The Implied Volatility of Forward Starting Options: ATM Short-Time Level, Skew and Curvature

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Abstract

For stochastic volatility models, we study the short-time behaviour of the at-the-money implied volatility level, skew and curvature for forward-starting options. Our analysis is based on Malliavin Calculus techniques.

Keywords: Forward starting options, implied volatility, Malliavin calculus, stochastic volatility models
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1 Introduction

Consider two moment times $s > t$. A forward-start call option with maturity $T > s$ allows the holder to receive, at time s and with no additional cost, a call option expirying at T, with strike set equal to KS_s, for some $K > 0$. So, the option life starts at s, but the holder pays at time t the price of the option. Some classical applications of forward starting options include employee stock options and cliquet options, among others (see for example Rubinstein (1991)).

Under the Black-Scholes formula, a conditional expectation argument leads to show that the price of a forward starting option is the price of a plain vanilla

option with time to maturity $T - t$. In the stochastic volatility case, a change-of-measure links the price of the forward option with the price of a classical vanilla (see, for example, Rubinstein (1991), Musiela and Rutkowski (1997), Wilmott (1998), or Zhang (1998)). For stochastic volatility models, change of numeraire techniques can be applied to obtain a closed-form pricing formula in the context of the Heston model (see Kruse and Nögel (2005)).

The implied volatility surface for forward starting options exhibits substantial differences to the classical vanilla case (see for example Jacquier and Roome (2015)). This paper is devoted to the study of the at-the-money (ATM) short time limit of the implied volatility for forward starting options. More precisely, we will use Malliavin Calculus techniques to compute the ATM short-time limit of the implied volatility level, skew and curvature. In particular, we will see that -contrary to the classical vanilla case- the ATM short-time level depends on the correlation parameter (see Lemma 6 and Theorem 7). We will also prove that the skew depends of the Malliavin derivative of the volatility process in a similar way as for vanilla options, while the curvature (see Theorems 14 and 15) is of order $O(T - s)$.

The paper is organized as follows. Section 2 is devoted to introduce forward starting options and the main notation used troghout the paper. In Section 3 we obtain a decomposition of the option price that will allow us to compute, in Sections 4, 5 and 6, the limits for the ATM implied volatility level, skew and curvature.

2 Forward start option

We will consider the Heston model for stock price on a time interval $[0, T]$ under a risk neutral probability P^* :

$$dS_t = \hat{r} S_t dt + \sigma_t S_t \left(\rho dW_t^* + \sqrt{1 - \rho^2} B_t^* \right), \, t \in [0, T], \tag{1}$$

where $\hat{r}$ is the instantaneous interest rate (supposed to be constant), W^* and B^* are independent standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P^*)$ and σ is a positive and square-integrable process adapted to the filtration generated by W^*. In the following we will denote by $\mathcal{F}^{W^*}$ and $\mathcal{F}^{B^*}$ the filtrations generated by W^* and B^*, respectively. Moreover we define $\mathcal{F} := \mathcal{F}^{W^*} \vee \mathcal{F}^{B^*}$. It will be convenient in the following sections to make the change of variable $X_t = \log(S_t), \, t \in [0, T]$. Now, we consider a point $s \in [0, T]$. We want to evaluate the following option price:

$$V_t = e^{-\hat{r}(T-t)} E_t^* \left(e^{X_T} - e^{\alpha} e^{X_s} \right)_+, \tag{2}$$

where E^* denotes the conditional expectation given $\mathcal{F}_t$ and α is a real constant. Notice that if $t \geq s$ this is the payoff of a call option, while in the case $t < s$ this defines a forward start option.

We will make use of the following notation

- $BS(t, x, K, \sigma)$ will represent the price of a European call option under the classical Black-Scholes model with constant volatility σ, current log stock price x, time to maturity $T - t$, strike price K and interest rate $\hat{r}$. Remember that in this case

$$BS(t, x, K, \sigma) = e^x N(d_+) - e^{-\hat{r}(T-t)} K N(d_-),$$

where N is the cumulative probability function of the standard normal law and

$$d_{\pm} := \frac{x - \ln K + \hat{r}(T - t)}{\sigma\sqrt{T - t}} \pm \frac{\sigma}{2}\sqrt{T - t}.$$

- $\mathcal{L}_{BS}(\sigma)$ will denote the Black-Scholes differential operator (in the log variable) with volatility σ :

$$\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} + \left(\hat{r} - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x} - \hat{r}.$$

It is well known that $\mathcal{L}_{BS}(\sigma) BS(\cdot, \cdot; K, \sigma) = 0$.

- $G(t, x, K, \sigma) := (\partial_{xx}^2 - \partial_x)BS(t, x, K, \sigma)$.

- $H(t, x, K, \sigma) := \partial_x(\partial_{xx}^2 - \partial_x)BS(t, x, K, \sigma)$.

- $BS^{-1}(a) := BS^{-1}(t, x, K, a)$ denotes the inverse of BS as a function of the volatility parameter.

- $\alpha^* := \hat{r}(T - s)$

We recall the following result, that can be proved following the same arguments as in that of Lemma 4.1 in Alòs, León and Vives (2007)

Lemma 1 *Let $0 \le t \le s, u < T$ and $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_T^W$. Then for every $n \ge 0$, there exists $C = C(n, \rho)$ such that*

a) If $u \le s$

$$|E\left(\partial_x^n G\left(u, X_u, M_u, v_u\right)|\mathcal{G}_t\right)| \le CE\left(e^{X_u}|\mathcal{G}_t\right)\left(\int_s^T \sigma_s^2 ds\right)^{-\frac{1}{2}(n+1)}.$$

b) If $u \ge s$

$$|E\left(\partial_x^n G\left(u, X_u, M_u, v_u\right)|\mathcal{G}_t\right)| \le CE\left(e^{X_s}|\mathcal{G}_t\right)\left(\int_s^T \sigma_s^2 ds\right)^{-\frac{1}{2}(n+1)}.$$

3 A decomposition formula for forward option prices

We will stand for

- $M_t := E_t^* \left(e^\alpha e^{X_s} \right)$, $t \in [0, T]$. We observe that

$$
\begin{aligned}
M_t &= e^\alpha e^{\hat{r}s} e^{X_0} + \int_0^t \sigma_u 1_{[0,s]}(u) e^\alpha e^{\hat{r}(s-u)} e^{X_u} \left(\rho dW_u^* + \sqrt{1-\rho^2} dB_u^* \right) \\
&= M_0 + e^\alpha \int_0^{t \wedge s} \sigma_u e^{\hat{r}(s-u)} e^{X_u} \left(\rho dW_u^* + \sqrt{1-\rho^2} dB_u^* \right).
\end{aligned}
$$

- $v_t := \left(\frac{Y_t}{T-t} \right)^{\frac{1}{2}}$, with

$$
Y_t := \int_t^T \sigma_u^2 1_{[s,T]}(u) du = \int_{t \vee s}^T \sigma_u^2 du
$$

Notice that, if $t < s$, $v_t \sqrt{T-t} = v_s \sqrt{T-s}$.

