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Inequality Reducing Properties of Progressive Income Tax Schedules: The Case of Endogenous Income*

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Abstract

The case for progressive income taxation is often based on the classic result of Jakobsson (1976) and Fellman (1976), according to which progressive and only progressive income taxes—in the sense of increasing average tax rates on income—ensure a reduction in income inequality. This result has been criticized on the ground that it ignores the possible disincentive effect of taxation on work effort, and the resolution of this critique has been a long-standing problem in public finance. This paper provides a normative rationale for progressivity that takes into account the effect of an income tax on labor supply. It shows that a tax schedule is inequality reducing only if it is progressive—in the sense of increasing *marginal* tax rates on income, and identifies a necessary and sufficient condition on primitives under which progressive and only progressive taxes are inequality reducing.

Keywords: progressive taxation; income inequality; incentive effects of taxation.
JEL classifications: D63, D71.

1 Introduction

A prominent and largely studied normative rationale for progressive income taxation derives from the fundamental result of Jakobsson (1976) and Fellman (1976), which asserts that progressive and only progressive income taxes—in the sense of increasing average tax rates on income—reduce income inequality (regardless of the income distribution they are applied

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to) according to the relative Lorenz dominance criterion.[1] Jakobsson (1976) and Fellman (1976) were early contributors to a vast literature on the redistributive effects of tax systems initiated by Musgrave and Thin (1948).[2] This literature is for the most part framed in terms of exogenous income. In particular, while the work of Jakobsson (1976) and Fellman (1976) has been extended in various directions,[3] treatments that incorporate the disincentive effects of taxation tend to emphasize negative results, such as the existence of non-pathological consumer preferences for which progressive tax schedules increase income inequality. As shown in Ebert and Moyes (2007), the Jakobsson-Fellman result can be extended to the case of endogenous income, but in this case inequality reducing tax schedules are no longer completely characterized by average rate progressivity. Instead, the effect of a tax on gross incomes, in addition to its shape, determines the redistributive effect.[4] In particular, a progressive tax schedule may well increase income inequality if the associated elasticity of gross income with respect to non-taxed income is large enough. Allingham (1979) and Ebert and Moyes (2003, 2007) provide examples of such progressive tax schedules.

In this paper we recover a version of the Jakobsson-Fellman result that takes into account the incentive effects of taxation.[5] Our first result states that a piecewise linear tax schedule is inequality reducing (*i.e.*, it reduces income inequality whatever the distribution of abilities) only if it is *marginal-rate progressive* (*i.e.*, it exhibits nondecreasing marginal tax rates on income) (Theorem 1). A second result asserts that the set of all inequality reducing tax schedules is precisely the set of all marginal-rate progressive tax schedules if and only if the linear tax schedules are inequality reducing (Theorem 2). This property of linear taxes is then characterized in terms of first principles (essentially a condition on preferences), which allows us to identify a class of utility functions for which income taxes are inequality reducing if and only if they are marginal-rate progressive (Theorem 3 and Corollary 3). Some additional results and illustrative examples are also provided in Section 3.

The main results are formulated within the standard Mirrlees model (see Mirrlees, 1971), which provides a suitable framework for the analysis of nonlinear income taxation with endogenous labor supply, and rest on mild assumptions. First, the set of admissible tax schedules is defined as the set of all continuous, piecewise linear maps from pre-tax incomes to tax liabilities that are nondecreasing and preserve the pre-tax ranking of income (*i.e.*, marginal tax rates are less than unity). Second, consumers require an increasingly large

[1] A second normative rationale for income tax progressivity is based on the principle of equal sacrifice, which dates back to Samuelson (1947). See Young (1990); Berliant and Gouveia (1993); Ok (1995); Mitra and Ok (1996, 1997); D'Antoni (1999). Progressive income taxation has also been studied from a positive perspective as an equilibrium outcome of a voting game (*e.g.*, see Snyder and Kramer, 1988; Marhuenda and Ortuño-Ortín, 1995; Roemer, 1999; Carbonell-Nicolau and Klor, 2003; Carbonell-Nicolau and Ok, 2007).

[2] See Lambert (2002) for a survey.

[3] See, *e.g.*, Kakwani (1977); Hemming and Keen (1983); Eichhorn et al. (1984); Liu (1985); Formby et al. (1986); Thon (1987); Latham (1988); Thistle (1988); Moyes (1988, 1994); Le Breton et al. (1996); Ebert and Moyes (2000); Ju and Moreno-Ternero (2008).

[4] See also Onrubia et al. (2005) and Preston (2007).

[5] See Section 4 for a discussion of the relationship between our results and the Jakobsson-Fellman result.

and unbounded compensation for an extra unit of labor as their leisure time tends to zero.[6] Finally, consumer preferences satisfy *agent monotonicity*, a condition introduced by Mirrlees (1971, Assumption B, p. 182) and named by Seade (1982). This condition requires that the marginal rate of substitution of gross income for consumption be nonincreasing with productivity, and, as pointed out by Mirrlees (1971), is equivalent to the condition that, in the absence of taxation, consumption does not decrease as the wage rate increases.

The paper is organized as follows. Section 2 lays out the formal setting. The main results, together with sketches of their proofs, are presented in Section 3. Technical proofs are relegated to the Appendix.

2 The model

Consider an economy with a finite number $n \geq 2$ of individuals. The welfare of an individual is measured by a continuous utility function $u : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ defined over consumption-labor pairs $(c,l) \in \mathbb{R}_+ \times [0,1]$ such that $u(\cdot,l)$ is strictly increasing in c for each $l \in [0,1)$, and $u(c,\cdot)$ is strictly decreasing in l for each $c > 0$. It is assumed that u is strictly quasiconcave on $\mathbb{R}_{++} \times [0,1)$ and twice continuously differentiable on $\mathbb{R}_{++} \times (0,1)$.[7] For $(c,l) \in \mathbb{R}_{++} \times (0,1)$, let

$$MRS(c,l) := -\frac{u_l(c,l)}{u_c(c,l)}$$

denote the marginal rate of substitution of labor for consumption, where

$$u_c(c,l) := \frac{\partial u(c,l)}{\partial c} \quad \text{and} \quad u_l(c,l) := \frac{\partial u(c,l)}{\partial l}.$$

(Observe that $MRS(c,l) \geq 0$ for each (c,l).) We assume that, for each $c > 0$,

$$\lim_{l \to 1^-} MRS(c,l) = +\infty \quad \text{and} \quad \lim_{l \to 0^+} MRS(c,l) < +\infty. \tag{1}$$

The second condition is readily acceptable. According to the first condition, the compensation required by an individual for an extra unit of working time tends to infinity as the agent's leisure time approaches zero.[8]

Let $\mathscr{U}$ denote the set of all utility functions satisfying the above conditions.[9]

[6] In other words, the marginal rate of substitution of labor for consumption tends to infinity as leisure time vanishes.

[7] Following the tradition of the literature on optimal taxation, it is assumed that u is common across agents; accordingly, this map carries a connotation of "social norm."

[8] This condition is used in Lemma 2.

[9] All these conditions are readily acceptable, except, perhaps, the first limit condition in (1) (which we just described intuitively). Many standard utility functions belong to the class $\mathscr{U}$. See Remark 3 and footnote 12 for some concrete examples. For an instance of a utility function not in $\mathscr{U}$ (precisely because it violates the first limit condition in (1)), consider the case of a constant marginal rate of substitution of labor for consumption (*i.e.*, the case when labor and consumption are perfect substitutes).

Prior to formulating the agents' utility maximization problem, we need the formal definition of a tax schedule. Throughout the sequel we confine attention to nondecreasing and order-preserving piecewise linear tax schedules.

