Sorting in Public School Districts under the Boston Mechanism

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Abstract

We study the extent to which the widely used Boston Mechanism (BM) fosters ability and socioeconomic segregation across public schools. Our model encompasses an endogenous component of school quality –determined by the peer group– and an exogenous one, so that there is at least one bad school *ex-ante*. Even with no residential priorities, BM generates *ability sorting* between a priori equally good public schools: an elitist public school emerges. A richer model with some preference for closer schools and flexible residential choice does not eliminate this effect. It rather worsens the peer quality of the nonelitist good school. The existence of private schools makes the best public school more elitist, while the bad school loses peer

quality. Their presence may also engender *socioeconomic segregation*. The main alternative assignment mechanism, Deferred Acceptance, is resilient to such sorting effects.

Key-words: School choice, mechanism design, peer effects, local public goods.

JEL classification numbers: I21, H4, D78.

1 Introduction

School choice has expanded in many countries in recent decades. In particular, public school choice systems with centralized assignment of children to schools are currently used in a large fraction of OECD countries (Musset, 2012). Yet, we still have a limited understanding about how such systems impact educational and other social outcomes, and the debate is accordingly heated.

Advocates defend that expanding school choice could be a tide that lifts all boats, allowing equal access to higher quality schooling for all. Two important arguments in favor of this view are that choice introduces competition, pushing schools to be more productive,[1] and that affluent families always had choice –as they could afford private schooling or housing in expensive areas– so that expanding choice improves equity by allowing poor households to choose as well (e.g. Friedman, 1955; Hoxby, 2003).[2] In sharp contrast, critics argue that expanding choice could exacerbate educational inequality and harm vulnerable students, increasing segregation across schools, and leaving them behind in lower quality ones. Arguments in this side of the debate include that schools usually prefer students from better-off socioeconomic backgrounds, that better-off parents exercise choice more often, that they make better informed choices, and that low income households have their effective choice sets restricted, as they cannot so easily afford transport and

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[1]While several theoretical contributions explain why school competition may *harm* school productivity in the presence of reputation effects or asymmetric information (De Fraja and Landeras, 2004; MacMillan, 2004, MacLeod and Urquiola, 2008, but see also Hoxby, 1999), recent empirical evidence supports the existence of positive productivity effects of school competition (see Hoxby, 2000, 2003, 2007; Gibbons et al., 2010, OECD, 2014).

[2]Indeed, it has been shown that, under certain stylized conditions, specific forms of school choice could be the solution to school and neighborhood segregation (see Epple and Romano, 2003, 2008).

other indirect costs (e.g. Smith and Meier, 1995; Musset, 2012; OECD, 2012; Hastings, Kane and Staiger, 2010).[3]

Two different and disconnected literatures in economics study the impact of school choice on social outcomes: 1) the literature on multi-community models of local public good provision, inspired by Tiebout (1956), and 2) the mechanism design literature that studies the implications of using alternative mechanisms to assign children to schools, initiated by Abdulkadiroglu and Sonmez (2003). The former endogenizes school quality through the peer group effect, but oversimplifies the assignment problem assuming that children are assigned to their local school.[4] The choice of where to live embeds the choice of school and, since better-off households are willing to pay more for school quality, socioeconomic segregation across communities and their schools ensues.[5] In that setting, *frictionless* school choice (with no transport costs or capacity constraints) indeed prevents segregation within the public sector (Epple and Romano, 1998, 2003).[6] Epple and Romano (2003) conjecture that their results would extend to a model where public schools had limited capacity and overdemands were resolved through lotteries, but do not provide relevant details of the school choice mechanism used (e.g. what happens with children excluded from their first choice).[7]

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[3]Theoretical work has also underpinnded some of these concerns (e.g. Epple and Romano, 1998).

[4]Tiebout (1956) is written as a response to the result of Musgrave (1939) and Samuelson (1954) that, due to the preference revelation problem, "no "market-type" solution exists to determine the level of expenditures on public goods." Tiebout suggested that a solution could exist for local public goods, at least in large and decentralized multi-community systems. The idea is that in such settings households effectively "shop" for local public goods by choosing which community to live in, and so where to pay local taxes and consume local public goods (See Wooders, 1999).

[5]Early contributions explain how decentralized school finance can lead to income segregation across the school districts of a metropolitan area: school quality differences emerge and sustain segregation because: (i) higher-income communities, with a larger tax base, vote for larger levels of education spending, and (ii) the income elasticity of demand (e.g. Epple et al., 1984, 1993) . More recent contributions explain that the peer group effect and other neighbourhood externalities can be sufficient to trigger segregation across the schools or neighbourhoods of a single district, and explore different equity, efficiency and policy implications of that observation (e.g. De Bartolome, 1990; Bénabou, 1996; Epple and Romano, 2003; De Fraja and Martinez-Mora, 2014).

[6]Epple and Romano (2003) also study the effects of transport costs and find that full residential segregation and partial school segregation by income would emerge in equilibrium.

[7]Other important contributions to this literature include Bénabou (1993), which il-

The second literature takes school quality and residence as given and ignores the presence of private schools to focus on the role played by the details of the assignment mechanism. Started by Abdulkadiroglu and Sönmez (2003), that literature reveals the importance of the rules applied to resolve overdemands when limited school capacities preclude the immediate satisfaction of parents' first choices. It formally analyzes the game generated by a centralized system where families submit a ranking of schools and a set of rules determines who gets accepted in an overdemanded school and what options are left for rejected applicants. These rules define the so-called school choice mechanisms, which often include priorities for applicants living in the neighborhood of the school or having a sibling in the school. Abdulkadiroglu and Sönmez (2003), and a fruitful literature derived from it, define several properties that these mechanisms should satisfy and establish a trade-off between efficiency (satisfying parents preferences) and stability (respecting priorities).[8]

Our paper builds a bridge between these two strands of the literature. We construct a model of centralized public school choice with partly endogenous school quality,[9] with private schools and where families choose where to live. There is a continuum of agents that are characterized by their type (socioeconomic background or ability), with preferences for school quality, money and nearby schools.[10] Our framework allows us to gain a better un-

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lustrates how socioeconomic segregation may create poverty traps and ghettos; Durlauf (1996), which explains how socioeconomic segregation can perpetuate income inequality across generations; or Nechyba (2000), which shows how the existence of private schools may reduce socioeconomic segregation by severing the link between a household place of residence and the school the child attends.

[8]This tradeoff has been argued to be small –Chen and Sönmez (2006) question its relevance through lab experiments.

[9]In line with Epple and Romano (2011), we define peer effects as any influence that a student has in the learning of her class or school mates. There is a large and growing body of literature studying the empirical relevance of peer effects and the mechanisms through which they affect the educational process. A consensus exists that they matter, and that a "better" peer group enhances performance (Epple and Romano, 2011; Sacerdote, 2011). We also introduce observable exogenous quality differences that vertically differentiate the public school system.

[10]The robustness analysis presented in the Appendix proves that our results hold with a different characterization of exogenous quality differences and with an arbitrary number of districts and schools. Moreover, it contains an extension of the model in which households differ along two dimensions: parental income and child ability. Qualitative results do not change.

derstanding of the impact school choice design has on socioeconomic sorting into schools and neighborhoods. In particular, we show that ex-ante identical schools become unequal under the widely used Boston mechanism if a third school is exogenously worse. That is, we find that the *Boston mechanism facilitates the creation of elite schools* within the public system, where only sufficiently high types self-select into. The introduction of private schools in the model exacerbates this effect. Preference for residence location close to school does not qualitatively affect these results. It rather worsens the peer quality of nonelitist public schools. Segregation across public schools occurs both according to children's abilities and to family income. In our view, it is surprising that the assignment to public, tuition-free schools is affected by families' differences in both income and ability. Methodologically, our results are novel in the literature because we introduce a particular, widely used, well-defined assignment mechanism, instead of assuming flexible (perfect) school choice.

As a policy recommendation, Epple and Romano (2008) propose a voucher system to avoid segregation with private schools. In their proposal, private schools can select students and optimal vouchers compensate for externalities so that schools are indifferent among all students: equal quality ensues. Since our approach relies on the specifics of the assignment mechanism to public schools, we instead propose another popular mechanism as a solution: Deferred Acceptance. There has been a wide literature on the good properties of this mechanism: strategy-proofness, stability-constrained efficiency under strict priorities, and protection of nonstrategic families (see among others Gale and Shapley 1962, Roth 1985, Erdil and Sönmez 2006, and Pathak and Sönmez 2010.) Our paper adds another property to the list of good properties: resilience to segregation across a priori identically good public schools.

We cite two recent pieces of related theoretical work. An ongoing research by Cantillon (2014) suggests that group admission quotas can avoid the emergence of segregation when preferences are endogenously determined by peer quality. Avery and Pathak (2015) compare school heterogeneity when there is neighborhood-based assignment and when flexible choice is implemented when residential choice between local and an adjacent town is incorporated in the model.

Recent empirical results make it plausible that self selection by parents of different types outweighs the incentives for schools to compete for students through improving their quality. Work by Calsamiglia et al. (2014)

and Calsamiglia and Güell (2015) provides empirical evidence showing that priorities play a large role in the final allocation of students to schools when residential priorities exist.[11] Although they do not consider peer effects or residential choices explicitly, the sorting effects that we identify in BM seem empirically plausible in the light of their results. Similarly, Calsamiglia et al. (2014) finds that a substantial fraction of families taking risks in the city of Barcelona opt for a private school if they do not get the desired school, empirically validating the channel that private schools play when BM is used.

The rest of the paper is organized as follows. Section 2 studies the base model with no private schools nor preference for nearby schools. Section 3 includes the presence of private schools into the base model. Section 4 includes preference for geographical proximity into the base model. Section 5 includes a discussion about an alternative mechanism that mostly avoids segregation issues: Deferred Acceptance. Section 6 concludes. An Appendix includes long proofs as well as extended models including an arbitrary number of schools and bidimensional types (income and ability.)

2   The base model

The model represents a single school district with three equally sized public schools that provide tuition-free education.[12] Schools are indexed with $j = 1, 2, 3$. Total school capacity is assumed to be identical to the number of children residing in the city, and each school has a capacity $1/3$. A population of households (indistinctly called families, agents, children or students) with mass normalized to 1 lives in the city. Every household consists of a parent, who takes decisions, and a school-aged child. Household type is denoted with $t \in D \equiv \left[\underline{t}, \overline{t}\right]$, and it is distributed in the population according to a continuous and strictly increasing distribution function $\Phi(t)$ with full support D. We denote with t_ϕ the ϕ quantile of Φ, i.e. $\Phi(t_\phi) = \phi$. Types can be interpreted as parental human capital or wealth, which determines both household income and its ability to benefit from school quality (Bénabou, 1996).

Schools j differ along two dimensions: peer quality q_j and an exogenous school component, denoted Δ_j. Let Φ_j be the (nonatomic) distribution of

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[11] That is the case in most OECD countries (OECD, 2012).

[12] This is only for expositional simplicity. Our results can be generalized to an economy with an arbitrary number of schools (see Appendix 2).

students' types conditional on being assigned to school j. Denote the peer quality as:

$$q_j = q(\Phi_j)$$

[13]where $q(\cdot)$ is continuous[14] and *monotonic in the first-order stochastic dominance sense*:[15] Φ_j FOSD Φ_i implies $q_j > q_i$. This functional form accommodates simple setups such as quality as the average type in the distribution of students. It nevertheless covers much richer setups. It accommodates, for example, a setting where a smaller proportion of children with ability below a certain threshold and a larger proportion of children with ability above a larger threshold enhance quality, as in Summers and Wolfe (1977). The quality function is flexible in the way we rank two schools that are not ranked according to first-order domination. Then this function may extend as well to a setting where quality is affected by the degree of heterogeneity in abilities at the school, as in Bénabou (1996).

Preferences. In the base model, households care about their offsprings' human capital, which depends on peer quality, q_j and the exogenous school characteristics Δ_j, of the school the children attend to. A household of type t assigned to school j obtains a utility $V(q_j, \Delta_j, t) = h(q_j, t) + \Delta_j$. In the presence of uncertainty about the final allocation, household's payoff is simply the expected value of V. The exogenous quality element enters linearly and it affects all households equally.[16] We assume that h is strictly increasing, continuous, bounded above and strictly supermodular (if $q' > q$, then $h(q', t) - h(q, t)$ is increasing in t.)

We impose that there is a bad school and two equally good schools: $\Delta_1 = \Delta_2 = \Delta > 0 = \Delta_3$. The purpose of this assumption is to illustrate how the presence of a bad school can generate segregation even across schools that are a priori equally good. Furthermore, we will make either one of the following assumptions.

Assumption 1 *Ghetto School:* $\Delta > h(q(\Phi_{t \geq t_{1/3}}), \bar{t}) - h(q(\Phi_{t \leq t_{1/3}}), \bar{t})$

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[13]When Φ_j collapses to a degenerate distribution with a single value t, we use the notation $q(t)$.

[14]It is continuous according to the distance $d(\Phi, \Phi') = \int_{\underline{t}}^{\bar{t}} |\Phi(t) - \Phi'(t)| dt$.

[15]Φ_j FOSD Φ_i if for all $t \in D$ we have $\Phi_i(t) \geq \Phi_j(t)$, and the inquality is strict for some t.

[16]We consider an alternative specification of preferences for Δ in Appendix 5. In the alternative modelling, school 3 quality is discounted by a factor $\delta < 1$. Results are qualitatively identical.

where $\Phi_{t \geq t_\phi}$ is the truncated distribution of students' types above t_ϕ and $\Phi_{t \leq t_\phi}$ denotes the truncated distribution of students' types below t_ϕ.

Assumption 2 *Limited Complementarity:* $\forall \phi \leq 1/3$,

$$h(q(\Phi_{t \geq t_\phi}), \bar{t}) - h(q(3\phi\Phi_{t \leq t_\phi} + (1 - 3\phi)\Phi_{t \geq t_\phi}), \bar{t})$$
$$< \quad 2 \cdot [h(q(\Phi_{t \geq t_\phi}), t_\phi) - h(q(3\phi\Phi_{t \leq t_\phi} + (1 - 3\phi)\Phi_{t \geq t_\phi}), t_\phi)]$$

where $3\phi\Phi_{t \leq t_\phi} + (1 - 3\phi)\Phi_{t \geq t_\phi}$ is a convex combination between distributions $\Phi_{t \leq t_\phi}$ and $\Phi_{t \geq t_\phi}$, with respective weights 3ϕ and $1 - 3\phi$.

