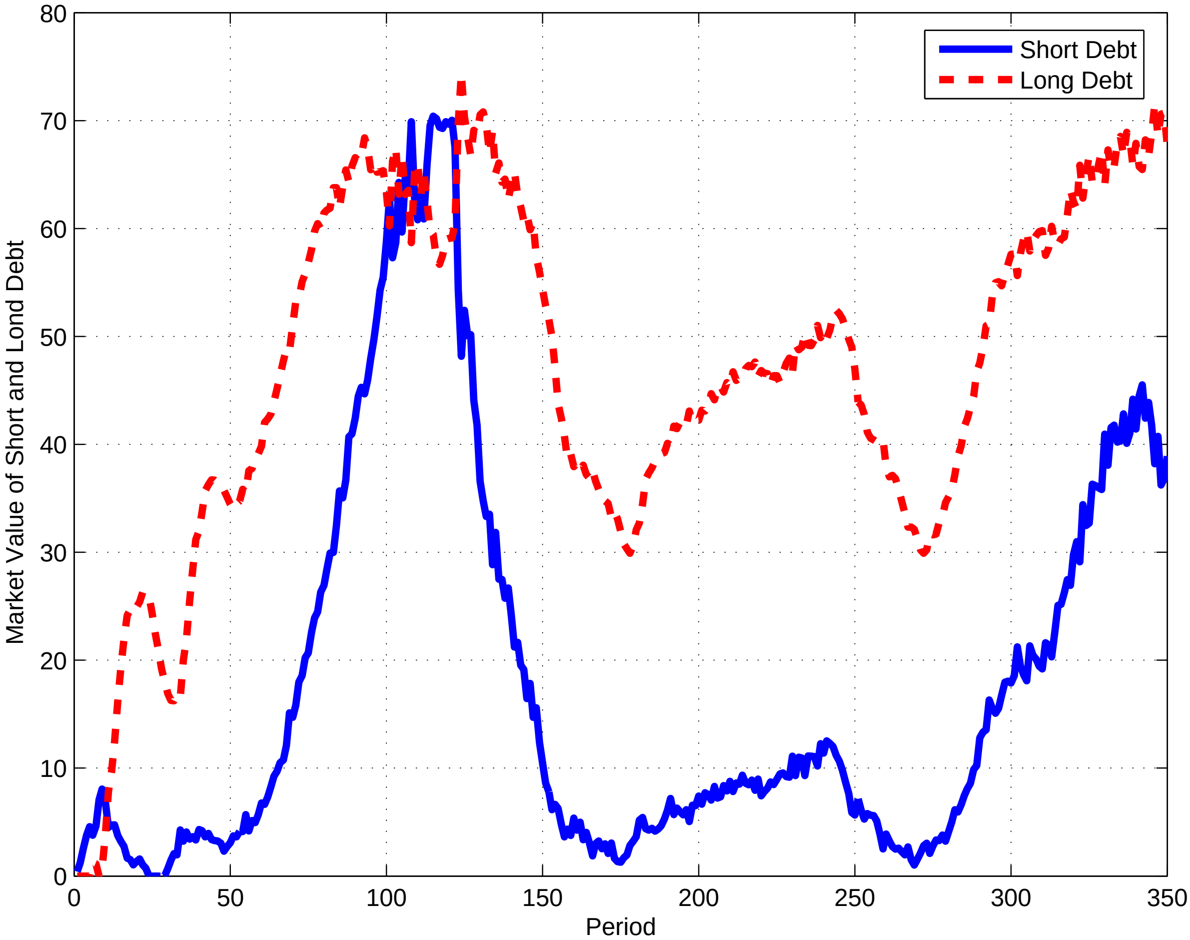
Notes: The Figure plots the tax rate in the single bond model without buyback presented in Section 3.4.4. The solid line corresponds to a bond of one year maturity. The dashed line sets the maturity to three years and the crossed line to 10 years.

Figure 6: **Model Simulations: Buyback and Lending**



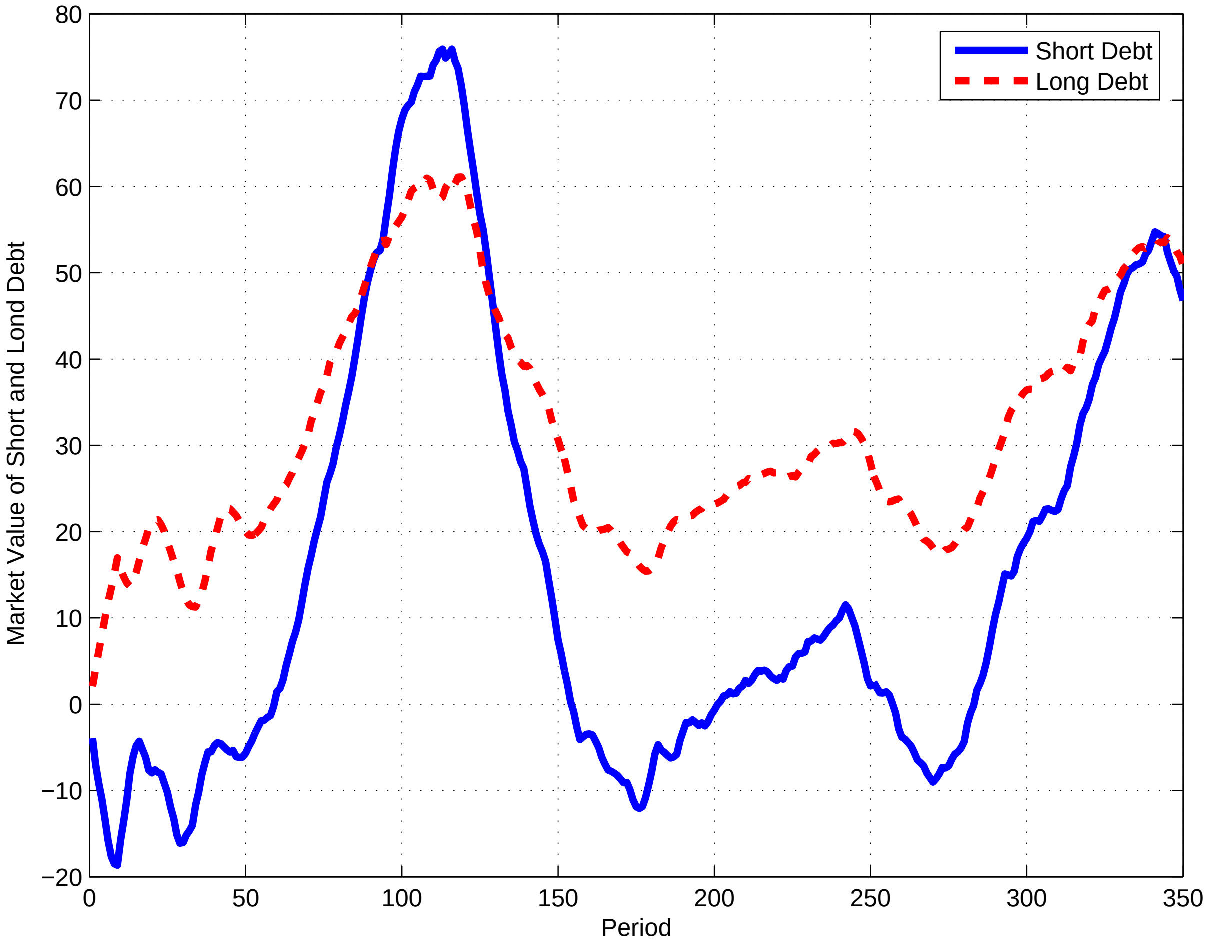
Notes: The Figure plots a typical sample of the optimal portfolio under buyback. The bound on each maturity is equal to 100% of steady state GDP. The lower bound constraint equals -100% of GDP.

Figure 7: **Model Simulations: Buyback and No Lending**



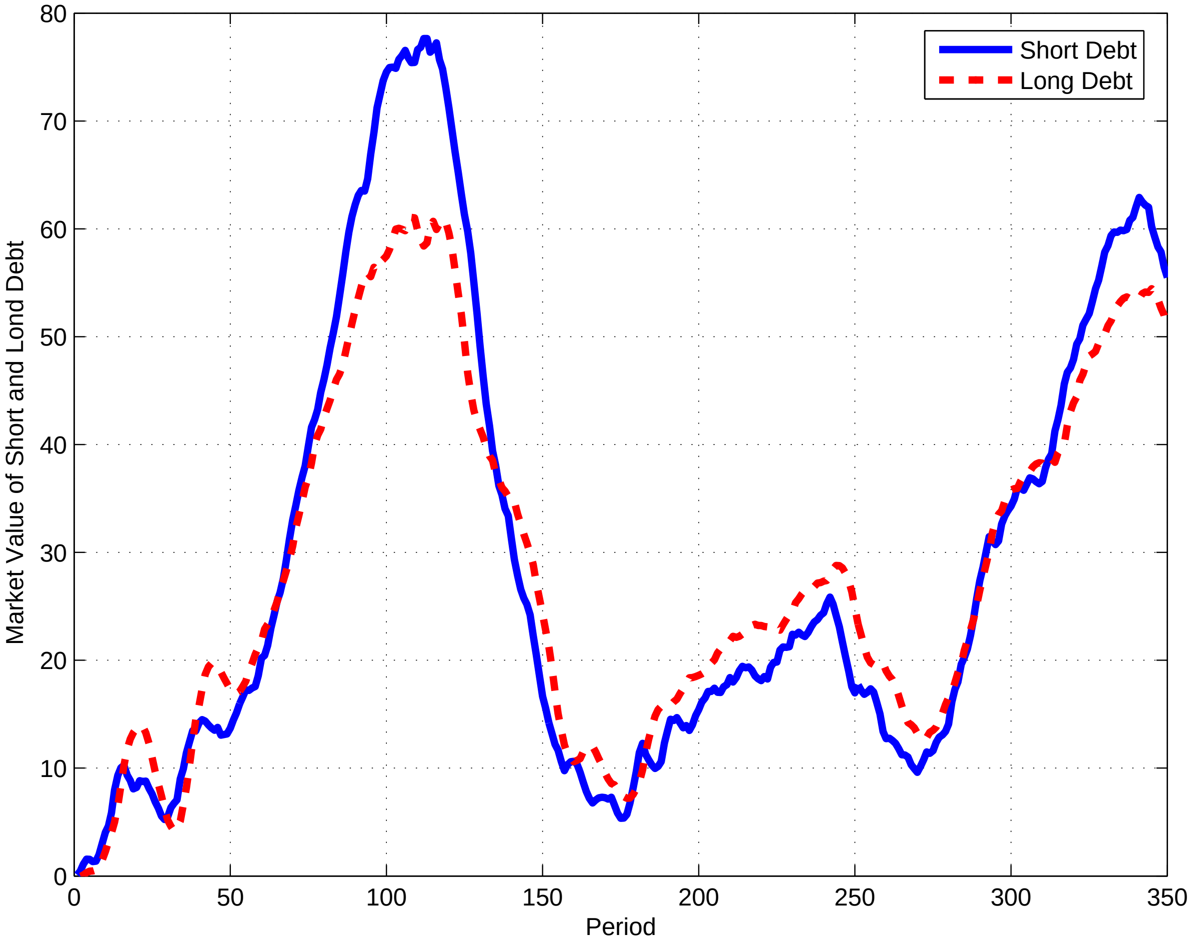
Notes: The Figure plots a typical sample of the optimal portfolio under buyback and no lending. The upper bounds on short and long maturities equal 100 percent of steady state GDP. The lower bounds equal 0.

Figure 8: **Model Simulations: No Buyback and Lending**



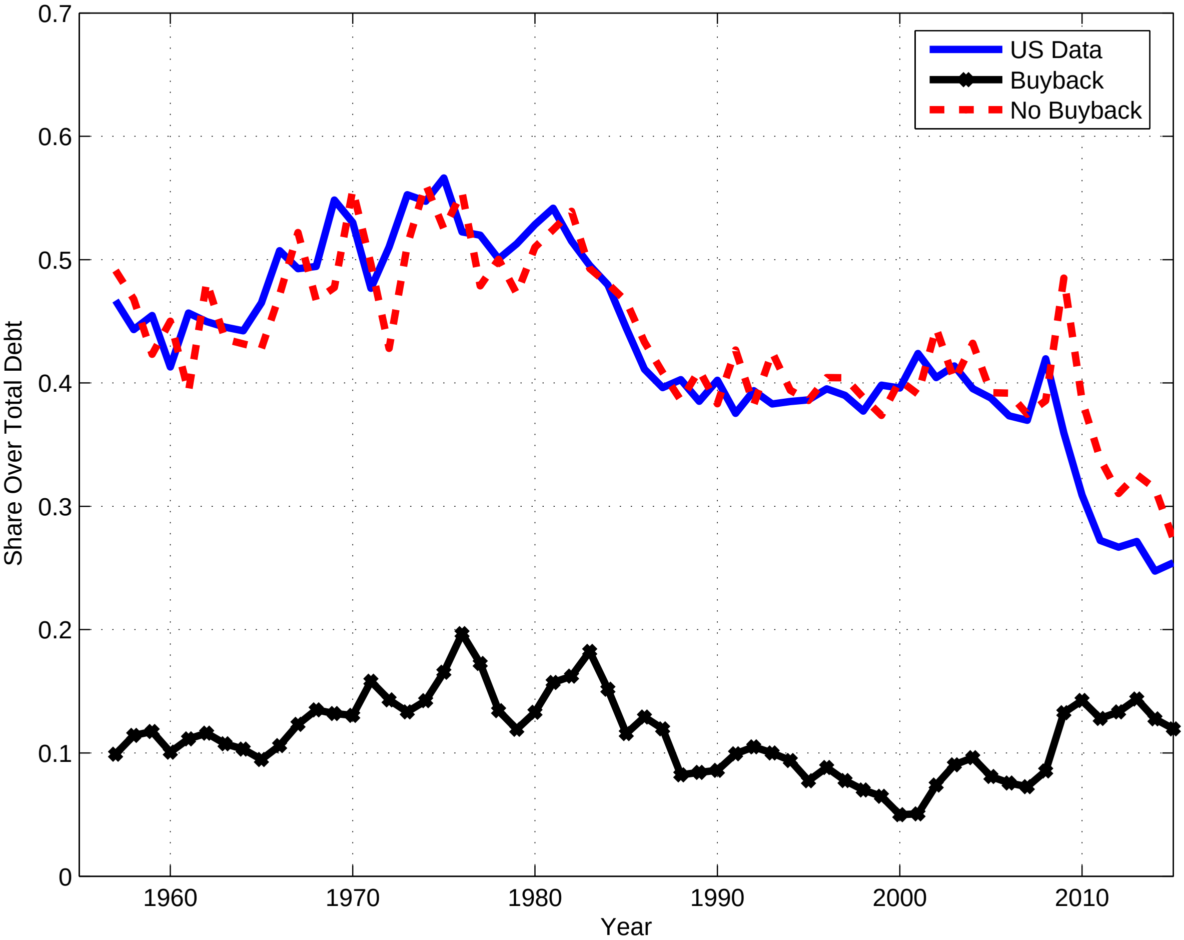
Notes: The Figure plots a typical sample of the optimal portfolio under no buyback and lending. The upper bounds imposed on the market value of short and long debt equal 100% of steady state GDP. The lower bounds are equal to -100% of GDP.