Definition 1. Let $(\alpha_0, \boldsymbol{t}, \overline{\boldsymbol{y}}) = \big(\alpha_0, (t_0, ..., t_K), (\overline{y}_0, ..., \overline{y}_K)\big)$, where $\alpha_0 \geq 0$, $K \in \mathbb{Z}_+$, $t_k \in [0,1)$ for each $k \in \{0, ..., K\}$, $t_k \neq t_{k+1}$ whenever $k \in \{0, ..., K-1\}$ and $K \geq 1$, and $0 = \overline{y}_0 < \cdots < \overline{y}_K$. A ***piecewise linear tax schedule*** is a real-valued map T on $\mathbb{R}_+$ uniquely determined by $(\alpha_0, \boldsymbol{t}, \overline{\boldsymbol{y}})$ as follows:

$$T(y) := \begin{cases} -\alpha_0 + t_0 y & \text{if } 0 = \overline{y}_0 \leq y \leq \overline{y}_1, \\ -\alpha_0 + t_0 \overline{y}_1 + t_1 (y - \overline{y}_1) & \text{if } \overline{y}_1 < y \leq \overline{y}_2, \\ \quad \vdots & \quad \vdots \\ -\alpha_0 + t_0 \overline{y}_1 + t_1 (\overline{y}_2 - \overline{y}_1) + \cdots + t_{K-1} (\overline{y}_K - \overline{y}_{K-1}) + t_K (y - \overline{y}_K) & \text{if } \overline{y}_K < y. \end{cases}$$

Here $T(y)$ is interpreted as the tax liability for gross income level y.

Letting $\overline{y}_{K+1} := +\infty$, the tax schedule corresponding to the vector $(\alpha_0, \boldsymbol{t}, \overline{\boldsymbol{y}})$ can be succinctly expressed as

$$T(y) = t_k y - \alpha_k \quad \text{for } \overline{y}_k \leq y \leq \overline{y}_{k+1}, \quad k \in \{0, \ldots, K\},$$

where $\alpha_k := \alpha_{k-1} + (t_k - t_{k-1}) \overline{y}_k$ for $k \in \{1, \ldots, K\}$.

We write $(\alpha_0, \boldsymbol{t}, \overline{\boldsymbol{y}})$ and the associated map T interchangeably. The set of piecewise linear tax schedules is denoted by $\mathscr{T}$.

Individuals differ in their abilities. An ***ability distribution*** is a vector $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{++}$ such that $a_1 \leq \cdots \leq a_n$ (that is, without loss of generality, individuals are sorted in ascending order according to their ability). The set of all ability distributions is denoted by $\mathscr{A}$.

An agent of ability $a > 0$ who chooses $l \in [0,1]$ units of labor and faces a tax schedule $T \in \mathscr{T}$ consumes $c = al - T(al)$ units of the good and obtains a utility of $u(c,l)$. Thus, the agent's problem is

$$\max_{l \in [0,1]} u\,(al - T(al), l). \tag{2}$$

Because the members of $\mathscr{U}$ and $\mathscr{T}$ are continuous, for given $u \in \mathscr{U}$, $a > 0$, and $T \in \mathscr{T}$, the optimization problem in (2) has a solution, although it need not be unique. A ***solution function*** is a map $l^u : \mathbb{R}_{++} \times \mathscr{T} \to [0,1]$ such that $l^u(a,T)$ is a solution to (2) for each $(a,T) \in \mathbb{R}_{++} \times \mathscr{T}$. The ***pre-tax*** and ***post-tax income functions*** associated to a solution function l^u, denoted by $y^u : \mathbb{R}_{++} \times \mathscr{T} \to \mathbb{R}_+$ and $x^u : \mathbb{R}_{++} \times \mathscr{T} \to \mathbb{R}_+$ respectively, are given by

$$y^u(a,T) := a l^u(a,T) \quad \text{and} \quad x^u(a,T) := a l^u(a,T) - T\big(a l^u(a,T)\big).$$

The superscript u will sometimes be omitted to lighten notation.

Given $a > 0$, let $U^a : \mathbb{R}_+ \times [0,a] \to \mathbb{R}$ be defined by $U^a(c,y) := u\left(c, \frac{y}{a}\right)$. For $(c,y,a) \in \mathbb{R}^3_{++}$ with $y < a$, define

$$U^a_c(c,y) := \frac{\partial U^a(c,y)}{\partial c}, \quad U^a_y(c,y) := \frac{\partial U^a(c,y)}{\partial y}, \quad \text{and } \eta^a(c,y) := -\frac{U^a_y(c,y)}{U^a_c(c,y)}.$$

Observe that $\eta^a(c,y)$ can be viewed as the marginal rate of substitution of gross income for consumption at (c,y) for an agent with ability a.

The following condition plays an important role in the proofs of our main results. It was introduced by Mirrlees (1971, Assumption B, p. 182) and termed *agent monotonicity* by Seade (1982).

Definition 2. A utility function $u \in \mathscr{U}$ satisfies ***agent monotonicity*** if $\eta^a(c,y) \geq \eta^{a'}(c,y)$ for each $(c,y) \in \mathbb{R}^2_+$ and $0 < a < a'$ with $y < a$.

Agent monotonicity is a single-crossing condition on the agents' indifference curves in the space of gross income-consumption pairs (y,c). It is equivalent to the condition that (in the absence of taxation) consumption is a nondecreasing function with respect to productivity, for any non-wage income (Mirrlees, 1971, p. 182). A sufficient condition for agent monotonicity is that consumption is not an inferior good (*i.e.*, it does not decrease as lump-sum income increases) (Myles, 1995, p. 136).

The set of all the members of $\mathscr{U}$ satisfying agent monotonicity is represented as $\mathscr{U}^*$.[10]

An ***income distribution*** is a vector $\boldsymbol{z} = (z_1, ..., z_n) \in \mathbb{R}^n_+$ of incomes arranged in increasing order, *i.e.*, $z_1 \leq \cdots \leq z_n$; here z_1 denotes the income of the poorest agent, z_2 denotes the income of the second poorest agent, and so on.

In this paper we use the standard relative Lorenz ordering to make inequality comparisons between income distributions. Given two income distributions $\boldsymbol{z} = (z_1, ..., z_n)$ and $\boldsymbol{z}' = (z'_1, ..., z'_n)$ with $z_1, z'_1 > 0$, we say that $\boldsymbol{z}$ is at least as equal as $\boldsymbol{z}'$ if $\boldsymbol{z}$ ***Lorenz dominates*** $\boldsymbol{z}'$, *i.e.*, if

$$\frac{\sum_{i=1}^k z_i}{\sum_{i=1}^n z_i} \geq \frac{\sum_{i=1}^k z'_i}{\sum_{i=1}^n z'_i}, \quad \text{for all } k \in \{1, ..., n\}.$$

For $u \in \mathscr{U}^*$, and given pre-tax and post-tax income functions y^u and x^u, an ability distribution $\boldsymbol{a} = (a_1, ..., a_n) \in \mathscr{A}$ and a tax schedule $T \in \mathscr{T}$ determine a ***pre-tax income distribution***

$$\boldsymbol{y}^u(\boldsymbol{a}, T) := \left(y^u(a_1, T), ..., y^u(a_n, T)\right)$$

and a ***post-tax income distribution***

$$\boldsymbol{x}^u(\boldsymbol{a}, T) := \left(x^u(a_1, T), ..., x^u(a_n, T)\right).$$[11]

[10] See Remark 3 and footnote 12 for examples of utility functions in $\mathscr{U}^*$. Agent monotonicity is a mild requirement, and it is not easy to find preferences that violate it. As per our previous remark, any such preferences would necessarily treat consumption as an inferior good.

[11] Under the agent monotonicity condition, in both cases the vector components are arranged in increasing order. See Lemma 1 below.

In the absence of taxation, *i.e.*, if $T \equiv 0$, one has $\boldsymbol{y}^u(\boldsymbol{a}, 0) = \boldsymbol{x}^u(\boldsymbol{a}, T)$.

Definition 3. Let $u \in \mathscr{U}$. A tax schedule $T \in \mathscr{T}$ is ***income inequality reducing with respect to*** u, which we denote as (u-iir), if $\boldsymbol{x}^u(\boldsymbol{a}, T)$ Lorenz dominates $\boldsymbol{y}^u(\boldsymbol{a}, 0)$ for each ability distribution $\boldsymbol{a} := (a_1, ..., a_n) \in \mathscr{A}$ and for each pre-tax and post-tax income functions y^u and x^u.

The functions y^u and x^u are uniquely determined at almost every point of their domain (*cf*. Lemma 1 below). The quantifier for y^u and x^u in Definition 3 deals with the cases when these maps are not uniquely defined.

3 The main results

We begin by defining two important subclasses of the set $\mathscr{T}$ of piecewise linear tax schedules.