Roughly speaking, the first assumption establishes a lower bound on the exogenous quality difference between good and bad schools. The second assumption instead caps the supermodularity of h, that is, the ability of higher types to profit more than lower types from peer quality differences.

No priorities. We consider setups where schools have no priorities over students in the school assignment procedure. When priorities for neighborhood exist the equilibrium analysis requires including residential choice prior to the school choice assignment. It can be easily shown that in such a model individuals segregate between neighborhoods and schools, with housing prices capitalizing the marginal valuation for having a better peers in that neighborhood school. This does not add much to standard results already known under neighborhood school assignment (e.g. Bénabou 1996, Epple and Romano 2003.) What we illustrate in this paper is that priorities are not necessary for the Boston Mechanism to generate segregation.

Strategies and timing. Each family is asked to submit a ranking of the three schools. We use "$i \succ j \succ k$" for the list where school i is ranked first, school j second and school k last. Let $R(t)$ be the (possibly mixed) ranking strategy of household t.

The base model has a single stage. Given an assignment mechanism M, each household type t submits a ranking of the three schools. The rules specified in M determine an assignment of children to schools. The allocation of children to schools in turn determines the peer groups and the academic quality of schools, q_1, q_2, q_3. Finally, payoffs are realized.

Equilibria. Our equilibrium concepts are the following:
An **equilibrium** given a mechanism M is a tuple of *beliefs* about qualities $(\hat{q}_1, \hat{q}_2, \hat{q}_3)$ and a ranking strategy profile R^* such that

8

1. *Rational choices*: Given the beliefs $(\hat{q}_1, \hat{q}_2, \hat{q}_3)$, no $t-$type household can increase utility by submitting a different ranking of schools other than $R^*(t)$.

2. *Consistent beliefs*: Given the assignment provided by M and R^*, induced qualities coincide with believed qualities: $\hat{q}_j = q_j$, $\forall j$.

A **stable equilibrium** given a mechanism M is an equilibrium such that for each converging sequence of beliefs $(q_1^n, q_2^n, q_3^n)_{n \in \mathbb{N}} \to (\hat{q}_1, \hat{q}_2, \hat{q}_3)$ there is a sequence of ranking strategy profiles R^n such that

1. For each type t, $R^n(t)$ is best responding to R^* given (q_1^n, q_2^n, q_3^n).

2. For almost every $t \in D$, $R^n(t) \to R^*(t)$ as $n \to \infty$.

The notion of stability captures the idea that arbitrarily small trembles in individual beliefs do not dramatically alter each agent's best response with respect to her equilibrium strategy. In most of the analysis we focus on pure ranking strategies. Schmeidler (1973) guarantees the existence of an equilibrium in pure strategies in this game.

Sorting. Our definitions of sorting are based on comparisons of the ex-post distribution of types between pairs of schools and neighborhoods. We say that there is **full sorting between schools** i **and** j if $\sup(supp(\Phi_j)) \leq \inf(supp(\Phi_i))$. That is, the maximum type assigned to school j lies weakly below the minimum type assigned to i. There is **partial sorting** between schools i and j if $\Phi_i FOSD \Phi_j$, implying $q_i > q_j$.[17] There is sorting if there is either full sorting or partial sorting. There is **no sorting** between schools i and j if $\Phi_i = \Phi_j$.

The Boston Mechanism (a.k.a. Immediate Acceptance). Before we proceed with the analysis, we need to explain how the Boston Mechanism (BM) assigns students to schools. First of all, parents are requested to report a complete ranking of the available schools to the school authority. A multi-round algorithm is used that assign children to schools round by round. In the first round, each student is considered for the school her parents ranked

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[17]Other forms of sorting could also be explored. For instance, sorting coming from second-order stochastic dominance, or from similarity to the unconditional distribution Φ (diversity). In this paper, however, our results concern the kinds of sorting defined in the main text and so we do not provide other definitions of sorting.

first. If the number of students considered for a school exceed the capacity of that school (i.e. the school is **overdemanded**,) some students will need to be rejected, following the school priorities (if any) and a tie-breaking lottery when necessary. Each student rejected in some round goes to the next round where she is considered for the highest-ranked school that has not rejected her yet. In BM, every accepted student keeps her slot at the school for which she was considered and both the student and the slot are removed from the assignment algorithm (definite acceptance).

While the way BM proceeds is easier to understand for parents, it also has its drawbacks. An assignment mechanism is strategy-proof if providing truthful information about one's own preferences when asked constitutes a weakly dominant strategy (i.e. it is always a best response to any profile of the other agents' strategies). In school choice problems, this property provides a valuable simplification of the strategic choice parents face, since they cannot do better than reporting their true ordinal preferences. This is not the case in BM. Given that slots are definitely given round by round, the opportunity cost of truthfully reporting preferences is the reduction of available slots in not-so-preferred schools in further rounds. Thus, each parent needs to balance her preferences with her chances. Consequently, she may rank a moderately good school with high acceptance chances in first position, and so on.

In an environment with peer effects, parents' preferences are also affected by the peer effect, an endogenous outcome. Thus, strategy-proofness is not a guarantee of strategic simplicity, since each parent still needs to take other parents' strategies into account in order to construct her own preferences. The value of strategy-proofness is then diminished. On the other hand, BM, precisely because it is not strategy-proof, manages valuable information about parents' preference intensities. In fact, parents with the same ordinal preferences may report different rankings if their preference intensities are different. This feature yields some efficiency properties for BM (Miralles, 2008; Abdulkadiroglu et al. 2011).

Results. The next proposition proves summarizes the results of our basic model.

Proposition 1 *Under the Boston Mechanism:*
(a) There is no stable equilibrium with no sorting between schools 1 and 2.

10

(b) If either the bad school is a ghetto school or h shows limited complementarity, there is a stable equilibrium with sorting between schools 1 and 2.

(c) Moreover, for Δ sufficiently high, all stable equilibria show full sorting between good schools.

The proof is found in Appendix 1. For a generalization to an arbitrary number of good schools, see Appendix 2. We provide some intuition in these lines. As soon as beliefs are such that peer qualities between good schools differ, say $\hat{q}_1 > \hat{q}_2$, best responses to any strategy profile take the form of a **cutoff strategy**. Types higher than some threshold $\hat{t}$ rank school 1 first, types below rank school 2 in first position.[18] Supermodularity of h produces this single-crossing property of best responses. These best responses indeed reinforce the original beliefs, eventually generating an equilibrium threshold $\tilde{t}$ under which $q_1 > q_2$. This equilibrium is stable because of its cutoff nature. Best responses change only around the equilibrium cutoff type $\tilde{t}$ when we tremble beliefs. Existence is illustrated in Figure 1. For a cutoff $\hat{t} = \underline{t}$, the cutoff type prefers to rank school 2 in first position (school 2 has identical ex-post peer quality as school 1 and it is underdemanded.) For a cutoff $\hat{t} = t_{1/2}$, the cutoff type prefers to rank school 1 on top (school 2 has as many applicants in the first round as school 1, and school 1 has higher peer quality.) A simple use of the intermediate value theorem gives us the existence of an indifferent cutoff $\tilde{t}$.

On the other extreme we consider possible equilibria where eventually $\Phi_1 = \Phi_2$, and consequently $q_1 = q_2$, a natural prediction since schools 1 and 2 are equally good a priori. Strategically, the game for families is simple. If they prefer school 3, they rank this school first. Otherwise they rank both schools 1 and 2 above school 3. About how to rank school 1 with respect to school 2, there are infinitely many profiles yielding $\Phi_1 = \Phi_2$ ex post. For each household, the chances of being admitted at either school 1 or school 2 are identical in equilibrium, no matter how school 1 was ranked with respect to school 2. Now, consider a sequence in beliefs such that $q_1^n > q_2^n$ (or $q_1^n < q_2^n$) converging to the equilibrium beliefs. Out of a sudden, best responding along the sequence implies ranking schools 1 above school 2 for all types (or doing

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[18] We assume that everyone ranks school 3 in last position, which is consistent with best-responding ex post. We note that this result holds even in cases in which school 3 is not the least-preferred by all types. Hence proposition 1 is not an artifact of an artificially created alignment of ordinal preferences.
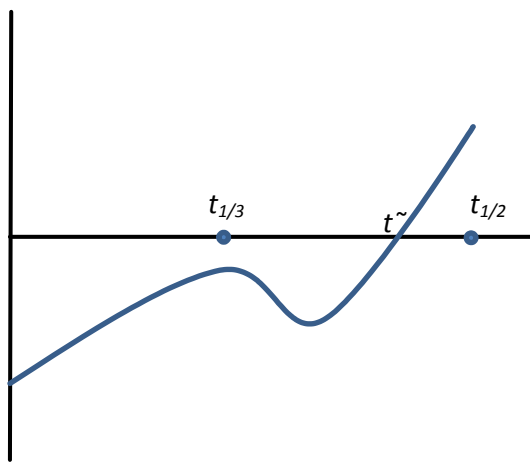
Figure 1: Change in cutoff type's expected payoff when ranking school 1 first instead of school 2, if the strategy profile follows a cutoff rule (higher types rank school 1 first, lower types rank school 2 first.)

the opposite if $q_1^n < q_2^n$). One can always choose one of these two sequences of beliefs such that the sequence of derived best responses do not converge to the equilibrium strategy profile.

Interestingly, in a segregation equilibrium in BM, the bad school has better ex-post peer quality than the second best public school ($\Phi_3 FOSD \Phi_2$), since school 3 collects students rejected from both schools 1 and 2. This partially compensates the effect of Δ. Moreover, the ex-post peer quality of school 3 under BM exceeds $q(\Phi)$. Both results easily fade away if private schools are available.

3   School choice with an outside option

This section introduces a private school in our baseline model in order to investigate the way outside options affect parental choices, the performance of the assignment mechanisms and the resulting allocation of children across public schools, when peer effects matter. This is especially relevant for school assignment mechanisms, since outside options are typically available in school markets. To the best of our knowledge, this is the first paper to study the workings of specific school choice mechanisms in the presence of both an outside option and peer effects.

For simplicity, we assume there is a single private school, which we call p. This label is also a number that denotes its price or tuition. Parents now have a choice between the public school the child is assigned to and a private school *after* the school assignment algorithm is completed. This school has endogenous peer quality $q_p = q(\Phi_p)$, *small* capacity η_p,[19] and exogenous quality Δ_p.

We assume that the private school sets the maximum price p that allows it to cover its capacity η_p, a reasonable assumption for a tightly capacity-constrained school that is maximizing tuition revenues. Since students assigned to the least-preferred public school have relatively higher willingness to pay for the private school, an outcome of our model will be that only students assigned to school 3 might leave it and enroll at the private school. This gives us a simplification about what to do in the assignment mechanism with the unoccupied slots. Nobody wants them ex post.

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[19]Private elementary schools not subject to the public school allocation system contain a very small percentage of the total student body in cities such as Barcelona, Spain (Calsamiglia and Güell, 2014.)

We include **two changes with respect to the basic model**. Firstly, the timing includes this possibility of opting for the private school ex post. Secondly, the utility function for a $t-$type household whose child enters a school with tuition fee p (recall that public schools are tuition-free), peer quality q and exogenous quality Δ is extended to

$$V(q, \Delta, p, t) = u(t - p) + h(q, t) + \Delta$$

where $u(\cdot)$ is the utility from consumption. Our extended utility function comprises these two extreme assumptions:

Assumption 3 *Type Represents Ability.* $u(t-p) = t-p$ *and h is strictly increasing and strictly supermodular (i.e. the base model with p entering linearly.)*

Assumption 4 *Type Represents Income.* u *is strictly increasing and strictly concave. h is additively separable: $h(q, t) = h_1(q) + h_2(t)$ with h_1 strictly increasing (hence h is not supermodular)*

For an extension to bidimensional types t and y and $V(q, \Delta, p, t, y) = u(y - p) + h(q, t) + \Delta$, see Appendix 4.

An **equilibrium** in this extended model follows the previous definition, jut adding that 1) equilibrium beliefs extend to the private school, $\hat{q}_p$, 2) the ranking decision profile $R^*(t), t \in D$, is accompanied by a profile of decisions on joining the private school or not contingent on the assigned public school, $P^*(t) \in \{public, private\}^3$, $t \in D$, and 3) that there is a price p (tuition fee) so that there is no excess demand for the private school. We impose a **consistency condition on equilibrium beliefs**: these are confirmed not only ex post but they are also *ordinally interim confirmed* when the assignment is done and before families decide whether to join the private school. That is, if $q_i \geq q_j$ just after the public school assignment takes place, it cannot be that $q_i < q_j$ after families decide whether to enrol at the private school. This avoids unnatural cases of self-confirmed beliefs that nothing have to do with the outcome of the public school assignment. Thus the presence of a private school just shrinks or enlarges a peer quality difference between schools, yet it does not create previously nonexistent differences. An equilibrium is **stable** if for each converging sequence of beliefs $(q_1^n, q_2^n, q_3^n, q_p^n)_{n \in \mathbb{N}} \rightarrow (\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_p)$ there is a sequence of ranking strategy profiles R^n, P^n such that 1) for each

type t, $R^n(t)$, $P^n(t)$ is best responding to R^*, P^* given $(q_1^n, q_2^n, q_3^n, q_p^n)$ and p,[20] and 2) for almost every $t \in D$, $R^n(t), P^n(t) \to R^*(t), P^*(t)$ as $n \to \infty$.

The availability of a private school affects the outcome of the BM in two ways. On the one hand, the complementarity between school quality and type is no longer necessary for the emergence of segregation when agents have an outside option: an equilibrium displaying segregation exists if the marginal utility of income is decreasing. On the other hand, if school quality and the child's type are complements in the production of human capital, the availability of a private alternative leads to an equilibrium with a more elitist best public school. This gives rise to a new source of unfairness: *top types have higher chances of admission at the best public school when a private school is present.*

We stay with the type-represents-ability model in the main text, noticing that Appendix 3 contains a similar result for the type-represents-income model. The former model is the most similar to the base model, so that the additional effects of introducing a private school become more apparent.