We will need the following hypotheses:

(H1) There exist two constants $0 < c < C$ such that $c < \sigma_t < C$, for all $t \in [0, T]$, with probability one.

(H2) $\sigma, \sigma e^X \in \mathbb{L}^{2,2} \cap \mathbb{L}^{1,4}$.

Now we are in a position to prove the following theorem, that allows us to identify the impact of correlation in the forward option price

Theorem 2 *Consider the model (1) and assume that hypotheses (H1) and (H2) hold. Then, for all $0 \leq t \leq s \leq T$,*

$$
\begin{aligned}
V_t = \; & E_t^* \Bigg[\exp(X_t) BS\left(s, 0, e^\alpha, v_s\right) \\
& + \frac{\rho}{2} \int_s^T e^{-\hat{r}(u-t)} H(u, X_u, M_u, v_u) \sigma_u \Lambda_u^{W^*} du \\
& + \frac{\rho}{2} G(s, 0, e^\alpha, v_s) \int_t^s e^{-\hat{r}(u-t)} e^{X_u} \sigma_u \Lambda_u^{W^*} du \Bigg].
\end{aligned}
$$

where $\Lambda_u^{W^*} := \int_{u \vee s}^T D_u^{W^*} \sigma_\theta^2 d\theta$.

Proof. Remember that $M_t := E_t^* \left(e^\alpha e^{X_s} \right)$ and notice that

$$
\begin{aligned}
V_t &= e^{-\hat{r}(T-t)} E_t^* \left(e^{X_T} - e^\alpha e^{X_s} \right)_+ = e^{-\hat{r}(T-t)} E_t^* \left(e^{X_T} - M_T \right)_+ \\
&= e^{-\hat{r}(T-t)} BS(T, X_T, M_T, \nu_T).
\end{aligned}
$$

Therefore, the anticipating Itô's formula for the Skorohod integral (see for example Nualart (2006)) allows us to write

$$E_t^* \left(e^{-\hat{r}T} BS(T, X_T, M_T, v_T) \right)$$

$$= E_t^* \left[e^{-\hat{r}t} BS(t, X_t, M_t, v_t) - \hat{r} \int_t^T e^{-\hat{r}u} BS(u, X_u, M_u, v_u) du \right.$$

$$+ \int_t^T e^{-\hat{r}u} \frac{\partial BS}{\partial u}(u, X_u, M_u, v_u) du$$

$$+ \int_t^T e^{-\hat{r}u} \frac{\partial BS}{\partial x}(u, X_u, M_u, v_u) \left(\hat{r} - \frac{\sigma_u^2}{2} \right) du$$

$$+ \frac{1}{2} \int_t^T e^{-\hat{r}u} \frac{\partial^2 BS}{\partial x^2}(u, X_u, M_u, v_u) \sigma_u^2 du$$

$$+ \int_t^T e^{-\hat{r}u} \frac{\partial^2 BS}{\partial x \partial K}(u, X_u, M_u, v_u) d\langle X, M \rangle_u$$

$$+ \frac{1}{2} \int_t^T e^{-\hat{r}u} \frac{\partial^2 BS}{\partial K^2}(u, X_u, M_u, v_u) d\langle M, M \rangle_u$$

$$+ \frac{1}{2} \int_t^T e^{-\hat{r}u} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u) \left(v_u^2 - \sigma_u^2 1_{]s,T]}(u) \right) du$$

$$+ \frac{1}{2} \int_t^T e^{-\hat{r}u} \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u) \sigma_u \rho \Lambda_u^{W^*} du$$

$$+ \frac{1}{2} \int_t^T e^{-\hat{r}u} \frac{\partial}{\partial K} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u) \sigma_u e^{\alpha} e^{\hat{r}(s-u)} e^{X_u} \rho \Lambda_u^{W^*} 1_{[0,s]}(u) du \left. \right].$$

That is, using $t \leq s$,

$$
\begin{aligned}
V_t \;=\; & E_t^* \Bigg[BS(t, X_t, M_t, v_t) \\
& + \int_t^T e^{-\hat{r}(u-t)} \mathcal{L}_{BS}(v_u)\, BS(u, X_u, M_u, v_u)\, du \\
& + \frac{1}{2} \int_t^T e^{-\hat{r}(u-t)} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u)\, (\sigma_u^2 - v_u^2)\, du \\
& + \int_t^T e^{-\hat{r}(u-t)} \frac{\partial^2 BS}{\partial x \partial K}(u, X_u, M_u, v_u)\, d\langle X, M\rangle_u \\
& + \frac{1}{2} \int_t^T e^{-\hat{r}(u-t)} \frac{\partial^2 BS}{\partial K^2}(u, X_u, M_u, v_u)\, d\langle M, M\rangle_u \\
& + \frac{1}{2} \int_t^T e^{-\hat{r}(u-t)} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u)\, \left(v_u^2 - \sigma_u^2 1_{]s,T]}(u) \right) du \\
& + \frac{1}{2} \int_t^T e^{-\hat{r}(u-t)} \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u) \sigma_u \rho \Lambda_u^{W^*}\, du \\
& + \frac{1}{2} \int_t^s e^{-\hat{r}(u-t)} \frac{\partial}{\partial K}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u) \sigma_u e^{\alpha} e^{\hat{r}(s-u)} e^{X_u} \rho \Lambda_u^{W^*}\, du \Bigg].
\end{aligned}
$$

Thus, we get, for $t \leq s$,

$$
\begin{aligned}
V_t \;=\; & E_t^* \Bigg[BS(t, X_t, M_t, v_t) \\
& + \int_t^T e^{-\hat{r}(u-t)} \mathcal{L}_{BS}(v_u)\, BS(u, X_u, M_u, v_u)\, du \\
& + \frac{1}{2} \int_t^T e^{-\hat{r}(u-t)} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u)\, \left(\sigma_u^2 - \sigma_u^2 1_{]s,T]}(u) \right) du \\
& + \int_t^T e^{-\hat{r}(u-t)} \frac{\partial^2 BS}{\partial x \partial K}(u, X_u, M_u, v_u)\, d\langle X, M\rangle_u \\
& + \frac{1}{2} \int_t^T e^{-\hat{r}(u-t)} \frac{\partial^2 BS}{\partial K^2}(u, X_u, M_u, v_u)\, d\langle M, M\rangle_u \\
& + \frac{1}{2} \int_t^T e^{-\hat{r}(u-t)} \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u) \sigma_u \rho \Lambda_u^{W^*}\, du \\
& + \frac{1}{2} \int_t^s e^{-\hat{r}(u-t)} \frac{\partial}{\partial K}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u) \sigma_u e^{\alpha} e^{\hat{r}(s-u)} e^{X_u} \rho \Lambda_u^{W^*}\, du \Bigg].
\end{aligned}
$$

Now, taking into account the facts that

$$
\mathcal{L}_{BS}(v_u)(BS)(u, X_u, M_u, v_u) = 0,
$$

$$
d\langle M, X\rangle_u = \sigma_u^2 e^{\alpha} e^{\hat{r}(s-u)} e^{X_u} 1_{[0,s]}(u)\, du,
$$