Definition 4. A tax schedule $T \in \mathscr{T}$ is ***marginal-rate progressive*** if it is a convex function.

The set of all marginal-rate progressive tax schedules in $\mathscr{T}$ is denoted by $\mathscr{T}_{prog}$.

Definition 5. A tax schedule $T \in \mathscr{T}$ is ***linear*** if $T(y) = -\alpha_0 + t_0 y$ for all $y \in \mathbb{R}_+$ and some $\alpha_0 \geq 0$ and $t_0 \in [0, 1)$.

The set of all linear tax schedules in $\mathscr{T}$ is denoted by $\mathscr{T}_{lin}$.

We now present the first two main results of the paper. Theorem 1 states that only marginal-rate progressive tax schedules can be inequality reducing.[12] Theorem 2 asserts that if the linear members of $\mathscr{T}$ are inequality reducing, then the set of all inequality reducing tax schedules is precisely the set of marginal-rate progressive tax schedules.

Theorem 1. *Given $u \in \mathscr{U}^*$, a tax schedule in $\mathscr{T}$ is u-iir only if it is marginal-rate progressive.*

Theorem 2. *Given $u \in \mathscr{U}^*$, the set of all u-iir tax schedules in $\mathscr{T}$ equals $\mathscr{T}_{prog}$ if and only if the members of $\mathscr{T}_{lin}$ are u-iir.*

Remark 1. The reader may wish to consider tax schedules that are inequality reducing not only for a *fixed* utility function $u \in \mathscr{U}^*$ but rather *for all* utility functions u in some subdomain $\mathscr{U}^{**}$ of $\mathscr{U}$. Given $\mathscr{U}^{**} \subseteq \mathscr{U}$, call a tax schedule $T \in \mathscr{T}$ ***income inequality reducing with respect to*** $\mathscr{U}^{**}$ ***($\mathscr{U}^{**}$-iir)*** if $(x^u(a_1, T), ..., x^u(a_n, T))$ Lorenz dominates $(y^u(a_1, 0), ..., y^u(a_n, 0))$ for each pre-tax and post-tax income functions y^u and x^u, each $(a_1, ..., a_n) \in \mathscr{A}$, and every $u \in \mathscr{U}^{**}$. Theorems 1 and 2 immediately give the following variants in terms of this strengthening of Definition 3.

Corollary 1 (to Theorem 1)**.** *Given $\mathscr{U}^{**} \subseteq \mathscr{U}^*$, a tax schedule in $\mathscr{T}$ is $\mathscr{U}^{**}$-iir only if it is marginal-rate progressive.*

[12] It is easy to find preferences and taxes for which the converse of Theorem 1 is false. Consider, for example, the utility function $u(c, l) = c - \frac{1}{1-l}$, together with the linear tax $T(y) = 0.5y$, with associated labor supply function $l(a, T) = \max\{0, 1 - ((1-t)a)^{-0.5}\}$. While T is marginal-rate progressive, the ratios $\frac{x(3,T)}{y(3,0)} = 0.217 < 0.293 = \frac{x(4,T)}{y(4,0)}$ increase as the ability goes from 3 to 4, implying that T is not u-iir (according to Lemma 3).

Corollary 2 (to Theorem 2)**.** *Given $\mathscr{U}^{**} \subseteq \mathscr{U}^{*}$, the set of all $\mathscr{U}^{**}$-iir tax schedules in $\mathscr{T}$ equals $\mathscr{T}_{prog}$ if and only if the members of $\mathscr{T}_{lin}$ are $\mathscr{U}^{**}$-iir.*[13]

The main proofs of Theorem 1 and Theorem 2 are furnished in Subsection 3.1. Before presenting these proofs, we set ourselves the important task of providing necessary and sufficient conditions on primitives under which marginal-rate progressive and only marginal-rate progressive taxes are inequality reducing. This will be accomplished by first characterizing the preferences for which the members of $\mathscr{T}_{lin}$ are inequality reducing (in Theorem 3 below); this characterization will then allow us to present a variant of Theorem 2 in terms of first principles (see Corollary 3 below). A number of additional results and illustrative examples will also be provided.

In the particular case when T is a member of $\mathscr{T}_{lin}$ with $T(y) = -b$, where $b \geq 0$, we write $l^u(a,b)$ for $l^u(a,T)$, $y^u(a,b)$ for $y^u(a,T)$, and $x^u(a,b)$ for $x^u(a,T)$. In this case, $l^u(a,b)$ can be viewed as the labor supply for an agent of ability a who is endowed with a monetary sum b prior to her choice of labor income. For each $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$, $l^u(a,b)$ is a solution to the problem

$$\max_{l \in [0,1]} u(al+b, l). \tag{3}$$

Since u is strictly quasiconcave on $\mathbb{R}_{++} \times [0,1)$, for each $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$, there is a unique solution $l^u(a,b)$ to (3).[14] It is not difficult to show that for given $b \geq 0$, the derivative of the map

$$a \mapsto l^u(a,b) \tag{4}$$

exists for all but perhaps one $a > 0$.[15]

[13] Observe that in the particular case when $\mathscr{U}^{**} = \mathscr{U}^{*}$, Corollary 2 is vacuous, for we know (as per the last example in Remark 3 below (see footnote 22)) that there exist preferences for which no proportional tax is inequality reducing.

[14] More generally, for $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}$ such that $a + b > 0$, call $l^u(a,b)$ the (unique) solution to the problem

$$\max_{l \in [0,1]} u(al+b, l).$$

The assumptions on u ensure that $l^u(a,b) \in [0,1)$ for all $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}$ with $a + b > 0$.

[15] For each $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}$ with $a + b > 0$, the solution $l^u(a,b)$ satisfies

$$MRS(al^u(a,b)+b, l^u(a,b)) = -\frac{\frac{\partial u(al^u(a,b)+b, l^u(a,b))}{\partial l}}{\frac{\partial u(al^u(a,b)+b, l^u(a,b))}{\partial c}} \geq a,$$

with equality if $l^u(a,b) > 0$. Define $F : \{(a,b,s) \in \mathbb{R}_{++} \times \mathbb{R} \times (0,1) : as + b > 0\} \to \mathbb{R}$ by

$$F(a,b,s) = -\frac{\frac{\partial u(as+b,s)}{\partial l}}{\frac{\partial u(as+b,s)}{\partial c}} - a.$$

Because u is twice continuously differentiable on $\mathbb{R}_{++} \times (0,1)$, F is continuously differentiable. Furthermore, given $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}$ with $a + b > 0$, if there exists $s^* \in (0,1)$ such that $F(a,b,s^*) = 0$, and since it is straightforward to verify that $\frac{\partial F(a,b,s^*)}{\partial s} \neq 0$, it follows from the Implicit Function Theorem that the partial derivatives of the map $(a,b) \mapsto l^u(a,b)$ exist for every $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$ such that $l^u(a,b) > 0$. In addition, for $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$ with $l^u(a,b) = 0$ and $MRS(al^u(a,b)+b, l^u(a,b)) > a$, there exists an open set V_a in $\mathbb{R}_{++}$ containing a such that $l^u(a',b) = 0$ for all $a' \in V_a$, implying that $\frac{\partial l^u(a,b)}{\partial a}$ is well-defined. Because for given $b \geq 0$, there is at most one

For $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$, define

$$\zeta^u(a,b) := \frac{\partial(al^u(a,b)+b)}{\partial a} \cdot \frac{a}{al^u(a,b)+b} \quad \text{and} \quad \xi^u(a,b) := \frac{\partial l^u(a,b)}{\partial a} \cdot \frac{a}{l^u(a,b)}; \tag{5}$$

these are, respectively, the *elasticity of income with respect to ability* and the *elasticity of labor supply with respect to ability* at ability level a and endowment b.[16]

Let $\widehat{\mathscr{U}}$ denote the universe of utility functions $u \in \mathscr{U}^*$ satisfying the following two conditions:

- $\zeta^u(a,b) \leq \zeta^u(a,0)$ for all $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$; and
- the map $a \mapsto \zeta^u(a,0)$ defined on $\mathbb{R}_{++}$ is nondecreasing.[17]

Remark 2. Since $\zeta^u(a,0) = 1 + \xi^u(a,0)$ (see the proof of Proposition 1), the second bullet point in the previous definition can be equivalently stated as follows: the map $a \mapsto \xi^u(a,0)$ defined on $\mathbb{R}_{++}$ is nondecreasing.

