



## **A Generalized Model of Sales**

**Sandro Shelegia  
Chris M. Wilson**

**October 2016**

*Barcelona GSE Working Paper Series*

*Working Paper n° 934*

# A Generalized Model of Sales

Sandro Shelegia and Chris M. Wilson\*

October 10, 2016

## Abstract

To provide a more flexible workhorse model of temporary price reductions or ‘sales’, this paper presents a substantially generalized ‘clearinghouse’ sales framework. Our framework permits multiple dimensions of firm heterogeneity, and views firms as competing directly in utility rather than prices. The paper i) reproduces and extends many equilibria from the existing literature, ii) offers a range of new results on how firm heterogeneity affects market outcomes, iii) provides original insights into the number and type of firms that use sales, and iv) extends a ‘cleaning’ procedure that is commonly used in empirical studies of sales and price dispersion.

Keywords: Sales; Price Dispersion; Advertising; Clearinghouse; Heterogeneity

JEL Codes: L13; D43; M3

---

\*Shelegia: Department of Economics and Business, Universitat Pompeu Fabra, and Barcelona GSE, sandro.shelegia@upf.edu. Wilson: School of Business and Economics, Loughborough University, c.m.wilson@lboro.ac.uk. We especially thank Mark Armstrong, Mike Baye, Andrew Rhodes, Yossi Spiegel, Matthijs Wildenbeest, and seminar participants at EEA-ESEM (Geneva), EARIE (Lisbon), the Workshop on Consumer Search and Switching Costs (Indiana), the Workshop on the Economics of Advertising and Marketing (Vienna), the Industrial Organization: Theory, Empirics and Experiments Conference (Alberobello), the University of Vienna, Universitat Autònoma de Barcelona, and Brunel University.

# 1 Introduction

The evidence of price dispersion within markets is overwhelming, even when products are homogeneous (as reviewed by Baye et al. (2006)). Empirical findings suggest that much of this dispersion arises from temporary price reductions or ‘sales’. Such sales activity is often accompanied by informative advertising and accounts for 20-50% of retail price variation, and 38% of all packaged consumer good purchases in the US.<sup>1</sup>

One major theoretical literature explains how sales can arise in the form of mixed strategies due to variation in consumers’ search frictions and/or the existence of moderate advertising costs.<sup>2</sup> This literature has offered deep insights into sales and provided an analytical foundation for many broader topics, including price comparison platforms, obfuscation, choice complexity, and even macroeconomic fluctuations.<sup>3</sup> Such mixed strategy sales are often consistent with empirical evidence; however, current models struggle to fully explain the observed differences in firms’ pricing and advertising behaviors due to their restricted ability to allow for firm heterogeneity.<sup>4</sup> This limitation constrains the theoretical and empirical understanding of sales, and inhibits wider literatures. Indeed, as Baye and Morgan (2009, p.1151) state “...little is known about asymmetric models within this class. Breakthroughs on this front would not only constitute a major theoretical advance, but permit a tighter fit between the underlying theory and empirics”.

In response, this paper presents a substantially generalized and fully asymmetric ‘clearinghouse’ sales framework (e.g. Baye and Morgan (2001), Baye et al. (2004a), Baye et al. (2006)). At the theoretical level, while its modeling assumptions sometimes differ to existing research, our framework can neatly reproduce many of the past literature’s sales predictions and extend them to more complex markets with multiple dimensions of firm heterogeneity. Moreover, we show how the framework can offer a range of new results on the effects of firm asymmetries on market outcomes, and the factors that determine the number and type of firms that use sales. At the empirical level, we then use the framework to assess and extend a ‘cleaning’ procedure that is commonly used within the large empirical literature on sales and price dispersion. Overall, we hope that our framework can provide a convenient workhorse model of sales and open up new sales research areas where

---

<sup>1</sup>See Nakamura and Steinsson (2008), Hosken and Reiffen (2004); and Steenkamp et al. (2005).

<sup>2</sup>For example, Varian (1980), Burdett and Judd (1983), Stahl (1989), Janssen and Moraga-González (2004); Robert and Stahl (1993), Baye and Morgan (2001). See Baye et al. (2006) for a review.

<sup>3</sup>For reviews, see Moraga-González and Wildenbeest (2012), Armstrong (2015), and Spiegler (2015). For an example of a macroeconomic model using sales, see Kaplan and Menzio (2016).

<sup>4</sup>For example evidence, see Lach (2002), Baye et al. (2004a), Baye et al. (2004b), Lewis (2008), Chandra and Tappata (2011), Wildenbeest (2011), Giuliatti et al. (2014) and Pennerstorfer et al. (2015).

firm heterogeneity is important, including the incentives for platforms to charge different advertising fees, and individual firm’s choice of brand loyalty, advertising channel, price presentation, and investment strategies.

The original clearinghouse framework considers a symmetric market with a single homogeneous good. Consumers are potentially split into ‘non-shoppers’ that are only willing to buy from a designated firm, and ‘shoppers’ that can buy from any firm. Firms choose their price, and whether to inform consumers of this price via some advertising channel. As consistent with observed sales behavior, the equilibrium involves each firm randomizing between selecting a high price without advertising, and advertising a lower price drawn from a common support. The sales model by Varian (1980) is obtained as a special case when advertising costs are zero.

We modify the clearinghouse framework in two important respects. First, we recast the firms as competing in (net) utility rather than prices. By drawing on the seminal (symmetric, pure-strategy) model of competition in the utility space by Armstrong and Vickers (2001), we let each firm  $i$  compete directly in utility,  $u_i$ , with an associated per-consumer profit function,  $\pi_i(u_i)$ , that depends upon the firm’s underlying demand, products, costs, and pricing technology. With little increase in computation, this facilitates a high level of generality across otherwise complex market settings including downward-sloping demand, multiple products, and two-part tariffs.

Second, we make a subtle change to the tie-break rule. The existing literature assumes shoppers i) trade exclusively with advertising firms in any tie between advertising and non-advertising firms, and ii) mix between the tied firms with equal probability in any other form of tie. While consistent with a symmetric ‘gatekeeper’ version of the clearinghouse model where the advertising channel involves a price-comparison platform and where shoppers face additional visit costs to buy from non-listed firms (Baye and Morgan (2001)), this form of tie-break rule complicates and limits any analysis under firm heterogeneity.<sup>5</sup> To resolve this problem, we introduce a different tie-break rule within the ‘advertising’ version of the clearinghouse model where shoppers receive all adverts before making their visit decisions (Baye et al. (2004a) and Baye et al. (2006)). Here, shoppers should be willing to buy from any advertising or non-advertising firm with the same expected utility and so we are free to determine the assignment of shoppers between any tied firms as part

---

<sup>5</sup>For instance, even in a duopoly with unit demand where the only form of heterogeneity involves firms’ shares of non-shoppers, Arnold et al. (2011) provide an equilibrium which exhibits mass points in advertised prices, and does not converge to standard equilibria as advertising costs tend to zero (e.g. Narasimhan (1988)). In more complex asymmetric settings, equilibria with this tie-break rule are often intractable.

of equilibrium. In particular, we adopt a tie-break rule where shoppers mix between any tied firms in a unique way that partially offsets any firm heterogeneities and ensures that all firms have the same incentive to advertise a common upper utility bound. This modification makes no difference in symmetric settings, but offers significant tractability in asymmetric settings by eliminating any mass points in advertised utility distributions. Specifically, it allows us to simultaneously permit i) any variation in firms' shares of non-shoppers, ii) any variation in firms' advertising costs, and iii) considerable variation in firms' profit functions. In equilibrium, the firms then differ in advertising probabilities, utility distributions, and profits depending on the level and form of heterogeneity.<sup>6</sup>

In Sections 2 and 3, we present the framework under duopoly. After deriving the equilibrium, we demonstrate how it can reproduce many predictions from the existing literature and generalize them to more complex market environments with multiple forms of heterogeneity.<sup>7</sup> We also show how the framework can enable characterizations of common forms of sales that have previously remained unstudied within the clearinghouse literature, including cases where firms use two-part tariffs or non-price variables such as package size (e.g 'X% Free').

Sections 4 and 5 then provide a number of new theoretical results regarding the effects of firm-level characteristics and advertising costs on sales and market performance. These remain untested empirically because existing empirical studies often focus on different factors, including market information, competition, and rivals' behavior.<sup>8</sup> Section 4 offers some comparative statics on the effects of non-shoppers, advertising costs, and profit functions. For instance, standard results suggest that an industry-wide increase in advertising costs deters the use of sales and raises firms' profits. However, by isolating the increase in a single firm's advertising costs, we show that both firms still reduce their use of sales, but that it is rival rather than own advertising costs that matter in determining profits. Similarly, while an industry-wide increase in firm profitability enhances the use of sales, we find that an increase in an individual firm's profitability can either increase or decrease its use of sales depending on whether its profits have increased more at upper or lower utility levels.

---

<sup>6</sup>The variation in profit functions is subject to a condition that is implicit within the existing literature whereby each firm would offer the same utility level under monopoly,  $u_i^m = u^m \geq 0$ . The condition places no restriction on each firm's monopoly profits, and is innocuous under some market conditions, including unit demand.

<sup>7</sup>Among many others, these include symmetric models such as Varian (1980), Baye et al. (2004a), Baye et al. (2006), and Simester (1997), and asymmetric models, such as Narasimhan (1988), Baye et al. (1992), Kocas and Kiyak (2006), and Wildenbeest (2011).

<sup>8</sup>For example, Lewis (2008), Chandra and Tappata (2011); Shankar and Bolton (2004), Ellickson and Misra (2008).

Section 5 offers further new insights by extending the framework to markets with  $n > 2$  firms. Here, the existing literature with heterogeneous firms is particularly scant - in a setting with heterogeneous firms, unit demand, and zero advertising costs, it suggests that only two firms can ever engage in sales behavior (Baye et al. (1992), Kocas and Kiyak (2006) and Shelegia (2012)). In contrast, and in better line with typical empirical findings (e.g. Lach (2002), Lewis (2008), Chandra and Tappata (2011)), we show that any number of heterogeneous firms  $k^* \in [2, n]$  can engage in equilibrium sales once advertising costs are allowed to be positive. Intuitively, despite the direct cost increase, higher advertising costs can prompt *more* firms to use advertised sales by softening competition for the shoppers. Our results also provide a broad characterization of which types of firms are likely to use advertised sales. Ceteris paribus, these are firms with relatively low shares of non-shoppers, low advertising costs, and either high or low profitability depending on market conditions.

Finally, Section 6 uses our framework to assess and extend current methodologies within the large empirical literature on sales and price dispersion. Without a general theoretical foundation, typical papers are forced to ‘clean’ their raw price data from firm-level heterogeneities by using the residuals from a price regression with firm-level fixed-effects.<sup>9</sup> However, this approach is known to be restrictive. In a rare theoretical justification, Wildenbeest (2011) verifies the procedure’s validity in a setting of unit demand and zero advertising costs where the firms differ in quality and costs, but share the same value-cost margin. Under our more general framework, we show two results. First, under downward-sloping demand, we identify a related value-cost condition but find that the fixed-effects approach is insufficient. Second, under unit demand, we suggest a modified procedure than can be used for a broader range of heterogeneities than those considered by Wildenbeest (2011).

## Related Literature

Armstrong and Vickers (2001) introduced the concept of competition in the utility space to study price discrimination in a symmetric, pure-strategy equilibrium setting. In contrast, we transfer their utility approach into an asymmetric (clearinghouse) model to study mixed strategy sales. Some past sales papers have also referred to competition in utility

---

<sup>9</sup>Papers then use the residuals in i) reduced-form studies of price dispersion, e.g. Sorensen (2000), Lach (2002), Brown and Goolsbee (2002), Barron et al. (2004), Lewis (2008), Chandra and Tappata (2011), Pennerstorfer et al. (2015), and Sherman and Weiss (2015), or ii) structural estimations, e.g. Wildenbeest (2011), Moraga-González et al. (2013), Giuliatti et al. (2014), Allen et al. (2014), and An et al. (2015).

or ‘surplus’ (Simester (1997), Hosken and Reiffen (2007), Wildenbeest (2011), Dubovik and Janssen (2012), Anderson et al. (2015)). However, these papers only use it to compute sales equilibria in specific market settings, and do not use the associated profit function,  $\pi(u)$ , to explore any general results or implications.