6

$$d \langle M, M \rangle_u = \sigma_u^2 e^{2\alpha} e^{2\hat{r}(s-u)} e^{2X_u} 1_{[0,s]}(u) du,$$

$$\frac{\partial^2 BS}{\partial x \partial K}(t, x, K, \sigma) = -\frac{1}{K} \left(\frac{\partial^2 BS}{\partial x^2} - \frac{\partial BS}{\partial x} \right)(t, x, K, \sigma)$$

and

$$\frac{\partial^2 BS}{\partial K^2}(t, x, K, \sigma) = \frac{1}{K^2} \left(\frac{\partial^2 BS}{\partial x^2} - \frac{\partial BS}{\partial x} \right)(t, x, K, \sigma),$$

it follows that

$$
\begin{aligned}
V_t = E_t^* \Bigg[& BS(t, X_t, M_t, v_t) \\
& + \frac{1}{2} \int_t^T e^{-\hat{r}(u-t)} \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u) \sigma_u \rho \Lambda_u^{W^*} du \\
& + \frac{1}{2} \int_t^s e^{-\hat{r}(u-t)} \frac{\partial}{\partial K} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u) \sigma_u e^{\alpha} e^{\hat{r}(s-u)} e^{X_u} \rho \Lambda_u^{W^*} du \Bigg] \\
= E_t^* \Bigg[& BS(t, X_t, M_t, v_t) \\
& + \frac{1}{2} \int_s^T e^{-\hat{r}(u-t)} \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u) \sigma_u \rho \Lambda_u^{W^*} du \\
& + \frac{1}{2} \int_t^s e^{-\hat{r}(u-t)} \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u) \sigma_u \rho \Lambda_u^{W^*} du \\
& + \frac{1}{2} \int_t^s e^{-\hat{r}(u-t)} \frac{\partial}{\partial K} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(u, X_u, M_u, v_u) \sigma_u M_u \rho \Lambda_u^{W^*} du \Bigg].
\end{aligned}
$$

Since we have

$$\frac{\partial}{\partial x} \left(\frac{\partial^2 BS}{\partial x^2} - \frac{\partial BS}{\partial x} \right)(t, x, k, \sigma) = \frac{e^x N'(d_+)}{\sigma \sqrt{T-t}} \left(1 - \frac{d_+}{\sigma \sqrt{T-t}} \right)$$

and

$$\frac{\partial}{\partial K} \left(\frac{\partial^2 BS}{\partial x^2} - \frac{\partial BS}{\partial x} \right)(t, x, k, \sigma) = \frac{e^x N'(d_+)}{K \sigma \sqrt{T-t}} \left(\frac{d_+}{\sigma \sqrt{T-t}} \right),$$

then it is easy to see that

$$
\begin{aligned}
V_t = E_t^* \Bigg[& BS(t, X_t, M_t, v_t) \\
& + \frac{\rho}{2} \int_s^T e^{-\hat{r}(u-t)} H(u, X_u, M_u, v_u) \sigma_u \Lambda_u^{W^*} du \\
& + \frac{\rho}{2} \int_t^s e^{-\hat{r}(u-t)} G(u, X_u, M_u, v_u) \sigma_u \Lambda_u^{W^*} du \Bigg].
\end{aligned}
$$

Hence, due to

$$
\begin{aligned}
& BS(t, X_t, M_t, v_t) \\
= \; & \exp(X_t) N \left(\frac{-\alpha + \hat{r}(T-s)}{v_s\sqrt{T-s}} + \frac{v_s\sqrt{T-s}}{2} \right) \\
& - e^\alpha \exp(X_t) e^{-\hat{r}(T-s)} N \left(\frac{-\alpha + \hat{r}(T-s)}{v_s\sqrt{T-s}} - \frac{v_s\sqrt{T-s}}{2} \right) \\
= \; & \exp(X_t) BS \left(s, 0, e^\alpha, v_s \right)
\end{aligned}
$$

and, for all $u < s$,

$$
\begin{aligned}
G(u, X_u, M_u, v_u) \; & = \; \frac{e^{X_u} N' \left(\frac{-\alpha + \hat{r}(T-s)}{v_s\sqrt{T-s}} + \frac{v_s\sqrt{T-s}}{2} \right)}{v_s\sqrt{T-s}} \\
& = \; e^{X_u} G(s, 0, e^\alpha, v_s),
\end{aligned}
$$

the proof is complete. ∎

Remark 3 *We have chosen Hypotheses (H1) and (H2) for the sake of simplicity, which can be substituted by adequate integrability conditions.*

Remark 4 *Notice that, if $t = s$ we recover the decomposition formula for call option prices presented in Alòs, León and Vives (2007).*

Remark 5 *If the volatility process is constant ($\sigma_u = \sigma$, for some positive constant σ), then $v_s = \sigma$ and $\Lambda^W \equiv 0$, which implies that, for $t \leq s$,*

$$
V_t = \exp(X_t) BS \left(s, 0, e^\alpha, \sigma \right). \tag{3}
$$

Consequently, we recover the well-known option pricing formula for forward start options under the Black-Scholes model (see for example Rubinstein (1991), Wilmott (1998), or Zhang (1998)).

4   The ATM short-time limit of the implied volatility.

For $t \in [0, s]$, we define the implied volatility $I(t, s; \alpha)$ as the $\mathcal{F}_t$-adapted process satisfying

$$
V_t = \exp(X_t) BS \left(s, 0, e^\alpha, I(t, s; \alpha) \right). \tag{4}
$$

Notice that, from (3), in the constant volatility case $\sigma_u = \sigma$, $I(t, s; \alpha) = \sigma$.

For the sake of simplicity, we will consider first the uncorrelated case $\rho = 0$. In this case, we have the following result

Lemma 6 *Consider the model (1) with $\rho = 0$ and assume that hypotheses in Theorem 2 hold. Then, for all $t < s$,*

$$\lim_{T \to s} I(t, s; \alpha^*) = E_t^* (\sigma_s).$$

Proof. We know that

$$
\begin{aligned}
I(t, s; \alpha^*) \\
&= BS^{-1}(E_t^*(BS\,(s, 0, \exp(\alpha^*), v_s))) \\
&= E_t^* \left[BS^{-1}(BS\,(s, 0, \exp(\alpha^*), v_s)) \right. \\
&\quad \left. + BS^{-1}(E_t^*(BS\,(s, 0, \exp(\alpha^*), v_s))) - BS^{-1}(BS\,(s, 0, \exp(\alpha^*), v_s)) \right] \\
&= E_t^* \left[v_s + BS^{-1}(E_t^*(BS\,(s, 0, \exp(\alpha^*), v_s))) \right. \\
&\quad \left. - BS^{-1}(BS\,(s, 0, \exp(\alpha^*), v_s)) \right].
\end{aligned}
$$

Notice that a direct application of Clark-Ocone's formula gives us that

$$BS(s, 0, \exp(\alpha^*), v_s) = E_t^* \left(BS(s, 0, \exp(\alpha^*), v_s)\right) + \int_t^T U_r dW_r,$$

with

$$U_u = E_u^* \left(\left(\frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), v_s) \right) \frac{D_u^{W^*} \int_s^T \sigma_r^2 dr}{2(T - s) v_s} \right). \tag{5}$$