The following result states that the members of $\widehat{\mathscr{U}}$, and only the members of $\widehat{\mathscr{U}}$, render linear tax schedules inequality reducing.[18]

Theorem 3. *For $u \in \mathscr{U}^*$, the members of $\mathscr{T}_{lin}$ are u-iir if and only if $u \in \widehat{\mathscr{U}}$.*

The proof of Theorem 3 is relegated to Appendix A.

Theorem 3 can now be combined with Theorem 2 to obtain a necessary and sufficient condition on primitives under which marginal-rate progressive and only marginal-rate progressive taxes are inequality reducing.

Corollary 3. *For $u \in \mathscr{U}^*$, the set of all u-iir tax schedules in $\mathscr{T}$ is precisely $\mathscr{T}_{prog}$ if and only if $u \in \widehat{\mathscr{U}}$.*

Remark 3. The set $\widehat{\mathscr{U}}$ is nonempty. Indeed, any utility function of the Cobb-Douglas form $u(c,l) := Ac^{\alpha}(1-l)^{\beta}$, where A, α, and β are positive constants, is a member of $\widehat{\mathscr{U}}$.[19] In addition, the so-called *GHH preferences* (*cf.* Greenwood et al., 1988), given by

$$u(c,l) := \frac{1}{1-\sigma}\left(c - \frac{l^{1+\chi}}{1+\chi}\right)^{1-\sigma},$$

$a > 0$ such that $l^u(a,b) = 0$ and $MRS(al^u(a,b)+b, l^u(a,b)) = a$, it follows that the derivative of the map in (4) exists for all but perhaps one $a > 0$.

[16] As per the assertion in the previous paragraph, these elasticities are well-defined almost everywhere.

[17] See footnote 16.

[18] The following version of Theorem 3 can be proven for the case of proportional taxation: a proportional tax schedule of the form $T(y) = t_0 y$ is u-iir if and only if the map $a \mapsto \zeta^u(a,0)$ defined on $\mathbb{R}_{++}$ is nondecreasing. Because $\zeta^u(a,0) = 1 + \xi^u(a,0)$, the latter condition can be equivalently stated as follows: the map $a \mapsto \xi^u(a,0)$ defined on $\mathbb{R}_{++}$ is nondecreasing.

[19] In this case, the elasticity of income with respect to ability, $\zeta^u(a,b)$, is either zero (when $l^u(a,b) = 0$) or $\frac{a}{a+b}$ (if $l^u(a,b) > 0$), implying that conditions (ii) and (iii) are satisfied. In addition, because $al^u(a,b)+b = \frac{\alpha}{\alpha+\beta}(a+b)$ is increasing in a, it follows from Mirrlees (1971, p. 182) that u satisfies condition (i), agent monotonicity.

where σ and χ are positive constants and $\sigma \neq 1$, also satisfy the conditions (i)-(iii).[20,21]

Finally, we should point out that there are utility functions that do not belong to $\widehat{\mathscr{U}}$. For example, the map $u(c,l) := \sqrt{c} + \sqrt{1-l}$ satisfies (i)-(ii) but not (iii).[22]

A subset of $\widehat{\mathscr{U}}$ can be characterized in terms of the elasticity of labor supply with respect to ability, as defined in (5). Let $\widehat{\widehat{\mathscr{U}}}$ denote the universe of utility functions $u \in \mathscr{U}^*$ satisfying the following two conditions:

- $\xi^u(a,b) \leq \xi^u(a,0)$ for all $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_+$; and
- the map $a \mapsto \xi^u(a,0)$ defined on $\mathbb{R}_{++}$ is nondecreasing.[23]

The next result states that $\widehat{\widehat{\mathscr{U}}}$ is a strict subset of $\widehat{\mathscr{U}}$.

Proposition 1. $\widehat{\widehat{\mathscr{U}}} \subsetneq \widehat{\mathscr{U}}$.

The proof of Proposition 1 is relegated to Appendix B.

Proposition 1, combined with Corollary 3, immediately gives the following result.

Corollary 4. *Given $u \in \widehat{\widehat{\mathscr{U}}}$, the set of all u-iir tax schedules in $\mathscr{T}$ equals $\mathscr{T}_{prog}$.*

3.1 Proofs of Theorem 1 and Theorem 2

In this subsection, we present the main arguments for the proofs of Theorem 1 and Theorem 2 and relegate technical details to the Appendix. The proof of Theorem 3, which is technical in nature, is furnished in Appendix A. We begin with three preparatory lemmas. The first lemma says that pre-tax and post-tax income functions are monotone in a, and that the solution to the agents' maximization problem (2) is almost always unique. This result is well-known (see Mirrlees, 1971, Theorem 1 and the ensuing discussion on p. 183).

Lemma 1. *Let $u \in \mathscr{U}^*$, $T \in \mathscr{T}$. For every pre-tax and post-tax income functions y^u and x^u, the maps $a \mapsto y^u(a,T)$ and $a \mapsto x^u(a,T)$ are nondecreasing on $\mathbb{R}_{++}$. Moreover, given $T \in \mathscr{T}$, there is a unique solution to (2) for all $a > 0$, except for a set of measure zero.*

[20] As defined in Greenwood et al. (1988), the domain for l is the set of all the nonnegative reals. Thus, this specification does not exactly conform to the condition imposed here that the range of acceptable values for l should be the unit interval. This is a minor point, for our results extend to the case when the upper bound for l is $+\infty$.

[21] For this specification, $y^u(a,b) = a\, a^{\frac{1}{\chi}} + b$ is increasing in a, implying that u satisfies the agent-monotonicity, condition (i). Moreover, $\zeta^u(a,b) = \frac{\theta a^\theta}{a^\theta + b}$, where $\theta := \frac{1+\chi}{\chi}$, which yields

$$\zeta^u(a,0) = \theta \geq \theta \frac{a^\theta}{a^\theta + b} = \zeta^u(a,b) \quad \text{and} \quad \frac{\partial \zeta^u(a,b)}{\partial a} = \theta^2 \frac{b a^{\theta-1}}{(a^\theta + b)^2} \geq 0,$$

implying that conditions (ii) and (iii) hold.

[22] For this utility function, we have $\frac{\partial \zeta^u(a,0)}{\partial a} = -\frac{1}{(a+1)^2}$, implying that no proportional tax schedule is u-iir (recall footnote 18).

[23] See footnote 16.

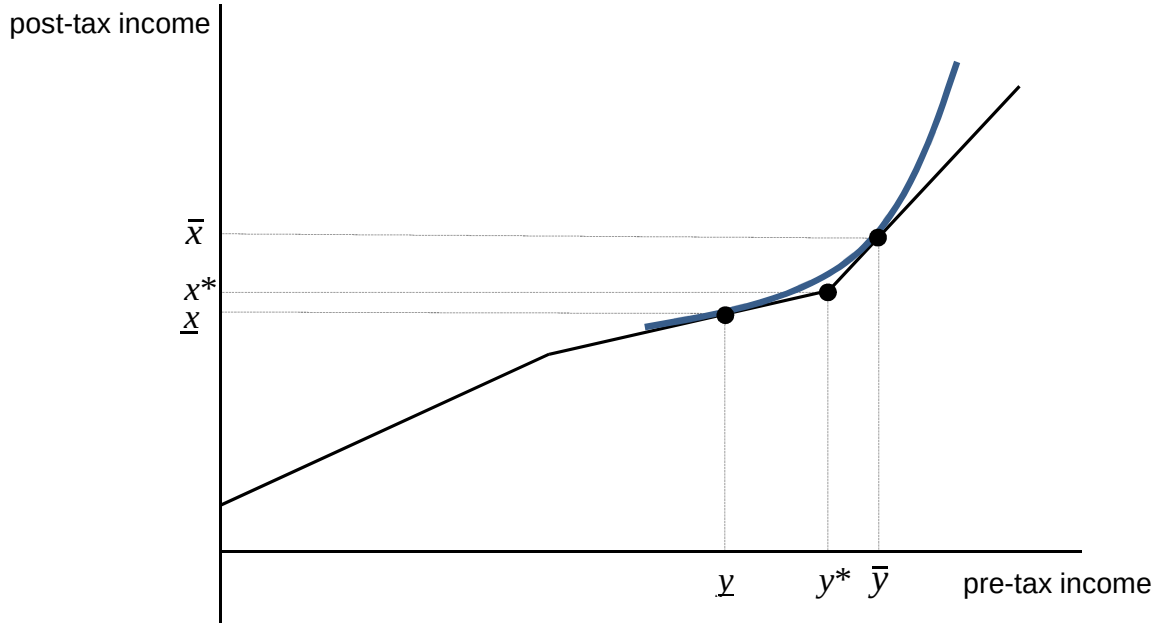

Figure 1: Figure for Theorem 1

Lemma 2. *Given $u \in \mathscr{U}$, $(c,y) \in \mathbb{R}^2_{++}$, and $q \in (0,+\infty)$, there exists an $a > y$ such that $\eta^a(c,y) = q$.*[24]

The proof of Lemma 2 is relegated to Appendix C.