Proposition 2 *Assume either that the bad school is a ghetto school or that h shows limited complementarity. In the type-represents-ability model, let $\tilde{t}_{\max}$ denote the maximum equilibrium cut-off in the BM game without private schools. With a private school with sufficiently low capacity η_p there exists a stable equilibrium characterized by a cutoff ranking profile with threshold $\tilde{t}^p > \tilde{t}_{\max}$.*

Corollary 1 *In a (maximum cutoff) equilibrium in BM with a private school and ability types, top-type students have a higher probability of accessing the best school than when a private school does not exist. Furthermore, the ex-post quality of the ex-post best public school increases with respect to the case without private school.*

Key here is that the cut-off type becomes worse-off with respect to the base model in case she ends up allocated at the bad school. Why does she become worse-off in such event? Because high types refuse to enrol at the bad school. Instead they pay for private schooling. This lowers the quality of the bad school. Therefore ranking the most popular best school (say school 1) in first position becomes riskier than in the base model. Rejection endorses a higher cost (see Figure 2.) [21]
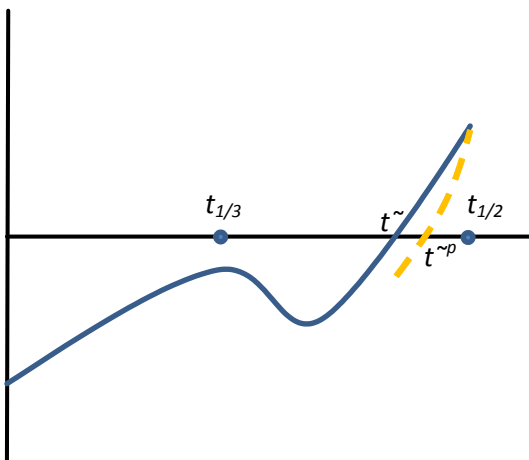

Figure 2: The difference in expected payoffs of figure 1 swifts down when we introduce a private school (orange dashed line.)

Remark 1 *We could have considered a "direct" model in which only high types (with t above some threshold t_3 close to $\bar{t}$) had an outside option that is better than the bad school yet worse than any good public school. An example of this could be the case of a selective private school. Conclusions would be identical.*

4 Preference for nearby schools and endogenous location

This section introduces one of the most natural extensions to our base model. Geographical proximity is many times a relevant factor in families' preferences over schools. Our extended model is also considering that the families' location decisions are endogenous and hence drawn to some extent from competitive residential prices.

Our **extended model** in this section departs for the base model in several ways. First, each school $j \in \{1, 2, 3\}$ belongs to a neighborhood j whose

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school 1) to the private school. We also set that p is low enough so that high types prefer the private school to school 3. This is possible for sufficiently low capacities η_p.

residential capacity equals 1/3 each. A strategy is a pair formed by a location decision over neighborhoods and a ranking decision over schools.[22] Preferences differ from the base model in two elements: 1) if the child gets enrolled at a school located in a different neighborhood from where the family lives, the household's utility is reduced by a small positive amount c,[23] and 2) a family that chooses to live in a neighborhood j obtains a utility loss equivalent to residential prices in the area, π_j.

An **equilibrium** is a pair of profiles $L^*(t), R^*(t)$, $t \in D$ of respectively location and ranking decisions, a vector of beliefs over peer qualities $(\hat{q}_1, \hat{q}_2, \hat{q}_3)$ and a vector of residential prices (π_1, π_2, π_3) such that 1) $L^*(t), R^*(t)$ is best-responding to L^*, R^* given the beliefs and the prices, for a.e. $t \in D$, 2) beliefs are confirmed ex-post, 3) residential market clears at all three neighborhoods. We will apply the following **stability** requirement as in the base model, that is, for each converging sequence of beliefs $(q_1^n, q_2^n, q_3^n, q_p^n)_{n \in \mathbb{N}} \rightarrow (\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_p)$ there is a sequence of ranking strategy profiles R^n, L^n such that 1) for each type t, $R^n(t), L^n(t)$ is best responding to R^*, L^* given (q_1^n, q_2^n, q_3^n) and (π_1, π_2, π_3), [24] and 2) for almost every $t \in D$, $R^n(t), L^n(t) \rightarrow R^*(t), L^*(t)$ as $n \rightarrow \infty$.

Proposition 3 *Assume that either the bad school is a ghetto school, or that h shows limited complementarity.*

(a) If the base model contains a stable cutoff equilibrium with cutoff $\tilde{t} < t_{1/3}$, then for c low enough the extended model has a stable equilibrium with cutoff $\tilde{t}^c \in (\tilde{t}, t_{1/3}]$.

(b) If on the contrary the base model contains only stable cutoff equilibria with thresholds $\tilde{t} \geq t_{1/3}$, then for any $c < \Delta$ and for every such equilibrium threshold $\tilde{t}$ there is a stable equilibrium with cutoff $\tilde{t}^c \in [t_{1/3}, \tilde{t})$ (when $\tilde{t} = t_{1/3}$ we instead have $\tilde{t}^c = t_{1/3}$.)

Corollary 2 *Consequently, school 2 has lower peer quality than in the base model, and school 1 has higher quality than in the base model if $\tilde{t} < t_{1/3}$.*

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[22] A natural timing in this model would require households to take location decisions first and then play the school choice game. We make this double decision profile simultaneous for the sake of simplicity. We do not lose generality because each equilibrium in our simplified model represents an equilibrium path in the extended game.

[23] When c is very large, we have the same (not surprising) outcome as with the presence of neighborhood priorities, that is, full sorting across schools, again as in Bénabou (1996) and Epple and Romano (2003.) In this paper we rather assume that geographical preferences play a secondary role as compared to the concern for offspring's human capital.

[24] This definition admits disequilibria in the residential market along the sequence.
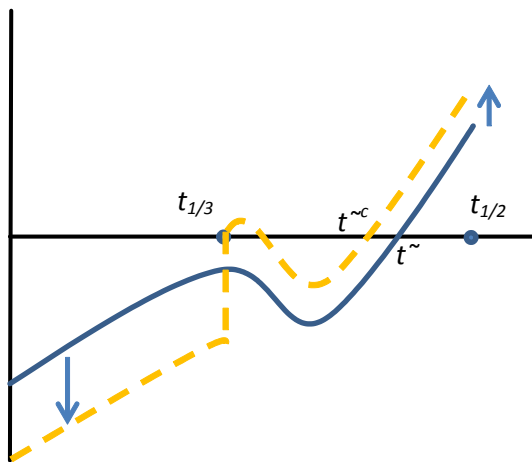
Figure 3: Changes in figure 1 when we introduce a preference for nearby schools (dashed orange line.)

The corollary is true because one can easily check that school 2 decreases peer quality as the cutoff type approaches $t_{1/3}$ (this is the lowest cutoff such that all types assigned to school 2 are below the cutoff.) The presence of preference for nearby schools creates a discontinuity in the cut-off type's evaluation of ranking strategies (namely, ranking the popular good school 1 in first position versus ranking school 2 first.) To understand this, we recall that location choice is not exogenous. Location decision depends on prices and on the intended ranking strategy. Interestingly, introducing endogenous location decision simplifies the problem. When household location is exogenously given, we would have to calculate a cut-off for each neighborhood. Here, since the ranking decision is optimally linked to the location decision, only one cutoff is relevant.

When the cut-off is below $t_{1/3}$, all students who rank school 1 first must be located along all neighborhoods. Then they must be indifferent among locations, making neighborhood 2 the cheapest one regarding residential prices. This gives an advantage to top-ranking school 2 (and living in neighborhood 2), as shown in Figure 3. When the cut-off is above $t_{1/3}$, no neighborhood $j = 1, 2$ can be the cheapest one. Otherwise all students ranking school j would choose to live there, leading to excess demand. Thus neighborhood 3 must be the cheapest one. This gives an advantage to ranking school 1

18

first, since in such a case ending up in the bad school is more likely. This is also seen in Figure 3. A cutoff equal to $t_{1/3}$ is an intermediate case in which any convex combination of the two aforementioned effects may arise. This creates this tendency towards the worst equilibrium for school 2.

5   Deferred Acceptance

In light of these results, a natural step forward in the analysis would consist of exploring alternatives to the Boston Mechanism. In the debate on school choice, Deferred Acceptance has been suggested as a valid alternative. In this section, we argue that Deferred Acceptance is more resilient to inter-school segregation than the Boston Mechanism.

As a mechanism, Deferred Acceptance works almost identically as the Boston Mechanism. There is only one difference, which is eventually quite relevant. In the Deferred Acceptance assignment algorithm, when a student is accepted at some school, this *acceptance is tentative* as opposed to definite. In other words, a student accepted at some school at some round only gains the right to continue applying for the same school in subsequent rounds. It is possible that this student gets rejected at a further round, since in each round the school selects students from a different cohort.

While this property of the mechanism is difficult to understand for parents, it turns out that it endorses nice properties. Mainly, Deferred Acceptance is strategy-proof. Parents have no gain in misrepresenting preferences, no matter what other parents do. We will see that strategy-proofness have nonnegligible implications regarding segregation.

We summarize these results in the following proposition.

Proposition 4 *Under Deferred Acceptance:*

(a) There is no sorting between good schools in the base model, nor there is sorting between a good school and the bad school.

(b) For Δ sufficiently large, there is no sorting between goods schools in the extended model with private schools.

(c) There is an equilibrium with sorting between good schools in the extended model with preference for nearby schools and sufficiently low c, characterized by a cutoff $\tilde{t}^D$ such that types above prefer school 1 to school 2 and types below prefer school 2 over school 1. As c approaches zero, $\tilde{t}^D$ tends to $\underline{t}$. Moreover, for any equilibrium cutoff $\tilde{t}^c$ in the Boston Mechanism, we have $\tilde{t}^D < \tilde{t}^c$.

19

The resilience of Deferred Acceptance with respect to segregation can be explained as follows. Suppose that parents expect that two a priori identical schools are going to have two different peer qualities. Now, strategy-proofness is key: everyone would rank these two schools according to these expectations. But this leads to contradiction. There is no distinction between the distribution of those students applying for the higher-quality school and those who are rejected and hence apply of the lower-quality school.

As for part (c) of the proposition above, the presence of some preference for nearby schools gives rise to segregation. The reason is simple. Low types may care less about peer quality than about geographical proximity. However, part (c) also states that even in that case, segregation vanishes as c tends to zero. Moreover, this segregation is "lower" than that generated under BM (where peer quality of school 1 is higher.)

6 Concluding remarks

This paper has introduced a theory of sorting into public schools with school choice. It is, to the best of our knowledge, the first study on school choice mechanisms that endogenizes preferences and priorities. We showed that the choice of the assignment mechanism, along with the details of the institutional context in which it is applied, are crucial for the resulting distribution of children across public schools and for the degree of equality of opportunities offered by the education system. The welfare implications of segregating or mixing students are well known (Arnott and Rowse, 1987; Bénabou, 1996). We thus provided a theoretical underpinning for the equity concerns expressed by the OECD (2012) and others, even when there are no informational asymmetries. Our analysis also offers guidance about how to guarantee equality of opportunities in a context with public sector school choice.

7 Appendix

7.1 Proofs

Proof. Proposition 1.

(1) No stable equilibria with no sorting between schools 1 and 2. Suppose

on the contrary that an equilibrium with no sorting between these schools exists ($\Phi_1 = \Phi_2$.) In such a case, $\hat{q}_1 = \hat{q}_2$ and both schools are de facto identical for all students. Best responses involve either ranking school 3 on top or at the bottom, depending on whether school 3 is preferred to the other two or not (since students are choosing among two types of schools, there is no risk in revealing the true position of school 3.) If there are types that prefer school 3 to the other schools, it must be that $\hat{q}_3 > \hat{q}_1 = \hat{q}_2$ in order to compensate for the exogenous quality advantage Δ. Therefore, by supermodularity of h, there is a cutoff $t' \in D$ such that types above prefer school 3 to the others, and the opposite happens when the type lies below the cutoff. (In case $\hat{q}_3 - \hat{q}_1$ is low enough or negative, we obtain $t' = \bar{t}$.)

It is not possible that $t' = \underline{t}$. In such a case we obtain $\Phi_3 = \Phi$, and since by no sorting we have $\Phi_1 = \Phi_2$ we must have $\Phi_1 = \Phi_2 = \Phi = \Phi_3$ and hence $q_1 = q_2 = q_3$ contradicting the fact that everyone prefers school 3 to the other schools. Thus $t' > \underline{t}$. We focus on types below t'. Those types rank school 3 at the bottom, and they choose whether to rank school 1 or school 2 in first position. Let p_{ij} be the equilibrium probability of entering school $i = 1, 2$ if school $j = 1, 2$ is ranked in first position.

Since $\hat{q}_1 = \hat{q}_2$, an equilibrium condition is $p_{11} + p_{21} = p_{12} + p_{22}$. (Suppose $p_{1j} + p_{2j} > p_{1,3-j} + p_{2,3-j}$. Consequently all student types below t' would optimally rank school j in first position. But then school $3 - j$ is underdemanded in the first round of the assignment procedure, implying $p_{3-j,3-j} = 1 \geq p_{1j} + p_{2j}$, a contradiction.) We argue that a necessary condition for stability is $p_{11} = p_{12}$ and $p_{22} = p_{21}$. Suppose on the contrary that $p_{ij} > p_{i,3-j}$. By the equilibrium condition this implies that $p_{3-i,j} < p_{3-i,3-j}$. Construct a sequence of beliefs satisfying $q_i^n > q_{3-i}^n$, $n \in \mathbb{N}$, converging to the equilibrium beliefs. For all types below t', the best response given the equilibrium probabilities implies ranking school i in first position, along the whole sequence. Construct another sequence of beliefs satisfying $q_i^n < q_{3-i}^n$, $n \in \mathbb{N}$, converging to the equilibrium beliefs. The best response for all types $t < t'$ given the equilibrium probabilities involves ranking school $3 - i$ in first position, along the whole sequence. Obviously these two sequences of beliefs induce two sequences of best response profiles that do not converge to the same profile, a violation of stability.