Figure 9: **Model Simulations: No Buyback and No Lending**



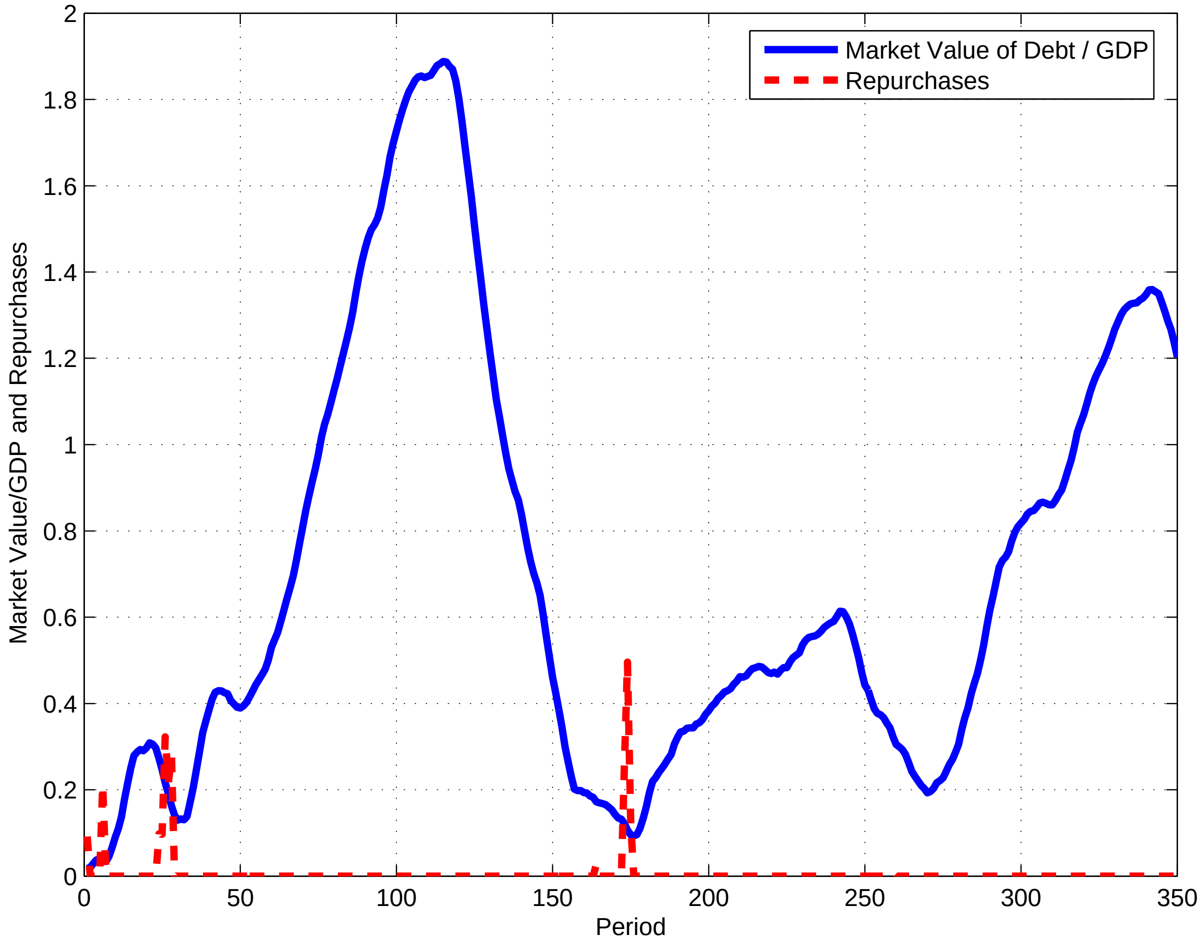
Notes: The Figure plots a typical sample of the optimal portfolio under no buyback and lending. The upper bounds imposed on the market value of short and long debt equal 100% of steady state GDP. The lower bounds are equal to 0.

Figure 10: **Models vs Data**



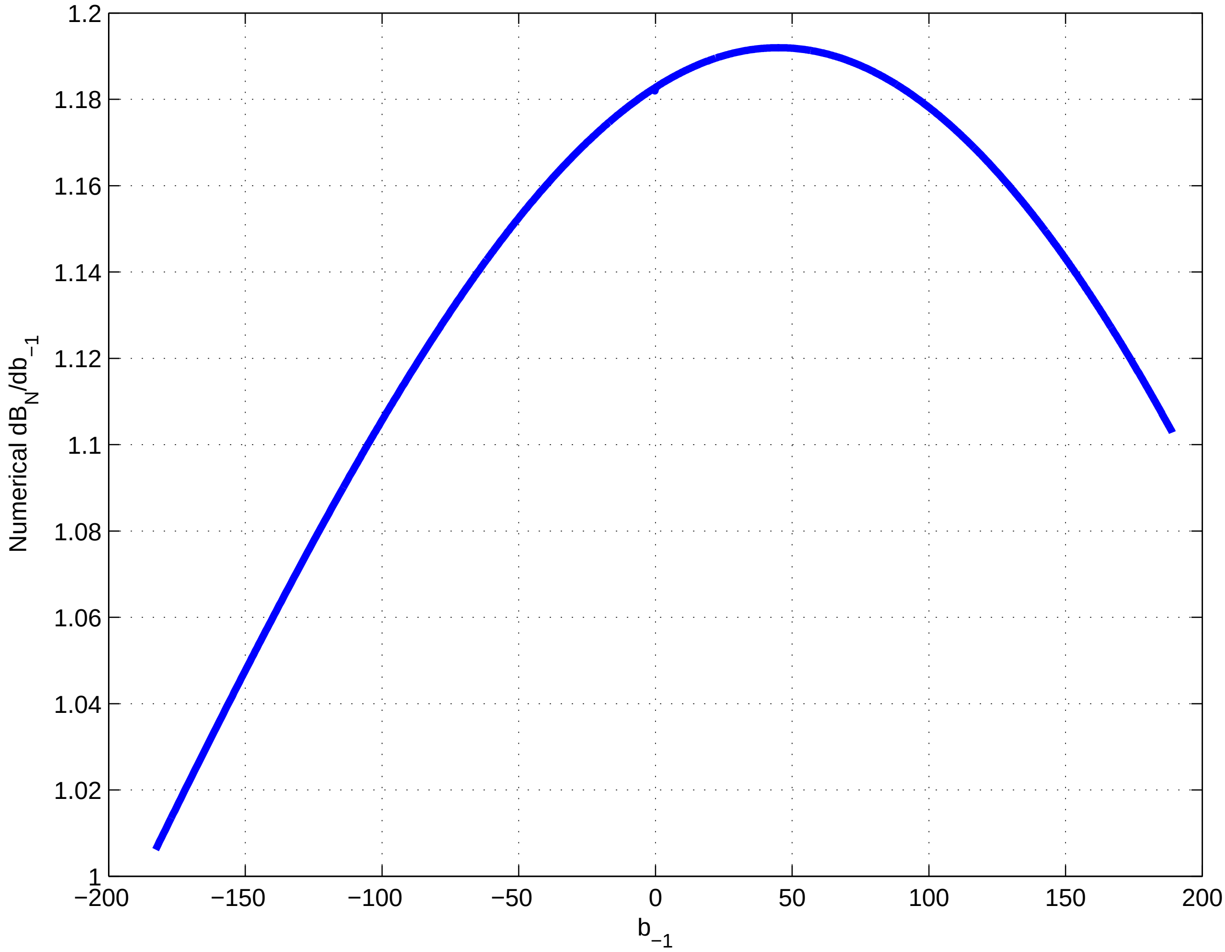
Notes: The solid line shows the average share of short term debt which was also shown in Figure 1. The dashed line shows the fit of the no buyback model. As described in text we constructed this figure through the following steps: First, we used model simulations to estimate the 'short debt share' policy function, regressing the share of short debt on its first and second lags, two lags of the market value of total debt over GDP and the current value and first lag of the spending to GDP series. We also included higher order terms of these variables to produce a 'good fit' of the model's policy function. Second, we fed the 'data state variables' to the model's estimated policy rule, and obtained the dashed line shown. For the buyback model (crossed line) we applied the same procedure. This gave us the crossed line.
Because of the inclusion of two lags of the share in the regressions, the years covered in the figure are 1957-2015.

Figure 11: **Model Simulations: Debt to GDP ratio and Repurchases**



Notes: The Figure plots the debt to GDP ratio (solid line) and the absolute level of repurchases (dashed line) in model of Section 6.3. We used the sample of spending as in Figures 6 to 9. The upper bounds imposed on the market value of short and long debt equal 100% of steady state GDP. Hence the market value of total government debt can be as high as 200% of steady state GDP. The lower bounds of short and long bonds are equal to 0.

Figure 12: **Numerical Derivative**



Notes: The Figure plots the numerical derivative $\frac{\partial B_N^{BB}}{\partial b_{-1}^g}$ in the vertical axis and initial debt b_{-1}^g in the horizontal axis. See Appendix B for details.

Online Appendix to Government Debt Management: The Long and the Short of It*

Faraglia, Elisa † Marcet, Albert‡ Oikonomou, Rigas § Scott, Andrew ¶

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†University of Cambridge and CEPR.

‡Institut d'Anàlisi Econòmica-CSIC, ICREA, Barcelona GSE, UAB, MOVE and CEPR.

§Université Catholique de Louvain. Email: Rigas.Oikonomou@uclouvain.be

¶London Business School and CEPR.

This appendix has two sections. Section A contains some analytic results that have been left out of the text and are given here for completeness. We first give some additional details about the complete markets model under buyback and no buyback. Then we set up the Lagrangeans that have not been included in the main text, namely for the model with optimal repurchases and transaction costs of section 6.3, the model with coupons of section 7.1 and the 'callable bonds' model of section 7.3. We derive the first order optimality conditions for each model.

Section B contains the Numerical Appendix. It discusses in detail the implementation of the 'Forward States' and 'Condensed PEA' algorithms, and several practical issues on solving portfolio models with incomplete markets with the PEA. We discuss how we selected the state variables of the core vector, $\mathbf{X}_t^{core}$ and of the 'out'-vector, $\mathbf{X}_t^{out}$, as well as the order of the polynomials of the states that were used. Moreover, we report how many linear combinations of state variables were added to the approximations. Finally, we discuss how we constructed approximations for the shadow cost calculation presented in Section 6.2 and we report on the accuracy of the simulated models.

A Some Theoretical Results

For simplicity we take the case $S = 1$ throughout this section.

A.1 Complete Markets and Buyback

We describe in this section the debt management strategy under complete financial market assuming buyback. This provides more details for the calculations in section 3.4.1 in the main text.

Let $\mathbf{g}^t = (g_0, g_1, ..., g_t)$ be the history of government spending shocks up to date t. As in ABN and as in the main text of this paper z represents the present discounted value of the government surplus contingent on $\mathbf{g}^{t-1}$ and the current realization of spending g_t. Substituting for equilibrium taxes this is:

$$z_t(\mathbf{g}^{t-1}, g_t) = E_t \sum_{i=0}^{\infty} \frac{\beta^i}{u_{c,t}} \left[(u_{c,t+i} - v_{x,t+i})(c_{t+i} + g_{t+i}) - g_{t+i} u_{c,t+i} \right]. \tag{1}$$

We assume for simplicity that government expenditure follows a two step Markov process taking values $g^H > g^L$ with probabilities μ_{HH} and μ_{LL} of remaining in the same state. The government debt is given initially by b_{-1}^1 and b_{-i}^N for $i = 1, ..., N$, at $t = 0$.

Following standard arguments as in Chari and Kehoe (1999) the equilibrium conditions if there are complete markets for Arrow securities is given by the implementability constraint at date 0

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[(u_{c,t} - v_{x,t})(c_t + g_t) - u_{c,t} g_t \right] = E_0 \sum_{i \in \{1,N\}} \sum_{t=0}^{i-1} \beta^t u_{c,t} b_{t-i}^i.$$

The corresponding Lagrangean that gives the Ramsey equilibrium under complete markets is

$$\mathcal{L}_{CM} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(T - c_t - g_t) \right. \right. \tag{2}$$

$$\left. + \Lambda((u_{c,t} - v_{x,t})(c_t + g_t) - u_{c,t} g_t) \right] - \Lambda \sum_{i \in \{1,N\}}^{N} \sum_{t=0}^{i-1} \beta^t u_{c,t} b_{t-i}^i \Bigg\}$$

where Λ is the Lagrange multiplier of the implementability constraint.