Then, applying Itô's formula to the process $BS^{-1}(A_u)$, where

$$A_u := E_t^* \left(BS(s, 0, \exp(\alpha^*), v_s)\right) + \int_t^u U_r dW_r$$

and taking expectations, we get

$$
\begin{aligned}
&E_t^* \left[BS^{-1}(E_t^*(BS\,(s, 0, \exp(\alpha^*), v_s))) - BS^{-1}(BS\,(s, 0, \exp(\alpha^*), v_s)) \right] \\
&= -\frac{1}{2} E_t^* \int_t^T (BS^{-1})''(E_r(BS\,(s, 0, \exp(\alpha^*), v_s))) U_r^2 dr.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\lim_{T \to s} I(t, s, \alpha^*) \\
&= E_t^* (\sigma_s) \\
&\quad - \lim_{T \to s} \frac{1}{2} E_t^* \int_t^T (BS^{-1})''(E_r(BS\,(s, 0, \exp(\alpha^*), v_s))) U_r^2 dr.
\end{aligned}
$$

Now, considering that

$$
\begin{aligned}
&(BS^{-1})''(E_r(BS\,(s, 0, \exp(\alpha^*), v_s)) \\
&= \frac{1}{\left(N'(d_1)\sqrt{T - s}\right)^2} \left(\frac{BS^{-1}(E_r(BS\,(s, 0, \exp(\alpha^*), v_s))(T - s)}{4} \right),
\end{aligned}
$$

where $d_1 := \frac{BS^{-1}(E_r(BS(s,0,\exp(\alpha^*),v_s))\sqrt{T-s}}{2}$, we get that

$$\lim_{T \to s} \frac{1}{2} E_t^* \int_t^T (BS^{-1})''(E_r(BS\,(s,0,\exp(\alpha^*),v_s))U_r^2 dr = 0.$$

Therefore, the proof is complete. $\blacksquare$

As a consequence of the previous result, we have the following limit in the correlated case.

Theorem 7 *Consider the model (1) and assume that hypotheses in Theorem 2 hold. Then, for all $t < s$,*

$$\lim_{T \to s} I(t,s;\alpha^*) = E_t^*(\sigma_s) + \frac{\rho}{2} e^{-X_t} E_t^* \left(\frac{1}{\sigma_s} \int_t^s e^{-\hat{r}(u-t)} e^{X_u} \sigma_u D_u^{W^*} \sigma_s^2 du \right).$$

Proof. We know, by Theorem 2, that

$$
\begin{aligned}
I(t,s;\alpha^*) \\
&= BS^{-1} [E_t^*(BS\,(s,0,\exp(\alpha^*),v_s)) \\
&\quad + \frac{\rho}{2} \exp(-X_t) E_t^* \int_s^T e^{-\hat{r}(u-t)} H(u,X_u,M_u,v_u)\sigma_u \Lambda_u^{W^*} du \\
&\quad + \frac{\rho}{2} \exp(-X_t) E_t^* \left(G(s,0,\exp(\alpha^*),v_s) \int_t^s e^{-\hat{r}(u-t)} e^{X_u} \sigma_u \Lambda_u^{W^*} du \right)].
\end{aligned}
$$

By the mean value theorem, we can find θ between $E_t^*(BS(s,0,\exp(\alpha^*),v_s))$ and

$$
\begin{aligned}
E_t^* &\left(BS\,(s,0,\exp(\alpha^*),v_s) \right. \\
&\quad + \frac{\rho}{2} \exp(-X_t \int_s^T e^{-\hat{r}(u-t)} H(u,X_u,M_u,v_u)\sigma_u \Lambda_u^{W^*} du \\
&\quad + \left. \frac{\rho}{2} \exp(-X_t G(s,0,\exp(\alpha^*),v_s) \int_t^s e^{-\hat{r}(u-t)} e^{X_u} \sigma_u \Lambda_u^{W^*} du \right)
\end{aligned}
$$

such that

$$
\begin{aligned}
I(t,s;\alpha^*) &- BS^{-1} \left(E_t^*(BS(s,0,\exp(\alpha^*),v_s)) \right) \\
&= \frac{\sqrt{2\pi} \exp(\frac{BS^{-1}(\theta)}{8}(T-s))}{\sqrt{T-s}} \left[\frac{\rho}{2} \exp(-X_t) \right. \\
&\qquad \times E_t^* \left(\int_s^T e^{-\hat{r}(u-t)} H(u,X_u,M_u,v_u)\sigma_u \Lambda_u^{W^*} du \right) \\
&\qquad + \left. \frac{\rho}{2} E_t^* \left(G(s,0,\exp(\alpha^*),v_s) \int_t^s e^{-\hat{r}(u-t)} e^{X_u} \sigma_u \Lambda_u^{W^*} du \right) \right] \\
&= I_1 + I_2.
\end{aligned}
\tag{6}
$$

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From Lemma 1, we have

$$
\begin{aligned}
\lim_{T \to s} |I_1| & \leq & C \lim_{T \to s} \frac{\exp(-X_t)}{\sqrt{T-s}} \int_s^T E^* \left(e^{X_s} \,|\mathcal{G}_t\right) (T-s)^{-1} \left| \Lambda_u^{W^*} \right| du \\
& \leq & C \exp(-X_t) \lim_{T \to s} (T-s)^{1/2} \left(E^* \left(e^{2X_s} \,|\mathcal{G}_t\right) \right)^{1/2} = 0.
\end{aligned}
$$

Finally, (6) and Lemma 6 imply

$$
\begin{aligned}
& \lim_{T \to s} I(t,s;\alpha^*) \\
& = & E_t^*(\sigma_s) + \frac{\rho \exp(X_t)}{2} \lim_{T \to s} E_t^* \left(\frac{\exp(\frac{v_s^2(T-s)}{8})}{v_s(T-s)} \int_t^s e^{-\hat{r}(u-t)} e^{X_u} \sigma_u \Lambda_u^{W^*} du \right) \\
& = & E_t^*(\sigma_s) + \frac{\rho \exp(X_t)}{2} E_t^* \left(\frac{1}{\sigma_s} \int_t^s e^{-\hat{r}(u-t)} e^{X_u} \sigma_u D_u^{W^*} \sigma_s^2 du \right).
\end{aligned}
$$

Therefore, the proof is complete. ∎

Remark 8 *Notice that, in the case $s = t$, we recover the results by Durrleman (2008) for classical vanilla options. Also, contrary to the classical vanilla case, the ATM short-time limit depends on the correlation parameter.*

We will need the following lemma later on.

Lemma 9 *Consider the model (1) and assume that $E \left(\exp \left(\frac{\int_0^T \sigma_u^2 du}{2} \right) \right) < \infty.$ Then*

$$
I(t,s;\alpha^*)\sqrt{T-s} \to 0
$$

as $T \to s$.