However, the Boston Mechanism precludes the accomplishment of the condition $p_{11} = p_{12}$ and $p_{22} = p_{21}$. Suppose $t' > t_{2/3}$. Then a positive mass of students with types below t' must end up in school 3, implying $p_{11} + p_{21} = p_{12} + p_{22} < 1$. This implies for instance $p_{11} < 1$, thus school 1 is overdemanded

in the first round, and $p_{12} = 0 < p_{11}$. Now suppose $t' \leq t_{2/3}$. In such a case, there must be a school $i = 1, 2$ that is not overdemanded in the first round of the assignment procedure, that is, $p_{ii} = 1$. Since $p_{3-i,3-i} > 0$ we must have $p_{i,3-i} < 1 = p_{ii}$. This completes the proof.

(2) Existence of stable equilibrium with sorting between schools 1 and 2. Assume that agents play either one of the following strategies: ranking 1 above 2 and 2 above 3 ("$1 \succ 2 \succ 3$" from now on,) and ranking 2 above 1 and 1 above 3 ("$2 \succ 1 \succ 3$" hereafter.) Furthermore, assume that equilibrium beliefs are such that $\hat{q}_1 \geq \hat{q}_3 > \hat{q}_2$. (We later check that these assumptions are accomplished in equilibrium.)

Let p_{ij} denote the probability of enrolling into school i if the student ranks school $j = 1, 2$ in first position. We claim that $p_{i1} > p_{i2}$ for $i = 1, 3$ and that $p_{22} > p_{21}$ in equilibrium. That $p_{jj} > p_{j(3-j)}$, $j = 1, 2$ is just a consequence of the mechanism: one obtains higher chances at school j if she ranks it above rather than below. $p_{31} > p_{32}$ is an equilibrium condition. Suppose otherwise that $p_{31} \leq p_{32}$, implying $p_{11} \geq p_{12} + p_{22} + p_{21}$. Now, for a.a. $t \in D$

$$
\begin{aligned}
& p_{11}(h(\hat{q}_1, t) + \Delta) + p_{21}(h(\hat{q}_2, t) + \Delta) + p_{31}h(\hat{q}_3, t) \\
& -p_{12}(h(\hat{q}_1, t) + \Delta) - p_{22}(h(\hat{q}_2, t) + \Delta) - p_{32}h(\hat{q}_3, t) \\
= & p_{11}(h(\hat{q}_1, t) - h(\hat{q}_3, t) + \Delta) + p_{21}(h(\hat{q}_2, t) - h(\hat{q}_3, t) + \Delta) \\
& -p_{12}(h(\hat{q}_1, t) - h(\hat{q}_3, t) + \Delta) - p_{22}(h(\hat{q}_2, t) - h(\hat{q}_3, t) + \Delta) \\
\geq & (p_{12} + p_{22} + p_{21})(h(\hat{q}_1, t) - h(\hat{q}_3, t) + \Delta) + p_{21}(h(\hat{q}_2, t) - h(\hat{q}_3, t) + \Delta) \\
& -p_{12}(h(\hat{q}_1, t) - h(\hat{q}_3, t) + \Delta) - p_{22}(h(\hat{q}_2, t) - h(\hat{q}_3, t) + \Delta) \\
= & (p_{22} - p_{21})(h(\hat{q}_1, t) - h(\hat{q}_2, t)) > 0
\end{aligned}
$$

where the first equality comes from $p_{3j} = 1 - p_{1j} - p_{2j}$, the first inequality comes from $p_{11} \geq p_{12} + p_{22} + p_{21}$ and $h(\hat{q}_1, t) \geq h(\hat{q}_3, t)$, and the last inequality comes from $p_{22} > p_{21}$ and $h(\hat{q}_1, t) > h(\hat{q}_2, t)$. But then, almost all students best respond with strategy "$1 \succ 2 \succ 3$". This would contradict the assumption $\hat{q}_1 > \hat{q}_2$ since the distribution of student types would be identical across schools ex post (an hence $q_1 = q_2$.) This proves the claim.

A first consequence is that, under any strategy profile such that the beliefs $\hat{q}_1 \geq \hat{q}_3 > \hat{q}_2$ are confirmed ex post, the difference

$$
\begin{aligned}
& p_{11}(h(\hat{q}_1, t) + \Delta) + p_{21}(h(\hat{q}_2, t) + \Delta) + p_{31}h(\hat{q}_3, t) \\
& -p_{12}(h(\hat{q}_1, t) + \Delta) - p_{22}(h(\hat{q}_2, t) + \Delta) - p_{32}h(\hat{q}_3, t) \\
= & (p_{11} - p_{12})(h(\hat{q}_1, t) - h(\hat{q}_2, t)) + (p_{31} - p_{32})(h(\hat{q}_3, t) - h(\hat{q}_2, t) - \Delta)
\end{aligned}
$$

is increasing in t, due to the supermodularity of h. Therefore, any equilibrium ranking profile R must be characterized by a cut-off $\hat{t}$ such that types above $\hat{t}$ use strategy "$1 \succ 2 \succ 3$" and types below $\hat{t}$ use strategy "$2 \succ 1 \succ 3$". We focus attention on cut-off ranking profiles.

Using $p_{ij}(\hat{t})$ for the probability of ending enrolled in school i when ranking school $j = 1, 2$ in first position given a cut-off ranking profile with threshold $\hat{t}$, and $q_i(\hat{t})$ for the ex post peer quality of school i under a cut-off ranking profile with threshold $\hat{t}$, we construct the function

$$
\begin{aligned}
G(\hat{t}) &= (p_{11}(\hat{t}) - p_{12}(\hat{t}))(h(q_1(\hat{t}), \hat{t}) - h(q_2(\hat{t}), \hat{t})) \\
&\quad + (p_{31}(\hat{t}) - p_{32}(\hat{t}))(h(q_3(\hat{t}), \hat{t}) - h(q_2(\hat{t}), \hat{t}) - \Delta)
\end{aligned}
$$

which measures the difference in expected payoffs for the cut-off type between playing strategy "$1 \succ 2 \succ 3$" and playing strategy "$1 \succ 2 \succ 3$".

Both assignment probabilities and peer qualities evolve continuously with $\hat{t}$. Therefore, G is a continuous function. Notice that $G(\underline{t}) < 0$ since $q_i(\underline{t}) = q(\Phi)$ for all $i = 1, 2, 3$ and $p_{31}(\underline{t}) = 1/3 > p_{32}(\underline{t}) = 0$. Note as well that $G(t_{1/2}) > 0$ since $p_{31}(t_{1/2}) = 1/3 = p_{32}(t_{1/2})$, $p_{11}(t_{1/2}) = 2/3 > p_{12}(t_{1/2}) = 0$ and $q_1(t_{1/2}) = q(\Phi_{t \geq t_{1/2}}) > q_2(t_{1/2}) = q(\Phi_{t \leq t_{1/2}})$. The intermediate value theorem allows us to state that there exists $\tilde{t} \in (\underline{t}, t_{1/2})$ such that $G(\tilde{t}) = 0$. That is, there is an equilibrium cut-off ranking profile with threshold $\tilde{t}$.

We check that the assumptions we took are correct in equilibrium. We first check that $q_1 \geq q_3 > q_2$. In effect, if $\tilde{t} \geq t_{1/3}$ we have $q_1 = q(\Phi_{t \geq \tilde{t}}) > q_3 = q\left(\frac{2/3 - \Phi(\tilde{t})}{1/3}\Phi_{t \geq \tilde{t}} + \frac{\Phi(\tilde{t}) - 1/3}{1/3}\Phi_{t \leq \tilde{t}}\right) > q_2 = q(\Phi_{t \leq \tilde{t}})$. And if $\tilde{t} \leq t_{1/3}$ we obtain $q_1 = q(\Phi_{t \geq \tilde{t}}) = q_3 > q_2 = q\left(\frac{1/3 - \Phi(\tilde{t})}{1/3}\Phi_{t \geq \tilde{t}} + \frac{\Phi(\tilde{t})}{1/3}\Phi_{t \leq \tilde{t}}\right)$.

Then we check that only strategies "$1 \succ 2 \succ 3$" and "$2 \succ 1 \succ 3$" are played in equilibrium.

Consider $\tilde{t} \geq t_{1/3}$. Since both schools 1 and 2 are overdemanded in the first assignment round, the only part of the submitted ranking is the school ranked in first position. So the only relevant alternative to "$1 \succ 2 \succ 3$" and "$2 \succ 1 \succ 3$" is ranking school 3 first ("$3 \succ ...$".) Since $\hat{q}_1 \geq \hat{q}_3$ and $\Delta > 0$, all families prefer school 1 to school 3. No student would rank school 3 above school 1 in equilibrium. Therefore "$1 \succ 2 \succ 3$" is better than any of the alternative strategies for all types. Since those who play "$2 \succ 1 \succ 3$" prefer it to "$1 \succ 2 \succ 3$", all alternative strategies are discarded.

We consider further the case $\hat{t} < t_{1/3}$. Now submitting an alternative ranking message "$1 \succ 3 \succ 2$" is better than "$3 \succ ...$" for all types, since all

families prefer school 1 to school 3 and acceptance at school 3 is guaranteed at all rounds of the assignment process. We compare As for schools 2 and 3, we first consider the case in which school 3 is a ghetto school. If $\hat{t} \leq t_{1/3}$, $q_2(\hat{t}) = q\left(\frac{1/3-\Phi(\tilde{t})}{1/3}\Phi_{t \geq \hat{t}} + \frac{\Phi(\tilde{t})}{1/3}\Phi_{t \leq \hat{t}}\right)$ is decreasing in $\hat{t}$ and $q_3(\hat{t}) = q(\Phi_{t \geq \hat{t}})$ is increasing in $\hat{t}$. If $\hat{t} \geq t_{1/3}$, $q_2(\hat{t}) = q\left(\Phi_{t \leq \hat{t}}\right)$ is increasing in $\hat{t}$ and $q_3(\hat{t}) = q(\frac{2/3-\Phi(\tilde{t})}{1/3}\Phi_{t \geq \tilde{t}} + \frac{\Phi(\tilde{t})-1/3}{1/3}\Phi_{t \leq \tilde{t}})$ is decreasing in $\hat{t}$. Then $\hat{t} = t_{1/3}$ minimizes $q_2(\hat{t})$ and maximizes $q_3(\hat{t})$. Thus $\forall t \in D$, $h(q_3(\tilde{t}), t) - h(q_2(\tilde{t}), t) \leq h(q_3(\tilde{t}), \bar{t}) - h(q_2(\tilde{t}), \bar{t}) \leq h(q_3(t_{1/3}), \bar{t}) - h(q_2(t_{1/3}), \bar{t}) < \Delta$ (the first and second inequalities by supermodularity of h, the last inequality by the ghetto school assumption.) Everyone prefers school 2 to school 3 in equilibrium. No student would rank school 3 above school 2 in equilibrium. Overall, only the assumed strategies are played in equilibrium.

In a second scenario, we consider the limited complementarity assumption. Notice that $G(\tilde{t}) = 0$ implies $h(q_3, \tilde{t}) - h(q_2, \tilde{t}) < \Delta$ thus the equilibrium cutoff type (and all types below) prefers school 2 to school 3. Consequently, no household using the ranking "$2 \succ 1 \succ 3$" has incentive to rank school 3 other than last. As for those households declaring "$1 \succ 2 \succ 3$". If $\tilde{t} \geq t_{1/3}$, then school 2 gives all its slots in the first round, so ranking school 3 over school 2 (instead of 2 over 3) has no effect. If $\tilde{t} \leq t_{1/3}$ we have $p_{11}(\tilde{t}) = p_{31}(\tilde{t}) = \frac{1/3}{1-\Phi(\tilde{t})}$ and $p_{12}(\tilde{t}) = p_{22}(\tilde{t}) = 0$. Together with $q_1 = q_3$ this makes $G(\tilde{t}) = p_{11}(\tilde{t})(2h(q_3, \tilde{t}) - 2h(q_2, \tilde{t}) - \Delta) = 0$. Then, for all $t \in D$, $h(q_3, t) - h(q_2, t) - \Delta \leq h(q_3, \bar{t}) - h(q_2, \bar{t}) - \Delta = h(q_3, \bar{t}) - h(q_2, \bar{t}) - 2(h(q_3, \tilde{t}) - h(q_2, \tilde{t})) < 0$. The first inequality comes from supermodularity of h knowing that $q_3 > q_2$. The last inequality comes from limited complementarity. We conclude that all types prefer school 2 to school 3, thus they have no incentive to rank school 3 above school 2. With this we have checked that only strategies "$1 \succ 2 \succ 3$" and "$2 \succ 1 \succ 3$" are played in equilibrium.

This equilibrium is stable. A best response profile to R given qualities q_1^n, q_2^n, q_3^n arbitrarily close to equilibrium beliefs is characterized by a threshold $\tilde{t}^n$ arbitrarily close to $\tilde{t}$.

Moreover, this equilibrium entails sorting between schools 1 and 2, since in equilibrium Φ_1 first-order stochastically dominates Φ_2.

(c) Only full sorting if Δ big enough. For cutoffs $\hat{t} \leq t_{1/3}$ we know (from previous paragraphs) that $G(\hat{t}) = p_{11}(\hat{t})(2h(q_3(\hat{t}), \hat{t}) - 2h(q_2(\hat{t}), \hat{t}) - \Delta)$. We also know that in this interval $q_3(\hat{t})$ is increasing and $q_2(\hat{t})$ is decreasing. By supermodularity of h, there is an equilibrium cutoff $\tilde{t} \leq t_{1/3}$ if and only if

$2h(q_3(t_{1/3}), t_{1/3}) - 2h(q_2(t_{1/3}), t_{1/3}) \geq \Delta$. If instead Δ is higher than the former left-hand side of the inequality, no such equilibrium exists. Therefore only equilibria with $\tilde{t} > t_{1/3}$ exist in such a case, where $\Phi_1 = \Phi_{t \geq \tilde{t}}$ and $\Phi_2 = \Phi_{t \leq \tilde{t}}$ (full sorting.) ∎

Proof. Proposition 2.

As in the proof of proposition 1 (b), we assume that agents play either one of the following ranking decisions: ranking 1 above 2 and 2 above 3 ("$1 \succ 2 \succ 3$") and ranking 2 above 1 and 1 above 3 ("$2 \succ 1 \succ 3$") Furthermore, assume that equilibrium beliefs are such that $\hat{q}_p > \hat{q}_1 \geq \hat{q}_3 > \hat{q}_2$. We also assume that the price p is low enough to allow the private school to attract some students who were assigned to school 3. Yet it is high enough to deter students assigned to either school 1 or 2 from enrolling in the private school. (We later check that these assumptions are accomplished in equilibrium.)