As later detailed, the few existing models of sales with firm heterogeneity often assume single products, unit demand, and zero advertising costs (e.g. Narasimhan (1988), Baye et al. (1992), Kocas and Kiyak (2006), and Wildenbeest (2011)). Our framework can reproduce and substantially extend such equilibria. Outside these market conditions, there are only the papers by Arnold et al. (2011), which we previously noted in footnote 5, and Anderson et al. (2015). Anderson et al. (2015) allow for firm heterogeneity in a non-clearinghouse model where firms must advertise to earn positive profits, and where all consumers are shoppers. Contrary to our model, they find that only two firms can ever use advertised sales when firms are heterogeneous. As such, they cannot analyze how market factors affect the number and type of firms that use sales, or connect to the larger theoretical or empirical clearinghouse literature. Instead, they focus on some interesting results regarding equilibrium selection and welfare.

## 2 Model

Let there be two firms,  $i = a, b$ , and a unit mass of risk-neutral consumers with a zero outside option. Firm  $i$  competes by choosing a utility offer (net of any associated payments),  $u_i \in \mathbb{R}_{\geq 0}$ . As standard within the mixed strategy sales literature, the consumers have identical preferences. Here, this implies that all consumers value firm  $i$ ’s offering at exactly  $u_i$ . The maximum possible profit that firm  $i$  can extract per consumer when providing  $u_i$  is defined as  $\pi_i(u_i)$ . The exact source of utility and form of the associated profit function can depend upon a rich set of demand, product, and cost conditions. However, to provide a simple illustrative example, let firm  $i$  sell a single good at price  $p_i$  with marginal cost  $c_i$ , to consumers with unit demand and a willingness to pay  $V_i$ . Firm  $i$ ’s utility offer then equals  $u_i = V_i - p_i$ , while its profits per consumer are  $\pi_i(u_i) = V_i - c_i - u_i$  for  $u_i \geq 0$ , and  $\pi_i(u_i) = 0$  otherwise.

We assume that  $\pi_i(u_i)$  is independent of the number of consumers served, and strictly quasi-concave in  $u_i$  with a unique maximizer at firm  $i$ ’s monopoly utility level,  $u_i^m \geq 0$ . For each firm  $i$ , we further assume i)  $\pi_i(u_i^m) \equiv \pi_i^m > 0$ , ii)  $\pi_i(u_i)$  is continuously differentiable for all  $u_i > u_i^m$ , and iii) there exists a finite break-even utility  $\hat{u}_i > u_i^m$  where  $\pi_i(\hat{u}_i) = 0$ .

Consumers are initially uninformed about the firms’ utility offers. Each firm can

choose whether or not to inform consumers of its offer by advertising under the following assumptions. In line with previous ‘advertising’ versions of the clearinghouse model (Baye et al. (2004a), Baye et al. (2006)), we assume that i) any advert is observed by all relevant consumers, and ii) advertising costs are exogenous. However, we further assume that iii) advertising costs can differ across firms, as consistent with different advertising capabilities or channels, and iv) each advertising cost is strictly positive with  $A_i > 0$  for  $i = a, b$ .<sup>10</sup>

There are two types of consumers, ‘non-shoppers’ and ‘shoppers’, in proportions,  $\theta \geq 0$  and  $(1 - \theta)$ . Non-shoppers ignore all adverts. They simply buy from their designated ‘local’ firm according to their underlying demand function, or exit. Our framework allows the firms to have different shares of non-shoppers,  $\theta_i \geq 0$ , with  $\theta_a + \theta_b = \theta$ . In contrast, the remaining ‘shopper’ consumers pay attention to adverts and can buy from any firm. However, to simplify exposition, we assume that shoppers can only visit one firm. Hence, shoppers choose between i) visiting an advertising firm to buy from its known utility offer, ii) visiting a non-advertising firm to discover its utility offer and potentially buy, or iii) exiting the market immediately.<sup>11</sup>

We analyze the following game. In Stage 1, each firm chooses its utility offer,  $u_i \in \mathbb{R}_{\geq 0}$ , and its advertising decision,  $\eta_i \in \{0, 1\}$ . To allow for mixed strategies, define i)  $\alpha_i \in [0, 1]$  as firm  $i$ ’s advertising probability, ii)  $F_i^A(u)$  as firm  $i$ ’s utility distribution when advertising, and iii)  $F_i^N(u)$  as firm  $i$ ’s utility distribution when not advertising, both on support  $\mathbb{R}_{\geq 0}$ . In Stage 2, consumers observe any adverts and then make their visit and purchase decisions in order to maximize their utility in accordance with the strategies outlined above. The solution concept is subgame perfect Nash equilibrium (SPE).

If shoppers are indifferent over which firm to visit, they randomize between firms with given probabilities (or, equivalently, visit the firms in given proportions). In particular, in the event of a utility tie, let  $x_i$  denote the probability that the shoppers visit firm  $i$ , with  $x_a + x_b = 1$ . The pair  $\{x_a, x_b\}$  will be determined as part of equilibrium. More precisely, and as later formalized, we select the unique  $\{x_a, x_b\}$  that partially offsets any firm heterogeneities and ensures that the firms have the same incentive to advertise a common upper utility bound. This tie-break rule makes two implicit assumptions. First,  $\{x_a, x_b\}$  is independent of the tied utility level; this is innocuous as we later show that ties can only occur at one utility level in equilibrium. Second,  $\{x_a, x_b\}$  is independent of

---

<sup>10</sup>iv) ensures that our tie-break rule is well-defined. However, when  $A_i = A_j \rightarrow 0$ , our equilibrium converges to that in a parallel model that allows for  $A_i = A_j = 0$  explicitly. See footnote 12 for more.

<sup>11</sup>These assumptions can be substantially generalized by allowing shoppers to visit the firms sequentially provided that i) the costs of any first visit are not too large, and ii) each shopper may only purchase from a single firm (‘one-stop shopping’). For technical details see Appendix C.



the firms' actual advertising decisions,  $\eta_a$  and  $\eta_b$ . As explained in the introduction, this assumption differs from the previous literature despite being natural in our context where the shoppers receive all adverts before making their visit decisions.

While our framework provides a large increase in generality, it cannot avoid an assumption that is implicit within the existing literature. We are the first to state it:

$$u_a^m = u_b^m = u^m \quad (\text{Assumption U})$$

Assumption U requires the firms to offer a common level of monopoly utility. Under unit demand or two-part tariffs, Assumption U is trivially satisfied because  $u_i^m$  is always zero. However, under downward-sloping demand and linear prices, one must restrict attention to a symmetric profit function  $\pi_a(u) = \pi_b(u)$ , (or introduce some binding lower bound on firms' utility offers, as consistent with a price ceiling or some unmodeled competitive fringe in some circumstances). Outside Assumption U, the power provided by our tie-break rule is lost: shoppers strictly prefer one firm in a tie when neither firm advertises and any mixed strategy equilibrium loses significant tractability, although some qualitative features remain.

Finally, we assume that both firms have some potential incentive to advertise. Assumption A makes a minimal restriction to ensure that firm  $i$ 's profits from not advertising with  $u_i = u^m$  and selling only to its non-shoppers,  $\theta_i \pi_i^m$ , are less than the profits it could obtain by advertising an offer just above  $u^m$  to gain all the shoppers,  $(1 - \theta_j) \pi_i^m - A_i$ . This assumption is relatively innocuous but allows us to focus on equilibria where both firms advertise.

$$A_i \leq (1 - \theta) \pi_i^m \quad \forall i = a, b \quad (\text{Assumption A})$$

### 3 Equilibrium Analysis

We proceed in a series of steps. First, any non-advertising firm will optimally set the monopoly utility level,  $u^m$ , with probability one because i) the firm has monopoly power over its non-shoppers, and ii) any visiting shopper cannot visit elsewhere. Thus, in any SPE, shoppers must expect any firm with  $\eta_i = 0$  to offer  $u^m$ .

Next, advertising is strictly dominated for firm  $i$  when  $u_i = u^m$  because advertising has no effect on the shoppers' tie-break decision, yet costs  $A_i > 0$ . Hence, firm  $i$  will only ever advertise  $u_i > u^m$ . This implies that firm  $i$ 's lowest advertised utility is strictly greater

than its non-advertised utility,  $u^m$ . Therefore, from this point forward, we can simply refer to firm  $i$ 's utility distribution *unconditional on advertising*,  $F_i(u)$ , with associated support,  $[u^m, \bar{u}_i]$ , where firm  $i$  sets  $u^m$  without advertising with probability  $1 - \alpha_i = F_i(u^m)$ , and uses advertised sales on  $(u^m, \bar{u}_i]$  with probability  $\alpha_i = 1 - F_i(u^m)$ .

By selecting  $u^m$  and not advertising, firm  $i$  will only ever possibly trade with the  $(1 - \theta)$  shoppers if i) firm  $j$  also chooses not to advertise, which occurs with probability  $(1 - \alpha_j)$ , and ii) the shoppers visit  $i$  rather than  $j$  in the subsequent tie, which occurs with probability,  $x_i$ . Consequently, when combined with the revenues from its  $\theta_i$  non-shoppers, firm  $i$  can always guarantee the following expected profits by not advertising, for any given  $x_i$  and  $\alpha_j$ :

$$\pi_i^m[\theta_i + x_i(1 - \alpha_j)(1 - \theta)] \quad (1)$$

There can be no equilibrium where both firms advertise with probability one because one firm would always deviate to avoid its positive advertising cost. However, as now formalized, the equilibrium takes one of two forms: when advertising costs are sufficiently large, neither firm advertises, otherwise both firms advertise with interior probabilities.

We first consider the latter form of equilibrium. Suppose, as later derived, that the firms advertise with probabilities,  $\alpha_i \in (0, 1)$  for  $i = a, b$ , and that the tie-breaking probabilities equal  $x_a \in (0, 1)$  and  $x_b = 1 - x_a$ . Then, by adapting standard arguments, one can show that no equilibrium exists with pure utility strategies. Instead, Lemma 1 follows (where all proofs are in Appendix A unless stated otherwise).

**Lemma 1.** *In a mixed strategy equilibrium, whenever a firm advertises, it randomizes its utility offer from a common interval  $(u^m, \bar{u}]$  without gaps or point masses.*

For firm  $i$  to advertise with interior probability  $\alpha_i \in (0, 1)$ , its profits from not advertising in (1) must equal its profits from advertising an offer slightly higher than  $u^m$ , where for a cost of  $A_i$  it can win the shoppers outright with the probability that its rival does not advertise,  $1 - \alpha_j$ . Hence, for both firms, we require  $\pi_i^m[\theta_i + x_i(1 - \alpha_j)(1 - \theta)] = \pi_i^m[\theta_i + (1 - \alpha_j)(1 - \theta)] - A_i$ . For a given  $x_j = 1 - x_i$ , we can then state:

$$\alpha_i = 1 - \frac{A_j}{x_i(1 - \theta)\pi_j^m}. \quad (2)$$

In a mixed strategy equilibrium, each firm must always expect to earn its equilibrium profits,  $\bar{\Pi}_i$ . By substituting  $\alpha_i$  from (2) into the profits from not advertising (1), and using  $x_j = 1 - x_i$ , we obtain (3). Equilibrium profits are therefore strictly positive, and

derive from two potential channels: non-shoppers,  $\theta_i$ , and costly advertising,  $A_i$ .

$$\bar{\Pi}_i = \theta_i \pi_i^m + \frac{x_i}{1 - x_i} A_i \quad (3)$$

Next, we derive the utility distributions. By advertising any sale offer within  $(u^m, \bar{u}]$ , firm  $i$  gains expected profits,  $\Pi_i(u) = \pi_i(u)[\theta_i + (1 - \theta)F_j(u)] - A_i$ . Intuitively, firm  $i$  always collects its non-shopper profits but also wins the profits of the  $(1 - \theta)$  shoppers with the probability that firm  $j$  does not advertise a higher utility,  $F_j(u)$ . For each firm to be indifferent over the support, their respective expected profits must equal their equilibrium profits for all  $u \in (u^m, \bar{u}]$ . As there are no mass points within  $(u^m, \bar{u}]$ , this requires  $\pi_i(u)[\theta_i + (1 - \theta)F_j(u)] - A_i = \bar{\Pi}_i$  for  $i = a, b$ . By substituting from (3) and reversing subscripts, firm  $i$ 's utility distribution can be expressed by (4), where  $F_i(u^m) = 1 - \alpha_i$ .

$$F_i(u) = \frac{\bar{\Pi}_j - \theta_j \pi_j(u) + A_j}{(1 - \theta)\pi_j(u)} = \frac{x_i \theta_j [\pi_j^m - \pi_j(u)] + A_j}{x_i (1 - \theta)\pi_j(u)} \quad (4)$$