The first order conditions for an optimum are given by:

$$u_{c,t} - v_{x,t} - \Lambda[u_{cc,t} c_t + u_{c,t} - v_{x,t} + v_{xx,t}(c_t + g_t)] = 0 \quad \text{for} \quad t \geq N \tag{3}$$

$$u_{c,t} - v_{x,t} + \Lambda[u_{cc,t} c_t + u_{c,t} - v_{x,t} + v_{xx,t}(c_t + g_t)] - \Lambda u_{cc,t} \sum_{i \in \{1,N\}} b_{t-i}^i = 0 \ \text{ for } \quad t \leq N - 1. \tag{4}$$

For $t \geq N$ the first order conditions above are time-invariant so that $c_t(\mathbf{g}^t) = c_t^{CM}$ takes two possible values $c_t(\mathbf{g}^{t-1}, g^i) = c^i$ for $i = H, L$. Therefore, given the Markov assumption on g, there are two possible values for $z_t(\mathbf{g}^{t-1}, g^i) = z^i$ and $p^{N-1}(\mathbf{g}^{t-1}, g^i) = p_i^{N-1}$ for $i = H, L$ and for all $t \geq N$.

It is clear that the first order conditions (14)-(15) in the main text coincide with (3) in this appendix for constant $\lambda_t = \Lambda$. Therefore c_t^{CM} satisfies the first order conditions of the incomplete markets model. All that is left to show is that the budget constraints under incomplete markets are satisfied.

As shown, for example, in Angeletos (2002) a necessary and sufficient condition for the period-t budget constraints to hold is that (17) in the main text holds. Therefore all we need to show is that if we implement $\{c_t^{CM}\}$ then z takes only values z^H and z^L an optimal portfolio needs to satisfy

$$b_{t-1}^1 \left(\mathbf{g}^{t-1}\right) + p_i^{N-1} \left(\mathbf{g}^{t-1}\right) b_{t-1}^N \left(\mathbf{g}^{t-1}\right) = z^i \ \text{ for } i = H, L \ \ \forall t. \tag{5}$$

So that

$$\begin{pmatrix} 1 & p_H^{N-1} \\ 1 & p_L^{N-1} \end{pmatrix} \begin{pmatrix} b_{t-1}^1 \left(\mathbf{g}^{t-1}\right) \\ b_{t-1}^N \left(\mathbf{g}^{t-1}\right) \end{pmatrix} = \begin{pmatrix} z^H \\ z^L \end{pmatrix}. \tag{6}$$

yielding

$$\begin{pmatrix} b_{t-1}^1 \left(\mathbf{g}^{t-1}\right) \\ b_{t-1}^N \left(\mathbf{g}^{t-1}\right) \end{pmatrix} = \begin{pmatrix} \frac{p_H^{N-1} z^L - p_L^{N-1} z^H}{p_H^{N-1} - p_L^{N-1}} \\ \frac{z^H - z^L}{p_H^{N-1} - p_L^{N-1}} \end{pmatrix} = \begin{pmatrix} B_1^{BB} \\ B_N^{BB} \end{pmatrix}. \tag{7}$$

To see that $B_N^{BB} > 0$ note first that clearly surpluses are higher when g^L occurs, therefore $z_H < z_L$.

We now argue that generically $p_H^{N-1} - p_L^{N-1} < 0$. Assume a symmetric g process such that $\mu = \mu_{HH} = \mu_{LL}$ and CRRA utility we have $p_H^{N-1} = \beta^{N-1} \left[\mu_{N-1} + (1 - \mu_{N-1}) \left(\frac{c^L}{c^H} \right)^{\gamma_c} \right]$ for risk aversion $-\gamma_c$, where $\mu_j = \Pr ob(g_{t+j} = g^i \mid g_t = g^i)$. Hence

$$p_H^{N-1} - p_L^{N-1} = \beta^{N-1}(1 - \mu_{N-1}) \left(\left(\frac{c^L}{c^H} \right)^{\gamma_c} - \left(\frac{c^H}{c^L} \right)^{\gamma_c} \right).$$

As long as consumption is a normal good $c^H < c^L$ hence $p_H^{N-1} - p_L^{N-1} < 0$ for $\gamma_c < 0$. This shows that, generically, $B_N^{BB} > 0$.

To see that $B_1^{BB} < 0$ note that since $B_1^{BB} = z^H - p_H^{N-1} B_N^{BB}$ and $B_N^{BB} > 0$, as long as initial debt is close to zero, z^H is close to zero and B_1^{BB} is negative.

As long as debt limits are sufficiently loose to contain B_1^{BB}, B_N^{BB} this gives the equilibrium under incomplete markets.

A.2 Complete Markets and No Buyback

Under no buyback the period t budget constraint of the consumer is given by equation (3) in the main text. In this case the government issues two kinds of bonds, but it really holds N kinds of bonds every period: in addition to the bonds that mature and produce income at t, namely $(b_{t-1}^1 + b_{t-N}^N)$, the government also holds long bonds that have not yet matured: namely, $b_{t-N+1}^N, ..., b_{t-1}^N$. Even though these non-maturing bonds do not show up in equation (3) they do appear in equation (20), stating that total wealth equals discounted surpluses z. We now give more details to show that equation (20), along with equilibrium prices, is necessary and sufficient for a competitive equilibrium under incomplete markets and no buyback.

To prove that (20) is necessary, proceed as suggested in the paragraph preceding equation (20). Let s_t be the government primary surplus as in the main text, define total wealth as $W_t = b_{t-1}^1 + \sum_{j=1}^N p_t^{N-j} b_{t-j}^N$ and using equilibrium prices we can show

$$s_t + \beta E_t \left[\frac{u_{c,t+1}}{u_{c,t}} W_{t+1} \right] = W_t. \tag{8}$$

Iterating forward and assuming no Ponzi games in the value of bonds W_t yields (20) in the main text for all $t = 0, 1, ...$ hence this is a necessary condition.

To show that this is a sufficient condition the previous steps can be reversed to show that any portfolio satisfying (20) and a no Ponzi game condition in W_t also satisfies (3).

Hence an allocation is an incomplete market equilibrium if and only if (20) holds for $p_t^j = E_t \beta^j \frac{u_{c,t+j}}{u_{c,t}}$ and the corresponding debt limits. As we show in the main text, the complete market allocations do not satisfy all these requirements: if we have bond limits then (20) can not hold for all t.

A.3 Optimal Repurchases: the Ramsey Program

In the optimal repurchase (OR) model of Section 6.3 the government maximizes the utility of the household subject to the following constraints

$$\sum_{i \in \{S,N\}} p_t^i b_t^i (1 - \mathcal{T}^i(b_t^i)) = b_{t-S}^S + b_{t-N}^N - R_{t-N+1} + p_t^{N-1} R_t (1 + \mathcal{T}^R(R_t)) + g_t - \tau_t (T - x_t) \tag{9}$$

$$T - x_t = c_t + g_t + \mathcal{TC}_t \tag{10}$$

$$0 \le b_t^i \le \frac{\overline{M}_i}{\sum_{j=1}^i \beta^j}, \quad 0 \le R_t \le b_{t-1}^N \tag{11}$$

where $\mathcal{TC}_t \equiv \sum_{i \in \{S,N\}} p_t^i b_t^i \mathcal{T}^i(b_t^i) + p_t^{N-1} R_t \mathcal{T}^R(R_t)$ represents total transaction costs.

To simplify the solution of this model we assume that the government treats as exogenous the function $\mathcal{TC}_t$, in other words it does not take derivatives of $\mathcal{TC}_t$ with respect to consumption and the bonds.[1]

The Lagrangian of the OR model is:

$$\mathcal{L} = E_0 \sum_t \beta^t \Bigg[u(c_t) + v(T - c_t - g_t - \mathcal{TC}_t) + \lambda_t [\sum_{i \in \{S,N\}} b_t^i \beta^i u_{c,t+i} (1 - \mathcal{T}(b_t^i)^i) \tag{12}$$

$$-\beta^{N-1} u_{c,t+N-1} R_t (1 + \mathcal{T}^R(R_t)) - (b_{t-S}^S + b_{t-N}^N - R_{t-N+1}) u_{c,t}$$

$$- g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t)] + \sum_{i \in \{S,N\}} \xi_{U,t}^i (\frac{\overline{M}_i}{\sum_{j=1}^i \beta^j} - b_t^i) + \sum_{i \in \{S,N\}} \xi_{L,t}^i b_t^i + \xi_{U,t}^R (b_{t-1}^N - R_t) + \xi_{L,t}^R R_t \Bigg].$$

The FONC are given by:

$$u_{c,t} - v_{x,t} + \lambda_t(-u_{cc,t} g_t + u_{c,t} + u_{cc,t}(T - x_t) + v_{xx,t}(T - x_t) - v_{x,t}) - u_{cc,t} \left[B_{t-S} \lambda_t - B\lambda_{t-S} \right] = 0$$
$$E_t \beta(-u_{c,t+1} \lambda_{t+1} + u_{c,t+1} \lambda_t (1 - \mathcal{T}_t^1 - \mathcal{T}_{b_t^1}^1 b_t^1)) + \xi_{L,t}^1 - \xi_{U,t}^1 = 0 \quad \text{for} \quad i = S$$
$$E_t \beta^N(-u_{c,t+N} \lambda_{t+N} + u_{c,t+N} \lambda_t (1 - \mathcal{T}_t^N - \mathcal{T}_{b_t^N}^N b_t^N)) + E_t \beta(\xi_{U,t+1}^R) + \xi_{L,t}^N - \xi_{U,t}^N = 0 \quad \text{for} \quad i = N$$
$$E_t \beta^{N-1}(u_{c,t+N-1} \lambda_{t+N-1} - u_{c,t+N-1} \lambda_t (1 + \mathcal{T}_t^R + \mathcal{T}_{R_t}^R R_t)) + \xi_{L,t}^R - \xi_{U,t}^R = 0$$

where

$$\begin{aligned} B_t &\equiv b_t^S + B_{t-N+1+S}^{net} \\ B_t^{net} &\equiv b_{t-1}^N - R_t \\ B\lambda_t &\equiv \lambda_t(1 - \mathcal{T}_t^S) b_t^S + B\lambda_{t-N+1+S}^{net} \\ B\lambda_t^{net} &\equiv \lambda_{t-1}(1 - \mathcal{T}_{t-1}^N) b_{t-1}^N - \lambda_t (1 + \mathcal{T}_t^R) R_t. \end{aligned}$$

A.4 Coupon Bonds and No Buyback: the Ramsey Program

We solve the optimal policy problem under no buyback and coupons of section 7.1 assuming for simplicity that $S = 1$. As in the rest of the paper we introduce debt limits, these are parameterized as:

$$b_t^N \in \left[\frac{\underline{M}_N}{\sum_{j=1}^N \beta^j + \kappa \sum_{j=1}^N \sum_{i=1}^j \beta^i}, \frac{\overline{M}_N}{\sum_{j=1}^N \beta^j + \kappa \sum_{j=1}^N \sum_{k=1}^j \beta^k} \right] \equiv [\underline{\widetilde{M}}_N, \widetilde{\overline{M}}_N] \tag{13}$$

so that $\underline{\widetilde{M}}$'s are in terms of the value of debt.

[1] Without this assumption we would need to keep track of the fact that there is a conditional expectation in the determination of $\mathcal{TC}_t$, therefore the solution to the model would feature both current and lagged values of the multiplier on this constraint. This would add yet more state variables in the model but with minimal quantitative effects.

Note that another way to simplify the planner's program (avoid having to keep track of the resource constraint as a separate object in the Lagrangean) is to assume $\mathcal{TC}_t$ do not enter the feasibility constraint. In this case transaction costs do not impact the overall resources of the economy, this would correspond to a situation where a financial firm can charge transaction costs on bond issuances without actually spending labor resources in it. When we run the model under this assumption (which appears sometimes in the literature) we found virtually no effect on our results.

Letting $[\underline{\widetilde{M}}_1, \widetilde{\overline{M}}_1] \equiv [\frac{\underline{M}_1}{\beta}, \frac{\overline{M}_1}{\beta}]$ be the analogous constraint set for one year debt the planning problem is given by:

$$\mathcal{L} = E_0 \sum \beta^t \Big\{ u(c_t) + v(T - c_t - g_t) + \lambda_t \Big[b_t^1 \beta u_{c,t+1} + b_t^N \Big(\beta^N u_{c,t+N} + \sum_{j=1}^{N} \beta^j u_{c,t+j} \kappa \Big)$$
$$- b_{t-1}^1 u_{c,t} - b_{t-N}^N u_{c,t} - \kappa \sum_{j=1}^{N} b_{t-j}^N u_{c,t} - g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t) \Big]$$
$$+ \sum_{i \in \{1,N\}} \xi_{U,t}^i (\widetilde{\overline{M}}_i - b_t^i) + \sum_{i \in \{1,N\}} \xi_{L,t}^i (b_t^i - \underline{\widetilde{M}}_i) \Big\}.$$

The first order condition for consumption is:

$$u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t})$$
$$+ u_{cc,t} \kappa \sum_{j=1}^{N} (\lambda_{t-j} - \lambda_t) b_{t-j}^N + u_{cc,t} \sum_{i \in \{1,N\}} (\lambda_{t-i} - \lambda_t) b_{t-i}^i = 0$$

and off corners the analogous conditions for b_t^1 and b_t^N are:

$$\lambda_t E_t(u_{c,t+1}) = E_t(\lambda_{t+1} u_{c,t+1}) \tag{14}$$

$$\lambda_t E_t(\kappa \sum_{j=1}^{N} \beta^j u_{c,t+j} + \beta^N u_{c,t+N}) = E_t(\kappa \sum_{j=1}^{N} \beta^j u_{c,t+j} \lambda_{t+j} + \beta^N u_{c,t+N} \lambda_{t+N}). \tag{15}$$

For brevity we summarized the properties of this model in Table 4 in the main text. In Figure 1 of this appendix we show a typical sample of long and short debt (analogous to Figures 6-9 in the main text). As was claimed in the text, when bonds pay positive coupons the properties of the model remain very close to the no buyback and zero coupons case.