The third lemma gives an alternative characterization of an inequality reducing tax schedule (recall Definition 3); its proof is relegated to Appendix D.[25]

Lemma 3. *Given $u \in \mathscr{U}^*$, a tax schedule $T \in \mathscr{T}$ is u-iir if and only if for any ability distribution $\boldsymbol{a} \in \mathscr{A}$ and for any pre-tax and post-tax income functions y^u and x^u,*

$$\frac{x^u(a_i,T)}{y^u(a_i,0)} \geq \frac{x^u(a_{i+1},T)}{y^u(a_{i+1},0)} \qquad \forall i \in \{1,\dots,n-1\}.$$

3.1.1 Proof of Theorem 1

The following lemma plays an essential role in the proof of Theorem 1.

Lemma 4. *Let $u \in \mathscr{U}^*$ and $T \in \mathscr{T}$, and let x^u be a post-tax income function. Then the map $a \mapsto x^u(a,T)$ is continuous on $\mathbb{R}_{++}$ if and only if T is marginal-rate progressive.*

The formal proof of Lemma 4 is given in Appendix E. Here we provide intuition for this result. Consider the agents' budget line in the space of pre-tax and post-tax income pairs

[24] Feasibility requires $a \geq y$ (recall that $l = \frac{y}{a}$ and $l \leq 1$), and the lemma states that there is an $a > y$ such that $\eta^a(c,y) = q$.

[25] Lemma 3 is analogous to Lemma 1 in Jakobsson (1976), Proposition 2.1 in Moyes (1994), and Lemma 2 in Ebert and Moyes (2007).

(y,x) for a given tax schedule. If agents face a marginal-rate progressive tax schedule T, this budget line is concave, and since preferences satisfy strict quasiconcavity, there is a unique optimal pre-tax and post-tax income pair for each agent; in this case the continuity of the map $a \mapsto x^u(a,T)$ follows from Berge's Maximum Theorem. Under a non-convex tax T, on the other hand, the budget line must be non-concave somewhere, as the black line in Figure 1. In this case there are multiple optimal pre-tax and post-tax income pairs for some ability level, say a^* (points $(\underline{y},\underline{x})$ and $(\overline{y},\overline{x})$ in Figure 1). Given the agent monotonicity condition (recall Definition 2), this multiplicity generates a discontinuity of the map $a \mapsto x^u(a,T)$ at a^*. Thus, continuity of the map $a \mapsto x^u(a,T)$ implies convexity of T.

Given Lemma 4, Theorem 1 can be concisely proven as follows. Take $u \in \mathscr{U}^*$. By Lemma 3, we only need to find, for each $T \in \mathscr{T}$ that is not marginal-rate progressive, $\boldsymbol{a} \in \mathscr{A}$ and pre-tax and post-tax income functions y^u and x^u violating (3). Note that given y^u and x^u, an ability distribution $\boldsymbol{a} \in \mathscr{A}$ will violate (3) if the map $a \mapsto x^u(a,T)$ defined on $\mathbb{R}_{++}$ has a discontinuity point and the map $a \mapsto y^u(a,0)$ defined on $\mathbb{R}_{++}$ is continuous. Indeed, in this case, letting $a^* > 0$ be a discontinuity point for the map $a \mapsto x^u(a,T)$,

$$\lim_{a \uparrow a^*} x^u(a,T) < \lim_{a \downarrow a^*} x^u(a,T) \quad \text{and} \quad \lim_{a \uparrow a^*} y^u(a,0) = \lim_{a \downarrow a^*} y^u(a,0),$$

since x^u is nondecreasing (Lemma 1), implying

$$\lim_{a \uparrow a^*} \frac{x^u(a,T)}{y^u(a,0)} < \lim_{a \downarrow a^*} \frac{x^u(a,T)}{y^u(a,0)}.$$

Thus, Theorem 1 is a consequence of Lemma 4. (Observe that $x^u(\cdot,T) = y^u(\cdot,0)$ whenever $T \equiv 0$.)

3.1.2 Proof of Theorem 2

To lighten notation, we will omit the superscript u throughout the proof.

It is clear that if the set of all *u-iir* tax schedules in $\mathscr{T}$ equals $\mathscr{T}_{prog}$, then the members of $\mathscr{T}_{lin}$ are *u-iir*.

Suppose that the members of $\mathscr{T}_{lin}$ are *u-iir*. By Theorem 1 it follows that the set of all *u-iir* tax schedules in $\mathscr{T}$ is contained in $\mathscr{T}_{prog}$. It remains to show the reverse containment. Let $T = (\alpha_0, \boldsymbol{t}, \overline{\boldsymbol{y}}) \in \mathscr{T}$ be marginal-rate progressive (recall Definition 1). By Lemma 3, we only need to show that condition (3) holds for any ability distribution $\boldsymbol{a} \in \mathscr{A}$ and for any pre-tax and post-tax income functions y^u and x^u.

Now, for each income threshold $\overline{y}_k$ of T, define the linear tax schedule $T_k(y) := t_k y - \alpha_k$ for $k \in \{0,...,K\}$, where $\alpha_k := \alpha_{k-1} + (t_k - t_{k-1})\overline{y}_k$ for $k \in \{1,\ldots,K\}$.

Pre-tax and post-tax income functions, y and x, are uniquely defined, since preferences are strictly quasiconcave and the tax function T is convex. For $k \in \{1,...,K\}$, define the abilities a_k^- and a_k such that

$$a_k^- := \min\{a : y(a,T_{k-1}) = \overline{y}_k\} \quad \text{and} \quad a_k := \max\{a : y(a,T_k) = \overline{y}_k\}$$

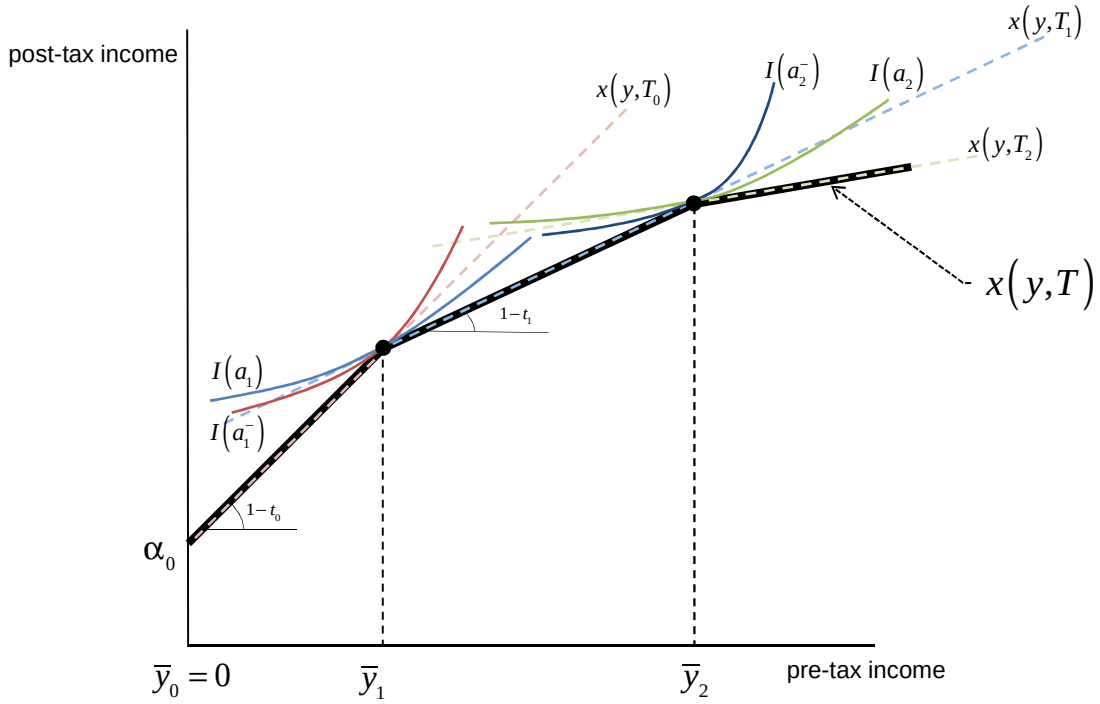

Figure 2: Figure for Theorem 2

(see Figure 2). Lemma 2 guarantees that a_k^- and a_k exist and are well-defined for all $k \in \{1,\dots,K\}$.