We now find the upper bound,  $\bar{u}$ , and the equilibrium tie-breaking probabilities,  $x_i$  and  $x_j$ . As there is no point mass at  $\bar{u}$ , a firm that advertises  $\bar{u}$  will win the shoppers with probability one. Hence, we require  $\bar{\Pi}_i = \Pi_i(\bar{u}) = (1 - \theta_j)\pi_i(\bar{u}) - A_i$ . By substituting from (3), we then specify  $x_i$  with the first equality in (5) below. The second equality then follows by using  $1 - x_i = x_j$  and reversing all the subscripts.

$$x_i = 1 - \frac{A_i}{\pi_i(\bar{u})(1 - \theta_j) - \theta_i \pi_i^m} = \frac{A_j}{\pi_j(\bar{u})(1 - \theta_i) - \theta_j \pi_j^m} \quad (5)$$

Intuitively, the equilibrium tie-breaking probabilities are chosen to partially offset the firm heterogeneities such that each firm has the same incentive to advertise the common upper utility bound  $\bar{u}$ . This prevents any mass points in the distribution of advertised utilities and creates significant tractability. In a symmetric market, it follows that  $x_a = x_b = 0.5$ . More generally, as later verified in Section 4,  $x_i$  is i) decreasing in factors that discourage firm  $i$  from advertising higher utilities, such as firm  $i$ 's share of non-shoppers,  $\theta_i$ , advertising costs,  $A_i$ , and monopoly profits,  $\pi_i^m$ , but ii) increasing in factors that encourage firm  $i$  to advertise higher utilities, such as firm  $i$ 's per-consumer profits at  $\bar{u}$ ,  $\pi_i(\bar{u})$ . Now, as  $x_a + x_b = 1$ , one can sum (5) over  $i = a, b$  and set equal to one:

$$\frac{A_a}{\pi_a(\bar{u})(1 - \theta_b) - \theta_a \pi_a^m} + \frac{A_b}{\pi_b(\bar{u})(1 - \theta_a) - \theta_b \pi_b^m} = 1 \quad (6)$$

Provided advertising costs are sufficiently low, this provides a unique solution for  $\bar{u}$  which is bounded between the monopoly utility level,  $u^m$ , and the lowest break-even

utility,  $\hat{u} = \min_i \{\hat{u}_i\}$ . This follows because i) the LHS of (6) is above 1 for  $\bar{u}$  sufficiently close to  $\hat{u}$ , and ii) the LHS of (6) is strictly increasing in  $\bar{u}$ ,  $A_a$  and  $A_b$ , over the relevant range. Therefore, to ensure the solution for  $\bar{u}$  is greater than  $u^m$ , an upper bound on advertising costs can be found by substituting  $u^m$  for  $\bar{u}$  in (6) and rearranging. This restriction is tighter than that under Assumption A:

$$\frac{A_a}{\pi_a^m} + \frac{A_b}{\pi_b^m} \leq 1 - \theta \quad (7)$$

This solution further ensures a unique set of interior tie-breaking probabilities  $x_a$  and  $x_b$  in (5), because it implies  $\pi_i(\bar{u})(1 - \theta_j) - \theta_i \pi_i^m > 0$  such that the two elements on the LHS of (6) are strictly between zero and one.<sup>12</sup>

When advertising costs are too high to satisfy (7), the game has a different form of equilibrium. Here, for a relevant range of  $x_a$  and  $x_b$ , both firms simply select  $u^m$  and refrain from advertising. Proposition 1 formally summarizes our equilibrium results:

**Proposition 1.** *Under our tie-breaking rule, the game has the following unique equilibrium:*

1. If  $\frac{A_a}{\pi_a^m} + \frac{A_b}{\pi_b^m} \leq 1 - \theta$ , each firm  $i$  offers  $u_i = u^m$  and does not advertise with probability  $(1 - \alpha_i) \in (0, 1)$  according to (2), and advertises a sale offer  $u_i$  from the interval  $(u^m, \bar{u}]$  according to (4) with probability  $\alpha_i$ , where  $\bar{u}$  solves (6) and where  $x_a = 1 - x_b$  is given by (5).
2. If  $\frac{A_a}{\pi_a^m} + \frac{A_b}{\pi_b^m} \geq 1 - \theta$ , both firms offer  $u_i = u^m$  and never advertise, while shoppers visit firm  $a$  with a probability  $x_a \in \left[1 - \frac{A_a}{\pi_a^m(1-\theta)}, \frac{A_b}{\pi_b^m(1-\theta)}\right]$ .

Henceforth, we focus only on the equilibrium with sales behavior. In the next section, we detail the effects of firm heterogeneities by examining some comparative statics. However, in the remainder of this section, we now briefly outline how our framework can reproduce and substantially extend many equilibria from the existing literature through further specification of utility offers,  $u_i$ , and profits per consumer,  $\pi_i(u_i)$ .<sup>13</sup>

<sup>12</sup>In an extreme case where the firms are asymmetric but  $A_i = A_j \rightarrow 0$ , the only way for the firms to share a common upper utility bound is for  $x_i \rightarrow 1$  and  $x_j \rightarrow 0$ . Firm  $j$  then advertises with  $\alpha_j \rightarrow 1$  and firm  $i$  advertises with  $\lim_{A \rightarrow 0} \alpha_i \in (0, 1)$ . This limit equilibrium converges to the equilibrium of a model that allows for  $A = 0$  explicitly without our tie-break rule. There, both firms advertise with probability one and use equivalent utility distributions except that firm  $i$  advertises  $u^m$  with a probability mass equivalent to  $\lim_{A \rightarrow 0} (1 - \alpha_i)$ . See Appendix B1 for full technical details.

<sup>13</sup>More precisely, while our modeling assumptions sometimes differ, we show how we can reproduce and extend the literature's key predictions for pricing, advertising, and purchasing behavior.

**Unit demand:** Following our previous unit demand example, suppose  $u_i = V_i - p_i$  and  $\pi_i(u_i) = V_i - c_i - u_i$ , where  $u_i^m = 0$  and  $\pi_i^m = V_i$ . Under symmetry, this produces a simple clearinghouse equilibrium, with  $x_i = 0.5$ ,  $\bar{\Pi} = \frac{\theta(V-c)}{2} + A$ ,  $1 - \alpha = \frac{2A}{(1-\theta)(V-c)}$ , and  $\bar{u} = \frac{2(1-\theta)(V-c)-4A}{2-\theta}$ . By using  $F(p) = 1 - F(u)$ , one can further derive  $1 - F(p) = \frac{\theta(V-p)+4A}{2(1-\theta)(p-c)}$ , with  $\bar{p} = V - u^m = V$  and  $\underline{p} = V - \bar{u} = c + \frac{\theta(V-c)+4A}{2-\theta}$ . This collapses to the (popularized) equilibrium of Varian (1980) when  $A \rightarrow 0$ . Under firm heterogeneity, the previous literature has largely focused on considering various combinations of asymmetries in terms of non-shopper shares, product values and/or costs under the restriction that  $A_i = A_j = 0$ . As detailed in Appendix B1, our framework can obtain these equilibria in the limit when  $A_i = A_j \rightarrow 0$  by allowing for any  $\theta_i$ ,  $c_i$ , and  $V_i$ . Moreover, our framework can also extend them to allow for positive and asymmetric advertising costs.<sup>14</sup>

**Downward-sloping demand:** Suppose firm  $i$  has  $K_i$  products, where  $\mathbf{c}_i$ ,  $\mathbf{p}_i$  and  $\mathbf{q}_i(\mathbf{p}_i)$  denote the associated vectors of marginal costs, prices, and product demand functions per consumer. The utility at firm  $i$  is then given its associated consumer surplus,  $u_i = S(\mathbf{p}_i, \mathbf{q}_i(\mathbf{p}_i))$ . To ensure Assumption U holds with  $u_a^m = u_b^m$ , we restrict attention to  $\pi_a(u) = \pi_b(u) = \pi(u)$ . Beyond cases with  $K_a = K_b = K$ ,  $\mathbf{q}_a = \mathbf{q}_b = \mathbf{q}$  and  $\mathbf{c}_a = \mathbf{c}_b = \mathbf{c}$ , this restriction also permits some specific cases with asymmetric demand and costs (as later detailed in Section 6). Given the sales equilibrium, each firm  $i$  then chooses its price vector to maximize its profits subject to supplying its required utility draw,  $u_i$ , with  $\mathbf{p}_i^*(u_i) = \operatorname{argmax}_{\mathbf{p}_i} \pi(\mathbf{p}_i)$  subject to  $S(\mathbf{p}_i, \mathbf{q}(\mathbf{p}_i)) = u_i$ . In a fully symmetric context, this set-up reproduces versions of i) the standard clearinghouse equilibrium (e.g. Baye et al. (2004a) and Baye et al. (2006)) when  $K = 1$ , and ii) the equilibrium of Simester (1997) when marginal costs are zero,  $K \geq 1$ , and  $A \rightarrow 0$ . More substantially, for any marginal costs and any  $K$ , our framework extends these equilibria to permit positive asymmetric advertising costs, and asymmetric shares of non-shoppers. See Appendix B2 for more formal details.

**Two-part tariffs:** As consistent with the markets for energy and telecommunications, consider a market where firms employ two-part tariffs. Existing work on sales in such markets is very limited. Theoretically, we know of only Hendel et al. (2014) who show how sales with non-linear prices can emerge in a dynamic context with storable goods. In

---

<sup>14</sup>For example, among others, we can reproduce and extend i) the sales equilibrium of Narasimhan (1988) with vertically differentiated products and asymmetric shares of non-shoppers (pp.439-440), and ii) the second stage equilibrium of Gu and Wenzel (2014)'s two-stage obfuscation game and the advertising games of Ireland (1993) and Roy (2000) which all allow for asymmetric  $\theta_i$ , and iii) the second stage equilibrium of Jing (2007)'s two-stage quality-investment game which allows for asymmetric  $V_i$ ,  $c_i$  and  $\theta_i$  (when non-shoppers and shoppers share common preferences).

contrast, our framework can analyze two-part tariff sales within a simpler clearinghouse setting. As detailed in Appendix B3, any firm with a two-part tariff that wishes to provide utility,  $u'$ , will optimally use marginal cost pricing and a suitably adjusted fixed fee. Therefore, our framework predicts that equilibrium sales will involve marginal cost pricing, and firms mixing between not advertising a high fixed fee, and advertising a stochastic lower fixed fee. While there is very little empirical analysis of two-part tariff sales, these predictions seem consistent with several anecdotal examples and some wider forms of evidence.<sup>15</sup>

**Non-price sales:** Finally, suppose firms hold prices constant but engage in sales by using some non-price variable. This setting covers a broad set of marketing practices, including i) temporary extensions to package size or quantity, such as ‘X% Free’ offers and ‘bonus packs’, ii) temporary increases in product quality or content, such as the inclusion of free items or ‘premiums’, and iii) other temporary increases in product value, such as the use of consumer finance deals, prize draws, or charity donations. Despite these non-price sales becoming increasingly common due to fears that price discounts can lead to weaker brand image, there are few theoretical studies (see the discussions in Chen et al. (2012), Palazon and Delgado-Ballester (2009)). However, as illustrated in Appendix B4, our framework can easily characterize such behavior as part of a clearinghouse equilibrium where firms mix between not advertising with a minimum ‘regular’ package size/product value, and advertising a sale with an increased package size/product value.

## 4 Comparative Statics

This section provides a range of new comparative statics results. For symmetric market cases, our findings extend standard clearinghouse results to a generalized market setting. More substantially, for asymmetric market cases where the existing literature has offered limited results, our framework can offer several new findings by isolating the effects of individual firm characteristics on sales behavior and market performance.

---

<sup>15</sup>For instance, many major UK suppliers of broadband, land-line and TV services use sales with reduced monthly fees but unchanged prices for charged telephone calls. More widely, our predictions are also consistent with a finding in Giulietti et al. (2014) which suggests that firms play mixed strategies with the implied ‘final bill’ for an average consumer in the British electricity market where suppliers often employ two-part tariffs.