Under these assumptions, we know from the proof of proposition 1 that best ranking response profiles are characterized by a threshold $\hat{t}$ such that types above the threshold play the ranking decision "$1 \succ 2 \succ 3$" whereas types below play "$2 \succ 1 \succ 3$".

Due to supermodularity of h, we can set a cutoff $t_3 < \bar{t}$ (the tuition fee p will later be calculated so as to make this type indifferent between school 3 and the private school) such that all types above t_3 prefer the private school over school 3 while all types below prefer school 3 to the private school. We select t_3 close to $\bar{t}$, thus for any cut-off type $\hat{t} < 1/2$ we have $t_3 > \hat{t}$. The cut-off type never uses the private school option in the event she is assigned to school 3. The function

$$
\begin{aligned}
\tilde{G}(\hat{t}, t_3) &= (p_{11}(\hat{t}) - p_{12}(\hat{t}))(h(q_1(\hat{t}), \hat{t}) - h(q_2(\hat{t}), \hat{t})) \\
&\quad + (p_{31}(\hat{t}) - p_{32}(\hat{t}))(h(\tilde{q}_3(\hat{t}, t_3), \hat{t}) - h(q_2(\hat{t}), \hat{t}) - \Delta)
\end{aligned}
$$

has analogous meaning to $G(\hat{t})$ in the proof of proposition 1 (indeed, $G(\hat{t}) = \tilde{G}(\hat{t}, \bar{t})$.) However, the new explanatory variable t_3 lowers the quality of school 3. The distribution of student types at school 3 is a truncation below t_3 of the distribution of student types at school 3 if no private school existed. That is, $\tilde{q}_3(\hat{t}, t_3) < q_3(\hat{t}) = \tilde{q}_3(\hat{t}, \bar{t})$. Consequently, $\tilde{G}(\hat{t}, t_3) < G(\hat{t})$ for all $\hat{t} \in (\underline{t}, t_{1/2})$.

Now, take the maximum equilibrium cutoff in the base model, $\tilde{t}_{\max} \in (\underline{t}, t_{1/2})$. Since $G(\tilde{t}_{\max}) = 0$ we have $\tilde{G}(\tilde{t}_{\max}, t_3) < 0$. On the other hand $\tilde{G}(t_{1/2}, t_3) = G(t_{1/2}) > 0$ (since $p_{31}(t_{1/2}) = p_{32}(t_{1/2}) = 1/3$.) Given that $\tilde{G}$ is a continuous function, there must exist $\tilde{t}^p \in (\tilde{t}_{\max}, t_{1/2})$ such that $\tilde{G}(\tilde{t}^p, t_3) = 0$.

This is our candidate to equilibrium cutoff.

We know check that the assumptions are correct in equilibrium. Given that $q_3(\tilde{t}^p) > q_2(\tilde{t}^p)$ (as we checked in the proof of proposition 1), and provided that $\tilde{q}_3$ is continuous in t_3, we can set t_3 close enough to $\bar{t}$ so that $\tilde{q}_3(\tilde{t}^p, t_3) > q_2(\tilde{t}^p)$. This is not even necessary if $\tilde{t}_{\max} \geq t_{1/3}$, since in that case $\tilde{q}_3(\tilde{t}^p, t_3) > q_2(\tilde{t}^p)$ for every selected t_3. $q_1(\tilde{t}^p) \geq \tilde{q}_3(\tilde{t}^p, t_3)$ follows from $q_1(\tilde{t}^p) \geq q_3(\tilde{t}^p) = \tilde{q}_3(\tilde{t}^p, \bar{t})$ where the latter inequality was shown in the proof of proposition 1.

That only ranking decisions "$1 \succ 2 \succ 3$" and "$2 \succ 1 \succ 3$" are used in equilibrium is an immediate consequence of the analogous result we check in proposition 1. In the model with private school, school 3 has lower peer quality than in the base model with the same cutoff ($\tilde{q}_3(\hat{t}, t_3) < q_3(\hat{t})$ for every cutoff $\hat{t}$.)

That the cutoff ranking strategy profile constitutes a stable equilibrium stems from its cutoff nature, as we showed in the proof of proposition 1.

We show that there are prices p that accomplish with the initial assumptions. Fix t_3, which determines $\tilde{t}^p$, then

$$p = h(q(\Phi_{t \geq t_3}), t_3) + \Delta_p - h(\tilde{q}_3(\tilde{t}^p, t_3), t_3)$$

where $q(\Phi_{t \geq t_3})$ is the quality of the private school in equilibrium.

We show that p is high enough to deter students who are assigned to either school 1 or 2 from enrolling at the private school. This is immediate both under the ghetto school assumption and under the limited complementarity assumption with $\tilde{t}^p \leq t_{1/3}$ because school 2 is strictly preferred to school 3 for all students in these setups. Therefore

$$h(q(\Phi_{t \geq t_3}), \bar{t}) + \Delta_p - h(q_2(\tilde{t}^p), \bar{t}) - \Delta$$
$$< \quad h(q(\Phi_{t \geq t_3}), \bar{t}) + \Delta_p - h(\tilde{q}_3(\tilde{t}^p, t_3), \bar{t})$$

and for t_3 sufficiently close to $\bar{t}$, we have $h(q(\Phi_{t \geq t_3}), \bar{t}) + \Delta_p - h(q_2(\tilde{t}^p), \bar{t}) - \Delta < p$. As for the remaining case (limited complementarity with $\tilde{t}^p \geq t_{1/3}$,) we first notice that no student finally assigned to school 2 has a type above $\tilde{t}^p$. Then

$$
\begin{aligned}
p \quad &= \quad h(q(\Phi_{t \geq t_3}), t_3) + \Delta_p - h(\tilde{q}_3(\tilde{t}^p, t_3), t_3) \\
&> \quad h(q(\Phi_{t \geq t_3}), \tilde{t}^p) + \Delta_p - h(\tilde{q}_3(\tilde{t}^p, t_3), \tilde{t}^p) \\
&> \quad h(q(\Phi_{t \geq t_3}), \tilde{t}^p) + \Delta_p - h(q_2(\tilde{t}^p), \bar{t}) - \Delta
\end{aligned}
$$

where the first inequality comes from supermodularity of h and the second inequality is due to $\tilde{G}(\tilde{t}^p, t_3) = 0$, which implies $h(\tilde{q}_3(\tilde{t}^p, t_3), \tilde{t}^p) - h(q_2(\tilde{t}^p), \tilde{t}^p) -$

$\Delta < 0$. Consequently no student assigned to school 2 chooses to enrol at the private school (by supermodularity of h, the inequality holds for all $t \leq \tilde{t}^p$.)

Only types above $\tilde{t}^p$ are assigned to school 1. We check that the highest type $\bar{t}$ prefers school 1 to the private school (that would suffice because 1) the peer quality of the higher school is higher than that of school 1, and 2) h is supermodular.) Indeed, since $q_1(\tilde{t}^p) \geq \tilde{q}_3(\tilde{t}^p, t_3)$, we have

$$h(q(\Phi_{t \geq t_3}), \bar{t}) + \Delta_p - h(q_1(\tilde{t}^p), \bar{t}) - \Delta$$
$$< \quad h(q(\Phi_{t \geq t_3}), \bar{t}) + \Delta_p - h(\tilde{q}_3(\tilde{t}^p, t_3), \bar{t})$$

and for t_3 sufficiently close to $\bar{t}$, we have $h(q(\Phi_{t \geq t_3}), \bar{t}) + \Delta_p - h(q_1(\tilde{t}^p), \bar{t}) - \Delta < p$.

We have seen that there is an interval of values $t_3 \in (t_3^*, \bar{t})$ for which all the initial assumptions hold. We complete the proof by defining $\eta_p^* = 1/3[1 - \Phi_3(t_3^*)]$, where Φ_3 is the distribution of student types at school 3 under the equilibrium where $t_3 = t_3^*$. For any capacity $\eta_p < \eta_p^*$ the result in proposition 2 holds true.

Proof. Proposition 3.

As in the proof of proposition 1 (b), assume that agents play either one of the following ranking decisions: ranking 1 above 2 and 2 above 3 ("$1 \succ 2 \succ 3$" from now on,) and ranking 2 above 1 and 1 above 3 ("$2 \succ 1 \succ 3$" hereafter.) Furthermore, assume that equilibrium beliefs are such that $\hat{q}_1 \geq \hat{q}_3 > \hat{q}_2$. (We later check that these assumptions are accomplished in equilibrium.) ∎

Let p_{ij} denote the probability of enrolling into school i if the student ranks school $j = 1, 2$ in first position. We know from that preceding proof that $p_{i1} > p_{i2}$ for $i = 1, 3$ and that $p_{22} > p_{21}$ in equilibrium. Since the utility loss due to residential misallocation of families does not depend on their ability types, the difference in expected payoffs for the cut-off type between playing a ranking decision "$1 \succ 2 \succ 3$" and playing "$1 \succ 2 \succ 3$" is increasing in the type t, again as in the aforementioned proof. We can then restrict attention to cut-off ranking profiles.

The analysis hereafter assumes that, conditional on a ranking decision, the location decision is optimal, and prices accommodate so as to preclude excess residential demands.

Consider a case in which the mass of families ranking school 1 first exceeds $2/3$ and needs to be located in all three neighborhoods. Residential prices should then make these individuals indifferent among the three neighborhoods, thus neighborhood 2 must be the cheapest one, since $p_{21} = \frac{1/3 - \Phi(\hat{t})}{1 - \Phi(\hat{t})} <$

$p_{11} = p_{31} = \frac{1/3}{1-\Phi(\hat{t})}$. Consequently, all those families who rank school 2 first optimally choose to live in neighborhood 2. The cut-off type $\hat{t}$ without loss of generality lives in neighborhood 2 as well. Then the expected payoff of the cut-off type in case of ranking school 1 first minus the expected payoff of the cut-off type in case of ranking school 2 first is reduced by an amount $(1-p_{21})c = \frac{2/3}{1-\Phi(\hat{t})}c$, linked to the probability of not being assigned to school 2. Residential prices would be $\pi_1 = \pi_3 = (p_{31} - p_{21})c = \frac{\Phi(\hat{t})}{1-\Phi(\hat{t})}c$ when π_2 is normalized to zero.

Consider the case in which both schools 1 and 2 are overdemanded in the first round. In such an event, neither neighborbood 1 nor neighborhood 2 could host the lowest residential price. If neighborhood 1 were the cheapest, all the mass of types ranking school 1 first, exceeding $1/3$, would choose to live in neighborhood 1, leading to excess demand. Same argument follows for neighborhood 2. Given that neighborhood 3 is the cheapest, no one ranking school 1 (2) in first position would ever optimally reside in neighborhood 2 (1) (a neighborhood whose school will never admit this student when both schools 1 and 2 are overdemanded in the first round.) So it is without loss of generality that the cut-off type $\hat{t}$ (who in equilibrium is indifferent between strategies) lives in neighborhood 3. Then the expected differences in payoffs for the cut-off type in case of ranking school 1 instead of school 2 in first position is increased by an amount $(p_{31} - p_{32})\,c = 1/3\frac{1-2\Phi(\hat{t})}{\Phi(\hat{t})(1-\Phi(\hat{t}))}c$, linked to the probabilities of being assigned to the school ranked first. Notice that in both cases the residential prices do not determine the nature of the equilibrium cut-off. Residential prices would be $\pi_1 = (p_{11} - p_{31})c = \frac{\Phi(\hat{t})-1/3}{1-\Phi(\hat{t})}c$ and $\pi_2 = (p_{22} - p_{32})c = \frac{2/3-\Phi(\hat{t})}{\Phi(\hat{t})}c$, after normalizing π_3 to zero.

Finally, consider the case in which $\hat{t}$ equals $t_{1/3}$. If the mass of families playing "$1 \succ 2 \succ 3$" spread locations among all neighborhoods, we have seen that in that case neighborhood would hold the lowest residential prices. But then all those who use "$2 \succ 1 \succ 3$", with mass $1/3$, would optimally choose to live in neighborhood 2, leading to excess residential demand in such neighborhood. Suppose now that families playing "$1 \succ 2 \succ 3$" locate on neighborhoods 1 and 2, so that families playing "$2 \succ 1 \succ 3$" have to occupy neighborhood 3. For that we need $\pi_2 - \pi_3 \geq c$, to compensate for the misallocation of the latter players. Moreover, since $p_{11} = 1/2 > p_{21} = 0$ we need $\pi_1 - \pi_2 = c/2$, to make the former families indifferent between neighborhoods 1 and 2. But then $\pi_1 > \pi_3$ and since $p_{11} = p_{31} = 1/2$

no student playing "$1 \succ 2 \succ 3$" would optimally live in neighborhood 1 (being neighborhood 3 a better option.) Using the same reasoning we can also discard that families playing "$1 \succ 2 \succ 3$" locate on neighborhoods 3 and 2. The only remaining option is that families playing "$1 \succ 2 \succ 3$" locate on neighborhoods 1 and 3, and hence $\pi_1 = \pi_3$, while families who use "$2 \succ 1 \succ 3$" locate in neighborhood 2. For the former group ("$1 \succ 2 \succ 3$") to avoid living in neighborhood 2 we require $\pi_1 - \pi_2 \leq c/2$. For the latter group ("$2 \succ 1 \succ 3$") to optimally choose to live in neighborhood 2 we require $\pi_1 - \pi_2 \geq -c$.

Pick any $\pi_1 - \pi_2 \in [-c, c/2]$. If the cutoff type chooses to play "$1 \succ 2 \succ 3$" (and hence to live in either neighborhood 1 or 3), her extra expected payoff as compared to what she obtains when playing "$2 \succ 1 \succ 3$" (and living in neighborhood 2) is increased by $-(\pi_1 - \pi_2) - c/2$ (the latter element being c times the probability of misallocation given ranking decision "$1 \succ 2 \succ 3$"), which belongs to the interval $[-c, c/2]$. Any effect in this interval can be supported by market-clearing residential prices.