Proof. We know that, in this case,

$$
V_t = e^{-\hat{r}(T-t)} E_t^* \left[\left(e^{X_T} - e^{\hat{r}(T-s)} e^{X_s} \right)_+ \right],
$$

which converges to zero as $T \to s$. Indeed, we have

$$
\left(e^{X_T} - e^{\hat{r}(T-s)} e^{X_s} \right)_+ \to 0,
$$

as $T \to s$ with probability 1. Moreover, by Novikov theorem (see for example Karatzas and Shreve (1991)), we have

$$
\left(e^{X_T} - e^{\hat{r}(T-s)} e^{X_s} \right)_+ \leq e^{X_T} + e^{\hat{r}(T-s)} e^{X_s}
$$

and

$$
E^* \left(e^{X_T} + e^{\hat{r}(T-s)} e^{X_s} \right) = 2e^{\hat{r}T} \to 2e^{\hat{r}s}
$$

as $T \to s$. Thus, the dominated convergence theorem implies that $V_t \to 0$ as $T \to s$, with probability one. On the other hand,

$$
\begin{aligned}
V_t &= \exp(X_t) \left(N \left(\frac{I(t, s; \alpha^*)\sqrt{T-s}}{2} \right) - N \left(-\frac{I(t, s; \alpha^*)\sqrt{T-s}}{2} \right) \right) \\
&= \exp(X_t) \left(1 - 2N \left(-\frac{I(t, s; \alpha^*)\sqrt{T-s}}{2} \right) \right),
\end{aligned}
$$

which allows us to complete the proof. ∎

5 An expression for the derivative of the implied volatility

Equation (4) leads us to get

$$
\begin{aligned}
\frac{\partial V_t}{\partial \alpha} &= \exp(X_t)\frac{\partial BS}{\partial k} (s, 0, e^\alpha, I(t, s; \alpha)) \\
&\quad + \exp(X_t)\frac{\partial BS}{\partial \sigma} (s, 0, e^\alpha, I(t, s; \alpha)) \frac{\partial I}{\partial \alpha}(t, s; \alpha),
\end{aligned}
$$

where $\frac{\partial BS}{\partial k} := \frac{\partial BS}{\partial (\ln K)}$.

Then, from Theorem 2 we are able to write

$$
\begin{aligned}
&\frac{\partial I}{\partial \alpha}(t, s; \alpha) \\
&= \frac{\frac{\partial V_t}{\partial \alpha} - \exp(X_t)\frac{\partial BS}{\partial k} (s, 0, e^\alpha, I(t, s; \alpha))}{\exp(X_t)\frac{\partial BS}{\partial \sigma} (s, 0, e^\alpha, I(t, s; \alpha))} \\
&= \frac{E_t^* \left[\frac{\partial BS}{\partial k} (s, 0, e^\alpha, v_s) \right] - \frac{\partial BS}{\partial k} (s, 0, e^\alpha, I(t, s; \alpha))}{\frac{\partial BS}{\partial \sigma} (s, 0, e^\alpha, I(t, s; \alpha))} \\
&\quad + \frac{\rho}{2} \frac{E_t^* \left[\int_s^T e^{-\hat{r}(u-t)} \frac{\partial H}{\partial k} (u, X_u, e^\alpha e^{X_s}, v_u) \sigma_u \Lambda_u^{W^*} du \right]}{\exp(X_t)\frac{\partial BS}{\partial \sigma} (s, 0, e^\alpha, I(t, s; \alpha))} \\
&\quad + \frac{\rho}{2} \frac{E_t^* \left[\frac{\partial G}{\partial k} (s, 0, e^\alpha, v_s) \int_t^s e^{X_u} e^{-\hat{r}(u-t)} \sigma_u \Lambda_u^{W^*} du \right]}{\exp(X_t)\frac{\partial BS}{\partial \sigma} (s, 0, e^\alpha, I(t, s; \alpha))}.
\end{aligned}
\qquad (7)
$$

Remark 10 *Note that in the uncorrelated case (i.e., $\rho = 0$), equality (7) implies*

$$
\frac{\partial I}{\partial \alpha}(t, s; \alpha) = \frac{E_t^* \left[\frac{\partial BS}{\partial k} (s, 0, e^\alpha, v_s) \right] - \frac{\partial BS}{\partial k} (s, 0, e^\alpha, I(t, s; \alpha))}{\frac{\partial BS}{\partial \sigma} (s, 0, e^\alpha, I(t, s; \alpha))}.
$$

In this case, for $\alpha = \alpha^*$, we get, by Theorem 2,

$$E_t^* \left[\frac{\partial BS}{\partial k} (s, 0, \exp(\alpha^*), v_s) \right]$$

$$= E_t^* \left[-N \left(-\frac{v_s \sqrt{T-s}}{2} \right) \right]$$

$$= E_t^* \left[\frac{N \left(\frac{v_s \sqrt{T-s}}{2} \right) - N \left(-\frac{v_s \sqrt{T-s}}{2} \right) - 1}{2} \right]$$

$$= E_t^* \left[\frac{BS(s, 0, \exp(\alpha^*), v_s) - 1}{2} \right]$$

$$= \frac{V_t \exp(-X_t) - 1}{2}$$

and

$$\frac{\partial BS}{\partial k} (s, 0, \exp(\alpha^*), I(t, s; \alpha^*))$$

$$= -N \left(-\frac{I(t, s; \alpha^*) \sqrt{T-s}}{2} \right)$$

$$= \frac{V_t \exp(-X_t) - 1}{2}.$$

So, we conclude that

$$E_t^* \left[\frac{\partial BS}{\partial k} (s, 0, \exp(\alpha^*), v_s) \right] - \frac{\partial BS}{\partial k} (s, 0, \exp(\alpha^*), I(t, s; \alpha^*)) = 0$$

and, consequently, $\frac{\partial I}{\partial \alpha}(t, s; \alpha^*) = 0$.

We will need the following hypothesis

(H3) There exist two constants $C > 0$ and $\delta \geq 0$ such that, for all $t \leq \theta < u < r$

$$E_t^* \left(\left(D_u^{W^*} \sigma_r \right)^2 \right) \leq C(r - u)^{2\delta} \tag{8}$$

and

$$E_t^* \left(\left(D_\theta^{W^*} D_s^{W^*} \sigma_r \right)^2 \right) \leq C (r - s)^{2\delta} (r - \theta)^{-2\delta}. \tag{9}$$

Theorem 11 *Consider the model (1), and assume that hypotheses (H1), (H2) and (H3) hold and that here exists a $\mathcal{F}_s$-measurable random variable $D_s^+ \sigma_s$ such that*

$$E_s^* \left(\sup_{s \leq u \leq \theta \leq T} \left| E_u^* \left(D_u^{W^*} \sigma_\theta^2 - D_s^+ \sigma_s^2 \right) \right|^4 \right) \to 0 \text{ as } T \to s. \tag{10}$$

Then, for all $s < t$,

$$\lim_{T \to s} \frac{\partial I}{\partial \alpha}(t, s; \alpha^*) = \frac{\rho \exp(-\hat{r}(s - t))}{4 \exp(X_t)} E_t^* \left(e^{X_s} \frac{D_s^+ \sigma_s^2}{\sigma_s^2} \right).$$