## 4.1 Changes in a Firm's Share of Non-Shoppers

Under symmetry, our framework produces a generalized form of the standard clearing-house result - an increase in the proportion of non-shoppers,  $\theta$ , (and associated reduction in the proportion of shoppers,  $1 - \theta$ ), leads to a lower probability of using advertised sales,  $\alpha$ , higher equilibrium profits,  $\bar{\Pi}$ , and lower expected utility offers,  $E(u)$ . More interestingly, we can analyze the effects from a change in an individual firm's share of non-shoppers,  $\theta_i$ . As these are difficult to characterize, we focus on evaluating them at the point of symmetry. To proceed, one must also stipulate whether the increase in  $\theta_i$  is associated with a reduction in shoppers,  $1 - (\theta_i + \theta_j)$ , or rival non-shoppers,  $\theta_j = \theta - \theta_i$ . We first consider the latter:

**Proposition 2.** *In an otherwise symmetric market, consider an increase in firm  $i$ 's non-shoppers  $\theta_i$  (and associated reduction in  $\theta_j$ ). Starting from  $\theta_i = \theta_j$ , this decreases  $x_i$ , *i*) increases  $\bar{\Pi}_i$ , *ii*) decreases  $\bar{\Pi}_j$ , *iii*) decreases  $\alpha_i$  and  $E(u_i)$ , and *iv*) increases  $\alpha_j$  and  $E(u_j)$ .*

Ceteris paribus, an increase in  $\theta_i$  makes firm  $i$  less willing to offer higher utilities. However, to maintain a common  $\bar{u}$  in equilibrium, this is partially offset by a reduction in firm  $i$ 's tie-break share,  $x_i$  (and an associated increase in  $x_j$ ). Hence, when combined, these effects lead firm  $i$  (firm  $j$ ) to use advertised sales with a lower (higher) probability, gain higher (lower) equilibrium profits, and set lower (higher) average utility offers. While intuitive, the last result about average utility offers differs to a finding in Arnold et al. (2011) which considers asymmetric  $\theta_i$  with unit demand and  $A > 0$  under the past literature's different tie-break rule. Instead, they suggest an increase in  $\theta_i$  leads firm  $i$  to become *more* aggressive in its advertised prices and so offer *higher* average utility offers. In contrast to our results, this finding conflicts with standard results under  $A = 0$  such as Narasimhan (1988).<sup>16</sup>

## 4.2 Changes in a Firm's Advertising Costs

As before, one can verify a generalized form of the standard clearinghouse result under symmetry - an increase in advertising costs,  $A$ , leads to a lower probability of using advertised sales,  $\alpha$ , higher equilibrium profits,  $\bar{\Pi}$ , and lower expected utility offers,  $E(u)$ . More

---

<sup>16</sup>With two exceptions, our findings remain robust in the alternative case where the increase  $\theta_i$  comes from a reduction in shoppers. First, an increase in  $\theta_i$  now raises  $\bar{\Pi}_j$  because there is no reduction in  $\theta_j$ . Second, an increase in  $\theta_i$  can provide reversed effects on  $\alpha_j$  and  $E(u_j)$  if advertising costs are relatively high. This arises because of the conflicting effects between a decrease in shoppers, and an increase in  $x_j$  which varies in  $A$ . However, it remains that firm  $i$  still offers a lower average utility than firm  $j$ . (Full details on request).

substantially, our framework can now isolate the effects from a change in an individual firm's advertising cost,  $A_i$ .

**Proposition 3.** *In an otherwise symmetric market, an increase in  $A_i$  leads to a lower  $x_i$ , i) no change in  $\bar{\Pi}_i$ , ii) an increase in  $\bar{\Pi}_j$ , and iii) a reduction in  $\alpha_k$  and  $E(u_k)$  for both firms,  $k = i, j$ .*

Ceteris paribus, an increase in  $A_i$  reduces the incentives for firm  $i$  to advertise higher utilities. However, to maintain a common  $\bar{u}$ , this is partially offset by a reduction in firm  $i$ 's tie-break share,  $x_i$ , (and increase in  $x_j$ ). Indeed, after expanding  $x_i$ , firm  $i$ 's profits can be written as  $\bar{\Pi}_i = \frac{\theta}{2}\pi^m + (\frac{A_i}{A_i})A_i$ . Therefore, an increase in  $A_i$  has no aggregate effect on  $\bar{\Pi}_i$  because the direct effect from  $A_i$  is exactly offset by the indirect effect from  $x_i$ . However, an increase in  $A_i$  raises firm  $j$ 's profits,  $\bar{\Pi}_j$ , and industry profits,  $\bar{\Pi}_i + \bar{\Pi}_j$ , because the indirect effect raises  $x_j$ . Hence, in contrast to the standard symmetric findings, our results show that it is rival rather than own advertising costs that matter in determining firm profits. Finally, the increase in  $A_i$  reduces both firms' use of sales, and prompts a subsequent reduction in their expected utility offers.

### 4.3 Changes in a Firm's Profit Function

A firm's profit function,  $\pi_i(u)$ , may vary due to many factors including costs, products, demand, or pricing technologies. Understanding the associated comparative statics is difficult at a general level, not least because a change can affect a firm's profits differently at different utility levels. Hence, to proceed, we focus on the following functional form,  $\pi_i(u) = \pi(u, e_i)$ , where  $\pi(u, e)$  is common across firms, and  $e_i > 0$  is a parameter representing firm  $i$ 's profitability. We assume that  $\pi(u, e)$  is twice continuously differentiable and increasing in  $e$  for all  $u \geq u^m$ , where  $u^m$  is defined as the maximizer of  $\pi(u, e)$ . In line with Assumption U, we require  $u^m$  to be independent of  $e$ . These assumptions are consistent with unit demand or two-part tariffs, as well as an additive and multiplicative case,  $\pi_i(u) = \pi(u) + e_i$  and  $\pi_i(u) = e_i\pi(u)$ .

Under symmetry, our framework shows that an increase in the common profitability parameter,  $e$ , raises the firm's equilibrium profits,  $\bar{\Pi}$ , and increases the probability of using advertised sales,  $\alpha$ . However, the effects on  $F(u)$  are more ambiguous, and depend upon how the increase in profitability varies over different utility levels.<sup>17</sup>

<sup>17</sup>In particular, an increase in profitability leads to an increase (decrease) in  $F(u)$  for a given  $u$  if  $\frac{\theta\pi(u)}{\theta\pi(u^m)+4A} - \frac{\pi_e(u)}{\pi_e(u^m)}$  is positive (negative). The condition is always negative for  $u$  close to  $u^m$ , but may be positive for higher  $u$ , such that higher profitability can lead to either better or worse utility offers.



We now isolate the effects from a change in an individual firm's profitability,  $e_i$ . As detailed in Section 3, the existing literature has only been able to consider a few specific cases involving changes in marginal costs or product values under unit demand and zero advertising costs. Some related technical difficulties are also still present in our framework. However, by evaluating the comparative statics at the point of symmetry, we can substantially improve on past results:

**Proposition 4.** *In an otherwise symmetric market, consider an increase in  $e_i$ . Starting from  $e_i = e_j = e$ , this leads to a higher i)  $\bar{\Pi}_i$  and ii)  $\bar{\Pi}_i + \bar{\Pi}_j$ . Further, if  $(1 - \frac{\theta}{2})\pi_e(\bar{u}, e) - \frac{\theta}{2}\pi_e(u^m, e)$  is positive (negative), then this also leads to a higher (lower)  $x_i$ , iii) a lower (higher)  $\bar{\Pi}_j$ , iv) a higher (lower)  $\alpha_i$ , and v) higher (lower)  $E(u_i)$ .*

An increase in firm  $i$ 's profitability,  $e_i$ , unambiguously increases firm  $i$ 's equilibrium profits and overall industry profits. However, the remaining effects on firm  $i$ 's use of sales and expected utility can go in either direction and depend on whether the increase in  $e_i$  increases firm  $i$ 's total profits at the upper utility bound,  $(1 - \frac{\theta}{2})\pi_e(\bar{u}, e)$ , by more than it increases firm  $i$ 's total profits at the lower utility bound,  $\frac{\theta}{2}\pi_e(u^m, e)$ . Suppose the condition holds. Then, ceteris paribus, firm  $i$  has an increased incentive to advertise higher utilities and  $x_i$  must increase (and  $x_j$  decrease) to maintain a common  $\bar{u}$ . On balance, the increase in  $e_i$  increases firm  $i$ 's use of advertised sales,  $\alpha_i$ , raises firm  $i$ 's average utility offer,  $E(u_i)$ , and lowers firm  $j$ 's equilibrium profits,  $\bar{\Pi}_j$ , by reducing  $x_j$ .<sup>18</sup> These results are reversed in the alternative case where the condition does not hold, such that an increase in firm  $i$ 's profitability can actually reduce its use of sales.

## 5 More Than Two Firms

In this section, we now illustrate the framework's ability to offer further new results by analyzing the number and type of firms that use sales in markets with  $n > 2$  firms. Here, the sales literature with heterogeneous firms is particularly scant because existing models quickly become intractable. Most notably, as part of their broader analysis, Baye et al. (1992, Lemmas 7'-14') establish that only two firms can ever engage in sales behavior in a unit-demand clearinghouse model with zero advertising costs when firms strictly differ in their shares of non-shoppers. Intuitively, the remaining firms with relatively larger shares of non-shoppers are less willing to lower their price and prefer to price highly to their

---

<sup>18</sup>The effects on  $\alpha_j$  and  $E(u_j)$  are more nuanced. Nevertheless, for  $\bar{u}$  sufficiently close to  $u^m$ , an increase in  $e_i$  leads both firms to increase their use of advertised sales and average utility offers.

non-shoppers. This finding has been extended to allow firms to vary in their product values (Kocas and Kiyak (2006)) or costs (Shelegia (2012)). It has also been used as the foundation for a number of studies, such as those aiming to endogenize consumer loyalty (e.g. Chioveanu (2008)). However, this ‘two-firm’ prediction contrasts to common empirical findings where multiple heterogeneous sellers exhibit sales behavior (e.g. Lach (2002), Lewis (2008), Chandra and Tappata (2011)). Instead, within our more general framework, we now show that any number of heterogeneous firms  $k^* \in [2, n]$  can engage in equilibrium sales once advertising costs are allowed to be positive. We also provide a broad characterization of the types of firms that are likely to use sales in equilibrium, depending upon firms’ advertising costs, non-shopper shares, and profit functions.

Unlike our duopoly case, there are two potential sources of equilibrium multiplicity when  $n > 2$ . First, similar to an insight by Baye et al. (1992) for zero advertising costs, the equilibrium distributions and supports may no longer be unique. Provided at least two firms mix in any given interval within  $(u^m, \bar{u}]$ , there may now be equilibria where other firms do not mix within that interval. Second, each firm’s tie-break share,  $x_i$ , may not be uniquely defined if one or more firms never advertise,  $\alpha_i = 0$ . Therefore, in addition to our previous assumptions, we focus on sales equilibria under the following two restrictions. First, similar to Chioveanu (2008)), we consider equilibria where all advertising firms advertise over the same convex support  $(u^m, \bar{u}]$ , such that  $F'_i(u) = f_i(u) > 0$  for all  $u \in (u^m, \bar{u}]$  if  $\alpha_i > 0$ . Second, we focus on equilibria where within any tie, shoppers disregard firms that *never* advertise, such that  $x_i = 0$  if  $\alpha_i = 0$ .<sup>19</sup>

After denoting  $\theta_{-i} = \theta - \theta_i$  as the total share of non-shoppers that are not associated with firm  $i$ , we now define  $\tilde{u}_i \in (u^m, \hat{u}_i)$  as the highest utility that firm  $i$  could possibly be willing to advertise: the level of utility at which firm  $i$ ’s highest possible profits from advertising,  $\pi_i(u_i)(1 - \theta_{-i}) - A_i$ , are equal to its lowest possible profits from not advertising (with  $x_i = 0$ ),  $\theta_i \pi_i^m$ .<sup>20</sup>

$$\tilde{u}_i \equiv \pi_i^{-1} \left( \frac{\theta_i \pi_i^m + A_i}{1 - \theta_{-i}} \right) \quad (8)$$

Without loss, we then index the firms in (weakly) decreasing order of  $\tilde{u}_i$  from 1 to  $n$  and focus on two settings: i) a quasi-symmetric setting where  $\tilde{u}_i = \tilde{u} > u^m$  for all  $i$ , and

<sup>19</sup>These restrictions may be less necessary within a symmetric setting. Indeed, within a symmetric  $n$ -firm clearinghouse model with unit demand and positive advertising costs, Arnold and Zhang (2014) show that the symmetric equilibrium is unique and that asymmetric equilibria do not exist.