Furthermore, since T is marginal-rate progressive (and hence $t_{k-1} < t_k$ for all $k \in \{1,\dots,K\}$), agent monotonicity (Definition 2) implies that $a_k^- \le a_k < a_{k+1}^-$.[26]

Next, define the following family of sets covering $(0,+\infty)$:

$$\mathfrak{A} := \left\{(0,a_1^-],\{[a_k^-,a_k]\}_{k=1}^{K},\{[a_k,a_{k+1}^-]\}_{k=1}^{K-1},[a_K,+\infty)\right\}.$$

We first show that condition (3) is satisfied for ability distributions contained in each element of the family $\mathfrak{A}$.

(*i*) Consider first the interval $(0,a_1^-]$. Observe that $y(a,T) = y(a,T_0)$ for all $a \le a_1^-$. Because T_0 is a linear tax, it is *u-iir*, and so Lemma 3 gives

$$\frac{x(a,T)}{y(a,0)} = \frac{x(a,T_0)}{y(a,0)} \ge \frac{x(a',T_0)}{y(a',0)} = \frac{x(a',T)}{y(a',0)} \qquad \forall a \le a' \le a_1^-. \tag{6}$$

(*ii*) For $[a_K,+\infty)$, a symmetric argument shows that

$$\frac{x(a,T)}{y(a,0)} \ge \frac{x(a',T)}{y(a',0)} \qquad \forall a_K \le a \le a'. \tag{7}$$

[26] Observe that, letting $\bar{x}_k = x(a_k^-,T) = x(a_k,T)$, $\eta^{a_k^-}(\bar{x}_k,\bar{y}_k) = 1-t_{k-1} > 1-t_k = \eta^{a_k}(\bar{x}_k,\bar{y}_k)$. On the other hand, $\eta^{a_k}(\bar{x}_{k+1},\bar{y}_{k+1}) > 1-t_k = \eta^{a_{k+1}^-}(\bar{x}_{k+1},\bar{y}_{k+1})$.

(*iii*) Now consider the interval $[a_k^-, a_k]$ for $k \in \{1,\dots,K\}$. Observe that

$$y(a_k,T) = y(a_k,T_k) = \overline{y}_k = y(a_k^-,T_{k-1}) = y(a_k^-,T).$$

Hence, by monotonicity of the map $a \mapsto y(a,T)$ (Lemma 1) , $y(a,T) = \overline{y}_k$ for all $a \in [a_k^-, a_k]$. Therefore, because $y(a',0) \geq y(a,0)$ for all $a_k^- \leq a \leq a' \leq a_k$ by Lemma 1,

$$\frac{x(a,T)}{y(a,0)} = \frac{\overline{y}_k - T(\overline{y}_k)}{y(a,0)} \geq \frac{\overline{y}_k - T(\overline{y}_k)}{y(a',0)} = \frac{x(a',T)}{y(a',0)} \qquad \forall a,a' \in [a_k^-, a_k],\ a \leq a'. \tag{8}$$

(*iv*) Finally, consider the interval $[a_k, a_{k+1}^-]$ for $k \in \{1,\dots,K-1\}$. By construction, we have $y(a,T) = y(a,T_k)$ for all $a \in [a_k, a_{k+1}^-]$. Therefore, since T_k is a linear (hence *u-iir*) tax, Lemma 3 gives

$$\frac{x(a,T)}{y(a,0)} = \frac{x(a,T_k)}{y(a,0)} \geq \frac{x(a',T_k)}{y(a',0)} = \frac{x(a',T)}{y(a',0)} \qquad \forall a,a' \in [a_k, a_{k+1}^-],\ a \leq a'. \tag{9}$$

Combining equations (6)-(9) we obtain (3) for every $\boldsymbol{a} \in \mathscr{A}$.

4 Concluding remarks

This paper provides a normative foundation for progressive income taxes. Our work—which goes beyond the classic results of Jakobsson (1976) and Fellman (1976) in that it takes into account the disincentive effects of taxation on work effort—relies on three basic conditions: the agent monotonicity condition, which is standard in the literature on nonlinear taxation with endogenous labor supply; piecewise linearity of admissible tax schedules, an ubiquitous feature of actual statutory tax schedules; and an increasingly large marginal rate of substitution of labor for consumption for vanishingly small amounts of leisure time. The latter condition fails in the Jakobsson-Fellman setting, viewed in the Mirrlees framework as the particular case of costless work effort.

Theorem 1 implies that tax schedules aimed at reducing income inequality must be marginal-rate progressive. In other words, only marginal-rate progressive tax schedules can secure a reduction in consumption inequality compared to a situation with no taxes.[27] Theorem 2 can be combined with Theorem 3 to obtain a necessary and sufficient condition on primitives under which the members of $\mathscr{T}_{prog}$, and only the members of $\mathscr{T}_{prog}$, are inequality reducing (see Corollary 3).

We conclude with a philosophical comment. This paper focuses on the reduction of *income* inequality through the tax system, as does virtually all the related literature. Some authors have suggested that considering instead the *welfare* inequality reducing properties of taxes might be more reasonable. After all, individuals ultimately care about their well-

[27] When individual incomes are subjected to different rounds of taxation, an analyst may wish to study the inequality reducing properties of the composition of different tax schedules. Le Breton et al. (1996) address this issue in the context of exogenous income. The extension of their analysis to the case of endogenous income is left for future research.

being. While we believe that this idea deserves further investigation, we would like to point out that a meaningful characterization of the welfare inequality reducing properties of progressive tax schedules seems problematic even in the standard context of exogenous incomes. Indeed, consider the marginal-rate progressive (hence income inequality reducing) tax function $T(y) := y - \ln(y+1)$, together with the utility function $u(x) := \ln(x)$; since the ratio $\frac{u(y-T(y))}{u(y)} = \frac{\ln(\ln(y+1))}{\ln(y)}$ is strictly increasing in y, it follows from Lemma 1 in Jakobsson (1976) that T is welfare inequality *increasing*.[28,29] The point raised here pertains to Lorenz-based measures ranking "welfare" distributions, according to some "social norm." A different, but related, issue concerns the characterization of the Lorenz ordering in terms of classes of social norms. This characterization draws on the link between statistical measures and social welfare, and takes different forms. For income distributions with the same mean, the Lorenz ordering used in this paper is equivalent to welfare dominance in terms of any *Schur-concave* (resp. quasiconcave) social norm (*cf.* Kolm (1969), Atkinson (1970), Dasgupta et al. (1973), and Rothschild and Stiglitz (1973)). For income distributions with different means, welfare dominance characterizes the ranking induced by the *generalized Lorenz curve* (*cf.* Shorrocks (1983)), which scales up the ordinary Lorenz curve by the mean of the distribution. The generalized Lorenz ordering is not generally equivalent to the relative Lorenz ordering.

Appendix

This section presents the proofs of Theorem 3, Proposition 1, Lemma 2, Lemma 3, and Lemma 4. For the convenience of the reader, each proof is preceded by a restatement of the corresponding result.