We construct a (possibly set-valued) function $\bar{G}(\hat{t}, c)$ that measures the difference in expected payoffs for the cut-off type between playing strategy "$1 \succ 2 \succ 3$" (and a corresponding optimal location choice given market clearing prices) and playing strategy "$1 \succ 2 \succ 3$" (and a corresponding optimal location choice given market clearing prices) as

$$
\bar{G}(\hat{t}, c) = \begin{cases} G(\hat{t}) + 1/3 \frac{1 - 2\Phi(\hat{t})}{\Phi(\hat{t})(1 - \Phi(\hat{t}))} c, \hat{t} \in (t_{1/3}, t_{1/2}) \\ [G(\hat{t}) - c, G(\hat{t}) + c/2], \hat{t} = t_{1/3} \\ G(\hat{t}) - \frac{2/3}{1 - \Phi(\hat{t})} c, \hat{t} \in (\underline{t}, t_{1/3}) \end{cases}
$$

where $G(\hat{t})$ was defined in the proof or proposition 2 (b). Notice that $\bar{G}(\hat{t}, c)$ is upper-hemicontinuous. An equilibrium cutoff $\tilde{t}^c$ accomplishes with $0 \in \bar{G}(\tilde{t}^c, c)$.

Consider an equilibrium cutoff $\tilde{t}$ in the Boston Mechanism without preferences for nearby schools.

Assume that $\tilde{t} < t_{1/3}$. Then $\bar{G}(\tilde{t}, c) < 0$. Notice as well that the sign of $G(\hat{t})$ for $\hat{t} < t_{1/3}$ is the sign of $2(h(q_3(\hat{t}), \hat{t}) - h(q_2(\hat{t}), \hat{t})) - \Delta$, which is increasing in $\hat{t} < t_{1/3}$ (we know this from the proof of proposition 1.) Hence $G(t_{1/3}) > 0$ and we know that $G(t_{1/3}) \in \bar{G}(t_{1/3}, c)$. By upper-hemicontinuity of h there must be some $\tilde{t}^c \in (\tilde{t}, t_{1/3}]$ such that $0 \in \bar{G}(\tilde{t}^c, c)$. Now consider $\tilde{t} = t_{1/3}$. Then $0 = G(t_{1/3}) \in \bar{G}(t_{1/3}, c)$ thus we set $\tilde{t}^c = t_{1/3}$.

Consider the case in which every equilibrium threshold $\tilde{t}$ satisfies $\tilde{t} > t_{1/3}$. Since $G(\underline{t}) < 0$ and $2(h(q_3(\hat{t}), \hat{t}) - h(q_2(\hat{t}), \hat{t})) - \Delta$ is increasing in $\hat{t} < t$, we must have $G(t_{1/3}) < 0$. Now, since $\bar{G}(\tilde{t}^c, c) > G(\tilde{t}) = 0$ and $G(t_{1/3}) \in \bar{G}(t_{1/3}, c)$, by upper-hemicontinuity of h there must be some $\tilde{t}^c \in [t_{1/3}, \tilde{t})$ such that $0 \in \bar{G}(\tilde{t}^c, c)$.

Stability of such equilibria is immediate as seen in previous proofs with cutoff ranking profiles.

We now show that the initial assumptions are correct in equilibrium. Clearly $q_1 \geq q_3 > q_2$ ex post. We skip the proof, for it mimics the analogous proof in proposition 1 (b).

As for alternative strategies: there are two main alternative ranking decisions, "$1 \succ 3 \succ 2$" and "$3 \succ ...$". The former is irrelevant when $\tilde{t}^c \geq t_{1/3}$ (it leads to the same assignment as "$1 \succ 2 \succ 3$".) So in case $\tilde{t}^c \geq t_{1/3}$, we only consider "$3 \succ ...$", leading to sure assignment to school 3, which is optimally accompanied by living in neighborhood 3 (the cheapest neighborhood when $\tilde{t}^c > t_{1/3}$, and the preferred choice in case $\tilde{t}^c = t_{1/3}$ even under the worst price $\pi_3 = c/2$.) Since $q_1 \geq q_3$, it is clear that "$1 \succ 2 \succ 3$" is better than "$3 \succ ...$" for all $t \in D$ provided $c < \Delta$. Those families who instead play "$2 \succ 1 \succ 3$" prefer it to "$1 \succ 2 \succ 3$", which is again preferred to "$3 \succ ...$".

We analyze the case $\tilde{t}^c < t_{1/3}$. Both "$1 \succ 3 \succ 2$" and "$3 \succ ...$" are optimally accompanied by living in neighborhood 3. To see that, we first discard neighborhood 1 as an optimal residential choice, since $\pi_1 = \pi_3$ and being assigned to school 3 is more likely than being assigned to school 1. Knowing that the probability of entering school 3 is at least $1 - p_{11} = p_{21} + p_{31}$ and that $\pi_3 - \pi_2 = (p_{31} - p_{21})c$, living in neighborhood 2 is also discarded (being assigned to school 2 is impossible under these alternative ranking decisions.) We then observe that, conditional again on $c < \Delta$, the ranking decision "$1 \succ 3 \succ 2$" is better than "$3 \succ ...$" for all types. We then compare "$1 \succ 3 \succ 2$" against "$1 \succ 2 \succ 3$". Is such a case we see that the latter is better option than the former for a $t-$type family if and only if $h(q_3(\tilde{t}^c), t) < h(q_2(\tilde{t}^c), t) + \Delta - c$. (The $-c$ comes from the fact that under the decision "$1 \succ 2 \succ 3$", the individual is indifferent among residential locations, hence she chooses neighborhood 3 for easiness in the comparison: being assigned to school 2 carries then a misallocation utility loss c.)

Under the ghetto school assumption we have $\Delta > h(q_3(\tilde{t}^c), \bar{t}) - h(q_2(\tilde{t}^c), \bar{t}) \geq h(q_3(\tilde{t}^c), t) - h(q_2(\tilde{t}^c), t), \forall t \in D$ (by supermodularity of h.) For c low enough we have $h(q_3(\tilde{t}^c), t) < h(q_2(\tilde{t}^c), t) + \Delta - c$ as desired.

Under the limited complementarity assumption we have $h(q_3(\tilde{t}^c), \bar{t}) - h(q_2(\tilde{t}^c), \bar{t}) < 2(h(q_3(\tilde{t}^c), \tilde{t}^c) - h(q_2(\tilde{t}^c), \tilde{t}^c))$, and for c low enough we still have $h(q_3(\tilde{t}^c), \bar{t}) - h(q_2(\tilde{t}^c), \bar{t}) + c < 2(h(q_3(\tilde{t}^c), \tilde{t}^c) - h(q_2(\tilde{t}^c), \tilde{t}^c) - c)$. Now, the equilibrium condition for $\tilde{t}^c < t_{1/3}$ is $2(h(q_3(\tilde{t}^c), \tilde{t}^c) - h(q_2(\tilde{t}^c), \tilde{t}^c) - c) = \Delta$. With this an supermodularity of Δ we obtain $h(q_3(\tilde{t}^c), t) < h(q_2(\tilde{t}^c), t) + \Delta - c$ $\forall t \in D$ as desired. ∎

Proof. Proposition 4.

(a) If sorting between schools 1 and 2 happens, we have $q_1 > q_2$ ex post. But under the correct beliefs, every student prefers school 1 to school 2. Since Deferred Acceptance is strategy-proof, every student optimally ranks school 1 above school 2. But then, the distribution of students finally assigned to school 1 is indistinguishable from that of school 2, contradicting $q_1 > q_2$. Same reasoning denies sorting between school 1 (or 2) and school 3.

(b) For Δ sufficiently large, school 3 is the least-preferred one for all students regardless the peer qualities in all schools. The question is whether to rank school 1 in first position or instead school 2. Again the same argument as in (a) leads to $q_1 = q_2$ just after the assignment has taken place and before families take a decision on enrolling at the private school. The consistency condition on beliefs imposed in the main text precludes the rise of peer quality differences between these two schools ex post.

(c) Select c small. There is a cutoff value $\tilde{t}^D > \underline{t}$ close to $\underline{t}$ defined as $h(q(\Phi_{t \geq \tilde{t}^D}), \tilde{t}^D) - c = h(q\left(2\Phi(\tilde{t}^D)\Phi_{t \leq \tilde{t}^D} + (1 - 2\Phi(\tilde{t}^D))\Phi_{t \geq \tilde{t}^D}\right), \tilde{t}^D)$. Suppose beliefs are $\hat{q}_1 = q(\Phi_{t \geq \tilde{t}^D}) > \hat{q}_3 = q(\Phi) > \hat{q}_2 = q\left(2\Phi(\tilde{t}^D)\Phi_{t \leq \tilde{t}^D} + (1 - 2\Phi(\tilde{t}^D))\Phi_{t \geq \tilde{t}^D}\right)$. c is sufficiently small so that $h(\hat{q}_2, t) + \Delta > h(\hat{q}_3, t)$ for all $t \in D$. Assume that families with type $t > \tilde{t}^D$ report "$1 \succ 2 \succ 3$", whereas families with type below $\tilde{t}^D$ report "$2 \succ 1 \succ 3$". The mass of families reporting "$1 \succ 2 \succ 3$" exceeds $2/3$, so they have to spread along all neighborhoods. We set residential prices $\pi_1 = 2/3\frac{\Phi(\tilde{t}^D)}{1 - \Phi(\tilde{t}^D)}c > \pi_3 = 1/3\frac{\Phi(\tilde{t}^D)}{1 - \Phi(\tilde{t}^D)}c > \pi_2 = 0$ so that all families who report "$1 \succ 2 \succ 3$" are indifferent among all possible residential choices (notice that $p_{11} = \frac{1/3}{1 - \Phi(\tilde{t}^D)}, p_{21} = 2/3 - p_{11}, p_{31} = 1/3$.) Under these circumstances, all families who report "$2 \succ 1 \succ 3$" optimally choose to leave in neighborhood 2 ($p_{12} = 0, p_{22} = 2/3, p_{32} = 1/3$ —note that Deferred Acceptance leads to all students having equal chances to end up in the worst school, regardless the chosen ranking decision.) It is without loss of generality that we assume a family lives in neighborhood 2. In such a case, type $\tilde{t}^D$ is indifferent between schools 1 and 2, types above prefer school 1 and types below prefer school 2. By strategy-proofness, the cutoff profile with thresh-

old $\tilde{t}^D$ (types above report "$1 \succ 2 \succ 3$" and types below report "$2 \succ 1 \succ 3$") constitutes an equilibrium ranking profile. It is easy to see that beliefs are confirmed ex post and that there is sorting between schools 1 and 2. Also, it can be readily checked that $\tilde{t}^D$ converges to $\underline{t}$ as c converges to 0, leading to limit equivalent distributions of student types (Φ) across schools. As for the comparison with an equilibrium cutoff $\tilde{t}^c$ under BM. If $\tilde{t}^c \geq t_{1/3}$ we obviously have $\tilde{t}^D < \tilde{t}^c$ for c low enough. Thus we assume $\tilde{t}^c < t_{1/3}$. The equilibrium condition is then

$$
\begin{aligned}
& h(q(\Phi_{t \geq \tilde{t}^c}), \tilde{t}^c) - h(q\left(2\Phi(\tilde{t}^c)\Phi_{t \leq \tilde{t}^c} + (1 - 2\Phi(\tilde{t}^c))\Phi_{t \geq \tilde{t}^c}\right), \tilde{t}^c) \\
= \; & c + \Delta/2 > c = \\
= \; & h(q(\Phi_{t \geq \tilde{t}^D}), \tilde{t}^D) - h(q\left(2\Phi(\tilde{t}^D)\Phi_{t \leq \tilde{t}^D} + (1 - 2\Phi(\tilde{t}^D))\Phi_{t \geq \tilde{t}^D}\right), \tilde{t}^D)
\end{aligned}
$$

By supermodularity of h, the fact that $q(\Phi_{t \geq \tilde{t}})$ increases in $\tilde{t}$ and the fact that $q\left(2\Phi(\tilde{t})\Phi_{t \leq \tilde{t}} + (1 - 2\Phi(\tilde{t}))\Phi_{t \geq \tilde{t}}\right)$ decreases in $\tilde{t}$, we conclude that $\tilde{t}^D < \tilde{t}^c$. ∎

7.2 The base model with more than 3 schools

Suppose that we have $J > 3$ equally sized schools and that school J is bad (being assigned there entails a utility loss equal to $\Delta > 0$). Is there an equilibrium with (full) sorting?

Proposition 5 *In BM as in the base model yet with $J > 3$ equally sized schools and being school J the only bad school, if Δ is sufficiently large, there only exist stable equilibria with full sorting between every pair of good schools i, j such that $i < j$.*

Proof. We assume that h is differentiable (h_t denotes its partial derivative with respect to type). The first step is to show that with Δ sufficiently large, all schools apart from the bad one give all their slots in the first round of the assignment algorithm in equilibrium. Suppose not. If a strict subset $S \subset \{1, ..., J - 1\}$ of good schools do not give all its slots in the first round, then any student who ranks a school from S in first position avoids the punishment Δ for sure. Conditional on ranking any school in the complement of S first, the probability of being assigned to the worst school (hence suffering the utility loss Δ) is on average at least $\frac{1}{J - \#S}$. Setting $\Delta > (J - \#S)[h(q(\Phi_{t | t \geq t_{(J-1)/J}}), \bar{t}) - h(q(\Phi_{t | t \leq t_{1/J}}), \bar{t})]$, some types who were

not ranking a school from S in first position would be strictly better-off ex ante by doing so. Hence we did not have a best-response profile, a contradiction.