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Proof. From (7) we obtain

$$\frac{\partial I}{\partial \alpha}(t, s; \alpha^*)$$

$$= \frac{E_t^* \left[\frac{\partial BS}{\partial k}(s, 0, \exp(\alpha^*), v_s) \right] - \frac{\partial BS}{\partial k}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))}{\frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))}$$

$$+ \frac{\frac{\rho}{2} E_t^* \left[\int_s^T e^{-\hat{r}(u-t)} \frac{\partial H}{\partial k}(u, X_u, M_u, v_u) \sigma_u \Lambda_u^{W^*} du \right]}{\exp(X_t) \frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))}$$

$$+ \frac{\frac{\rho}{2} E_t^* \left[\frac{\partial G}{\partial k}(s, 0, \exp(\alpha^*), v_s) \int_t^s e^{X_u} e^{-\hat{r}(u-t)} \sigma_u \Lambda_u^{W^*} du \right]}{\exp(X_t) \frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))}$$

$$= T_1 + T_2 + T_3.$$

Proceeding as in Remark 10 and using Theorem 2, we get

$$E_t^* \left[\frac{\partial BS}{\partial k}(s, 0, \exp(\alpha^*), v_s) \right] - \frac{\partial BS}{\partial k}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))$$

$$= E_t^* \left[\frac{BS(s, 0, \exp(\alpha^*), v_s) - 1}{2} \right] - \frac{V_t \exp(-X_t) - 1}{2}$$

$$= -\frac{\exp(-X_t)}{2} E_t^* \left\{ \frac{\rho}{2} \int_s^T e^{-\hat{r}(u-t)} H(u, X_u, M_u, v_u) \sigma_u \Lambda_u^{W^*} du |_{\alpha=\alpha^*} \right.$$

$$\left. - \frac{\exp(-X_t) \rho}{4} G(s, 0, \exp(\alpha^*), v_s) \int_t^s e^{-\hat{r}(u-t)} \sigma_u e^{X_u} \Lambda_u^{W^*} du \right\}.$$

Then,

$$\lim_{T \to s} T_1 = \lim_{T \to s} I_1 + \lim_{T \to s} I_2,$$

where

$$I_1 = -\frac{\exp(-X_t) \rho E_t^* \left(\int_s^T e^{-\hat{r}(u-t)} H(u, X_u, M_u, v_u) \sigma_u \Lambda_u^{W^*} du \right) |_{\alpha=\alpha^*}}{4 \frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))}$$

and

$$I_2 = -\frac{\exp(-X_t) \rho E_t^* \left(G(s, 0, \exp(\alpha^*), v_s) \int_t^s e^{-\hat{r}(u-t)} \sigma_u e^{X_u} \Lambda_u^{W^*} du \right)}{4 \frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))}.$$

It is easy to see that, under (H3), $\lim_{T \to s} I_1 = 0$ due to Lemma 1. On the other hand,

$$G(s, 0, \exp(\alpha^*), v_s) = 2 \frac{\partial G}{\partial k}(s, 0, \exp(\alpha^*), v_s),$$

which yields

$$I_2$$

$$= -\frac{\rho}{2} \frac{\exp(-X_t) E_t^* \left(\frac{\partial G}{\partial k}(s, 0, \exp(\alpha^*), v_s) \int_t^s e^{-\hat{r}(u-t)} e^{X_u} \sigma_u \Lambda_u^{W^*} du \right)}{\frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))}$$

$$= -T_3.$$

This gives us that $I_2 + T_3 = 0$. On the other hand, using Lemmas 1 and 9, and the anticipating Itô's formula again, it follows that

$$
\begin{aligned}
\lim_{T \to s} T_2 &= \lim_{T \to s} \frac{\frac{\rho}{2} E_t^* \left[\int_s^T e^{-\hat{r}(u-t)} \frac{\partial H}{\partial k}(u, X_u, M_u, v_u) \sigma_u \Lambda_u^{W^*} du \right]}{\exp(X_t) \frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))} \\
&= \frac{\rho}{2} \lim_{T \to s} \frac{E_t^* \left[\frac{\partial H}{\partial k}(s, X_s, M_s, v_s) e^{-\hat{r}(s-t)} \int_s^T \sigma_u \Lambda_u^{W^*} du \right]}{\exp(X_t) \frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))} \\
&= \frac{\rho}{2} \lim_{T \to s} E_t^* \left[\frac{\exp(X_s) \exp(-\hat{r}(s-t))}{\exp(X_t) v_s^3 (T-s)^2} \int_s^T \sigma_u \Lambda_u^{W^*} du \right] \\
&= \frac{\rho}{2 \exp(X_t)} \lim_{T \to s} E_t^* \left[\frac{\exp(X_s) \exp(-\hat{r}(s-t))}{\sigma_s^3 (T-s)^2} \int_s^T \sigma_u \Lambda_u^{W^*} du \right].
\end{aligned}
$$

Now, (10) allows us to write

$$
\lim_{T \to s} T_2 = \frac{\rho \exp(-\hat{r}(s-t))}{4 \exp(X_t)} E_t^* \left[\frac{D_s^+ \sigma_s^2}{\sigma_s^2} \right],
$$

and now the proof is complete. ∎

Remark 12 *If $t = s$, this formula agrees with the at-the-money short-time limit skew proved in Alòs, Leòn and Vives (2007).*

6   The second derivative of the implied volatility

In this section we figure out $\frac{\partial^2 I}{\partial \alpha^2}(t, s; \alpha^*)$. Towards this end, note that

$$
\begin{aligned}
\frac{\partial V_t}{\partial \alpha} &= \exp(X_t) \frac{\partial BS}{\partial k}(s, 0, \exp(\alpha), I(t, s; \alpha)) \\
&\quad + \exp(X_t) \frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha), I(t, s; \alpha)) \frac{\partial I}{\partial \alpha}(t, s; \alpha)
\end{aligned}
$$

implies

$$
\begin{aligned}
\frac{\partial^2 V_t}{\partial \alpha^2} &= \exp(X_t) \frac{\partial^2 BS}{\partial k^2}(s, 0, \exp(\alpha), I(t, s; \alpha)) \\
&\quad + 2 \exp(X_t) \frac{\partial^2 BS}{\partial \sigma \partial k}(s, 0, \exp(\alpha), I(t, s; \alpha)) \frac{\partial I}{\partial \alpha}(t, s; \alpha) \\
&\quad + \exp(X_t) \frac{\partial^2 BS}{\partial \sigma^2}(s, 0, \exp(\alpha), I(t, s; \alpha)) \left(\frac{\partial I}{\partial \alpha}(t, s; \alpha) \right)^2 \\
&\quad + \exp(X_t) \frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha), I(t, s; \alpha)) \frac{\partial^2 I}{\partial \alpha^2}(t, s; \alpha). \qquad (11)
\end{aligned}
$$

6.1   The uncorrelated case

In this subsection we assume that $\rho = 0$.

We will need the following result, similar to Theorem 5 in Alòs and León (2015).