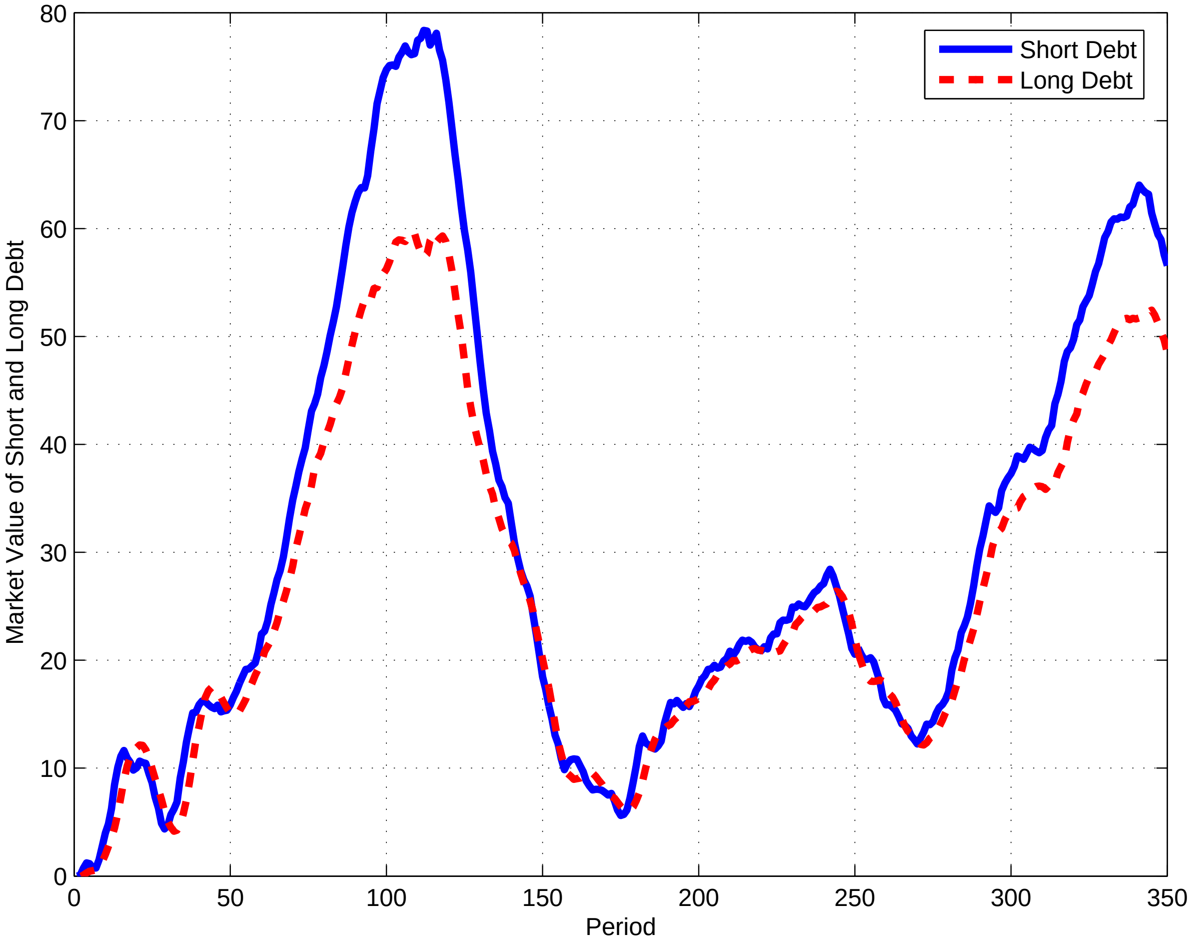
A.5 Callable Bonds: the Ramsey Program

As explained in Section 2 of the paper the US government issued callable bonds in the past. These types of securities give the issuer the option to buy them back after $m < N$ years, at every coupon date, until the bond matures. Their price at buyback is at par. We showed that historically the US government has repurchased callable bonds at the start of the call window.

In proposing the model of section 7.3, our intention is not to motivate the empirical observations of why the Treasury chooses to buyback at the first call date. Rather we seek to establish that removing debt from the market before, but close to, the maturity date is akin to the model of no buyback and that our findings about the importance of short term debt and the positive comovement between short and long debt still hold.

This is why we assume that the buyback of the N-year bond occurs automatically m years after issuance. We keep our calibration $N = 10$ and we set the recall date $m = 5$. Notice that a lower m makes the model closer to the buyback section, since buyback is equivalent with $m = 1$. If we were to find that even for a low m the model behaves similar to no buyback, higher m are likely to be

Figure 1: **Market Value of Short and Long Debt under no Buyback+Coupons**



Notes: The Figure plots a typical sample path from the no buyback model with positive coupons. As explained in text the value of the coupon κ is calibrated so that bonds trade on average at par. The upper bound on b_t^N is given in (13). The lower bound is zero. The value of short term debt in a given period t in the Figure, is constructed by adding the coupon payments and principals which are to mature in $t+1$ to the market value of one year bonds issued in t.

be even closer to no-buyback. The call window in the data for 10 year bonds starts 2 years before maturity, suggesting a buyback period of $m = 8$. Our model choice of a much lower $m = 5$ means that if we find that the role for short bonds is close to no buyback this suggests that the same is likely to happen in practice.

The budget constraint of the government is:

$$\sum_{i\in\{1,N\}} p_t^i b_t^i = b_{t-1}^1 + p_t^{N-m} b_{t-m}^N + g_t - \tau_t(T - x_t).$$

The ad hoc debt constraints for the N year bond are

$$b_t^N \in \left[\frac{\underline{M}_N}{\sum_{j=0}^{m-1}\beta^{N-j}}, \frac{\overline{M}_N}{\sum_{j=0}^{m-1}\beta^{N-j}}\right] \equiv [\underline{\widetilde{M}}_N, \overline{\widetilde{M}}_N].$$

Letting $[\underline{\widetilde{M}}_1, \overline{\widetilde{M}}_1] = [\frac{\underline{M}_1}{\beta}, \frac{\overline{M}_1}{\beta}]$ be the analogous constraints for one year debt and substituting the equilibrium expressions for the tax rate and the bond prices we represent the planning problem as follows:

$$\mathcal{L} = E_0 \sum_t \beta^t \Big\{ u(c_t) + v(T - c_t - g_t) + \lambda_t \Big[\sum_{i \in \{1,N\}} b_t^i \beta^i u_{c,t+i} - b_{t-1}^1 u_{c,t} - b_{t-m}^N \beta^{N-m} u_{c,t+N-m}$$

$$-g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t) \Big]$$

$$+ \sum_{i \in \{1,N\}} \xi_{U,t}^i (\widetilde{\overline{M}}_i - b_t^i) + \sum_{i \in \{1,N\}} \xi_{L,t}^i (b_t^i - \widetilde{\underline{M}}_i) \Big\}.$$

The first order conditions for the optimum are given by:

$$u_{c,t} - v_{x,t} + \lambda_t \Big(u_{cc,t} c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t} \Big) + u_{cc,t} \Big[(\lambda_{t-1} - \lambda_t) b_{t-1}^1 + (\lambda_{t-N} - \lambda_{t-N+m}) b_{t-N}^N \Big]$$

$$\beta E_t (u_{c,t+1} \lambda_t - u_{c,t+1} \lambda_{t+1}) + \xi_{L,t}^1 - \xi_{U,t}^1 = 0$$

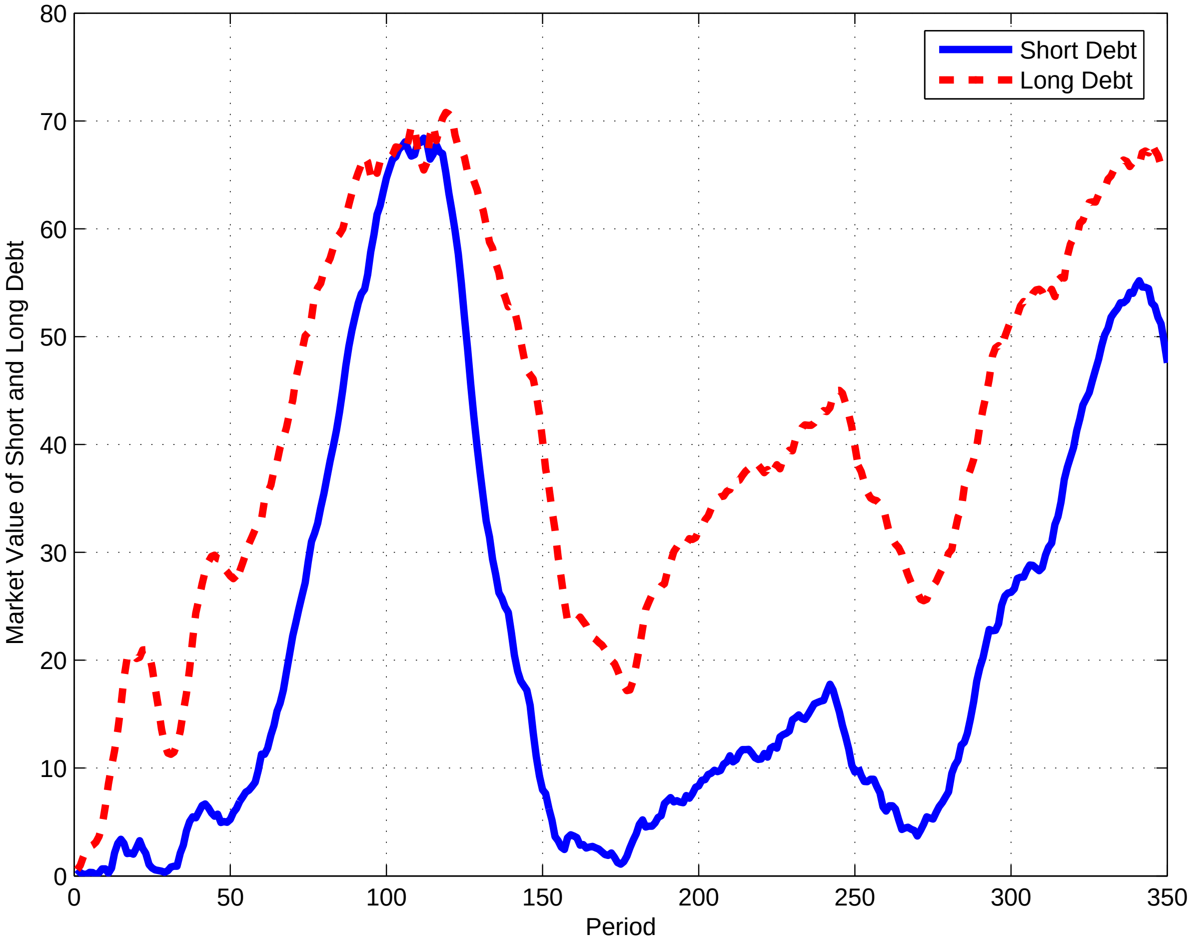
$$\beta^N E_t (u_{c,t+N} \lambda_t - u_{c,t+N} \lambda_{t+m}) + \xi_{L,t}^N - \xi_{U,t}^N = 0.$$

We assume $\widetilde{\underline{M}}_1 = \widetilde{\underline{M}}_N = 0$. In Figure 2 we plot a typical sample of the market value of short and long debt. Notice that assuming that the government repurchases debt from the market $m = 5$ years after issuance, does indeed reduce the share of short bonds in the portfolio compared to a model with no buyback. The portfolio is somewhere between the 'full buyback model' studied in the paper (i.e. when $m = 1$) and the no buyback model where $m = N$. In terms of the moments reported in Table 4 in the paper the model of this section gives us the following: $\overline{\mathcal{S}}_t = 18.5\%$, $\sigma_{\mathcal{S}_t} = 10.6\%$, $\rho_{\mathcal{S}_t, \mathcal{S}_{t-1}} = 0.88$ $\rho_{\widetilde{b}_t^S, \widetilde{b}_t^N} = 0.84$ $\%_{\mathcal{S}_t = 0} = 0.35\%$.

We view these results as encouraging because they confirm the hypothesis that even if the government buys back some of the debt before maturity there is still a role for short bonds. First, because the share of short debt is very rarely zero in simulations (e.g. $\%_{\mathcal{S}_t=0} = 0.35\%$ versus the analogous figure in the buyback no lending model in the paper of 13%). As we mentioned, the choice of $m = 5$ is quite conservative. The data suggest that $m = 8$ would be more appropriate. With $m = 8$ we expect the model to generate results very close to the no buyback ones.

This is only a partial study of callable bonds. Clearly, the modelling of callable bonds can be made closer to the data by introducing that they can be repurchased at par or that transaction costs are involved in their recall. A model taking all these features into account is beyond the scope of this paper. However the message that there is still a role for short bonds comes out clearly from the analysis.

Figure 2: **Market Value of Short and Long Debt under 'Callable' Bonds**



Notes: The Figure plots a typical sample path from the model of Section A.5. We assume that government buybacks of 10 year bonds occur 5 years after issuance.

B Numerical Appendix

In section 4 of the paper we described the 'Condensed PEA' that deals with the high dimensionality of the state vector, and the 'Forward States PEA' that deals with the indeterminacy of the portfolio generated by the use of the PEA. This numerical appendix outlines in greater detail the two methods, their implementation and the steps we followed to approximate the conditional expectations in the different models. In particular, we report how we selected the state variables of the core vector, $\mathbf{X}_t^{core}$, the 'out'-vector, $\mathbf{X}_t^{out}$ and the order of the polynomials of the states that were used. Moreover, we report how many linear combinations of state variables were added to the approximations, and also discuss some practical features of our numerical procedure that can help the algorithm's convergence.