In that context, strategies can be simplified to "what good school to rank first": a total of $J - 1$ relevant strategies. The second step is to show how, when the second round assigns slots only to the worst school, an equilibrium with beliefs $\hat{q}_1 > \hat{q}_2 > ... > \hat{q}_{J-1}$ is characterized by cutoffs in the ranking profile R. Let p_i and p_j denote the probabilities of being accepted at schools i and j respectively, conditional on ranking respectively i or j first. Let also $q_i > q_j$. Conditional on ranking i first, the expected payoff for a t-type household is $p_i h(q_i, t) + (1 - p_i)[h(q_J, t) - \Delta]$. An analogous expected payoff form arises when ranking j first. A t-type household prefers to rank i first over ranking j first if $\frac{h(q_i,t) - h(q_J,t) + \Delta}{h(q_j,t) - h(q_J,t) + \Delta} > \frac{p_j}{p_i}$. The left-hand side ratio is increasing in t if Δ is high enough: its first derivative is positive when

$$h_t(q_i, t) - h_t(q_J, t) > [h_t(q_j, t) - h_t(q_J, t)] \frac{h(q_i, t) - h(q_J, t) + \Delta}{h(q_j, t) - h(q_J, t) + \Delta}$$

The fraction on the right-hand side is arbitrarily close to 1 as Δ becomes sufficiently large, and $h_t(q_i, t) > h_t(q_j, t)$ by supermodularity of h. Then for Δ sufficiently large the inequality above holds regardless the probabilities p_i and p_j. Since $\frac{h(q_i,t) - h(q_J,t) + \Delta}{h(q_j,t) - h(q_J,t) + \Delta}$ is increasing, there exists a threshold $\hat{t}_{ij}$ such that types above it prefer to rank i first over ranking j first while types below prefer the opposite. Moreover, for any triple of good schools i, j, k such that $q_i > q_j > q_k$, we have $\hat{t}_{ij} > \hat{t}_{jk}$ (otherwise no student would rank school j first, contradicting the fact that every good school gives all of its slots in the first round). Therefore we have proven that a series of thresholds $\bar{t} \equiv \hat{t}_{01} > \hat{t}_{12} > \hat{t}_{23} > ... > \hat{t}_{J-2,J-1} > \hat{t}_{J-1,J} \equiv \underline{t}$ (where types in $(\hat{t}_{j,j+1}, \hat{t}_{j-1,j})$ rank school j first) characterize a best-response profile that produces $q_1 > q_2 > ... > q_{J-1}$ ex post.

Existence: Note that $p_j = \frac{1/3}{\Phi(\hat{t}_{j-1,j}) - \Phi(\hat{t}_{j,j+1})}$ when best responses are characterized by cutoffs as depicted above. As an equilibrium feature we must have $p_j < p_{j+1}$ for every good school j (lower peer qualities must be compensated with higher admission chances.) The suggested equilibrium would sat-

isfy, for every $j = 1, .., J-2$, noting and using the notation $\mathcal{T} \equiv (\hat{t}_{01}, \hat{t}_{12}, \hat{t}_{23}, ..., \hat{t}_{J-2,J-1}, \hat{t}_{J-1,J})$:

$$
\begin{aligned}
0 &= G_{j,j+1}(\mathcal{T}) \\
&\equiv \; [\Phi(\hat{t}_{j,j+1}) - \Phi(\hat{t}_{j+1,j+2})]h(q_j(\hat{t}_{j,j+1}; \hat{t}_{j-1,j}), \hat{t}_{j,j+1}) \\
&\quad -[\Phi(\hat{t}_{j-1,j}) - \Phi(\hat{t}_{j,j+1})]h(q_{j+1}(\hat{t}_{j,j+1}; \hat{t}_{j+1,j+2}), \hat{t}_{j,j+1}) \\
&\quad +[\Phi(\hat{t}_{j-1,j}) - 2\Phi(\hat{t}_{j,j+1}) + \Phi(\hat{t}_{j+1,j+2})][h(q_J(\mathcal{T}), \hat{t}_{j,j+1}) - \Delta]
\end{aligned}
$$

where $q_j(\hat{t}_{j,j+1}; \hat{t}_{j-1,j}) \equiv q(\Phi_{t|\hat{t}_{j,j+1}\leq t\leq \hat{t}_{j-1,j}}) > q_{j+1}(\hat{t}_{j,j+1}; \hat{t}_{j+1,j+2}) \equiv q(\Phi_{t|\hat{t}_{j+1,j+2}\leq t\leq \hat{t}_{j,j+1}})$, and $q_J(\mathcal{T}) \equiv q(\Phi_J|\mathcal{T})$, where $\Phi_J|\mathcal{T}$ is a convex combination of all preceding $\Phi_{t|\hat{t}_{j,j+1}\leq t\leq \hat{t}_{j-1,j}}$, $j = 1, ..., J-1$.

Such an equilibrium exists because (1) $\displaystyle\lim_{\Phi(\hat{t}_{j,j+1})\to\Phi(\hat{t}_{j+1,j+2})+1/J} G_{j,j+1}(\mathcal{T}) < 0$ for Δ high enough and (2) $\displaystyle\lim_{\Phi(\hat{t}_{j,j+1})\to\frac{\Phi(\hat{t}_{j-1,j})+\Phi(\hat{t}_{j+1,j+2})}{2}} G_{j,j+1}(\mathcal{T}) > 0$. Since each function $G_{j,j+1}(\mathcal{T})$ is continuous we can make use of the intermediate value theorem to show existence. Its cutoff nature guarantees stability.

Finally, we show that an equilibrium with no sorting between good schools i and j (implying $q_i = q_j$) is not stable. Such an equilibrium belief cannot be confirmed ex post through a threshold-like strategy profile (where only types above some threshold $\hat{t}_{ij}$ can rank one of the schools first and only types below can rank the other school first.) Yet for any sequence $(q_i^n, q_j^n) \to (q_i, q_j)$ with $q_i^n \neq q_j^n$ we have that the best-response profile R^n is characterized by a threshold $\hat{t}_{ij}^n$ such that only types above the threshold can rank one of the schools first and only types below can rank the other school first. Then the limit of the sequence R^n cannot converge to the initial equilibrium strategies supporting $q_i = q_j$. ∎

7.3 The type-represents-income model

When types are linked to family incomes and h is not supermodular, it is clear that, in the absence of private schools, no segregation can arise. However, when private schools exist, segregation arise due to the differing willingness to pay for the outside option across types. We prove the following result.

Proposition 6 *Suppose that types represent incomes. Fix any cutoff $t_3 < t_{1/2}$ separating those who do not make use of the private school from those who do in case they are assigned to school 3. Then, for Δ sufficiently high, there is a capacity of the private school η_p such that there is a stable equilibrium and a*

fulfilling capacity price p (such that t_3 is indifferent between school 3 and the private school) in which there is full sorting between school 1 and school 2, full sorting between school 1 and school 3, and partial sorting between school 2 and school 3.

Segregation emerges in this scenario if all households who rank school 1 first have the back up of the private school, that is, if they have enough income to avoid ending up in the ghetto school if rejected from school 1. Put differently, segregation arises when those who cannot afford the private school misrepresent their preferences to play a safer strategy. The single-crossing condition then holds: because higher income agents have lower marginal utility of income, their utility cost of paying tuition fees p is smaller. Hence their relative valuation of the private school (and so of strategy 1) is larger *even if their kids do not benefit more from school quality than others.*[25]

Proof. Fix a threshold $t_3 \in (\underline{t}, t_{1/2})$ such that, by assumption, types below the threshold optimally stay in school 3 in case they are assigned there, and types above choose instead to pay the tuition fee p (that will be calculated later) for the private school. Conditional on t_3 for every $\hat{t} \in (t_3, t_{1/2})$ we study the cutoff ranking strategy profile in which types above the cutoff $\hat{t}$ adopt the ranking decision "$1 \succ 2 \succ 3$" whereas families with lower type declare "$2 \succ 1 \succ 3$" instead constitutes a (stable) equilibrium strategy profile.

We assume (we later fix parameter Δ to make this assumption correct) that none of the students assigned to either school 1 or 2 enrolls at the private school. Then the two cutoffs t_3 and $\hat{t}$ drive all ex-post peer qualities: $q_1(\hat{t}) = q(\Phi_{t \geq \hat{t}})$, $q_2(\hat{t}) = q(\Phi_{t \leq \hat{t}})$, $q_3(\hat{t}) = q(\Phi_{t \leq t_3})$, and

$$q_p(\hat{t}) = q\left(\frac{2/3 - \Phi(\hat{t})}{2/3 - \Phi(\hat{t}) + [\Phi(\hat{t}) - \Phi(t_3)]\frac{\Phi(\hat{t})-1/3}{\Phi(\hat{t})}} \Phi_{t \geq \hat{t}} + \frac{[\Phi(\hat{t}) - \Phi(t_3)]\frac{\Phi(\hat{t})-1/3}{\Phi(\hat{t})}}{2/3 - \Phi(\hat{t}) + [\Phi(\hat{t}) - \Phi(t_3)]\frac{\Phi(\hat{t})-1/3}{\Phi(\hat{t})}} \Phi_{t_3 \leq t \leq \hat{t}} \right)$$

. It can be readily checked that $q_1(\hat{t}) > q_p(\hat{t}) > q_2(\hat{t}) > q_3(\hat{t})$. Obviously beliefs will correspond to ex post qualities in equilibrium. They also drive public school assignment probabilities p_{ij} conditional on the school 1 that is ranked in first position (either school 1 or 2): $p_{11}(\hat{t}) = \frac{1/3}{1-\Phi(\hat{t})}$

Cutoff strategy profile. Given the cutoff $\hat{t}$, the tuition fee is determined by the equation $h_1(q_p(\hat{t})) + \Delta_p - h_1(q_3(\hat{t})) = u(t_3) - u(t_3 - p(\hat{t}))$. For types $t > t_3$,

---

[25]Otherwise, the relative valuation of strategies 1 and 2 does not change with type and the single-crossing condition only holds weakly.

being assigned to school 3 is followed by enrollment in the private school. Then the payoff from the ranking decision "$1 \succ 2 \succ 3$" is $p_{11}(\hat{t})(h_1(q_1(\hat{t})) + \Delta + u(t)) + (1 - p_{11}(\hat{t}))(h_1(q_p(\hat{t})) + \Delta_p + u(t - p(\hat{t})))$. The payoff from "$2 \succ 1 \succ 3$" is $p_{22}(\hat{t})(h_1(q_2(\hat{t})) + \Delta + u(t)) + (1 - p_{22}(\hat{t}))(h_1(q_p(\hat{t})) + \Delta_p + u(t - p(\hat{t})))$. The difference between payoffs is increasing in t (recall that for cutoffs in $(t_{1/3}, t_{1/2})$ we have $p_{22}(\hat{t}) > p_{11}(\hat{t})$ and that u is concave,) and for the cutoff type this difference is

$$\begin{aligned} \hat{G}(\hat{t}) &= p_{11}(\hat{t})(h_1(q_1(\hat{t})) + \Delta - h_1(q_p(\hat{t})) - \Delta_p) \\ &\quad - p_{22}(\hat{t})(h_1(q_2(\hat{t})) + \Delta - h_1(q_p(\hat{t})) - \Delta_p) \\ &\quad - (p_{22}(\hat{t}) - p_{11}(\hat{t}))(u(\hat{t}) - u(\hat{t} - p(\hat{t}))) \end{aligned}$$

In the lower limit of the cutoff interval, we show that $\hat{G}(t_3) < 0$ for Δ large enough. Provided that this type is indifferent between school 3 and the private school, we have $\hat{G}(t_3) = (h_1(q_1(t_3)) - h_1(q_2(t_3)) - \Delta)/2$ (after noticing that $p_{22}(t_3) = 1$ and $p_{11}(t_3) = 1/2$, and that $q_2(t_3) = q_3(t_3)$) Then there is $\Delta(t_3)$ such that for all $\Delta > \Delta(t_3)$ we obtain the desired result. We can also easily see that $\hat{G}(t_{1/2}) > 0$, based on $p_{22}(t_{1/2}) - p_{11}(t_{1/2}) = 2/3$. By continuity of $\hat{G}$ and the intermediate value theorem, for each $\Delta > \Delta(t_3)$ there is at least one cutoff $\tilde{t}(\Delta, t_3)$ such that $\hat{G}(\tilde{t}(\Delta, t_3)) = 0$.

It is clear that provided $\Delta \geq \Delta_p$ all students prefer school 1 to the private school for any tuition fee (school 1 is free and enjoys higher peer quality.) We check that all students prefer school 2 to the private school, that is $h_1(q_p(\tilde{t}(\Delta, t_3))) - h_1(q_2(\tilde{t}(\Delta, t_3))) + \Delta_p - \Delta < u(\tilde{t}(\Delta, t_3)) - u(\tilde{t}(\Delta, t_3) - p(\tilde{t}(\Delta, t_3)))$, for Δ large enough. This comes from the fact that all students prefer school 1 to the private school under $\Delta \geq \Delta_p$ and from $\hat{G}(\tilde{t}(\Delta, t_3)) = 0$. Thus for our fixed t_3, for any $\Delta > \max\{\Delta(t_3), \Delta_p\}$, there is a cutoff $\tilde{t}(\Delta, t_3)$ and a private school tuition fee $p = p(\tilde{t}(\Delta, t_3))$ so that our proposed cutoff strategy profile (types above the cutoff declare "$1 \succ 2 \succ 3$", types below decide "$2 \succ 1 \succ 3$" instead) constitutes an equilibrium strategy profile under beliefs $q_1(\tilde{t}(\Delta, t_3)) > q_p(\tilde{t}(\Delta, t_3)) > q_2(\tilde{t}(\Delta, t_3)) > q_3(\tilde{t}(\Delta, t_3))$. Capacity η_p is set to accommodate all students who are assigned to school 3 with types above t_3.

Stability is proven given the cutoff nature of the ranking decision profile. Sorting relations as depicted in the proposition are apparent from previous paragraphs ($\Phi_1 FOSD \Phi_p FOSD \Phi_2 FOSD \Phi_3$). ∎

7.4 Two-dimensional characteristics space

A two-dimensional type space is useful, since it not only considers ability (our "type" t) but also wealth (denoted by y). This subsection extends some results in considering a model with two income levels, H and L, where $H > L$. Conditional on the income level y, the ability distribution is $\Phi(t|y)$. We assume that there is positive correlation between income and ability in such a way that $\Phi(\cdot|H)$ $FOSD$ $\Phi(\cdot|L)$. A mass $\lambda \in (0, 1/2)$ of households has income H and the rest have income L. In order to talk about sorting of abilities across schools, we analyze each subpopulation (high- and low-income households separately). The definitions in the paper can be used for each subpopulation. $\Phi_j(t|y)$ would denote the distribution of ability types among those attending school j conditional on having income y. Accordingly, the ex-post school quality is $q_j = q(\lambda \cdot \Phi_j(\cdot|H) + (1 - \lambda) \cdot \Phi_j(\cdot|L))$. Utility is now defined as $V(q, \Delta, p; t, y) = u(y - p) + h(q, t) + \Delta$, where u is increasing and concave and h is increasing and supermodular. We further assume that $u(t) - u(t - p)$ tends to 0 as t grows large. We assume in this subsection that $h(q, \underline{t})$ is invariant in q (the lowest type does not benefit from peer qualities.) Finally, we assume that $\Delta > \Delta_p$. We analyze cutoff equilibria characterized by thresholds $\tilde{t}_H$ and $\tilde{t}_L$ for rich and poor families respectively (ability types above the threshold declare "$1 \succ 2 > 3$", types below use the ranking "$2 \succ 1 > 3$".)