<sup>20</sup>The value of  $\tilde{u}_i \in (u^m, \hat{u}_i)$  is unique as i)  $\pi_i(u_i)(1 - \theta_{-i}) - A_i$  is decreasing in  $u_i$ , ii)  $\pi_i^m(1 - \theta_{-i}) - A_i > \theta_i \pi_i^m$  at  $u_i = u^m$  by Assumption A, and iii)  $\pi_i(\hat{u}_i)(1 - \theta_{-i}) - A_i < \theta_i \pi_i^m$ .

ii) a strict asymmetric setting where  $\tilde{u}_1 > \tilde{u}_2 > \dots > \tilde{u}_n > u^m$  such that firm  $n$  is the least willing to advertise high utility levels.<sup>21</sup> After assuming that advertising costs are not prohibitively high,  $(n-1) > \sum_{i=1}^n \frac{A_i}{(1-\theta)\pi_i^m}$ , and denoting  $k^*$  as the number of firms that use advertised sales in equilibrium (with  $\alpha_i > 0$ ), we state the following preliminary result:

**Lemma 2.** *In any equilibrium that satisfies our restrictions, with upper utility bound,  $\bar{u}$ , firm  $i$  uses advertised sales with  $\alpha_i > 0$  if and only if  $\tilde{u}_i \geq \bar{u}$ . Hence, i) if  $k^* = n$  then  $\tilde{u}_n \geq \bar{u}$ , and ii) if  $k^* \in [2, n)$  then  $\tilde{u}_{k^*} \geq \bar{u} > \tilde{u}_{k^*+1}$ .*

This follows by contradiction. Under our restrictions, firms that never advertise receive zero shoppers in a tie,  $x_i = 0$  if  $\alpha_i = 0$ . Therefore, by definition of  $\tilde{u}_i$ , any firm  $i$  with  $x_i = 0$  makes the same profit from not advertising as it would if it advertised  $\tilde{u}_i$  and won all the shoppers. Hence, if any other firm  $j$  advertises on the interval  $(u^m, \bar{u}]$ , then any non-advertising firm  $i$  with  $\alpha_i = 0$  and  $\tilde{u}_i > \bar{u}$  would always strictly prefer to set  $\alpha > 0$  and advertise  $\bar{u}$  instead. Similarly, any advertising firm with  $\alpha_i > 0$  and  $\tilde{u}_i < \bar{u}$  would always strictly prefer to set  $\alpha_i = 0$  and refrain from advertising instead due to our restriction that advertising firms must advertise over the entire support  $u \in (u^m, \bar{u}]$ .

Using Lemma 2, we now derive the game equilibria under our restrictions. While we show that the equilibrium will always be unique, it is hard to demonstrate existence for the general case without further specifying the exact form of profit functions (e.g. when the firms are sufficiently symmetric or when the firms differ only in their advertising costs, see proof for details).

**Proposition 5.** *When an equilibrium exists under our restrictions, it is unique. In such an equilibrium, firms  $i = \{1, \dots, k^*\}$  advertise with interior probabilities over  $(u^m, \bar{u}]$ , while any remaining firms,  $i = \{k^* + 1, \dots, n\}$  set  $u_i = u^m$  and never advertise, where*

$$k^* = \begin{cases} n & \text{if } \sum_{i=1}^n \frac{A_i}{h_i(\tilde{u}_n)} > (n-1) \geq \sum_{i=1}^n \frac{A_i}{(1-\theta)\pi_i^m} \\ k \in [2, n) & \text{if } \sum_{i=1}^k \frac{A_i}{h_i(\tilde{u}_k)} > (k-1) \geq \sum_{i=1}^k \frac{A_i}{h_i(\tilde{u}_{k+1})} \end{cases} \quad (9)$$

and where  $h_i(u) \equiv \pi_i(u)(1 - \theta_{-i}) - \theta_i \pi_i^m$ .

First consider the simpler quasi-symmetric setting with  $\tilde{u}_i = \tilde{u} > u^m$  for all  $i$ . Using (8), note that  $\frac{A_i}{h_i(\tilde{u}_i)} = 1$  for any  $i$  such that  $\sum_{i=1}^k \frac{A_i}{h_i(\tilde{u})} = k$ . Therefore, from (9), the only

<sup>21</sup>A third setting where a subset of firms have the same  $\tilde{u}$  but where some remaining firms differ in  $\tilde{u}$  can also be analyzed but is omitted for brevity due to its unnecessary complications.

possible equilibrium under our restrictions must involve all firms using advertised sales, with  $k^* = n$ , provided advertising costs are sufficiently small,  $(n - 1) \geq \sum_{i=1}^n \frac{A_i}{(1-\theta)\pi_i^m}$ . As detailed in the proof, this equilibrium then strongly resembles that under duopoly; all firms engage in advertised sales on  $(u^m, \bar{u}]$ , where the utility distributions,  $F_i(u)$ , and advertising probabilities,  $\alpha_i \in (0, 1)$ , are only symmetric if the firms also have identical advertising costs, non-shopper shares, and profit functions. Hence, for the rest of the section, we focus on the strict asymmetric setting,  $\tilde{u}_1 > \tilde{u}_2 > \dots > \tilde{u}_n > u^m$ , by considering the number and type of firms that use equilibrium sales, in turn.

## 5.1 The Number of Firms that Use Advertised Sales

The conditions in (9) determine the unique number of firms that use advertised sales,  $k^*$ , by stipulating a set of upper and lower bounds on advertising costs. The upper bound ensures that  $\bar{u}$  is sufficiently large, with  $\bar{u} > \tilde{u}_{k^*+1}$ , such that all non-advertising firms have no incentive to advertise, while the lower bound ensures that  $\bar{u}$  is sufficiently small, with  $\bar{u} \leq \tilde{u}_{k^*}$ , such that all advertising firms are willing to advertise  $u_i = \bar{u}$  without requiring  $x_i < 0$ .

In general, the relationship between  $k^*$  and advertising costs is complex and potentially non-monotonic. However, some insights can be gained under a common advertising cost,  $A_i = A$  for all  $i$ , where any changes in  $A$  do not affect the ranking of firms,  $\tilde{u}_1 > \tilde{u}_2 > \dots > \tilde{u}_n$ . Using Proposition 5, we can then state:

**Corollary 1.** *Suppose  $\tilde{u}_1 > \tilde{u}_2 > \dots > \tilde{u}_n > u^m$  and  $A_i = A$  for all  $i$ . In the limit of our equilibrium, i)  $k^* = 2$  when  $A \rightarrow 0$ , and ii)  $k^* = n$  when  $A \rightarrow \frac{(n-1)(1-\theta)}{\sum_i 1/\pi_i^m}$  if the firms' profit functions are sufficiently symmetric,  $\sum_{i=1}^n \frac{1}{\pi_i^m} > \frac{n-1}{\pi_n^m}$ .*

When  $A \rightarrow 0$ , our findings generalize the existing literature's two-firm result (Baye et al. (1992), Kocas and Kiyak (2006), Shelegia (2012)) to a broad range of market settings. Intuitively, when  $A \rightarrow 0$ , competition for the shoppers is fierce. Hence, the only way for any firms to advertise up to a common utility upper bound is to give the firm with the highest incentive to advertise, firm 1, all the shoppers in a tie. In equilibrium, firms 1 and 2 then use advertised sales with  $\bar{u} = \tilde{u}_2$ , but the remaining firms refrain from sales and set  $u_i = u^m$ .

However, once we allow for higher advertising costs, the two-firm result becomes a special case of a new and more general relationship. Indeed, from (9), any number of heterogeneous firms  $k^* \in [2, n]$  may now use advertised sales in equilibrium.

At the extreme, Corollary 1 states that *all* firms can use advertised sales provided two conditions are met. The first condition requires advertising costs to be sufficiently moderate,  $A \rightarrow \frac{(n-1)(1-\theta)}{\sum_i 1/\pi_i^m}$ . This appears paradoxical at first sight but can be explained as follows. On the one hand, a movement from lower to moderate levels of  $A$  reduces the direct incentives for each firm to use advertised sales as evidenced by the associated reduction in  $\tilde{u}_i$ . However, on the other hand, the increase in  $A$  softens the competition for the shoppers and reduces  $\bar{u}$  in a way that prompts firms with lower  $\tilde{u}_i$  to start using advertised sales. Indeed, for sufficiently moderate  $A$ ,  $\bar{u}$  can fall below  $\tilde{u}_n$  such that *all* firms can use advertised sales.

The second condition requires the firms' profit functions to be sufficiently symmetric,  $\sum_{i=1}^n \frac{1}{\pi_i^m} > \frac{n-1}{\pi_n^m}$ , to ensure that advertising is individually rational for all firms (as consistent with Assumption A). This condition places no restrictions on firms' shares of non-shoppers,  $\theta_i$ , and is trivially satisfied when  $n = 2$ . However, it becomes increasingly stringent as  $n$  grows, requiring arbitrarily symmetric profit functions when  $n \rightarrow \infty$ . More generally, holding  $\pi_n^m$  fixed, the condition is most binding for a given  $n$  when  $\pi_i^m \approx \pi_1^m$  for all  $i \leq n-1$ . Thus, a sufficient condition is  $\pi_n^m > \frac{n-2}{n-1}\pi_1^m$ .

## 5.2 The Types of Firms that Use Advertised Sales

Having established  $k^*$ , we now examine which types of firms are likely to use advertised sales. The existing literature only considers some specific dimensions under unit demand and zero advertising costs (e.g. Baye et al. (1992), Kocas and Kiyak (2006), Shelegia (2012)). However, in our general setting, we can offer a broader characterization. In particular, when  $k^* < n$ , Proposition 5 implies that the firms using advertised sales will be the firms with the highest values of  $\tilde{u}_i$ , firms  $i = \{1, \dots, k^*\}$ . By focusing on the form of profit function introduced in Section 4.3,  $\pi(u, e_i)$ , we can then use (8) to state:

**Corollary 2.** *Suppose  $k^* < n$ . Ceteris paribus, the firms that use advertised sales in equilibrium will be those with the lowest  $A_i$  and  $\theta_i$ , and the highest (lowest) profitability  $e_i$  if  $(1 - \theta_{-i})\pi_e(\tilde{u}_i, e_i) - \theta_i\pi_e(u^m, e_i)$  is positive (negative).*

Intuitively, the firms with the lowest advertising costs and smallest shares of non-shoppers have the highest incentives to advertise high utilities, and will therefore be most likely to engage in advertised sales. However, the effects of firm profitability,  $e_i$ , on sales usage can go in either direction. More profitable firms are more likely to use sales if profitability has a greater impact at higher, rather than lower, utility levels such that an

increase in  $e_i$  raises firm  $i$ 's highest possible profits at  $\tilde{u}_i$ ,  $(1 - \theta_{-i})\pi_e(\tilde{u}_i, e_i)$ , by more than it increases firm  $i$ 's total profits at the lower utility bound,  $\theta_i\pi_e(u^m, e_i)$ . Alternatively, less profitable firms can be more likely to use sales if profitability has a greater impact at lower utility levels. Under the case of symmetric non-shopper shares, the proof shows that this condition hinges on the sign of  $\pi_{eu}$ . For instance, under unit demand, it follows that  $\pi_{eu} = 0$  such that more profitable firms are more likely to use advertised sales.

## 6 Implications for Empirical Procedures

Without the foundation of an adequate theoretical model, current empirical papers on sales behavior and price dispersion are commonly forced to resort to a restrictive ‘cleaning’ procedure. This section now uses our framework to better understand when such an approach is valid, and to provide the basis for modified methodologies that can be applied more widely.

As listed in the introduction, empirical studies often find that firms employ sales in ways that are consistent with mixed strategies, but observe that firms frequently differ in their pricing behaviors. This pattern is driven by two forms of price dispersion. The first ‘temporal’ form involves price differences that vary over time, such as those generated by sales behavior. The second ‘spatial’ form arises from inter-firm heterogeneities that remain over time, such as those arising from firms’ characteristics, products, or costs. To focus only on the temporal form, empirical papers typically ‘clean’ their raw price data by retrieving a set of price residuals from a price regression involving observable firm characteristics or firm-level fixed effects. The price residuals are then interpreted as resulting from a homogeneous symmetric market and used to either i) perform descriptive/reduced-form analysis of the features of temporal price dispersion, or ii) conduct structural estimations of market parameters.