B.1 Implementation of "Condensed PEA" and "Forward states PEA"

B.1.1 Selection of variables in the approximation

Recall that 'Condensed PEA' divides the state vector $\mathbf{X}$ into two subvectors: the core vector, $\mathbf{X}_t^{core}$, which includes variables that (we believe a priori) are of primary importance in the approximation, and the $\mathbf{X}_t^{out}$ vector, which includes the remaining state variables and possibly higher order polynomial terms. 'Forward States PEA' resolved the portfolio indeterminacy issue through approximating $E_t u_{c,t+i}$ with $E_t(\Phi^i(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^i))$ and $E_t \lambda_{t+1} u_{c,t+i}$ with $E_t(\Psi^i(\mathbf{X}_{t+1}, \boldsymbol{\delta}^i))$. To clearly show how we implemented these two methods, we bring them now together and in what follows we outline the 'Condensed PEA' using since the first iteration (i.e. with core state variables only) 'Forward States' to solve the portfolio choice.

The $\mathbf{X}^{core}$ vector:

In all the models presented in the paper, but the 3 bond model, we used the core vector

$$\mathbf{X}_{t+1}^{core} = \Big\{1,\ g_{t+1},\ \{b_t^i\}_{i=1,N},\ \lambda_t,\ \{(b_t^i)^2\}_{i=1,N},\ \{g_{t+1}b_t^i\}_{i=1,N}\Big\} \tag{16}$$

i.e. $\mathbf{X}_{t+1}^{core}$ is composed of a constant, the level of government spending in $t+1$, the levels of date t variables (the bonds and the multipliers), the square of the bonds and the interaction term between the bonds in t and g_{t+1}.

We solve the system of FONCs after integrating out the term g_{t+1} as discussed in the text. We use the analytical formula for the conditional expectation of g_{t+1} at time t given by:

$$\int g_{t+1} f_{g_{t+1}|g_t} dg_{t+1} = \omega_t + (\underline{g}-\omega_t)\Phi\left(\frac{\underline{g}-\omega_t}{\sigma_\epsilon}\right) + (\overline{g}-\omega_t)\left[1-\Phi\left(\frac{\overline{g}-\omega_t}{\sigma_\epsilon}\right)\right] - \sigma(\left[\phi\left(\frac{\overline{g}-\omega_t}{\sigma_\epsilon}\right) - \phi\left(\frac{\underline{g}-\omega_t}{\sigma_\epsilon}\right)\right] \tag{17}$$

where $\omega_t \equiv \rho_g g_t + (1-\rho_g) g_{ss}$, Φ (ϕ) is the standard normal cdf (pdf), $\overline{g}$ and $\underline{g}$ are upper and lower bounds on government spending[2]. σ_ϵ is the standard deviation of the spending shock.[3]

We have chosen to introduce higher order terms in the core vector for three reasons. First, approximating the conditional expectations $E_t u_{c,t+i}$ means that we are approximating the bond prices. If we include only the levels of the bonds, $\{b_t^i\}_{i=1,N}$, we are imposing that bonds are close substitutes in terms of their influence on prices.[4]. This is not a property that we are likely to find in equilibrium and non-linear terms may be potentially important. Secondly, if the ad hoc debt limits are occasionally binding this is known to introduce non-linearities so that higher order terms may be important. Finally the 'Condensed PEA' inclusion criterion suggested that these non linear terms are important for the approximations. Indeed when we solved the models without higher order terms and tested whether they should be included in the linear combinations, the percentage gains in R^2 were substantial. Higher order terms therefore should be included in the polynomials, either in $\mathbf{X}_{t+1}^{core}$ or in $\mathbf{X}_{t+1}^{out}$. We ultimately chose to introduce some higher order terms in the core vector and left others for the linear combinations (see below) finding this helpful for the stability of the algorithm.

The $\mathbf{X}^{out}$ vector:

The 'out' vector is different for each of the models presented in the paper. To identify the elements of $\mathbf{X}^{out}$ we use as guidance the FONC.

[2]As discussed in the calibration section, we assume that spending fluctuates between 15 and 35 percent of steady state GDP.

[3]The expression is reached as follows:

$$\int g_{t+1} f_{g_{t+1}|g_t} dg_{t+1} = \int_{-\infty}^{\underline{g}-\omega_t} \underline{g} dF(\epsilon_{t+1}) + \int_{\overline{g}-\omega_t}^{\infty} \overline{g} dF(\epsilon_{t+1}) + \int_{\underline{g}-\omega_t}^{\overline{g}-\omega_t} (\omega_t + \epsilon_{t+1}) dF(\epsilon_{t+1})$$
$$= \omega_t + \int_{-\infty}^{\underline{g}-\omega_t} (\underline{g} - \omega_t) dF(\epsilon_{t+1}) + \int_{\overline{g}-\omega_t}^{\infty} (\overline{g} - \omega_t) dF(\epsilon_{t+1}) + \int_{\underline{g}-\omega_t}^{\overline{g}-\omega_t} \epsilon_{t+1} dF(\epsilon_{t+1})$$

where F denotes the cdf of ϵ. Standard results give:

$$\int_{-\infty}^{\underline{g}-\omega_t} (\underline{g} - \omega_t) dF(\epsilon_{t+1}) = \int_{-\infty}^{\frac{\underline{g}-\omega_t}{\sigma_\epsilon}} (\underline{g} - \omega_t) \frac{1}{\sqrt{2\pi}} e^{-\frac{z_{t+1}^2}{2}} dz_{t+1} = (\underline{g} - \omega_t) \Phi(\frac{\underline{g} - \omega_t}{\sigma_\epsilon})$$

where z is a standard normal variable and Φ is the cdf. Analogously:

$$\int_{\overline{g}-\omega_t}^{\infty} (\overline{g} - \omega_t) dF(\epsilon_{t+1}) = (\overline{g} - \omega_t)(1 - \Phi(\frac{\overline{g} - \omega_t}{\sigma_\epsilon}))$$

Finally,

$$\int_{\underline{g}-\omega_t}^{\overline{g}-\omega_t} \epsilon_{t+1} dF(\epsilon_{t+1}) = \int_{\frac{\underline{g}-\omega_t}{\sigma_\epsilon}}^{\frac{\overline{g}-\omega_t}{\sigma_\epsilon}} \sigma_\epsilon \frac{1}{\sqrt{2\pi}} z_{t+1} e^{-\frac{z_{t+1}^2}{2}} d\epsilon_{t+1} = -\sigma(\phi(\frac{\overline{g} - \omega_t}{\sigma_\epsilon}) - \phi(\frac{\underline{g} - \omega_t}{\sigma_\epsilon}))$$

Putting everything together we get (17)

[4]To see this consider the approximation of $E_t u_{c,t+N} \approx \gamma_0^N + \gamma_1^N \int g_{t+1} f(g_{t+1}|g_t) dg_{t+1} + \gamma_2^N b_t^1 + \gamma_3^N b_t^N + \gamma_4^N \lambda_t$ under linear polynomials. Clearly there are (infinitely) many pairs (b_t^1, b_t^N) that give the same bond price (holding λ_t fixed).

Notice that the optimal portfolio is nonetheless identified under linear polynomials since b_t^i, $i = 1, N$ influence all conditional expectations and enter in a nonlinear fashion in the system of FONCs (for example in the budget constraint of the government).

Consider first the *buyback model*. The first order conditions, in the case where $S = 1$, are

$$u_{c,t} - v_{x,t} + \lambda_t\Big(u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t}\Big) + u_{cc,t}\sum_{i\in\{1,N\}}\Big(\lambda_{t-i} - \lambda_{t-i+1}\Big)b^i_{t-i} = 0 \tag{18}$$

$$\beta^i E_t\Big(u_{c,t+i}\lambda_t - u_{c,t+i}\lambda_{t+1}\Big) + \xi^i_{L,t} - \xi^i_{U,t} = 0 \quad \text{for} \quad i = 1, N. \tag{19}$$

When markets are incomplete, the term $\sum_{i\in\{1,N\}}\Big(\lambda_{t-i} - \lambda_{t-i+1}\Big)b^i_{t-i}$ summarises interest rate manipulation under commitment (see FMOS (2016)). Suppose that a positive spending shock arrives in period t and that $b^N_t > 0$. Since $\Big(\lambda_{t-1} - \lambda_t\Big)b^N_t$ becomes negative, the government finds optimal to promise a tax cut in $t + N - 1$ and lower the marginal utility of consumption in that period. It is then evident that the terms $\lambda_{t-1}b^N_t$ and $\lambda_t b^N_t$ are important determinants of $u_{c,t+N-1}$ and $u_{c,t+N}$ and hence they should be accounted for when we approximate the conditional expectations.[5]

Applying the above argument to determine which states potentially exert a significant influence to the expectations of date $t+1$, $t+N-1$ and $t+N$ variables in the buyback model, we include in $\mathbf{X}^{out}_{t+1}$ the following terms: $\lambda_t b^N_t$, $\lambda_t b^1_t$, $\lambda_t b^N_{t-1}$, $\lambda_{t-1}b^N_{t-1}$, $\lambda_{t-N+1}b^N_{t-N+1}$ and $\lambda_{t-N+2}b^N_{t-N+1}$.

Two more comments about this choice are necessary. Firstly, despite the fact that each of the above terms is potentially important for (some of) the conditional expectations we wish to approximate, it is unlikely that each term bears the same importance to each conditional expectation. For example, the term $\lambda_t b^N_t$ clearly exerts an influence on $u_{c,t+N}$ (through the FONC) but it is less likely to exert a significant influence on $u_{c,t+N-1}$. In this case the 'Condensed PEA' will assign a coefficient close to zero to $\lambda_t b^N_t$ in the approximation of $E_t u_{c,t+N-1}$ and a coefficient different from zero in the approximation of $E_t u_{c,t+N}$. This shows how convenient it is to include these terms in $\mathbf{X}^{out}$ where having coefficients close to zero for some state variables is not an issue, as opposed to including them in $\mathbf{X}^{core}$, in which case variables with close to zero coefficients may cause convergence problems.

Secondly, as explained before, (18) suggests that the cross terms between λ and b are potentially important for the solution. However, one may wonder whether the levels of these variables should also be included in the state vector. The FONCs show that the influence of λ_{t-N+1} on the optimal allocation in $t+1$ is close to zero if b^N_{t-N+1} is close to zero. The effect of changes in the value

[5] In the text the implementation of 'Forward States' to the buyback model was summarized in the following equations

$$\lambda_t = \frac{E_t(\Psi^i(\mathbf{X}_{t+1}, \boldsymbol{\delta}^i))}{E_t(\Phi^i(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^i))} \qquad \text{for} \qquad i = S, N \tag{20}$$

$$\sum_{i\in\{S,N\}} b^i_t\beta^i E_t(\Phi^i(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^i)) = \sum_{i\in\{S,N\}} b^i_{t-1}\beta^{i-1}\Phi^i(\mathbf{X}_t, \boldsymbol{\gamma}^i) + g_t u_{c,t} - (u_{c,t} - v_{x,t})(g_t + c_t) \tag{21}$$

Notice that in (21) we parameterize the term $E_t u_{c,t+N-1}$ as $\Phi^N(\mathbf{X}_t, \boldsymbol{\gamma}^N)$. In other words we apply the standard PEA to this term. An alternative is to define $E_t u_{c,t+N-2} = \Phi^N(\mathbf{X}_t, \boldsymbol{\gamma}^{N-1})$ and then use Forward States to get: $E_t u_{c,t+N-1} = E_t\Phi^N(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^{N-1})$. We follow the latter route in the numerical implementation. We therefore write (21) as follows

$$\begin{aligned} b^1_t\beta E_t(\Phi^1(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^1)) + b^N_t\beta^N E_t(\Phi^N(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^N)) &= b^1_{t-1}u_{c,t} + b^N_{t-1}\beta^{N-1}E_t(\Phi^{N-1}(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^{N-1})) \\ &\quad + g_t u_{c,t} - (u_{c,t} - v_{x,t})(g_t + c_t) \end{aligned} \tag{22}$$

i.e. when $S = 1$ and realizing that $E_t u_{c,t} = u_{c,t} = \Phi^1(\mathbf{X}_t, \boldsymbol{\gamma}^1)$.

The two ways of solving the model are obviously conceptually equivalent.

of the multiplier is felt more when government debt is high. This nonlinear influence seems to be (sufficiently) well captured in our specification by the cross terms and not by the levels since, as we verify in section 7.4 of the main text, we pass accuracy tests.[6].

We apply the above selection criterion to the other models. Consider the *no buyback model* and its first order conditions and budget constraint:

$$u_{c,t} - v_{x,t} + \lambda_t \Big(u_{cc,t} c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t} \Big) + u_{cc,t} \sum_{i \in \{1,N\}} \Big(\lambda_{t-i} - \lambda_t \Big) b^i_{t-i} = 0$$

$$\beta^i E_t \Big(u_{c,t+i}\lambda_t - u_{c,t+i}\lambda_{t+i} \Big) + \xi^i_{L,t} - \xi^i_{U,t} = 0 \quad \text{for} \quad i = 1, N$$

$$\sum_{i \in \{1,N\}} b^i_t \beta^i E_t u_{c,t+i} = g_t u_{c,t} + u_{c,t} \sum_{1,N} b^i_{t-i} - (u_{c,t} - v_{x,t})(c_t + g_t).$$

We include in the $\mathbf{X}^{out}$ vector: $\lambda_t b^N_t$, $\lambda_t b^1_t$, $\lambda_{t-N+1} b^N_{t-N+1}$, and b^N_{t-N+1}, as these appear directly on the FONC.

Next, consider the *no buyback model with coupons*. To solve the coupon model we need to approximate the term $\sum_{j=1}^N \beta^j E_t u_{c,t+j}\kappa + \beta^N u_{c,t+N}$ and the term $\sum_{j=1}^N \beta^j E_t u_{c,t+j}\lambda_{t+j}\kappa + \beta^N u_{c,t+N}\lambda_{t+N}$. From the FONC of consumption and the government budget constraint (omitted for brevity), it is easy to show that all the lags of b^N_{t-j} and $\lambda_{t-j} b^N_{t-j}$, for $j = 1, 2, .., N-1$ should be introduced in the out vector. The X^{out} vector is therefore composed by: $\{\lambda_{t-j} b^N_{t-j}\}_{j=0}^{N-1}$, $\lambda_t b^1_t$, $\{b^N_{t-j}\}_{j=1}^{N-1}$.