A Proof of Theorem 3

Theorem 3. *For $u \in \mathscr{U}^*$, the members of $\mathscr{T}_{lin}$ are u-iir if and only if $u \in \widehat{\mathscr{U}}$.*

Proof. Given $u \in \mathscr{U}^*$, the members $T(y) = -b + t_0 y$ of $\mathscr{T}_{lin}$ are u-iir if and only if the map

$$a \mapsto \frac{x^u(a,T)}{y^u(a,0)} = \frac{a(1-t_0)l^u(a,T)+b}{al^u(a,0)} = \frac{a(1-t_0)l^u((1-t_0)a,b)+b}{al^u(a,0)} \tag{10}$$

[28] According to Ebert and Moyes (2007, footnote 19), in the case of exogenous income, the Jakobsson-Fellman result can be stated in terms of welfare inequality if and only if the utility function is isoelastic, *i.e.*, it takes the form $u(x) = \chi x^{\xi}$, where χ and ξ are constants.

[29] A further complication is given by the strong cardinal nature of the notion of *welfare* inequality, which imposes conditions on sums and ratios of utility indices that are generally violated by order-preserving utility transformations. We thank an anonymous referee for raising this point.

defined on $\mathbb{R}_{++}$ is nonincreasing for every $(b,t_0)\in\mathbb{R}_+\times[0,1)$ (*cf.* Lemma 3).[30] Equivalently, the members of $\mathscr{T}_{lin}$ are u-iir if and only if

$$\frac{(1-t_0)\left((1-t_0)a'\frac{\partial l^u((1-t_0)a',b)}{\partial a}+l^u((1-t_0)a',b)\right)a'l^u(a',0)}{(a'l^u(a',0))^2} - \frac{((1-t_0)a'l^u((1-t_0)a',b)+b)\left(a'\frac{\partial l^u(a',0)}{\partial a}+l^u(a',0)\right)}{(a'l^u(a',0))^2}\leq 0 \tag{11}$$

for every $(a',b,t_0)\in\mathbb{R}_{++}\times\mathbb{R}_+\times[0,1)$.[31] Since the above inequality can be expressed as

$$\frac{(1-t_0)a'\left((1-t_0)a'\frac{\partial l^u((1-t_0)a',b)}{\partial a}+l^u((1-t_0)a',b)\right)}{(1-t_0)a'l^u((1-t_0)a',b)+b}\leq\frac{a'\left(a'\frac{\partial l^u(a',0)}{\partial a}+l^u(a',0)\right)}{a'l^u(a',0)},$$

or, equivalently, as

$$\zeta^u((1-t_0)a',b)\leq\zeta^u(a',0), \tag{12}$$

we see that the members of $\mathscr{T}_{lin}$ are u-iir if and only if (12) holds for every $(a',b,t_0)\in\mathbb{R}_{++}\times\mathbb{R}_+\times[0,1)$. This is equivalent to the following condition: for every $(a',b,t_0)\in\mathbb{R}_{++}\times\mathbb{R}_+\times[0,1)$,

$$\zeta^u((1-t_0)a',b)\leq\zeta^u((1-t_0)a',0)\leq\zeta^u(a',0). \tag{13}$$

Consequently, for $u\in\mathscr{U}^*$, the members of $\mathscr{T}_{lin}$ are u-iir if and only if $u\in\widehat{\mathscr{U}}$.[32] ■

B Proof of Proposition 1

Proposition 1. $\widehat{\widehat{\mathscr{U}}}\subsetneq\widehat{\mathscr{U}}$.

Proof. First, observe that, after some manipulation, we can write

$$\zeta^u(a,b)=\frac{al^u(a,b)}{al^u(a,b)+b}(1+\xi^u(a,b))=H(a,b)(1+\xi^u(a,b)), \tag{14}$$

where $H(a,b):=\frac{al^u(a,b)}{al^u(a,b)+b}$.

The (weak) containment is established in two steps.

Step 1. We show that if $\xi^u(a,0)$ is nondecreasing in a then so is $\zeta^u(a,0)$. Take $a\in\mathbb{R}_{++}$. Since $H(a,0)=1$, (14) becomes

$$\zeta^u(a,0)=1+\xi^u(a,0),$$

[30] The last equality follows from the fact that both $l^u(a,T)$ and $l^u((1-t_0)a,b)$ are solutions to the problem

$$\max_{l\in[0,1]}u((1-t_0)al+b,l).$$

[31] More precisely, the map defined in (10) is nonincreasing for every $(b,t_0)\in\mathbb{R}_+\times[0,1)$ if and only if for every $(b,t_0)\in\mathbb{R}_+\times[0,1)$, (11) holds for all but perhaps one $a'>0$.

[32] Ebert and Moyes (2007) use inequalities analogous to (12) and (13) to derive the sufficient conditions (b-1) and (b-2) in their Proposition 2.

implying that $\zeta^u(a,0)$ is nondecreasing in a if and only if $\xi^u(a,0)$ is nondecreasing in a.

Step 2. We show that if $\xi^u(a,b) \leq \xi^u(a,0)$ for all $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_+$, then $\zeta^u(a,b) \leq \zeta^u(a,0)$ for all $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_+$.

Take $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_+$. Since $H(a,0) = 1$, (14) gives

$$\frac{\zeta^u(a,b)}{\zeta^u(a,0)} = H(a,b)\frac{1+\xi^u(a,b)}{1+\xi^u(a,0)}.$$

Observe that $H(a,b) = \frac{al^u(a,b)}{al^u(a,b)+b} \leq 1$ and, by assumption, $\xi^u(a,b) \leq \xi^u(a,0)$. Therefore, both factors are less or equal to one, implying that $\zeta^u(a,b) \leq \zeta^u(a,0)$.

Finally, the map $u(c,l) := c(1-l)$ is a member of $\widehat{\mathscr{U}}$ (see Remark 3 and footnote 19) but does not belong to $\widehat{\widehat{\mathscr{U}}}$.[33] ■

C Proof of Lemma 2

Lemma 2. *Given $u \in \mathscr{U}$, $(c,y) \in \mathbb{R}^2_{++}$, and $q \in (0,+\infty)$, there exists an $a > y$ such that $\eta^a(c,y) = q$.*[34]

Proof. Observe that

$$\lim_{a\to\infty} \eta^a(c,y) = \lim_{a\to\infty} \frac{1}{a} MRS(c,y/a) = 0 \quad \text{and} \quad \lim_{a\downarrow y} \eta^a(c,y) = \lim_{a\downarrow y} \frac{1}{a} MRS(c,y/a) = +\infty.$$

Since the map $a \mapsto \eta^a$ is continuous, the lemma follows from the Intermediate Value Theorem. ■

D Proof of Lemma 3

Lemma 3. *Given $u \in \mathscr{U}^*$, a tax schedule $T \in \mathscr{T}$ is u-iir if and only if for any ability distribution $\boldsymbol{a} \in \mathscr{A}$ and for any pre-tax and post-tax income functions y^u and x^u,*

$$\frac{x^u(a_i,T)}{y^u(a_i,0)} \geq \frac{x^u(a_{i+1},T)}{y^u(a_{i+1},0)} \qquad \forall i \in \{1,\dots,n-1\}. \tag{15}$$

Proof. We adapt the proof of Lemma 2 in Ebert and Moyes (2007).[35]

($\Leftarrow$) A tax schedule is *u-iir* if condition (15) holds for any $\boldsymbol{a} \in \mathscr{A}$ and any pre-tax and post-tax income functions y^u and x^u. This is a direct consequence of Theorem 2.4 in Marshall et al. (1967), since for each $\boldsymbol{a} \in \mathscr{A}$, $x^u(a_i,T) > 0$ and $y^u(a_i,0) > 0$ for each $i \in \{1,\dots,n\}$.[36]

[33] Indeed, for $0 < b < a$, $\xi(a,b) = \frac{b}{a-b}$, and so $\xi(a,0) = 0 < \xi(a,b)$. Hence, $u \notin \widehat{\widehat{\mathscr{U}}}$.

[34] Feasibility requires $a \geq y$ (recall that $l = \frac{y}{a}$ and $l \leq 1$), and the lemma states that there is an $a > y$ such that $\eta^a(c,y) = q$.

[35] In their proof, Ebert and Moyes (2007) assume the existence of a unique solution to the agents' maximization problem in (2), while we allow for multiple maximizers. Our proof is otherwise identical in substance with that in Ebert and Moyes (2007).