Wildenbeest (2011) provides the only formal justification for this procedure within the literature under a specific set of market conditions. A version of Wildenbeest’s arguments can be derived within our  $n$ -firm clearinghouse framework, where in contrast, we generalize to positive advertising costs. Suppose firms sell a single product and that consumers have unit demand. Firms are allowed to vary in quality and costs subject to a common value-cost markup condition,  $V_i - c_i = \Psi$  for all  $i$ . In our model, this implies that the firms’ profit functions are symmetric with  $\pi_i(u) = V_i - c_i - u = \Psi - u$ . Under the additional assumption of symmetric non-shopper shares and advertising costs, all  $n$  firms then engage in a symmetric sales equilibrium where  $u^m = 0$ ,  $\bar{u} = \frac{n\Psi(1-\theta)-2nA}{n-(n-1)\theta}$ ,

$F(u) = \frac{\theta u + 2nA}{n(1-\theta)(\Psi - u)}$ , and  $\alpha = 1 - \left(\frac{nA}{(n-1)\Psi(1-\theta)}\right)^{\frac{1}{n-1}}$ . Moreover, if the firms play a stationary repeated game of finite horizon, each firm then chooses its utility level for each period as a draw from this utility distribution. As the utility distributions are symmetric and  $p_i(u_i) = V_i - u_i$ , firms' subsequent *price* distributions are simple iid translations of each other. Now suppose that the econometrician observes a panel of price observations for each firm. The econometrician can obtain measures of the firms' utility offers that are entirely cleaned of the effects of firm-heterogeneity by using one of two possible methods. First, one can use the observed maximum price for each firm to infer  $V_i$  directly, and then simply use the observed prices to recover  $u_i = V_i - p_i$ . However, as this method may be subject to data outliers, the literature typically favors the second method: given  $p_i = V_i - u_i$ , one can regress the raw price data on a set of firm-level fixed effects,  $p_{it} = \alpha + \delta_i + \varepsilon_{it}$ , to return a set of 'cleaned' price residuals that correctly proxy the utility draws up to a positive constant.<sup>22</sup>

## 6.1 Downward-Sloping Demand

Using our generalized framework, we now consider the procedure's validity under downward-sloping demand. Following the logic above, to ensure that the firms still employ a symmetric utility distribution in a single product market with symmetric non-shopper shares and advertising costs, the firms' profit functions also need to be symmetric,  $\pi_i(u) = \pi(u)$ . Under unit demand, this was guaranteed by the constant value-cost markup assumption. However, a different condition is now required if demand is downward-sloping.

While the following results can be extended to more general forms of downward-sloping demand, it is sufficient to present the case of linear demand. In particular, suppose each firm  $i$  has a linear per-consumer demand function that varies only in its intercept across firms,  $q_i(p) = a_i - bp$  where  $a_i \geq 0$  and  $b > 0$ . Further suppose that firm  $i$  has marginal cost  $c_i \geq 0$ . One can then use our past results from Section 3 and Appendix B2 to show that  $u_i = \frac{(a_i - bp_i)^2}{2b}$  and  $\pi_i(u) = \frac{1}{b}[a_i - bc_i - \sqrt{2bu}][\sqrt{2bu}]$  such that the firms have a symmetric profit function if and only if  $a_i - bc_i = \Psi$  for all  $i$ . Intuitively, this condition captures some sense of Wildenbeest's constant value-cost assumption.

Given the new condition, one would then aim to recover the firms' utility draws from the raw price data. However, unlike unit demand, the relationship between prices and

<sup>22</sup>In more detail, the estimated residuals can be expressed as  $\hat{\varepsilon}_{it} \equiv p_{it} - p_i^{ave}$  where  $p_i^{ave}$  is the average price chosen by firm  $i$ . Given unit demand and a symmetric equilibrium utility distribution with average utility offer,  $u^{ave}$ , it follows that  $p_{it} = V_i - u_{it}$  and  $p_i^{ave} = V_i - u^{ave}$ , such that the estimated residuals provide a negative measure of the utility draws,  $\hat{\varepsilon}_{it} \equiv -(u_{it} - u^{ave})$ .

utilities is non-linear,  $u = \frac{(a_i - bp)^2}{2b}$ . Therefore, despite the possibility of a symmetric utility distribution, the literature’s fixed effects cleaning procedure cannot be applied under downward-sloping demand. Instead, one would have to implement a more complex and data-intensive procedure to recover the utility draws. One way to do this would involve additional quantity data to estimate each firm’s demand function.

## 6.2 Asymmetric Utility Distributions

To further explore the validity of the cleaning procedure, we now return to the case of unit demand but depart from Wildenbeest’s constant value-cost condition. Firm heterogeneity is now substantial enough for the firms to use asymmetric utility distributions,  $F_i(u) \neq F(u)$ . As such, the fixed-effects procedure is clearly invalid because the firms’ price distributions are no longer simple iid translations of each other. However, one could draw on our framework to consider the following modified procedure.

Given  $F_i(u) \neq F(u)$ , one must now consider each firm separately to recover the firms’ utility draws. Instead of using the fixed-effects regression, one could think about estimating a set of firm-specific price regressions. However, this method is also invalid because the interpreted residuals from each regression,  $\hat{\varepsilon}_{it} \equiv p_{it} - p_i^{ave} \equiv u_{it} - u_i^{ave}$ , are no longer comparable across firms due to the differences in average utility levels,  $u_i^{ave}$ . Instead, one should employ the more direct method by using the observed maximum price of each firm to infer  $V_i$  and then calculate each firm’s utility offer with  $u_i = V_i - p_i$ . While this method offers super-consistent estimates of  $V_i$ , it is sensitive to possible data outliers. To reduce this sensitivity, one can i) assume  $V_i$  is measured with error and formally estimate it, or ii) establish each firm’s ‘regular’ price with a statistical procedure such as those proposed by Hosken and Reiffen (2004). Having recovered the utility draws, one can then use our theoretical insights to analyze the observed price dispersion or estimate a structural model while explicitly allowing for substantial firm heterogeneity. For instance, by using our theoretical predictions for the equilibrium utility distributions and advertising probabilities, one could use data on prices and advertising frequency to estimate each firm’s share of non-shoppers,  $\theta_i$ , and advertising cost,  $A_i$ .

## 7 Conclusions

Due to the apparent technical complexities, existing clearinghouse sales models are unable to fully consider the effects of firm heterogeneity. This restricts theoretical understand-



ing, empirical analysis, and policy guidance with regards to sales and price dispersion, and other wider topics in related literatures. The current paper has tried to fill this gap by providing a substantially generalized, fully asymmetric clearinghouse sales framework. The framework can i) neatly reproduce and extend many equilibria from the existing literature, ii) offer a range of new results on how firm heterogeneity affects market outcomes, iii) provide original insights into the number and type of firms that use advertised sales, and iv) offer a basis to assess and extend current empirical procedures. Moreover, by opening up the analysis of sales with firm heterogeneity, we hope that our framework can provide a convenient workhorse model for future research.

## Appendix A - Main Proofs

**Proof of Lemma 1.** The proof proceeds in a series of steps. First, by using the arguments in the text, all advertised utilities must be strictly above  $u^m$ . Second, we establish that there can be no point masses in advertised utilities. Assume the opposite such that firm  $i$  advertises some  $u > u^m$  with probability  $\beta_i > 0$ . If so, there cannot exist  $\varepsilon \in (0, u - u^m)$  such that firm  $j$  does not advertise in  $(u - \varepsilon, u)$  with  $F_j(u - \varepsilon) = F_j(u^-)$  because firm  $i$ 's total expected profits at  $u$ ,  $\Pi_i(u)$ , would then be strictly less than  $\Pi_i(u - \varepsilon)$ : a contradiction.<sup>23</sup> Therefore, given the mass point  $\beta_i > 0$ , it must be that  $F_j(u - \varepsilon) < F_j(u^-)$ . However, in that case,  $\Pi_j(u^-) < \Pi_j(u^+)$ , such that it would wish to move probability mass from just below  $u$  to just above  $u$ : another contradiction. We thus establish that there are no point masses in equilibrium advertised utilities.

Third, we prove that there are no gaps in firms' advertised utility supports. Assume the opposite so that there exists  $u_1$  and  $u_2$  with  $u^m < u_1 < u_2$  where some firm  $i$  has  $0 < F_i(u_1) = F_i(u_2) < 1$ ,  $F_i(u) < F_i(u_1)$  for all  $u < u_1$ , and  $F_i(u) > F_i(u_2)$  for all  $u > u_2$ . Then, it must be that  $F_j(u_2) < 1$ , otherwise firm  $i$  must be advertising some  $u > u_2$  such that  $\Pi_i(u) < \Pi_i(u_2)$ : a contradiction. Further, we must also have  $F_j(u_1) = F_j(u_2)$ , otherwise at some  $u \in (u_1, u_2)$  for firm  $j$ ,  $\Pi_j(u) < \Pi_j(u_1)$ : a contradiction. Hence, it must be that neither firm advertises in  $(u_1, u_2)$  and both firms advertise above  $u_2$ . But, in that case,  $\Pi_i(u_2^+) < \Pi_i(u_1)$ : another contradiction.

Fourth, firms must share a common advertised utility support in equilibrium. Assume the opposite such that  $i$  advertises with positive probability on all  $u$  in some interval

---

<sup>23</sup> $F_j(u^-)$  is defined as  $\lim_{w \rightarrow u^-} F_j(w)$ . Similarly,  $F_j(u^+) = \lim_{w \rightarrow u^+} F_j(w)$ . Analogous definitions apply for other functions.

$(u_1, u_2)$ , but firm  $j$  does not, with  $F_j(u_1) = F_j(u_2)$ . Then, as in the previous step,  $\Pi_i(u_2^-) < \Pi_i(u_1^+)$ : a contradiction.

Fifth, given the firms must randomize continuously on some common interval  $(\underline{u}, \bar{u})$  without mass points, it remains to show that  $\underline{u} = u^m$ . Assume the opposite such that  $\underline{u} > u^m$ . Then by using the arguments in the text, we know that  $\Pi_i(\underline{u}^+) < \Pi_i(u^{m+})$ : a contradiction with  $i$ 's profit maximization.  $\square$

**Proof of Proposition 1.** Part 1. First, we verify the equilibrium is well-behaved. When  $A_a$  and  $A_b$  satisfy the condition, we know from the text that there exists a unique  $\bar{u} \in (u^m, \hat{u})$  and a unique pair of interior tie-break probabilities,  $x_i = 1 - x_j$ . Each  $F_i(u)$  is also properly defined because (i)  $\frac{\partial F_i(u)}{\partial u} = -\frac{(A_j + \bar{\pi}_j)\pi_j'(u)}{(1-\theta)\pi_j(u)^2} > 0$  for all  $u^m < u \leq \bar{u}$ , (ii)  $F_i(\bar{u}) = 1$ , (iii)  $F_i(u^m) = \alpha_i$ , and (iv)  $F_i(u^m - \varepsilon) = 0$  for any  $\varepsilon > 0$ . As  $x_i \in (0, 1)$  and  $A_i > 0$ , equilibrium profits,  $\bar{\Pi}_i$ , are strictly positive and equal for all utility offers and advertising strategies in equilibrium. When not advertising, setting  $u \neq u^m$  can never be profitable because demand remains the same, but per consumer profit is maximized at  $u^m$ . Advertising  $u$  outside  $(u^m, \bar{u}]$  gives strictly lower profits than advertising inside the interval: for  $u < u^m$  no shoppers are attracted, and per consumer profits are lower than at  $u^m$ ; for  $u > \bar{u}$ , all shoppers are attracted with probability one, just as with  $\bar{u}$ , but per consumer profits are lower.

We now demonstrate that this equilibrium is unique given our tie-breaking rule. There are three other possibilities a) both firms do not advertise, b) only one firm advertises, or c) both firms advertise but not in the way described.

Consider the first possibility and suppose there is some allocation of non-shoppers to firm  $i$ , say  $z_i$ . Regardless of this allocation, both firms will set  $u = u^m$ . For this to be an equilibrium, we require that no firm can profitably deviate to advertising a utility slightly above  $u^m$  to capture all the non-shoppers. This deviation would give firm  $i$  a net benefit of  $\pi_i^m[\theta_i + (1 - \theta)] - A_i - \pi_i^m[\theta_i + x_i(1 - \theta)]$ . Therefore, the equilibrium would require  $(1 - \theta)(1 - x_i)\pi_i^m \leq A_i$  for  $i = a, b$ . After rewriting and summing up over  $i$ , this requires  $1 \geq 2 - \sum_i \frac{A_i}{(1-\theta)\pi_i^m} \Rightarrow \sum_i \frac{A_i}{\pi_i^m} \geq (1 - \theta)$ , which contradicts (7).

Now consider the second possibility where firm  $i$  advertises and firm  $j$  does not. Regardless of firm  $i$ , firm  $j$ 's optimal non-advertising strategy is to set  $u = u^m$ . Given this, firm  $i$  should advertise utility slightly above  $u^m$  and capture all the shoppers. In response, firm  $j$  will only be discouraged from also advertising a slightly higher utility if  $(1 - \theta)\pi_i^m < A_j$ , which contradicts Assumption A.

Finally, consider the third possibility where both firms advertise, but not according to our equilibrium. If so, following Lemma 1, they have to advertise with mixed strategies in a common utility support without point masses. As shown above, the lower bound of this support has to be  $u^m$ . However, as demonstrated in the text, the unique equilibrium on such a support is our original equilibrium - the only way such an equilibrium can differ from ours is if  $\alpha_a = \alpha_b = 0$ , which we have already shown not to be possible. This completes the proof of Part 1.