Similarly, when we consider the *callable bond* model the X^{out} vector includes:

$\{b^i_t \lambda_t\}_{i=1,N}$, $\lambda_t b^N_{t-N+m}$, $\lambda_{t-N+1} b^N_{t-N+1}$, $\lambda_{t-N+m} b^N_{t-N+1}$ and $b^N_{t-N+m+1}$, where m is the repurchase date.

Finally, for each of the above models we include in $\mathbf{X}^{out}$ other higher order terms of date t variables that have not been included in $\mathbf{X}^{core}$. In each approximation we add in $\mathbf{X}^{out}$ the following terms: λ^2_t, $(b^N_t)^3$, $(b^1_t)^3$, $b^N_t b^1_t$.

We now consider the *optimal repurchases model* of section 6.3 in the main text (see a previous subsection of this online appendix for the FONC of this model). The following expectations need to be approximated with PEA in this case:

$$E_t \xi^R_{U,t+1} \quad \text{and} \quad E_t u_{c,t+i}, \quad E_t \lambda_{t+i} u_{c,t+i}, \quad i = 1, N, N-1$$

where $\xi^R_{U,t}$ is the Langrange multiplier on the constraint $R_t \leq b^N_{t-1}$.

As discussed in the text, one way to reduce the total number of state variables in this model is to rewrite the state vector as:

$$\mathbf{X}_{t+1} = \Big\{ g_{t+1}, \;\; B_t, \;\; B\lambda_t, \;\; \{B^{net}_{t+1-i}, B\lambda^{net}_{t+1-i}\}_{i=1}^N, \;\; \lambda_{t-N}, \;\; b^N_{t-N} \Big\} \tag{23}$$

[6] Recall that $\mathbf{X}^{core}$ includes the variables λ_t,b^N_t and $b_{1,t}$ in levels. These first order terms, help us to identify the portfolio, but combined with their squares, cubes and so on can (practically speaking), explain part of the variability of some of the cross terms in $\mathbf{X}^{out}$. To avoid having residuals close to zero from the regressions of $\mathbf{X}^{out}$ on $\mathbf{X}^{core}$ when we compute linear combinations, we use an additional selection criterion that we describe in the next subsection.

where

$$B_t^{net} \equiv b_{t-1}^N - R_t$$
$$B_t \equiv b_t^S + B_{t-N+2}^{net}$$
$$B\lambda_t^{net} \equiv \lambda_{t-1}(1-\mathcal{T}^N)b_{t-1}^N + \lambda_t(1+\mathcal{T}^R)R_t$$
$$B\lambda_t \equiv \lambda_t(1-\mathcal{T}^1)b_t^1 + B\lambda_{t-N+2}^{net}.$$

As we did for the previous models we chose $\mathbf{X}^{core}$ and $\mathbf{X}^{out}$ in the optimal repurchase model to include the state variables which appear in the FONC and which therefore exert a direct influence on the conditional expectations. We specified $\mathbf{X}^{core}$ as in (16) and $\mathbf{X}^{out}$ as follows:

$$\mathbf{X}_{t+1}^{out} = \left\{ \{b_t^i \lambda_t\}_{i=1,N},\ b_{t-N+1}^N - R_{t-N+2},\ (b_{t-N+1}^N - R_{t-N+2})\lambda_{t-N+1},\ R_{t-N+2}\lambda_{t-N+2} \right\}. \tag{24}$$

Given the specification of $\mathbf{X}_{t+1}^{out}$ (and that of $\mathbf{X}_{t+1}^{core}$) the terms $B_t, B\lambda_t, \left(B_{t+1-i}^{net}, B\lambda_{t+1-i}^{net}\right)_{i=1}^N$ appear in the approximations. For example $B_t = b_t^1 + b_{t-N+1}^N - R_{t-N+2}$ and $B_{t-N+2}^{net} = b_{t-N+1}^N - R_{t-N+2}$ are part of the state vector, but we have chosen to separate the terms b_t^1 and $b_{t-N+1}^N - R_{t-N+2}$ in the approximations assigning b_t^1 to the core and $b_{t-N+1}^N - R_{t-N+2}$ to the out vector. We did this for convenience and most importantly to be able to use as an initial guess for our approximation the solution of the no buyback model.

Moreover, notice that though in principle we could introduce R_t as a variable in $\mathbf{X}^{core}$,[7] this is not necessary to identify the optimal path of R_t. Since this is a model where the government can repurchase only after one period and we assume positive transaction costs, we do not need R_t in the core states to determine the portfolio.[8]

Finally notice that in $\mathbf{X}^{core}$ and $\mathbf{X}^{out}$, the bond and repurchases variables are not multiplied by transaction costs. Since these variables (mostly) enter separately in the approximations and since the costs $\mathcal{T}$ are small, this does not influence the properties of the solution.

Let's now turn to the *model with three bonds*. When the government issues debt in three maturities $(1 < M < N)$ under no buyback the FONC are given by:

$$u_{c,t} - v_{x,t} + \lambda_t\Big(u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t+g_t) - v_{x,t}\Big) + u_{cc,t}\sum_{i\in\{1,M,N\}}\Big(\lambda_{t-i} - \lambda_t\Big)b_{t-i}^i = 0$$
$$\beta^i E_t\Big(u_{c,t+i}\lambda_t - u_{c,t+i}\lambda_{t+i}\Big) + \xi_{L,t}^i - \xi_{U,t}^i = 0 \quad \text{for} \quad i = 1, M, N$$
$$\sum_{i\in\{1,M,N\}} b_t^i \beta^i E_t u_{c,t+i} = g_t u_{c,t} + u_{c,t}\sum_{1,M,N} b_{t-i}^i - (u_{c,t} - v_{x,t})(c_t+g_t).$$

We need to approximate now 6 conditional expectations. We specify the 'core' and 'out' vectors as

[7] From (23) we know that $b_{t-1}^N - R_t$ is a state variable. However, this will not appear in the FONC in periods $t+1, t+N, t+N-1$ and for this reason we dropped it from the core state vector and from the out vector (24).

[8] In other words R_t can still be identified through the budget constraint or through the nonlinear transaction costs. Had we allowed the government to repurchase more than once and if the transaction costs were assumed independent of (the vector in the case of many repurchases) R we would need the control variables R to be in $\mathbf{X}^{core}$ in order to solve the model.

Moreover, since R_t is always close to zero introducing it as an independent variable in the core vector leads to convergence problems. We discuss this further below.

Table 1: **Variables used in approximations**

<table>
<tr><th></th><th>$\mathbf{X}^{core}$</th><th colspan="2">$\mathbf{X}^{out}$</th><th>total</th></tr>
<tr><th></th><th></th><th>common var.</th><th>ad hoc</th><th></th></tr>
<tr><td>BB</td><td rowspan="5">$1, g_{t+1}, \{b_t^i\}_{i=1,N},$
$\lambda_t, \{(b_t^i)^2\}_{i=1,N},$
$\{g_{t+1}b_t^i\}_{i=1,N}$</td><td rowspan="6">$\{(b_t^i)^3\}_{i=1,N},$
$\lambda_t^2, b_t^N b_t^1$</td><td>$\{b_t^i\lambda_t\}_{i=1,N}, \lambda_t b_{t-1}^N,$
$\lambda_{t-1}b_{t-1}^N, \lambda_{t-N+1}b_{t-N+1}^N,$
$\lambda_{t-N+2}b_{t-N+1}^N$</td><td>19</td></tr>
<tr><td>NBB</td><td>$\{b_t^i\lambda_t\}_{i=1,N},$
$\lambda_{t-N+1}b_{t-N+1}^N, b_{t-N+1}^N$</td><td>17</td></tr>
<tr><td>coupons</td><td>$\{\lambda_{t-i}b_{t-i}^N\}_{i=0}^{N-1},$
$\lambda_t b_t^1, \{b_{t-i}^N\}_{i=1}^{N-1}$</td><td>30</td></tr>
<tr><td>callables</td><td>$\{b_t^i\lambda_t\}_{i=1,N}, \lambda_t b_{t-N+m}^N,$
$\lambda_{t-N+1}b_{t-N+1}^N, \lambda_{t-N+m}b_{t-N+1}^N$
$b_{t-N+m+1}^N$</td><td>19</td></tr>
<tr><td>repurchases</td><td>$\{b_t^i\lambda_t\}_{i=1,N}, b_{t-N+1}^N - R_{t-N+2}$
$(b_{t-N+1}^N - R_{t-N+2})\lambda_{t-N+1}$
$R_{t-N+2}\lambda_{t-N+2}$</td><td>18</td></tr>
<tr><td>3 bonds</td><td>$1, g_{t+1}, \{b_t^i\}_{i=1,M,N},$
$\lambda_t, \{(b_t^i)^2\}_{i=1,M,N},$
$\{g_{t+1}b_t^i\}_{i=1,M,N}$</td><td>$\{b_t^i\lambda_t\}_{i=1,M,N}, \lambda_t^2,$
$(b_t^M)^3, b_t^1 b_t^M, b_t^M b_t^N$
$\{b_{t-i+1}^i\}_{i=M,N}, \{b_{t-i+1}^i\lambda_{t-i+1}\}_{i=M,N}$</td><td>26</td></tr>
</table>

follows:

$$\mathbf{X}_{t+1}^{core} = \left\{1, g_{t+1}, \lambda_t, \{b_t^i\}_{i=1,M,N}, \{(b_t^i)^2\}_{i=1,M,N}, \{g_{t+1}b_t^i\}_{i=1,M,N}\right\}$$

$$\mathbf{X}_{t+1}^{out} = \left\{\{b_t^i\lambda_t\}_{i=1,M,N}, \{(b_t^i)^3\}_{i=1,M,N}, \{b_t^i b_t^k\}_{i,k\in\{1,M,N\},k\neq i}, \lambda_t^2, \{b_{t-i+1}^i\}_{i=M,N}, \{b_{t-i+1}^i\lambda_{t-i+1}\}_{i=M,N}\right\}$$

Therefore we have 12 variables in the core vector and 14 variables in the out vector.

Table 1 summarises the previous discussion on our choices for $\mathbf{X}^{core}$ and $\mathbf{X}^{out}$.

B.1.2 An R^2 selection criterion for the elements of $\mathbf{X}^{out}$

Once we have chosen the composition of $\mathbf{X}^{core}$ and $\mathbf{X}^{out}$, we apply the following procedure:

1. We first regress each variable X_j^{out} on $\mathbf{X}_{-j}^{out}$ and $\mathbf{X}^{core}$ and compute the R-square of the regression, R_j^2.

2. We find the variable k with the highest R_k^2, that is $k = \arg\max_{j\in\{1,2,\ldots,\text{length}(\mathbf{X}^{out})\}}\{R_j^2\}$. If $R_k^2 > 0.995$ we set the coefficient $\alpha_k^1 = 0$. In other words, we set the coefficient of this variable in the first linear combination (and in all approximations) equal to zero.

3. We repeat Steps 1 and 2 removing the excluded variables from $\mathbf{X}^{out}$ until $R_k^2 < 0.995$.

4. We apply the 'Condensed PEA' to find the coefficients $\overline{\boldsymbol{\gamma}}^{i,f}$ and $\overline{\boldsymbol{\delta}}^{i,f}$, i.e. the new fixed point in the model, with the first linear combination of the elements of $\mathbf{X}^{out}$ which 'survive' Steps 1-3.

5. When we recover $\overline{\boldsymbol{\gamma}}^{i,f}$ and $\overline{\boldsymbol{\delta}}^{i,f}$, we repeat steps 1-4 to determine which of the variables in $\mathbf{X}^{out}$ have a non-zero coefficient in the second linear combination. We apply this procedure to all linear combinations we include to the model.

To understand why the above criterion is useful notice that when $R_j^2 > 0.995$, most of the variability of X_j^{out} is either explained by the core state variables and/or X_j^{out} is highly correlated with other variables in $\mathbf{X}^{out}$. In the first case the residuals of the regression of X_j^{out} on $\mathbf{X}^{core}$ (required to estimate the linear combination) will be close to zero so that the variable does not add almost anything to the approximation. In the second case, the residuals will be highly correlated with the residuals of other $\mathbf{X}^{out}$ variables. In both cases estimating the coefficients α becomes problematic and the convergence of the model with linear combinations becomes more difficult. Since a high R_k^2 denotes that the $k-$th variable is redundant, it helps the algorithm to converge if its coefficient is set to zero beforehand.

Number of linear combinations used in the approximation

Tables 2 to 4 summarize the number of linear combinations we add to the approximations of some of the models considered in the paper. Consider first Table 2 which reports the results for the buyback models (under 'no lending', top panel and under 'lending', bottom panel).

As described in the text, a new linear combination is added when it reduces significantly the residual sum of squares obtained from the regression of $u_{c,t+i}$, (for instance) on $\mathbf{X}^{core}$ and the linear combinations which were added in the approximation in previous rounds. Our criterion is based on the percentage gain in the coefficient of variation R^2 we get when we add the new linear combination.

The rows in the table summarise the gains in R^2 for each linear combination. R^2_{aug} is the value of the coefficient of variation we obtain when we include an additional linear combination to the model. R^2_{old} the coefficient of variation without the additional linear combination. The row labeled LC_1 corresponds to the 'Condensed PEA' test when we solve the model only with the $\mathbf{X}^{core}$ variables and test the inclusion of the first linear combination. LC_2 tests the significance of the second linear combination and so on.

We add a further linear combination to an approximation when

$$R^2_{diff} = \frac{R^2_{aug} - R^2_{old}}{R^2_{old}} * 100 > 0.05,$$

in other words when the gain in R^2 is greater than 0.05 percent.

As Table 2 shows the buyback model under 'no lending' requires one linear combination. The approximations of $E_t u_{c,t+1}$ and $E_t u_{c,t+N-1}$ include a linear combination in the first round and the approximation of $E_t u_{c,t+1}\lambda_{t+1}$ includes one in the second round. In the buyback 'lending' model the importance of the $\mathbf{X}^{out}$ variables is limited and so this model does not require any linear combinations.