Private school and no priorities

We want to explore the interesting case where the private school is overly expensive for poor families but affordable for richer families. In an extreme illustrative case we could assume $t_3^H = \underline{t}$. This could be done by properly increasing H so that $u(H - p) + \Delta_p \geq u(H)$ (recall that a $\underline{t}$−type household does not care about school quality differences, and that $u(H) - u(H - p)$ is decreasing in H.) But then, all rich households face less risk than poorer households since not being admitted in a good public school has as a consequence being enrolled in the private school, as compared to the bad school. Consequently, rich households would tend to bet for school 1 rather than the safer option of school 2. In equilibrium we would have $\tilde{t}_H < \tilde{t}_L < t_3^L$.

Key here is that in such equilibrium *the cutoff type among rich families makes use of the private school in case the student gets assigned to school 3, an option that is not used by the cutoff type among poor families.* The baseline model predicted an "ability elitization" of school 1 (top ability types get more chances at school 1), as compared to a scenario with no private school. When

we introduce income differences and non quasilinear utilities, there is also an "income elitization" effect.

Proposition 7 *Fix L and let $\Delta \geq h(q(\bar{t}), \bar{t}) - h(q(\Phi_{t|t \leq t_{1/3}}), \bar{t})]$. If H is high enough, then there is a capacity η_p associated to such H under which there exists a stable equilibrium characterized by cutoffs $\tilde{t}_H^p < \tilde{t}_L^p$ such that households with income $y \in \{L, H\}$ rank school 1 first if their ability types are above $\tilde{t}_y^p$, and they rank school 2 in first position otherwise.*

Proof. We restrict attention to cutoff strategy profiles as depicted in the proposition, and we assume peer qualities satisfy $\min\{q_p, q_1\} \geq q_3 > q_2$ ex post (a condition that will hold in equilibrium.) Fix $t_3^L > \frac{1/2 - \lambda}{1 - \lambda}$ so that we make sure that the equilibrium cut-off type $\tilde{t}_L^p$ for income L does not choose the private school against school 3 ($\tilde{t}_L^p > \frac{1/2 - \lambda}{1 - \lambda}$ would imply that more families are top-ranking school 2 than top-ranking school 1, impossible in equilibrium.) Setting H high enough, we make sure that $u(H - p) + \Delta_p \geq u(H)$ and then $t_3^H = \underline{t}$. In both income types, it can be checked that single crossing conditions apply: if an ability type chooses to rank school 1 first, so does a higher ability type; if an ability type chooses to rank school 2 first, so does a lower ability type. This allows us to search for income-dependent cut-off types $\tilde{t}_H^p$ and $\tilde{t}_L^p$ meeting $G_H^p(\tilde{t}_H^p, \tilde{t}_L^p) = G_L^p(\tilde{t}_H^p, \tilde{t}_L^p) = 0$ where $\blacksquare$

$$
G_y^p(\hat{t}_H, \hat{t}_L) = \begin{cases}
\begin{aligned}
& h(q_1(\hat{t}_H, \hat{t}_L), \hat{t}_y) - 2h(q_2(\hat{t}_H, \hat{t}_L), \hat{t}_y) - \Delta \\
& + I_{y=L} h(q_3^p(\hat{t}_H, \hat{t}_L), \hat{t}_y) \\
& + I_{y=L}[h(q_p(\hat{t}_H, \hat{t}_L), \hat{t}_y) + u(H - p) - u(H) + \Delta_p] \\
& \text{if } \lambda\Phi(\hat{t}_H|H) + (1 - \lambda)\Phi(\hat{t}_L|L) < 1/3
\end{aligned} \\
\begin{aligned}
& \frac{\lambda\Phi(\hat{t}_H|H) + (1-\lambda)\Phi(\hat{t}_L|L)}{1 - 2[\lambda\Phi(\hat{t}_H|H) + (1-\lambda)\Phi(\hat{t}_L|L)]} h(q_1(\hat{t}_H, \hat{t}_L), \hat{t}_y) - \\
& \frac{1 - [\lambda\Phi(\hat{t}_H|H) + (1-\lambda)\Phi(\hat{t}_L|L)]}{1 - 2[\lambda\Phi(\hat{t}_H|H) + (1-\lambda)\Phi(\hat{t}_L|L)]} h(q_2(\hat{t}_H, \hat{t}_L), \hat{t}_y) - \Delta \\
& + I_{y=L} h(q_3(\hat{t}_H, \hat{t}_L), \hat{t}_y) \\
& + I_{y=L}[h(q_p(\hat{t}_H, \hat{t}_L), \hat{t}_y) + u(H - p) - u(H) + \Delta_p] \\
& \text{if } \lambda\Phi(\hat{t}_H|H) + (1 - \lambda)\Phi(\hat{t}_L|L) \geq 1/3
\end{aligned}
\end{cases}
$$

where $I_{y=L}$ is the usual indicator function (1 if true, 0 if false), and $q_j(\hat{t}_H, \hat{t}_H)$'s is j's school ex-post peer quality the cutoffs are $\hat{t}_H$ and $\hat{t}_H$. $q_p(\hat{t}_H, \hat{t}_L)$ is the quality of the private school with these cutoffs. These qualities also depend on the fixed value t_3^H.

In case there exists a cut-off equilibrium, it cannot be the case that $\tilde{t}_H^p \geq \tilde{t}_L^p$ since we would have $G_H^p(\tilde{t}_H^p, \tilde{t}_L^p) > G_L^p(\tilde{t}_H^p, \tilde{t}_L^p)$. (Since $h(q_p(\tilde{t}_H^p, \tilde{t}_L^p), \tilde{t}_H^p) + u(H-p) - u(H) + \Delta_p \geq h(q_p(\tilde{t}_H^p, \tilde{t}_L^p), \tilde{t}_H^p) > h(q_3(\tilde{t}_H^p, \tilde{t}_L^p), \tilde{t}_H^p) \geq h(q_3(\tilde{t}_H^p, \tilde{t}_L^p), \tilde{t}_L^p).$) We then show that an equilibrium with $\tilde{t}_H^p < \tilde{t}_L^p$ exists. We first reduce our field of candidate cutoffs to those satisfying $\lambda\Phi\left(\hat{t}_H|H\right) + (1-\lambda)\Phi\left(\hat{t}_L|L\right) < 1/2$ (this condition is a natural equilibrium condition saying that the mass of applicants ranking school 2 first must be lower than the mass of students ranking school 1 is first position). Conditional on that we obtain $G_H^p(\underline{t}, \cdot) < 0$ for any $\hat{t}_L^p$ and $G_L^p(\cdot, \underline{t}) < 0$ for any $\hat{t}_H$ (immediate from the fact that a $\underline{t}$-type student does not care about peer quality.) Also, notice that in the limit of the aforementioned condition, i.e. $\lambda\Phi(\hat{t}_H|H) + (1-\lambda)\Phi(\hat{t}_L|L) = 1/2$, if $\hat{t}_H > \underline{t}$ then $G_H^p(\hat{t}_H, \hat{t}_L) = G_L^p(\hat{t}_H, \hat{t}_L) = \infty$. Select one such pair $(\hat{t}_H, \hat{t}_L)$ with $\lambda\Phi(\hat{t}_H|H) + (1-\lambda)\Phi(\hat{t}_L|L) = 1/2$. Continuity of G's almost everywhere and the intermediate value theorem imply that in the open segment with extremes $(\underline{t}, \underline{t})$ and $(\hat{t}_H, \hat{t}_L)$ there are two points $(t_H, f(t_H))$ and $(g(t_L), t_L)$ such that $G_L^p(t_H, f(t_H)) = G_H^p(g(t_L), t_L) = 0$. This defines two functions f and g which can be picked to be continuous almost everywhere due to the continuity of G's almost everywhere. We show that f and g intersect at some point $(\tilde{t}_H^p, \tilde{t}_L^p)$, a cut-off equilibrium. There is only one discontinuity of G_H^p around $(\underline{t}, t_L^m)$ where $(1-\lambda)\Phi(t_L^m|L) = 1/2$, thus $\lim_{t_L \to t_L^m} g(t_L) = \underline{t}$. Notice that $f(\underline{t}) < t_L^m$ since $G_L^p(\underline{t}, t_L^m) = \infty$. So when $\hat{t}_H \to \underline{t}$ (hence we approach a flat line from the origin $(\underline{t}, \underline{t})$), g lies at the right from f. If we go to the 45 degree line, it is easy to observe that g lies at the left from f on that line, since $G_H^p(\hat{t}, \hat{t}) > G_L^p(\hat{t}, \hat{t}) \ \forall t < t_{(1/2)}$. Continuity of f and g everywhere except for $(\underline{t}, t_L^m)$ ensures the existence of an intersection between f and g at some point $(\tilde{t}_H^p, \tilde{t}_L^p)$. $G_H^p(\tilde{t}_H^p, \tilde{t}_L^p) = G_L^p(\tilde{t}_H^p, \tilde{t}_L^p) = 0$ by definitions of f and g, therefore we have a cut-off equilibrium below the 45 degree line $(\tilde{t}_H^p < \tilde{t}_L^p)$.

7.5 An alternative ghetto effect

We consider a different modeling of the human capital loss produced by being assigned to a bad school. In the main model, the human capital loss is constant over all types. Here, we consider a reduction in quality: if the mean type across the students assigned to school 3 is q_3, then the school quality is δq_3, where $\delta \in (0,1)$ is a "degradation" factor that applies only to school 3. So the bad school effect affects school peer quality.

We assume that $\underline{t} > 0$ and that $h(\cdot, t)$ is defined on $(0, \bar{t}]$ for every t,

with $\lim_{q\to 0} h(q,t) = -\infty$ and $h(q(\underline{t}),t) \geq 0$. Going to a sufficiently degraded bad school produces an enormous human capital loss. Under these assumptions, one can see that the single-crossing conditions ensuring that the Boston Mechanism without priorities generates an equilibrium with sorting hold here, if the degradation factor is low enough.

Proposition 8 *If δ is sufficiently small, there is a stable equilibrium with full sorting between schools 1 and 2.*

Proof. The proof arises immediately because $h(\delta q_3, t)$ becomes negative and big in absolute terms. We assume beliefs $\hat{q}_1 > \hat{q}_2$ that will be confirmed in equilibrium. For δ low enough every student has school 3 as its least-preferred option in spite of the beliefs about q_3. Thus we restrict attention to strategies "$1 \succ 2 \succ 3$" and "$2 \succ 1 \succ 3$". No equilibrium in which school 2 is underdemanded in the first round (i.e. the mass of households submitting "$2 \succ 1 \succ 3$" is not higher than $1/3$) can arise. For strategy "$1 \succ 2 \succ 3$" gives negative expected payoff to everyone for δ low enough whereas strategy "$2 \succ 1 \succ 3$" ensures positive payoff to everyone (school 3 is avoided with certainty), regardless the type. The former strategy cannot be a best response. Hence we focus on strategy profiles where the mass of families playing "$2 \succ 1 \succ 3$" lies strictly between $1/3$ and $1/2$.

Strategy "$1 \succ 2 \succ 3$" is better than "$2 \succ 1 \succ 3$" for a $t-$ type family if $\frac{h(\hat{q}_1,t)-h(\delta\hat{q}_3,t)}{h(\hat{q}_2,t)-h(\delta\hat{q}_3,t)} > \frac{p_{22}}{p_{11}}$ (p_{jj} is the probability of being accepted at school j if j is ranked in first position.) The left-hand side of the inequality is increasing in t if $h_t(\hat{q}_1,t) - h_t(\delta\hat{q}_3,t) > [h_t(\hat{q}_2,t) - h_t(\delta\hat{q}_3,t)]\frac{h(\hat{q}_1,t)-h(\delta\hat{q}_3,t)}{h(\hat{q}_2,t)-h(\delta\hat{q}_3,t)}$. This inequality holds true provided δ is low enough, which makes $\frac{h(\hat{q}_1,t)-h(\delta\hat{q}_3,t)}{h(\hat{q}_2,t)-h(\delta\hat{q}_3,t)}$ arbitrarily close to 1. Since the LHS is increasing, the best response profile is characterized by a cutoff $\hat{t}$ (types above the cutoff play "$1 \succ 2 \succ 3$" and types below play "$2 \succ 1 \succ 3$".) Then we consider the function, for $\hat{t} \in [t_{1/3}, t_{1/2}]$

$$\breve{G}(\hat{t}) = \frac{1/3}{1-\Phi(\hat{t})}(h(q_1(\hat{t}),\hat{t}) - h(\delta q_3(\hat{t}),\hat{t})) - \frac{1/3}{\Phi(\hat{t})}(h(q_2(\hat{t}),\hat{t}) - h(\delta q_3(\hat{t}),\hat{t}))$$

where $q_j(\hat{t})$ is as defined in the proof of proposition 1. $\breve{G}(\hat{t})$ measures the payoff difference for the cutoff type between playing "$1 \succ 2 \succ 3$" and playing "$2 \succ 1 \succ 3$". $\breve{G}$ is continuous. Moreover $\breve{G}(t_{1/3}) < 0$ since $-h(\delta q_3(\hat{t}),\hat{t})$ is very big, and $\breve{G}(t_{1/2}) < 0$ since $q_1(\hat{t}) > q_2(\hat{t})$. Therefore there is $\tilde{t}^\delta \in (t_{1/3}, t_{1/2})$

accomplishing with $\breve{G}(\tilde{t}^\delta) < 0$. This characterizes our equilibrium strategy profile. The equilibrium is stable due to its cutoff nature. Moreover, this equilibrium shows full sorting between schools 1 and 2 since only types below $\tilde{t}^\delta$ go to school 2 whereas only types above attend school 1. ∎

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