Part 2 can be proven as follows. If neither firm advertises and shoppers are allocated according to  $x_a$  in the interval  $x_a \in \left[1 - \frac{A_a}{\pi_a^m(1-\theta)}, \frac{A_b}{\pi_b^m(1-\theta)}\right]$ , no firm would want to deviate and advertise, because even advertising slightly above  $u^m$  would not be profitable. We now need to show that no other equilibrium exists. There are two possibilities: only one firm advertises, or both firms advertise. The former cannot exist due to Assumption A (see proof of Part 1 above). The latter is also not possible because if both firms advertise, they have to do so in the fashion described in Part 1, and such equilibrium cannot be constructed because the advertising costs are too large to satisfy (7).  $\square$

**Proof of Proposition 2.** Let  $\pi_i(u) = \pi(u)$ ,  $A_i = A$  and  $\theta_j = \theta - \theta_i$ . From (6),  $\frac{\partial \bar{u}}{\partial \theta_i} = 0$  after we impose symmetry ex post with  $\theta_i = \theta_j$ . By using this with the derivative of (5), we gain  $\frac{\partial x_i}{\partial \theta_i} = -\frac{A[\pi^m - \pi(\bar{u})]}{[\pi(\bar{u})(1 - (\theta/2)) - (\theta/2)\pi^m]^2} < 0$ . These two results can then be used to help find the relevant derivatives. For i) and ii), using (3) gives  $\frac{\partial \bar{\Pi}_i}{\partial \theta_i} = \pi(\bar{u}) > 0$  and  $\frac{\partial \bar{\Pi}_j}{\partial \theta_i} = -\pi(\bar{u}) < 0$ . For iv) and v), using (2),  $\frac{\partial \alpha_i}{\partial \theta_i} = -\frac{[\pi^m - \pi(\bar{u})]}{(1-\theta)\pi^m} < 0$ , and  $\frac{\partial \alpha_j}{\partial \theta_i} = \frac{\pi^m - \pi(\bar{u})}{(1-\theta)\pi^m} > 0$ . Further, from (4),  $\frac{\partial F_i}{\partial \theta_i} = \frac{\pi(u) - \pi(\bar{u})}{(1-\theta)\pi(u)} > 0$  and  $\frac{\partial F_j}{\partial \theta_i} = -\frac{\pi(u) - \pi(\bar{u})}{(1-\theta)\pi(u)} < 0$  for all relevant  $u$ , such that expected utility at firm  $i$  (firm  $j$ ) decreases (increases).  $\square$

**Proof of Proposition 3.** Given  $\pi_i(u) = \pi(u)$  and  $\theta_i = \theta/2$ , first note from (6) and (5) that  $A_i + A_j = \pi(\bar{u})(1 - \frac{\theta}{2}) - \frac{\theta}{2}\pi^m = \frac{A_j}{x_i}$ , such that  $x_i = \frac{A_j}{A_i + A_j}$ . Clearly,  $x_i$  is decreasing in  $A_i$ . For i)-ii), substitute  $x_i$  into (3) to give  $\bar{\Pi}_i = \frac{\theta}{2}\pi^m + A_j$ . To derive iii), it is sufficient to substitute  $x_i$  into (2) to give  $\alpha_i = 1 - \frac{A_i + A_j}{(1-\theta)\pi^m}$ , and into (4) to obtain  $F_i(u) = \frac{[\pi^m - \pi(u)] + [A_i + A_j]}{(1-\theta)\pi(u)}$ . The results then follow because an increase in  $A_i$  produces a decrease in  $\alpha_i$  and  $\alpha_j$ , and an increase in  $F_i(u)$  and  $F_j(u)$  for all relevant  $u$ .  $\square$

**Proof of Proposition 4.** Given  $A_i = A$  and  $\theta_i = \theta/2$ , first note from (6) that  $\frac{\partial \bar{u}}{\partial e_i} |_{e_i=e_j=e} = \frac{(2-\theta)\pi_e(\bar{u},e) - \theta\pi_e(u^m,e)}{-2(2-\theta)\pi_u(\bar{u},e)}$ . As the denominator is positive, this has the same sign as the numerator: it is positive (negative) whenever  $(1 - (\theta/2))\pi_e(\bar{u}, e) > (<)(\theta/2)\pi_e(u^m, e)$ . Then, using (5) and the above,  $\frac{\partial x_i}{\partial e_i} = \frac{A[(2-\theta)\pi_e(\bar{u},e) - \theta\pi_e(u^m,e)]}{[(2-\theta)\pi(\bar{u},e) - \theta\pi(u^m,e)]^2}$ , which has the same sign as  $\frac{\partial \bar{u}}{\partial e_i} |_{e_i=e_j=e}$ .

Now, to prove i), note  $\bar{\Pi}_i = (1 - \frac{\theta}{2})\pi(\bar{u}, e_i) - A$ . At the point of symmetry, it then follows that  $\frac{\partial \bar{\Pi}_i}{\partial e_i} = (1 - \frac{\theta}{2}) \left( \pi_e(\bar{u}, e) + \frac{\partial \bar{u}}{\partial e_i} \pi_u(\bar{u}, e) \right)$  which equals  $\frac{1}{4}((2 - \theta)\pi_e(\bar{u}, e) + \theta\pi_e(u^m, e)) > 0$ . Similarly, note  $\bar{\Pi}_j = (1 - \frac{\theta}{2})\pi(\bar{u}, e_j) - A$ . iii) then follows as  $\frac{\partial \bar{\Pi}_j}{\partial e_i} = (1 - \frac{\theta}{2}) \frac{\partial \bar{u}}{\partial e_i} \pi_u(\bar{u}, e)$  which has the opposite sign of  $\frac{\partial \bar{u}}{\partial e_i} |_{e_i=e_j=e}$ . However, combining the two derivatives gives ii) as  $\frac{\partial(\bar{\Pi}_i + \bar{\Pi}_j)}{\partial e_i} = (1 - \frac{\theta}{2})\theta\pi_e(\bar{u}, e) > 0$ . Using (2), we can then prove iv) as  $\frac{\partial \alpha_i}{\partial e_i}$  has the same sign as  $\frac{\partial \bar{u}}{\partial e_i} |_{e_i=e_j=e}$ . Finally, to prove v), one can use the first part of (4) and previous results to show that  $\frac{\partial F_i}{\partial e_i}$  has the same sign as  $\frac{\partial \bar{\Pi}_j}{\partial e_i}$  and so the opposite sign to  $\frac{\partial \bar{u}}{\partial e_i} |_{e_i=e_j=e}$  for all relevant  $u$ .  $\square$

**Proof of Proposition 5.** The proof follows using some similar steps to the duopoly case. First, any advertising firms advertise in the same convex interval  $(u^m, \bar{u}]$  by assumption. There can be no point masses at any  $u'$  within this interval because this would create a profitable deviation for at least one other firm  $j$ ; either i) firm  $j$  also has a point mass at  $u'$  in which case it could increase profits by advertising a slightly higher utility, or ii) firm  $j$  does not, in which case it would be better off by moving probability from just below  $u'$  to just above, contrary to our assumption.

Now, using the results of Lemma 2, we can define the set of advertising firms as  $K^* = \{1, \dots, k^*\}$ . We require each firm  $i \in K^*$  to be indifferent between not advertising and advertising a utility slightly higher than  $u^m$  such that  $\pi_i^m[\theta_i + (1 - \theta)x_i \Pi_{j \neq i}(1 - \alpha_j)] = \pi_i^m[\theta_i + (1 - \theta)\Pi_{j \neq i}(1 - \alpha_j)] - A_i$ . This implies that the probability that all firms  $j \neq i$  do not advertise equals:

$$\Pi_{j \neq i}(1 - \alpha_j) = \frac{A_i}{(1 - x_i)(1 - \theta)\pi_i^m}. \quad (10)$$

After plugging this back into the previous equation, we gain  $\bar{\Pi}_i = \theta_i \pi_i^m + \frac{x_i}{1 - x_i} A_i$  for each  $i \in K^*$ . The same expression also applies to each firm that never advertises,  $l \notin K^*$ , because such firms have  $x_l = 0$  under our restrictions.

To ensure a common upper utility bound for each advertising firm,  $\bar{u}$ , we then require  $\bar{\Pi}_i = (1 - \theta_{-i})\pi_i(\bar{u}) - A_i$  for each  $i \in K^*$ . This provides an expression for  $x_i$  for each such firm, (11). Using  $\frac{A_i}{h_i(\bar{u}_i)} = 1$  from (8),  $\tilde{u}_{k^*} \geq \bar{u} > u^m$  from Lemma 2, and  $\hat{u}_i > \tilde{u}_i$ , it follows that each  $x_i$  is unique, and  $x_i \in [0, 1)$ . Summing (11) over  $i = 1, \dots, k^*$  and setting equal to 1, also provides (12). When combined, these provide  $k^* + 1$  equations to solve for  $k^* + 1$  unknowns,  $\{x_1, \dots, x_{k^*}\}$  and  $\bar{u}$ .

$$x_i = 1 - \frac{A_i}{\pi_i(\bar{u})(1 - \theta_{-i}) - \theta_i \pi_i^m} = 1 - \frac{A_i}{h_i(\bar{u})} \quad (11)$$

$$\sum_{i=1}^{k^*} \left[ 1 - \frac{A_i}{\pi_i(\bar{u})(1 - \theta_{-i}) - \theta_i \pi_i^m} \right] = 1, \quad (12)$$

First consider the strict asymmetric case,  $\tilde{u}_1 > \tilde{u}_2 > \dots > \tilde{u}_n > u^m$ . The LHS of (12) is strictly decreasing in  $\bar{u}$ , and thus reaches its maximum at  $\bar{u} = u^m$ , with value  $\bar{I}_k = \sum_{i=1}^k \left[ 1 - \frac{A_i}{(1-\theta)\pi_i^m} \right]$ . Using Assumption A,  $A_i \leq (1 - \theta)\pi_i^m$ ,  $\bar{I}_k$  is then weakly increasing in  $k$ . The minimum value of the LHS of (12) is reached at  $\bar{u} = \tilde{u}_k$ , which gives value  $\underline{I}_k = \sum_{i=1}^k \left[ 1 - \frac{A_i}{\pi_i(\tilde{u}_k)(1-\theta_{-i})-\theta_i\pi_i^m} \right]$ . From (8),  $\frac{A_i}{\pi_i(\tilde{u}_i)(1-\theta_{-i})-\theta_i\pi_i^m} = 1$ , therefore one can rewrite  $\underline{I}_k = (k - 1) - \sum_{i=1}^{k-1} \frac{A_i}{\pi_i(\tilde{u}_k)(1-\theta_{-i})-\theta_i\pi_i^m}$ . It can then be verified that  $\underline{I}_k$  is increasing in  $k$  and that  $\underline{I}_k < \bar{I}_{k-1}$  because  $\tilde{u}_{k-1} > \tilde{u}_k > u^m$ . We thus have a sequence of intervals  $[\underline{I}_k, \bar{I}_k)$  indexed by  $k$  that i) shift to the right as  $k$  increases, and ii) strictly overlap because  $\underline{I}_k < \bar{I}_{k-1}$ . Given  $(n - 1) > \sum_{i=1}^n \frac{A_i}{(1-\theta)\pi_i^m}$ , we know  $\bar{I}_n > 1$ . Hence, there is at least one  $k$  such that  $1 \in [\underline{I}_k, \bar{I}_k)$ , and therefore some  $\bar{u} \in (u^m, \tilde{u}_k]$  such that (12) holds. Notice that (12) may also hold for any  $k' < k$ . However,  $k'$  cannot be the equilibrium number of advertising firms because then firm  $k$  will be able to profitably deviate from not advertising by advertising  $\bar{u}$ . This follows from the fact that  $\underline{I}_k < 1$  and so the solution to (12) for  $k' < k$  has the property that  $\bar{u} > \tilde{u}_k$ . Thus the equilibrium  $k^*$  should be such that either  $\underline{I}_{k^*} < 1 \leq \underline{I}_{k^*+1}$  if  $k^* < n$  (where  $1 < \bar{I}_{k^*}$  follows automatically given  $\underline{I}_{k+1} < \bar{I}_k$ ), or  $\underline{I}_n < 1 \leq \bar{I}_n$  if  $k^* = n$ . Once simplified and rearranged, these give the conditions in (9).