[36] See also Chapter 5 in Marshall et al. (2011), where B.1.b only requires $\sum_{j=1}^n y^u(a_j,0) > 0$ instead of $y^u(a_i,0) > 0$ for each i.

($\Rightarrow$) Suppose that there exists $\overline{\boldsymbol{a}} \in \mathscr{A}$ and x^u and y^u such that

$$\frac{x^u(\overline{a}_h, T)}{y^u(\overline{a}_h, 0)} < \frac{x^u(\overline{a}_{h+1}, T)}{y^u(\overline{a}_{h+1}, 0)} \qquad \text{for some } h \in \{1, \ldots, n-1\}. \tag{16}$$

Choose $\boldsymbol{a}^* := (a_1^*, \ldots, a_n^*)$ where $a_1^* := \overline{a}_h$ and $a_i^* := \overline{a}_{h+1}$ for $i \in \{2, \ldots, n\}$. By definition, $a_1^* < a_2^* = \cdots = a_n^*$. It follows that $x^u(a_1^*, T) = x^u(\overline{a}_h, T)$ and $x^u(a_i^*, T) = x^u(\overline{a}_{h+1}, T)$ for $i \in \{2, \ldots, n\}$. And similarly for y. From (16),

$$\frac{x^u(a_1^*, T)}{y^u(a_1^*, 0)} < \frac{x^u(a_2^*, T)}{y^u(a_2^*, 0)} = \cdots = \frac{x^u(a_n^*, T)}{y^u(a_n^*, 0)}.$$

Appealing to Marshall et al. (2011, B.1.b in Chapter 5),

$$\left(\frac{x^u(a_1^*, T)}{\sum_j x^u(a_j^*, T)}, \ldots, \frac{x^u(a_n^*, T)}{\sum_j x^u(a_j^*, T)} \right)$$

is majorized by

$$\left(\frac{y^u(a_1^*, 0)}{\sum_j y^u(a_j^*, 0)}, \ldots, \frac{y^u(a_n^*, 0)}{\sum_j y^u(a_j^*, 0)} \right).^{37}$$

Therefore,

$$\sum_{i=1}^{k} \left(\frac{x^u(a_i^*, T)}{\sum_j x^u(a_j^*, T)} \right) \leq \sum_{i=1}^{k} \left(\frac{y^u(a_i^*, 0)}{\sum_j y^u(a_j^*, 0)} \right) \quad \forall k \in \{1, \ldots, n\}.$$

That is, $(y^u(a_1^*, 0), \ldots, y^u(a_n^*, 0))$ Lorenz dominates $(x^u(a_1^*, T), \ldots, x^u(a_n^*, T))$, and hence T is not *u-iir*. ■

E Proof of Lemma 4

Lemma 4. *Let $u \in \mathscr{U}^*$ and $T \in \mathscr{T}$, and let x^u be a post-tax income function. Then the map $a \mapsto x^u(a, T)$ is continuous on $\mathbb{R}_{++}$ if and only if T is marginal-rate progressive.*

Proof. Take $u \in \mathscr{U}^*$ and $T = (\alpha, \boldsymbol{t}, \overline{\boldsymbol{y}}) \in \mathscr{T}$. First observe that T is marginal-rate progressive (*i.e.*, convex) if and only if the map $y \mapsto x = y - T(y)$ defined on $\mathbb{R}_+$ is concave.

($\Leftarrow$) Let T be marginal-rate progressive. Because the map $y \mapsto x(y) := y - T(y)$ is concave and $u(c, l)$ is strictly quasiconcave (and strictly increasing (resp. decreasing) in c (resp. l)), for each $a > 0$ the problem

$$\max_{y \in [0, a]} u\left(y - T(y), \frac{y}{a}\right) \tag{17}$$

has a unique solution. Consequently, there is a unique map that assigns to each ability level $a > 0$ the pre-tax income $y(a)$ that solves (17), and by virtue of Berge's Maximum Theorem, this map is continuous. But then the map $a \mapsto y(a) - T(y(a))$ defined on $\mathbb{R}_{++}$ is continuous. In other words, for any post-tax income function x^u, the map $a \mapsto x^u(a, T)$ is continuous on $\mathbb{R}_{++}$.

[37] For $x, y \in \mathbb{R}^n$, in increasing arrangement, we say that x is majorized by y if $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ for $k = 1, \ldots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. (Marshall et al., 2011, p. 8.)

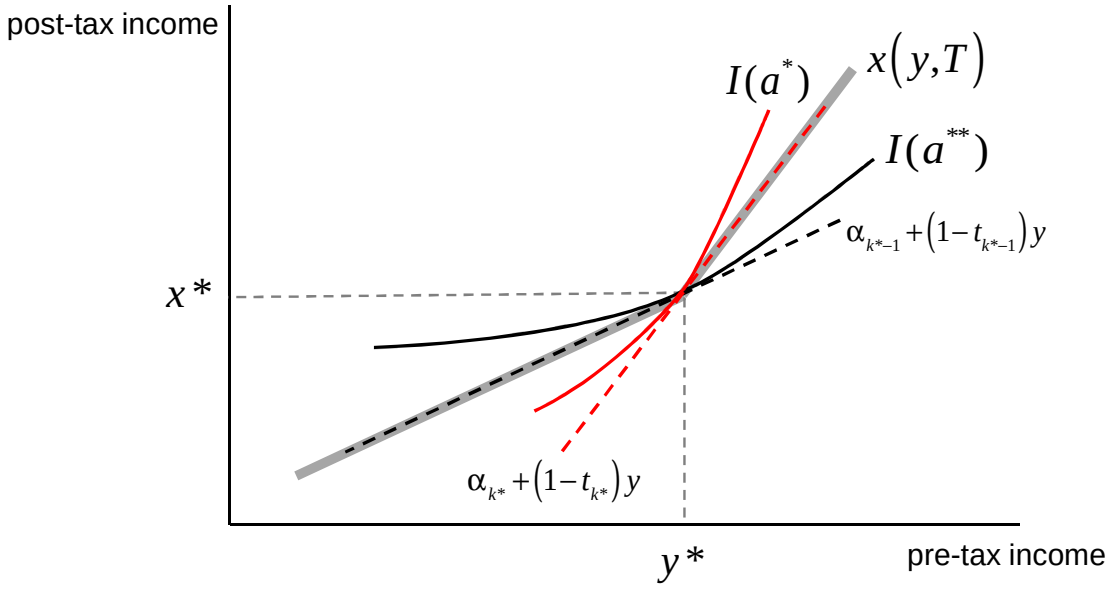

Figure 3: Figure for Lemma 4

(⇒) To prove the converse assertion we assume that T is not marginal-rate progressive and show that the map $a \mapsto x^u(a,T)$ has a discontinuity point in $\mathbb{R}_{++}$.

If T is not marginal-rate progressive, the map $x(y)$ is not concave. Define

$$y^* := \inf\{y \in \mathbb{R}_+ : x(y)|_{[0,y^*]} \text{ is concave}\}.$$

It is easy to see that $y^* = \overline{y}_{k^*}$ for some $k^* \in \{1,...,K\}$. In addition, the restriction of $x(y)$ to $[\overline{y}_{k^*-1}, \overline{y}_{k^*+1}]$ is convex (here $\overline{y}_{K+1} := +\infty$) and $(x^* := y^* - T(y^*), y^*) \gg 0$ and $0 < 1 - t_{k^*-1} < 1 - t_{k^*}$. Applying Lemma 2 it follows that there exist $0 < a^* < a^{**}$ such that

$$\eta^{a^*}(x^*, y^*) = 1 - t_{k^*} \quad \text{and} \quad \eta^{a^{**}}(x^*, y^*) = 1 - t_{k^*-1},$$

implying $x^u(a^*,T) < x^* < x^u(a^{**},T)$ and $x^u(a,T) \neq x^*$ for all $a \in (a^*, a^{**})$. (Refer to Figure 3.) If the map $a \mapsto x^u(a,T)$ were continuous on $\mathbb{R}_{++}$, the Intermediate Value Theorem would give $a \in (a^*, a^{**})$ with $x^u(a,T) = x^*$, a contradiction. We conclude that the map $a \mapsto x^u(a,T)$ has a discontinuity point in $\mathbb{R}_{++}$. ■

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