Lemma 13 *Consider the model (1) with $\rho = 0$ and assume that hypotheses in Theorem 11 hold. Then*

$$\frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*)) \frac{\partial^2 I}{\partial \alpha^2}(t, s; \alpha^*)$$

$$= \frac{1}{2} E_t^* \left[\int_t^T \frac{\partial^2 \Psi}{\partial a^2} \left(E_u^* \left(BS\left(s, 0, \exp(\alpha^*), v_s\right)\right)\right) U_u^2 du \right],$$

where

$$\Psi(a) := \frac{\partial^2 BS}{\partial k^2}\left(s, 0, \exp(\alpha^*), BS^{-1}(a)\right)$$

and U is given in (5).

Proof. This proof is similar to that in Alòs and León (2015) (Theorem 5), so we only sketh it. Notice that (11) and Remark 10 give us

$$\frac{\partial^2 V_t}{\partial \alpha^2}|_{\alpha = \alpha^*}$$

$$= \exp(X_t)\frac{\partial^2 BS}{\partial k^2}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))$$

$$+ \exp(X_t)\frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*)) \frac{\partial^2 I}{\partial \alpha^2}(t, s; \alpha^*).$$

Then, taking into account Theorem 2 and the fact that $I(t, s; \alpha^*) = BS^{-1}(\exp(-X_t)V_t)$, we are able to write

$$\exp(X_t)\frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*)) \frac{\partial^2 I}{\partial \alpha^2}(t, s; \alpha^*)$$

$$= \frac{\partial^2 V_t}{\partial \alpha^2}|_{\alpha = \alpha^*} - \exp(X_t)\frac{\partial^2 BS}{\partial k^2}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))$$

$$= \exp(X_t)E_t^* \left(\frac{\partial^2 BS}{\partial k^2}(s, 0, \exp(\alpha^*), v_s) - \frac{\partial^2 BS}{\partial k^2}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*)) \right)$$

$$= \exp(X_t)E_t^* \left[\frac{\partial^2 BS}{\partial k^2}\left(s, 0, \exp(\alpha^*), BS^{-1}\left(BS(s, 0, \exp(\alpha^*), v_s)\right)\right) \right.$$

$$\left. - \frac{\partial^2 BS}{\partial k^2}\left(s, 0, \exp(\alpha^*), BS^{-1}\left(E_t^*\left(BS(s, 0, \exp(\alpha^*), v_s)\right)\right)\right) \right].$$

Now, using (4), applying Itô's formula to the process

$$A_u := \frac{\partial^2 BS}{\partial k^2}\left(s, 0, \exp(\alpha^*), BS^{-1}\left(E_u^*\left(BS(s, 0, \exp(\alpha^*), v_s)\right)\right)\right)$$

and taking expectations, the result follows. ∎

Theorem 14 *Consider the model (1) with $\rho = 0$ and assume that hypotheses in Theorem 2 are satisfied. Then*

$$\lim_{T \to s} (T - s) \frac{\partial^2 I}{\partial \alpha^2}(t, s; \alpha^*) = \frac{1}{4} E_t^* \left(\int_t^s \left(E_u^* \left(\frac{D_u^{W^*} \sigma_s^2}{\sigma_s} \right) \right)^2 (E_u^*(\sigma_s))^{-3} du \right).$$

Proof. Lemma 13 yields

$$\lim_{T \to s} (T - s) \frac{\partial^2 I}{\partial \alpha^2}(t, s; \alpha^*)$$

$$= \lim_{T \to s} (T - s) \left(\frac{E_t^* \left[\int_t^s \frac{\partial^2 \Psi}{\partial a^2} \left(E_u^* (BS(s, 0, \exp(\alpha^*), v_s)) \right) U_u^2 du \right]}{2(\frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))} \right.$$

$$\left. + \lim_{T \to s} (T - s) \left(\frac{E_t^* \left[\int_s^T \frac{\partial^2 \Psi}{\partial a^2} \left(E_u^* (BS(s, 0, \alpha^*, v_s)) \right) U_u^2 du \right]}{2(\frac{\partial BS}{\partial \sigma}(s, 0, \exp(\alpha^*), I(t, s; \alpha^*))} \right)$$

$$= \lim_{T \to s} (T_1 + T_2).$$

Remember that we are assuming that there exist two positive constants $c, C > 0$ such that $c \le \sigma \le C$. Thus, the fact that $BS(s, 0, e^{\hat{r}(T-s)}, \cdot)$ is an increasing function, together with (H3) and

$$\frac{\partial^2 \Psi}{\partial a^2} \left(E_u^* (BS(s, 0, \alpha, v_s)) \right)$$

$$= \frac{2\sqrt{2\pi} \exp \left(\frac{\left(BS^{-1}(E*_u(BS(s,0,e^{\hat{r}(T-s)}, v_s))) \right)^2 (T-s)}{8} \right)}{\left(BS^{-1}(E*_u(BS(s, 0, e^{\hat{r}(T-s)}, v_s))) \right)^3 (T-s)^{3/2}},$$

allows us to deduce that, considering that C is a constant that may change from line to line,

$$0 < T_2 \le CE_t^* \left(\int_s^T \frac{\exp(\frac{C(T-s)}{8})}{c^3(T-s)} U_u^2 du \right) \le \frac{C}{(T-s)} \int_s^T E_t^*(U_u^2) du$$

$$\le C(T-s)^{1/2}.$$

which implies that $\lim_{T \to s} T_2 = 0$.

Finally, the dominated convergence theorem and Lemma 6 give us that

$$\lim_{T \to s} T_1 = \pi \lim_{T \to s} \int_t^s \frac{\exp(\frac{I(u,s;\alpha^*)^2(T-s)}{8})}{I(u, s; \alpha^*)^3(T-s)} E_t^*(U_u^2) du$$

$$= \frac{1}{4} \lim_{T \to s} E_t^* \int_t^s \frac{1}{I(u, s; \alpha^*)^3(T-s)} \left(E_u^* \left(\frac{\int_s^T (D_u^{W^*} \sigma_r^2) dr}{v_s \sqrt{T-s}} \right) \right)^2 du$$

$$= \frac{1}{4} E_t^* \left(\int_t^s \left(E_u^* \left(\frac{D_s^{W^*} \sigma_r^2}{\sigma_s} \right) \right)^2 (E_u^*(\sigma_s))^{-3} du \right)$$

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and this allows us to complete the proof. ∎

6.2 General case

Now we are ready to analyze the curvature in the correlated case. So we assume $\rho \neq 0$.