Table 3 reports the analogous findings in the no buyback models and Table 4 for the case of coupons. Each of these models is solved with linear combinations.

Table 2: **Linear Combinations: Buyback Model**

		BuyBack 'no Lending'				
		$u_{c,t+1}$	$u_{c,t+N}$	$u_{c,t+N-1}$	$u_{c,t+1}\lambda_{t+1}$	$u_{c,t+N}\lambda_{t+1}$
$\frac{R^2_{aug}-R^2_{old}}{R^2_{old}} * 100$	LC_1	**0.0757**	0.0169	**0.1677**	0.0441	0.0258
	LC_2	0.0026	0.0228	0.0043	**0.0547**	0.0417
	LC_3	0.0259	0.0232	0.0234	0.0060	0.0308
	Total	1	0	1	1	0
		BuyBack 'Lending'				
$\frac{R^2_{aug}-R^2_{old}}{R^2_{old}} * 100$	LC_1	0.0081	0.0451	0.0403	0.0385	0.0322
	Total	0	0	0	0	0

Note: The table shows the number of linear combinations in the buyback models ('no lending', top panel and 'lending, bottom panel). The columns list the conditional expectations we approximate in these models. The rows report the percentage gains in R^2 from adding a further linear combination to the model. Hence row LC_1 shows the gains when we compare the regressions with $\mathbf{X}^{core}$ only (R^2_{old}) to the regressions with $\mathbf{X}^{core}$ and one linear combination (R^2_{aug}.) In row LC_2 R^2_{old} derives from a regression on $\mathbf{X}^{core}$ and the first linear combination and R^2_{aug} adds a second linear combination and so on.
We denote in bold values of $\frac{R^2_{aug}-R^2_{old}}{R^2_{old}} * 100$ which exceed the 0.05 threshold (above which we introduce an additional linear combination to the model).

Table 3: **Linear Combinations: No-Buyback model**

		$u_{c,t+1}$	$u_{c,t+N}$	$u_{c,t+1}\lambda_{t+1}$	$u_{c,t+N}\lambda_{t+N}$
		No BuyBack 'No Lending'			
$\frac{R^2_{aug}-R^2_{old}}{R^2_{old}} * 100$	LC_1	0.0173	0.0166	**0.0561**	**0.0646**
	LC_2	**0.0578**	0.0112	0.0109	0.0035
	LC_3	0.0002	0.0194	0.0058	0.0001
	Total	1	0	1	1
		No BuyBack 'Lending'			
$\frac{R^2_{aug}-R^2_{old}}{R^2_{old}} * 100$	LC_1	0.0194	**0.0590**	0.0342	**0.1448**
	LC_2	0.0167	0.0007	0.0140	0.0006
	Total	1	0	0	1

Note: The table shows the number of linear combinations in the no buyback models ('no lending', top panel and 'lending, bottom panel). See Table 3 for details.

Table 4: **Linear Combinations: No-Buyback Model with Coupons**

		$u_{c,t+1}$	$q_{c,t}$	$u_{c,t+1}\lambda_{t+1}$	$q_{\lambda c,t}$
		No BuyBack Coupons			
$\frac{R^2_{aug}-R^2_{old}}{R^2_{old}} * 100$	LC_1	0.0067	0.0154	0.0213	**0.0654**
	LC_2	0.0048	0.0108	0.0077	0.0011
	Total	0	0	0	1

Note: The table shows the number of linear combinations in the no buyback model with coupons. See Table 3 for details. We define $q_{c,t} \equiv \sum_{j=1}^{N} \beta^j u_{c,t+j}\kappa + \beta^N u_{c,t+N}$ and $q_{\lambda c,t} \equiv \sum_{j=1}^{N} \beta^j u_{c,t+j}\lambda_{t+j}\kappa + \beta^N u_{c,t+N}\lambda_{t+N}$.

The picture is similar when we consider the models not included in the tables: the callable bonds, three maturities and the model with optimal repurchases. The callable bond model needs one linear combination to be added to get accurate solutions. For the three maturities model we find that $\mathbf{X}^{core}$ is sufficient and therefore we do not include any linear combinations. The optimal repurchase model requires two linear combinations to be accurately solved. The first linear combination is introduced to the approximations of $E_t u_{c,t+i}\lambda_{t+i}$, $i = 1, N, N-1$ and $E_t \xi^R_{U,t+1}$. The second linear combination is introduced to the approximation of $E_t u_{c,t+1}$.

B.2 Some practical features of the numerical implementation

B.2.1 Dealing with occasionally binding constraints on debt

As explained in the main text, we impose an upper and lower bound on the issuance of short and long bonds. In a two bond model we have in total four constraints. These constraints are only occasionally binding and in theory we could use the approach explained, for example, in Marcet and Singleton (1999) to deal with them. Suppose that the government can issue only one bond, whatever the maturity. Marcet and Singleton suggest for every period t first to solve the unconstrained problem and check whether one of the debt constraints is violated. If it is violated, the value of the bond is set equal to the value of the debt limit and c_t and λ_t are recalculated accordingly.

Unfortunately, this cannot be easily applied in the case of more than one maturity because of the number of constraints involved. If one of the constraints is violated when solving the unconstrained problem, we need to verify that forcing the constraint is not going to generate a violation of one of the constraints on the other bond. This problem presents too many cases to be checked one by one and the computational burden increases considerably when an additional maturity is introduced to the model. For this reason we impose the following (quadratic) costs when the bonds violate the limits in the buyback model:

$$C(b^i_t) = \begin{cases} \frac{\phi_1}{2}\left(b^i_t - \frac{\overline{M}_i}{\beta^i}\right)^2 & b^i_t > \frac{\overline{M}_i}{\beta^i} \\ \frac{\phi_1}{2}\left(\frac{\underline{M}_i}{\beta^i} - b^i_t\right)^2 & b^i_t < \frac{\underline{M}_i}{\beta^i} \\ 0 & otherwise \end{cases}$$

for $i = 1, N$. ϕ_1 governs the penalty from deviating from the debt limits $\frac{\underline{M}_i}{\beta^i}$ and $\frac{\overline{M}_i}{\beta^i}$. We choose a value of ϕ_1 equal to unity. Analogous cost functions are used in the no buyback and coupons models, the debt limits have to be adjusted in these cases as described in text.

In the optimal repurchase model we have an additional constraint on the level of repurchases: $0 \leq R_t \leq b^N_{t-1}$. In this case we continue to impose $C(b^i_t)$ for $i = 1, N$ however we use Marcet and Singleton's approach to deal with this extra constraint. When R violates a limit (either because $R_t < 0$ or $R_t > b^N_{t-1}$) we fix the value of R_t to the constraint and solve the FONC to determine the optimal portfolio and the value of the multipliers, $\xi^R_{L,t}$ and $\xi^R_{U,t}$.

B.2.2 Initial conditions and sample size

In order to generate a more precise approximation of the policy functions over the debt space we use PEA with oversampling. We choose 25 different initial conditions for the debt levels b^1_{-1} and b^N_{-j}, where $j = 1, 2, ..N - 1$ uniformly distributed in the interval $\left[\frac{\underline{M}_i}{\beta^i}, \frac{\overline{M}_i}{\beta^i}\right]$ (e.g. in the buyback model). We draw 10 samples of 500 periods for each initial condition. The total number of observations is then 125000.

Given the initial conditions for the portfolio, we also need to specify some initial values for the λ's. For this purpose we recover initial values $\lambda_{-N} = ..., \lambda_{-1}$ that would be consistent with the deterministic steady state. As is well known in steady state the debt level in these models is indeterminate and so we can obtain a different λ (consistent with a different c) for each bond vector. Under no buyback we obviously need to set $b^N_{-1} = b^N_{-2} = ... = b^N_{-N+1}$ to be in steady state.[9]

B.2.3 Rescaling

To improve the stability of the algorithm, we rescale the variables which enter in $\mathbf{X}^{core}$ and $\mathbf{X}^{out}$. For example we use $\frac{b^i_t}{\frac{\overline{M}_i}{\beta^i}}$ and $\frac{\lambda_t - \lambda^s}{\lambda^s}$ instead of b^i_t and λ_t. This is applied to every lag of the independent variables used in the approximation. We also rescaled the dependent variables in the PEA regressions by their steady state values such that their means are close to one in the approximations. For example, we regress $\frac{u_{c,t+1}}{u^s_c}$ and $\frac{u_{c,t+1}\lambda_{t+1}}{u^s_c \lambda^*}$ on $\mathbf{X}^{core}$ and the linear combinations, to obtain the approximations of $E_t\left(\frac{u_{c,t+1}}{u^s_c}\right)$ and $E_t\left(\frac{u_{c,t+1}\lambda_{t+1}}{u^s_c \lambda^*}\right)$ respectively. The same is done for the other expectations.

Rescaling is useful because some of the coefficients could be very small without it. For example, consider the buy back no lending model; b^N_t can fluctuate in simulations between 0 and $\frac{\overline{M}_i}{\beta^i} \approx 117$ and its square between 0 and 117^2. It is obvious that the estimated coefficients of these terms may be close to zero. This makes it difficult to find a reliable convergence criterion for the model.[10] Through rescaling the state variables fluctuate between 0 and 1. This improves significantly the stability of our algorithm (see also Judd et al (2011)).

B.2.4 Convergence of PEA - Finding Good initial conditions for the coefficients

Den Haan and Marcet (1990) show that PEA does not guarantee convergence. Convergence is more likely if we use good initial conditions for the coefficients. This is even more necessary in the context

[9] Notice that since sample sizes are sufficiently long (500 observations) our results do not change when we set $\lambda_{-1} = \lambda_{-2} = ... = 0$.

[10] To see this, denote the coefficient of variable b^N_t in the approximation of $E_t u_{c,t+i}$ by γ^i_3. Let $\gamma^i_{3,1}$ be the update of this coefficient and $\gamma^i_{3,0}$ the initial value. If we use a stopping rule

$$\text{Converge if} \quad \frac{|\gamma^i_{3,1} - \gamma^i_{3,0}|}{|\gamma^i_{3,0}|} < \epsilon \tag{25}$$

and $\gamma^i_{3,1}, \gamma^i_{3,0} \approx 0$, then the behavior of (25) will be very erratic (both very high and very low values are possible, and this does not tell us much about convergence of the model's quantities). Analogously, if we use the convention

$$\text{Converge if} \quad \frac{|\gamma^i_{3,1} - \gamma^i_{3,0}|}{1 + |\gamma^i_{3,0}|} < \epsilon \tag{26}$$

for some ϵ, then the algorithm may (wrongly) converge after a few iterations.

In our codes we employ the criterion (26), but since the variables are rescalled, we are sure that coefficients which are small in values, do not matter much for the optimal policy.

of the optimal portfolio problem under incomplete markets. If the initial coefficients constitute a very poor guess of the equilibrium of the model, then the algorithm may circle for a long time and subsequently diverge.

Good initial conditions for portfolio choice models can be obtained as follows:

1. Solve portfolio models with positive transaction costs.

   For example consider solving the Ramsey problem under buyback subject to the following government budget constraint:

$$\sum_{i=1,N} p_t^i b_t^i = \sum_{i=1,N} p_t^{i-1} b_{t-1}^i + g_t - \left(1 - \frac{v_{x,t}}{u_{c,t}}\right)(c_t + g_t) + \sum_{i=1,N} \omega_i (b_t^i)^2$$

   where $\omega_i(b_t^i)^2$ is a transaction cost paid by the government at issuance. It is obvious that in this model the optimal portfolio is determinate (even with the conventional PEA). In the limit when $\omega_i \to 0$ we obtain the buyback model considered in this paper, if $\omega_i \to \infty$ there is no trade in bonds. Hence, good initial conditions can be found from solving models with positive transactions costs and gradually reducing ω_i till 0.

2. Solve models under tight debt limits and gradually loosen them.

   We found that models with tight debt constraints converge more easily than models with looser ones. Generally speaking, models with very loose debt constraints can converge to a wrong equilibrium which features for example a constant λ as in the case of complete markets. This holds in particular because running the models with samples of 500 observations may imply that the debt limits are rarely hit, if they are very loose.[11] Moreover, when the bounds are loose, poor initial conditions may make the PEA circle or diverge. Assuming tight bounds helps the algorithm converge. The converged coefficients can then be used as initial conditions for models with looser bounds and so on.

B.2.5 Calculating the sample moments

As discussed in the text, to compute the moments reported in Table 4 in the paper, we simulated the model 1000 times using as initial conditions the values of $\mathcal{S}_t$ and the market value of debt, we recovered from the data. In 1955 the share of short debt equaled 39% and the initial debt to GDP ratio was 38% in the CRSP sample.

We then computed the values b_{-1}^1 and b_{-j}^N, $j = 1, 2, .., N$ in the deterministic steady state such that the initial share and market value of debt are consistent with these targets. For example in the

[11] To see this, consider the following example: Suppose that the initial guess for the polynomials is $E_t u_{c,t+i} = \gamma_0^i + \gamma_1^i E_t g_{t+1} + \gamma_2^i b_t^1 + \gamma_3^i b_t^N$ and $E_t \lambda_{t+1} u_{c,t+i} = \lambda^* E_t u_{c,t+i}$. Then, under very loose bounds (e.g. $-\underline{M}_i = \overline{M}_i = \infty$) for every t we get $\lambda_t = \lambda^*$ as a solution to the system of FONC, as in the case of complete markets.

Under tight bounds however, it is likely that $b_t^i = \frac{\overline{M}_i}{\beta^i}$ or $b_t^i = \frac{\underline{M}_i}{\beta^i}$ for some t, and in this case the condition $\lambda_t = \frac{E_t u_{c,t+i} \lambda_{t+1}}{E_t u_{c,t+i}}$ does not hold. λ_t is recovered from the FONC of consumption and generally it will be that $\lambda_t \neq \lambda^*$. This introduces variability in λ_t.