Theorem 15 *Under the assumptions of Theorem 2, we have*

$$
\lim_{T \to s} (T - s) \frac{\partial^2 I}{\partial \alpha^2} (t, s; \alpha^*)
$$

$$
= \frac{1}{4} E_t^* \left(\int_t^s \left(E_u^* \left(\frac{D_u^{W^*} \sigma_r^2}{\sigma_s} \right) \right)^2 (E_u^* (\sigma_s))^{-3} du \right)
$$

$$
+ \frac{1}{E_t^* (\sigma_s)} - \frac{1}{E_t^* (\sigma_s) + \frac{\rho}{2} \exp(-X_t) E_t^* \left(\frac{1}{\sigma_s} \int_t^s e^{-\hat{r}(u-t)} e^{X_u} \sigma_u D_u^{W^*} \sigma_s^2 du \right)}
$$

$$
- \frac{\rho}{2} \exp(-X_t) E_t^* \left(\frac{1}{\sigma_s^3} \int_t^s e^{-\hat{r}(u-t)} e^{X_u} \sigma_u \left(D_u^{W^*} \sigma_s^2 \right) du \right).
$$

Proof. By Theorem 2 and (11), we deduce

$$
\exp(X_t) \frac{\partial BS}{\partial \sigma} (s, 0, \exp(\alpha^*), I(t, s; \alpha)) \frac{\partial^2 I}{\partial \alpha^2} (t, s; \alpha)
$$

$$
= \exp(X_t) E_t^* \left(\frac{\partial^2 BS}{\partial k^2} (s, 0, \exp(\alpha^*), v_s) - \frac{\partial^2 BS}{\partial k^2} (s, 0, \exp(\alpha^*), I(t, s; \alpha)) \right)
$$

$$
- 2 \exp(X_t) \frac{\partial^2 BS}{\partial \sigma \partial k} (s, 0, \exp(\alpha^*), I(t, s; \alpha)) \frac{\partial I}{\partial \alpha} (t, s; \alpha)
$$

$$
- \exp(X_t) \frac{\partial^2 BS}{\partial \sigma^2} (s, 0, \exp(\alpha^*), I(t, s; \alpha^*)) \left(\frac{\partial I}{\partial \alpha} (t, s; \alpha^*) \right)^2
$$

$$
+ \frac{\rho}{2} E_t^* \left[\int_s^T e^{-\hat{r}(u-t)} \frac{\partial^2 H}{\partial k^2} (u, X_u, M_u, v_u) \sigma_u \Lambda_u^{W^*} du \bigg|_{\alpha = \alpha^*} \right.
$$

$$
\left. + \frac{\partial^2 G}{\partial k^2} (s, 0, \exp(\alpha^*), v_s) \int_t^s e^{-\hat{r}(u-t)} e^{X_u} \sigma_u \Lambda_u^{W^*} du \right]. \tag{12}
$$

Thus we can write

$$(T-s)\frac{\partial^2 I}{\partial \alpha^2}(t,s;\alpha^*)$$

$$= (T-s)\frac{E_t^*\left(\frac{\partial^2 BS}{\partial k^2}\left(s,0,\exp(\alpha^*),v_s\right)-\frac{\partial^2 BS}{\partial k^2}\left(s,0,\exp(\alpha^*),I^0(t,s;\alpha^*)\right)\right)}{\frac{\partial BS}{\partial \sigma}\left(s,0,\exp(\alpha^*),I(t,s;\alpha^*)\right)}$$

$$+(T-s)\frac{E_t^*\left(\frac{\partial^2 BS}{\partial k^2}\left(s,0,\exp(\alpha^*),I^0(t,s;\alpha^*)\right)-\frac{\partial^2 BS}{\partial k^2}\left(s,0,\exp(\alpha^*),I(t,s;\alpha^*)\right)\right)}{\frac{\partial BS}{\partial \sigma}\left(s,0,\exp(\alpha^*),I(t,s;\alpha^*)\right)}$$

$$-(T-s)\frac{2\frac{\partial^2 BS}{\partial \sigma \partial k}\left(s,0,\exp(\alpha^*),I(t,s;\alpha^*)\right)\frac{\partial I}{\partial \alpha}(t,s;\alpha^*)}{\frac{\partial BS}{\partial \sigma}\left(s,0,\exp(\alpha^*),I(t,s;\alpha^*)\right)}$$

$$-(T-s)\frac{\frac{\partial^2 BS}{\partial \sigma^2}\left(s,0,\exp(\alpha^*),I(t,s;\alpha^*)\right)\left(\frac{\partial I}{\partial \alpha}(t,s;\alpha^*)\right)^2}{\frac{\partial BS}{\partial \sigma}\left(s,0,\exp(\alpha^*),I(t,s;\alpha^*)\right)}$$

$$+(T-s)\left.\frac{\frac{\rho}{2}\exp(-X_t)E_t^*\left[\int_s^T e^{-\hat{r}(u-t)}\frac{\partial^2 H}{\partial k^2}(u,X_u,M_u,v_u)\sigma_u\Lambda_u^{W^*}\,du\right]}{\frac{\partial BS}{\partial \sigma}\left(s,0,\exp(\alpha^*),I(t,s;\alpha^*)\right)}\right|_{\alpha=\alpha^*}$$

$$+(T-s)\frac{\frac{\rho}{2}\exp(-X_t)E_t^*\left(\frac{\partial^2 G}{\partial k^2}(s,0,\exp(\alpha^*),v_s)\int_t^s e^{-\hat{r}(u-t)}e^{X_u}\sigma_u\Lambda_u^{W^*}\,du\right)}{\frac{\partial BS}{\partial \sigma}\left(s,0,\exp(\alpha^*),I(t,s;\alpha^*)\right)}$$

$$= T_1+T_2+T_3+T_4+T_5+T_6.$$

Here $I^0(t,s;\alpha)$ denotes the implied volatility in the uncorrelated case $\rho=0$. It is easy to see te definition of BS leads us to $\lim_{T\to s}(T_3+T_4+T_5)=0$. Moreover, we can easily see that

$$\lim_{T\to s}\frac{\frac{\partial BS}{\partial \sigma}\left(s,0,\exp(\alpha^*),I(t,s;\alpha^*)\right)}{\frac{\partial BS}{\partial \sigma}\left(s,0,\exp(\alpha^*),I^0(t,s;\alpha^*)\right)}=1.$$

Then, the proof of Lemma 13 yields

$$\lim_{T\to s}T_1=\lim_{T\to s}(T-s)\frac{\partial^2 I^0(t,s,\alpha^*)}{\partial k^2}.$$

By computing $\frac{\partial^2 BS}{\partial k^2}$ it is easy to see that

$$\lim_{T\to s}T_2=\lim_{T\to s}\left(\frac{1}{I^0(t,s;\alpha^*)}-\frac{1}{I(t,s;\alpha^*)}\right).$$

Therefore, we can use Lemma 6 and Theorem 7 to figure out this limit.

On the other hand

$$\lim_{T\to s}T_6$$

$$= \frac{\rho}{2}\exp(-X_t)\lim_{T\to s}(T-s)\frac{E_t^*\left(\frac{\partial^2 G}{\partial k^2}(s,0,\exp(\alpha^*),v_s)\int_t^s e^{-\hat{r}(u-t)}e^{X_u}\sigma_u\Lambda_u^{W^*}\,du\right)}{\frac{\partial BS}{\partial \sigma}\left(s,0,\exp(\alpha^*),I(t,s;\alpha^*)\right)}$$

$$= -\frac{\rho}{2}\exp(-X_t)E_t^*\left(\frac{1}{\sigma_s^3}\int_t^s e^{-\hat{r}(u-t)}e^{X_u}\sigma_u\left(D_u^{W^*}\sigma_s^2\right)du\right).$$

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Finally, the result is a consequence of Lemma 6, and Theorems 7 and 14. ∎

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