A similar argument, showing the importance of 'moving bounds' for convergence in the PEA, was made by Maliar and Maliar (2003).

buyback model we have

$$\frac{\beta b^1_{-1}}{\beta b^1_{-1} + \beta^N b^N_{-1}} = 0.39 \qquad \beta b^1_{-1} + \beta^N b^N_{-1} = 0.38 * 70$$

The analogous expressions for the other models are omitted for brevity.

Given the initial conditions for the bonds, we found the initial values of c and λ to satisfy the FONC of consumption and the government budget constraint in the deterministic steady state.

We then simulated the models and computed the market value of government debt and the share of short bonds. Notice that whereas in the buyback models to construct the market values for short and long bonds it is sufficient to use the approximations of $E_t u_{c,t+1}$ and $E_t u_{c,t+N}$, in the no buyback model this is not the case. In particular we need to compute the value of non-matured debt in period t. This requires all the prices p^j_t for $j = 2, 3, ..., N-1$. Since these prices do not affect the equilibrium properties of optimal allocations, we computed the approximations through simple regressions of $u_{c,t+i}$ on $\mathbf{X}_{t+1}$ once our algorithm has converged.[12]

Finally, note that because the model is solved with quadratic costs if the debt limits are violated, as described in subsection B.2.1, the market value of government debt can become (slightly) negative in the no lending models, in some periods and samples. The statistics reported in Table 4 in the paper are calculated after dropping samples where the market value becomes negative in 'no lending models'. For the same reason in order to avoid having a negative share of short debt in simulations (if say $b^1_t < 0, b^N_t > 0$) or greater than unity (i.e. when $b^N_t < 0, b^1_t > 0$) , we computed the moments using $\min\{\max\{\mathcal{S}_t, 0\}, 1\}$ in the no lending models: we forced the share to be equal to zero when it was negative and 1 when it exceeded unity. This adjustment obviously was much more frequent in the buyback 'no lending model' than in the no buyback model.[13]

B.2.6 Some limitations of the PEA

We cannot claim that the numerical algorithm we propose in this paper can solve every portfolio choice problem under incomplete markets. To make this point, we describe here a few cases where the approximation of conditional expectations under 'Forward States' may not compute accurately equilibria with multiple assets.

The first noteworthy difficulty of our methodology is that as the number of assets increases the optimal allocation may be close to the complete markets' one. Recall that in this case the portfolio and the multiplier λ are constant through time. Clearly, such equilibria cannot be approximated with polynomials of the form $E_t u_{c,t+i} = \gamma^i_0 + \gamma^i_1 E_t g_{t+1} + \gamma^i_2 b^1_t + \gamma^i_3 b^N_t + \gamma^i_3 \lambda_t + ...$; if the RHS variables are roughly constant, the estimation of the polynomial coefficients will not be reliable. Our algorithms are designed to deal with cases where markets are incomplete, this involves either few assets, or tight debt constraints or both.

[12] In these regressions we used all bonds and cross terms b^N_{t-j} and $b^N_{t-j}\lambda_{t-j}$ $j = 0, 1, ..., N-1$ as independent variables. We do not use the 'Condensed PEA' to approximate the bond prices since these approximations are performed when the algorithm has converged and thus the algorithms convergence properties do not depend on them.

[13] Recall that one of the main findings of the paper is that under no buyback long and short debt levels comove strongly. This property also holds for the issuances b^1_t and b^N_t. It is therefore rare that b^1_t is slightly negative and $b^N_t > 0$. If this occurs in our simulations it is likely that the overall market value is slightly negative in which case the sample is dropped as described previously.

However, under buyback and no lending, we frequently have $b^N_t >> 0$ and $b^1_t \approx 0$ so that small negative values of short debt can occur. In these cases we set the short term share to 0.

Second, even under incomplete markets if the government can trade three or more maturities we cannot rule out equilibria where some of the assets are roughly constant over time (and thus stable coefficients for the polynomials are hard to obtain). To see this assume that in the *buyback model* the government issues three different maturities: 1 year, M year and N year bonds where $1 < M < N$. To take advantage of fiscal hedging the government will likely adopt the following debt management strategy: set $b_t^1 = \frac{M_1}{\beta} < 0$, $b_t^N = \frac{\overline{M}_N}{\beta} > 0$ and use b_t^M to finance deficits and surpluses over the business cycle.[14] Therefore the values of b_t^1 and b_t^N will be roughly constant over time and thus it will be difficult to construct approximations of the conditional expectations when the polynomials contain b_t^1 and b_t^N as state variables.

Third, (we found that) it is not as easy to solve models where the government issues bonds of maturities close to one another (e.g. 1, 9 and 10 years or 1, 2 and 10 years) as it is to solve models where maturities are further from one another (e.g. 1, 5 and 10 years). As we have seen, no buyback models give us portfolios where all issuances have the tendency to comove strongly, however, when bonds are close substitutes, asset prices also comove strongly and so portfolios are difficult to pin down. The algorithm then tends to circle without converging.

Note that these difficulties are not relevant if the goal is to solve models with multiple assets and realistic frictions (e.g. imperfect substitutability among the assets). Small transaction costs, bond clienteles and preferences for short term (safe) assets, will give well defined demand curves for each maturity and these are realistic features of government debt markets. Our methodology is therefore broadly applicable to solve models with many assets, the limitations described in this subsection arise because our model is a simplistic one and abstracts from several realistic frictions.

B.3 Accuracy of the solutions

To check the accuracy of the solution of each model we compute the Euler Equation Errors (EEE) generated by our approximations (see for example Arouba et al (2006) for an exhaustive description of the methodology). Essentially this methodology checks that first order conditions hold with an acceptable degree of precision at many points in the state vector.

In particular, the test requires to numerically calculate each conditional expectation in the Euler equations. Ours is not a routine application of the standard accuracy test because we have expectations up to N leads, so the exact integration is not feasible. We use Monte Carlo integration to approximate the expectation integrals. In practice we draw 250 shock samples (for 25 initial conditions of public debt with 10 samples for each initial condition) of 450 periods each. We then simulate each model using our approximations. We discard the first 200 periods of each sample, and for each subsequent period [15], $\bar{t}$, we draw $k = 10000$ different shock paths of length N, the number of leads in the conditional expectations. We simulate our model for each shock path separately using our approximated policy functions and initial states given by the allocation in $\bar{t}$. We then compute the conditional expectations as the mean over the k samples. For example for the buy back and no buy

[14]If the debt limits on b_t^M are sufficiently loose, this strategy is feasible.

[15]For the 3 bond model we check the errors every 5 periods. This choice was made for computational purposes because in this model we have 125 initial conditions and 10 samples per initial condition.

back models we compute:

$$\Xi_{\bar{t},1} = E_t\left(u_{c,\bar{t}+1}\right) = \frac{\sum_{i=1}^{k} u^i_{c,\bar{t}+1}}{k}$$

$$\Xi_{\bar{t},2} = E_t\left(u_{c,t+N}\right) = \frac{\sum_{i=1}^{k} u^i_{c,\bar{t}+N}}{k}$$

$$\Xi_{\bar{t},3} = E_t\left(u_{c,t+1}\lambda_{t+1}\right) = \frac{\sum_{i=1}^{k} u^i_{c,\bar{t}+1}\lambda^i_{\bar{t}+1}}{k}$$

$$\Xi^{NBB}_{\bar{t},4} = E_t\left(u_{c,t+N}\lambda_{t+N}\right) = \frac{\sum_{i=1}^{k} u^i_{c,\bar{t}+N}\lambda^i_{\bar{t}+N}}{k} \text{ or } \Xi^{BB}_{\bar{t},4} = E_t\left(u_{c,t+N}\lambda_{t+1}\right) = \frac{\sum_{i=1}^{k} u^i_{c,\bar{t}+N}\lambda^i_{\bar{t}+1}}{k}$$

Since we have two Euler equations, we check separately each of them, calculating the value of the multiplier in period $\bar{t}$ implied by the expectations $\Xi_{\bar{t},}$ given the portfolio $b_{1,\bar{t}}, b_{N,\bar{t}}$[16]. In theory we could stop here and check the difference between the implied multiplier and the one generated by our simulation. However it is difficult to give an intuitive economic interpretation of this difference. Following the literature we then state our results in terms of of consumption deviations. To do this we calculate the implied consumption error using the FONC of c_t, given the implied multiplier.

In particular we compute the following quantities:

$$EEE^1_{\bar{t}} = \frac{\widetilde{c}^1_{\bar{t}} - c_{\bar{t}}}{\widetilde{c}^1_{\bar{t}}}$$

$$EEE^N_{\bar{t}} = \frac{\widetilde{c}^N_{\bar{t}} - c_{\bar{t}}}{\widetilde{c}^N_{\bar{t}}}$$

where $\widetilde{c}_{\bar{t}}$ is the consumption implied by the new approximation of the expectations, $\Xi_{\bar{t},}$ and $c_{\bar{t}}$ the one implied by our approximation. We compute the average EEE across all samples and initial conditions, the maximum error and the percentage of positive and negative errors. We average over 62500 errors.

As in Aruoba et al. (2006) we report the *absolute* errors using base 10 logarithms to make our findings comparable with the rest of the literature. A value of -3 means a 1$ mistake per 1000$, a value of -4 a mistake of $1 per $10000 and so on. Table 5 reports the results.

Table 5 shows that the average of the errors are between -3 and -4, that the percentage of positive errors is close to 50% and that the maximum errors are not large. Moreover, we find that it is quite unlikely that the region of the state space where the maximum error occurs is visited in simulations. These results are well within the range accepted by other authors (e.g. Aruoba et al (2006)). This suggests that the model solutions are accurate.

B.4 Shadow Cost Calculations

In section 6.2 of the paper we presented the results of an approximate "shadow cost" calculation of the loss in utility due to transaction costs. In this section we give details on how we proceeded in

[16] Since the optimal portfolio is determined through 'Forward States' it is not possible to use objects $\Xi_{\bar{t},i}$ to determined new values of $b_{1,\bar{t}}, b_{N,\bar{t}}$.

Table 5: **Euler Equation Errors**

		BB		NBB					
		lending	no lending	lending	no lending	coupons	callable	repurch.	3 bonds
EEE^1	ave	-3.97	-3.72	-3.84	-3.86	-3.89	-3.92	-3.84	-3.77
	max	-2.30	-2.28	-2.50	-2.64	-2.60	-2.51	-2.79	-2.32
	%+	0.41	0.51	0.48	0.44	0.43	0.44	0.48	0.51
EEE^N	ave	-3.18	-3.06	-3.53	-3.51	-3.30	-3.75	-3.18	-3.62
	max	-1.81	-1.93	-2.55	-2.47	-1.99	-2.49	-2.12	-2.74
	%+	0.50	0.54	0.54	0.49	0.57	0.49	0.61	0.42
EEE^{N-1}								-3.23	
								-1.94	
								0.40	
EEE^M	ave								-3.59
	max								-2.30
	%+								0.58

our calculations. As explained in the text we seek to find:

$$\chi = \frac{U^{BB} - U^{NBB}}{\mathcal{T}otal^{BB} - \mathcal{T}otal^{NBB}}$$

where $U^i = E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t^i) + v(x_t^i)\right]$ denotes the total welfare for each model $i = BB, NBB$. $\mathcal{T}otal^i = E_0 \sum_{t=0}^{\infty} \beta^t \left(\lambda_t^i u_{c,t}^i + v_{x,t}^i\right) \mathcal{T}ot_t^i$ is the total shadow transaction cost of buyback or no buyback in term of utility where $\mathcal{T}ot_t^i$ is the total transaction cost at time t for the optimal portfolio.

We then calculated numerically the four elements that determine χ using a mix of short and long run simulations in order to have a good approximation of the infinite sums. Let's take for example $U^{BB} = E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t^{BB}) + v(x_t^{BB})\right]$. In order to approximate this term we first run a long simulation of the buyback model with 100000 periods and calculated $U_{L,\bar{t}}^{BB} = \sum_{t=\bar{t}}^{\overline{T}} \beta^{t-\bar{t}} \left[u(c_t^{BB}) + v(x_t^{BB})\right]$ for every $\bar{t} = 1, ..., \overline{T}$, where $\overline{T} = 100000$. Starting from $\overline{T} = 100000$ we defined $U_{L,\overline{T}}^{BB} = \frac{\overline{U}^{BB}}{1-\beta}$ where $\overline{U}^{BB}$ is the average of $u(c^{BB}) + v(x^{BB})$ over the entire simulation. Then, iterating backwards one period we got $U_{L,\overline{T}-1}^{BB} = u(c_{\overline{T}-1}^{BB}) + v(x_{\overline{T}-1}^{BB}) + \beta U_{L,\overline{T}}^{BB}$. We continued to obtain $U_{L,\bar{t}}^{BB}$ up to $t = 1$.

After dropping the first 100 and the last 2000 periods, we regressed the generated sums on $b_{t-1}^1, b_{t-1}^N, .., b_{t-N}^N, \lambda_{t-1}, .., \lambda_{t-N}$ and g_t. This gave us an approximation $f(b_{t-1}^1, b_{t-1}^N, .., b_{t-N}^N, \lambda_{t-1}, .., \lambda_{t-N}, g_t)$ of the conditional expectation of U^{BB} in t based on the long run simulation. We used this as an 'end point' in the short run simulations.

The short run simulations were carried out as follows: We simulated our models 10000 times for 100 periods starting from the same initial condition. Continuing with the previous example of the buyback model, we calculated $U_{S,i}^{BB} = \sum_{t=1}^{100} \beta^t \left[u(c_{t,i}^{BB}) + v(x_{t,i}^{BB})\right] + \beta^{101} U_{L,i}^{BB}$ for every sample $i = 1, ..., 10000$ where $U_{L,i}^{BB} = f(b_{100,i}^1, b_{100,i}^N, .., b_{100-N,i}^N, \lambda_{100,i}, ..., \lambda_{100-N,i}, g_{100,i})$. Our approximation of U^{BB} is the average of $U_{S,i}^{BB}$ over the 10000 samples.

The above procedure was repeated for all four elements of χ, calibrating the transaction costs as explained in section 6.1 of the paper.

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