Now consider the quasi-symmetric case with  $\tilde{u}_i = \tilde{u} > u^m$  for all  $i$ . Here, given  $\frac{A_i}{h_i(\tilde{u}_i)} = 1$ , the conditions in (9) imply that any sales equilibrium must have  $k^* = n$ . Thus, the only requirement is  $\sum_{i=1}^n \frac{A_i}{h_i(\tilde{u}_n)} > (n - 1) \geq \sum_{i=1}^n \frac{A_i}{(1-\theta)\pi_i^m}$  where the first inequality is trivially satisfied because  $\sum_{i=1}^n \frac{A_i}{h_i(\tilde{u}_n)} = n$ , and where the second inequality is satisfied through our assumption on advertising costs.

Given  $K^*$ , all that now remains is to derive the unique equilibrium advertising probabilities and utility distributions for firms  $i \in K^*$ . To derive the advertising probabilities, plug (11) into (10), such that  $\prod_{j \neq i} (1 - \alpha_j) = \prod_{j \neq i \in K^*} (1 - \alpha_j) = \gamma_i(u^m)$  for  $i = 1, \dots, k^*$ , where  $\gamma_i(u) = \frac{\pi_i(\bar{u})(1-\theta_{-i})-\theta_i\pi_i(u)}{(1-\theta)\pi_i(u)} \leq 1$ . Then by multiplying each of these  $k^*$  equations together, we get  $\prod_{i=1}^{k^*} [\prod_{j \neq i \in K^*} (1 - \alpha_j)] = \prod_{i=1}^{k^*} \gamma_i(u^m)$ . On simplification, this equals

$\prod_{i=1}^{k^*} (1 - \alpha_i)^{k^*-1} = \prod_{i=1}^{k^*} \gamma_i(u^m)$  such that  $\prod_{i=1}^{k^*} (1 - \alpha_i) = [\prod_{i=1}^{k^*} \gamma_i(u^m)]^{\frac{1}{k^*-1}}$ . By now taking (10) and multiplying both sides by  $1 - \alpha_i$  we get  $\prod_{j=1}^{k^*} (1 - \alpha_j) = (1 - \alpha_i) \gamma_i(u^m)$ , which after substitution provides a unique solution,  $\alpha_i = 1 - \frac{[\prod_{j=1}^{k^*} \gamma_j(u^m)]^{\frac{1}{k^*-1}}}{\gamma_i(u^m)}$ . Similar steps can be they used to derive the unique utility distributions,  $F_i(u) = \frac{[\prod_{j=1}^{k^*} \gamma_j(u)]^{\frac{1}{k^*-1}}}{\gamma_i(u)}$ , where  $F_i(u^m) = 1 - \alpha_i$  and  $F_i(\bar{u}) = 1$  as required.

For equilibrium existence, we need  $\partial F_i(u)/\partial u > 0$  for relevant  $u$ . This is trivially satisfied for  $k^* = 2$  because  $F_i(u) = \gamma_j(u)$  and  $\gamma'_j(u) > 0$ . However, when  $n > 2$ , this is not always the case. Instead, as discussed in the text, one can demonstrate existence by further specifying the model. For example, consider the case where firms only differ in their advertising costs. We can then drop subscripts for  $\gamma_i$  and obtain  $\alpha_i = 1 - \gamma(u^m)^{\frac{1}{k^*-1}}$  and  $F_i(u) = \gamma(u)^{\frac{1}{k^*-1}}$ , the latter clearly satisfying  $\partial F_i(u)/\partial u > 0$  because  $\gamma'(u) > 0$ . Alternatively, if firms are sufficiently symmetric in that  $\theta_i$ ,  $\pi_i(u)$  and  $A_i$  are arbitrarily close (in case of  $\pi_i(u)$  uniformly) across firms,  $\partial F_i(u)/\partial u > 0$  holds because  $\gamma_i(u)$  can be made arbitrarily close to each other.  $\square$

**Proof of Corollary 1.** i) Given  $\frac{A}{h_i(\bar{u}_i)} = 1$ , the LHS term in the second line of (9) can be written as  $\sum_{i=1}^k \frac{A}{h_i(\bar{u}_k)} = 1 + \sum_{i=1}^{k-1} \frac{A}{h_i(\bar{u}_k)}$  which converges to 1 from above as  $A \rightarrow 0$ . It then follows that the only possible  $k^*$  that can satisfy (9) equals 2, with  $\bar{u} \rightarrow \bar{u}_2$ . ii) As  $A \rightarrow \frac{(n-1)(1-\theta)}{\sum_{i=1}^n \frac{1}{\pi_i^m}}$ , the solution to (12) converges to  $u^m < \tilde{u}_n$ . This follows because at  $\bar{u} = u^m$ , (12) becomes  $n - \sum_{i=1}^n \frac{A}{(1-\theta)\pi_i^m} = 1$ . However, for  $k^* = n$  to be an equilibrium, we also need Assumption A to hold which further requires  $A \leq (1-\theta)\pi_n^m$ .  $A \rightarrow \frac{(n-1)(1-\theta)}{\sum_{i=1}^n \frac{1}{\pi_i^m}}$  can still satisfy Assumption A provided  $\sum_{i=1}^n \frac{1}{\pi_i^m} > \frac{n-1}{\pi_n^m}$  as then  $\frac{(n-1)(1-\theta)}{\sum_{i=1}^n \frac{1}{\pi_i^m}} < (1-\theta)\pi_n^m$ .  $\square$

**Proof of Corollary 2.** The results about  $A_i$  and  $\theta_i$  follow from (8) as  $\frac{\partial \tilde{u}_i}{\partial A_i} < 0$  and  $\frac{\partial \tilde{u}_i}{\partial \theta_i} < 0$ . For the profitability result, rewrite (8) as  $(1 - \theta_{-i})\pi(\tilde{u}, e_i) - A_i = \theta_i \pi(u^m, e_i)$ . Then note that  $\frac{\partial \tilde{u}_i}{\partial e_i}$  is positive (negative) whenever  $\frac{1-\theta_{-i}}{\theta_i}$  is higher (lower) than  $\frac{\pi_e(u^m, e_i)}{\pi_e(\tilde{u}_i, e_i)}$  because  $\pi_u(\tilde{u}, e_i) < 0$ . Lastly, we note  $\frac{\partial \tilde{u}_i}{\partial e_i} > 0$  when  $\theta_i = \theta/n$  and  $\pi_{eu} \geq 0$  because i)  $\frac{1-\theta_{-i}}{\theta_i} > 1$  given  $n > 2$ , and ii)  $\frac{\pi_e(u^m, e_i)}{\pi_e(\tilde{u}_i, e_i)} \leq 1$  given  $\tilde{u}_i > u^m$ .  $\square$

## Appendix B - Further Technical Equilibrium Details

### B1. Market Equilibrium with Asymmetric Firms and $A_a = A_b \rightarrow 0$

When the firms are asymmetric but  $A_a = A_b = A \rightarrow 0$ , the equilibrium depends upon  $\pi_a^{-1} \left( \frac{\theta_a \pi_a^m}{1-\theta_b} \right) \geq \pi_b^{-1} \left( \frac{\theta_b \pi_b^m}{1-\theta_a} \right)$ . Without loss of generality, suppose  $\pi_i^{-1} \left( \frac{\theta_i \pi_i^m}{1-\theta_j} \right) < \pi_j^{-1} \left( \frac{\theta_j \pi_j^m}{1-\theta_i} \right)$  such that  $\pi_i(\bar{u})(1-\theta_j) - \theta_i \pi_i^m < \pi_j(\bar{u})(1-\theta_i) - \theta_j \pi_j^m$ . From (5), it must be that  $x_j > x_i$ . Moreover, with the additional use of (6), for  $\bar{u}$  to exist and for  $x_i$  and  $x_j$  to be well defined, it must be that  $\pi_i(\bar{u})(1-\theta_j) - \theta_i \pi_i^m \rightarrow 0$  such that  $x_i \rightarrow 0$ ,  $x_j \rightarrow 1$ , and  $\bar{u} \rightarrow \pi_i^{-1} \left( \frac{\theta_i \pi_i^m}{1-\theta_j} \right)$ . Given this, we know  $\lim_{A \rightarrow 0} \bar{\Pi}_i = \theta_i \pi_i^m$  and  $\lim_{A \rightarrow 0} \bar{\Pi}_j = \lim_{A \rightarrow 0} (1-\theta_i) \pi_j(\bar{u}) = (1-\theta_i) \pi_j \left( \pi_i^{-1} \left( \frac{\theta_i \pi_i^m}{1-\theta_j} \right) \right) > \theta_j \pi_j^m$ . Further, from (4), we know  $\lim_{A \rightarrow 0} F_i(u) = \lim_{A \rightarrow 0} \frac{\bar{\Pi}_j - \theta_j \pi_j(u)}{(1-\theta) \pi_j(u)}$  and  $\lim_{A \rightarrow 0} F_j(u) = \lim_{A \rightarrow 0} \frac{\bar{\Pi}_i - \theta_i \pi_i(u)}{(1-\theta) \pi_i(u)}$ . Finally, from (2),  $\alpha_j \rightarrow 1$ , while firm  $i$  advertises with probability  $\lim_{A \rightarrow 0} \alpha_i = 1 - \frac{\bar{\Pi}_j - \theta_j \pi_j^m}{(1-\theta) \pi_j^m} \in (0, 1)$ .

**Unit Demand Example:** Suppose  $u_i = V_i - p_i$  and  $\pi_i(u_i) = V_i - c_i - u_i$ , where  $u_i^m = 0$ , and  $\pi_i^m = V_i$ . By using the results above, the equilibrium then depends upon  $(1-\theta_a)(V_a - c_a) - (1-\theta_b)(V_b - c_b) \leq 0$ . For instance, when this is negative,  $x_a \rightarrow 0$  and  $x_b \rightarrow 1$ , such that  $\bar{\Pi}_a \rightarrow \theta_a(V_a - c_a)$ , and  $\bar{\Pi}_b \rightarrow (1-\theta_a)((V_b - c_b) - \bar{u})$ , where  $\bar{u} \rightarrow \left( \frac{(1-\theta)(V_a - c_a)}{1-\theta_b} \right)$ . By then denoting  $\Delta V = V_a - V_b$ , and noting that  $F_a(u_b) = Pr(u_a \leq u_b) = 1 - F_a(p_b + \Delta V)$  and  $F_b(u_a) = 1 - F_b(p_a - \Delta V)$ , it follows that  $F_a(p) = 1 - \left[ \frac{\bar{\Pi}_b - \theta_b(p - \Delta V - c_b)}{(1-\theta)(p - \Delta V - c_b)} \right] = 1 + \frac{\theta_b}{1-\theta} - \frac{(1-\theta_a)(\theta_a(V_a - c_a) + (1-\theta_b)(c_a - c_b - \Delta V))}{(1-\theta_b)(1-\theta)(p - \Delta V - c_b)}$  on  $[V_a - \bar{u}, V_a)$  and  $F_b(p) = 1 - \left[ \frac{\bar{\Pi}_a - \theta_a(p + \Delta V - c_a)}{(1-\theta)(p + \Delta V - c_a)} \right] = 1 - \left[ \frac{\theta_a(V_b - p)}{(1-\theta)(p + \Delta V - c_a)} \right]$  on  $[V_b - \bar{u}, V_b)$ , where  $\alpha_b \rightarrow 1$  but where firm  $a$  refrains from advertising with probability  $1 - \alpha_a = 1 - F_a(V_a) \in (0, 1)$ .

### B2. Equilibrium with Downward-Sloping Demand

Given  $\mathbf{p}_i = \{p_{i1}, \dots, p_{iK_i}\}$ , the individual consumer product demand functions at firm  $i$  can be permitted to be interrelated, as summarized by the demand vector  $\mathbf{q}_i(\mathbf{p}_i) = \{q_{i1}(\mathbf{p}_i), \dots, q_{iK_i}(\mathbf{p}_i)\}$ . One can then write  $u_i = S(\mathbf{p}_i, \mathbf{q}_i^*(\mathbf{p}_i))$ , where  $S(\mathbf{p}_i, \mathbf{q}_i(\mathbf{p}_i))$  denotes the indirect utility available at firm  $i$  for a given level of demand, and where  $\mathbf{q}_i^*(\mathbf{p}_i)$  denotes a consumer's optimal demand vector at firm  $i$ ,  $\mathbf{q}_i^*(\mathbf{p}_i) = \operatorname{argmax}_{\mathbf{q}_i} S(\cdot)$ . It then follows that  $\pi_i(\mathbf{p}_i) = \mathbf{q}_i^*(\mathbf{p}_i)'(\mathbf{p}_i - \mathbf{c}_i)$ , where  $\mathbf{c}_i = \{c_{i1}, \dots, c_{iK_i}\}$ . Under monopoly, firm  $i$  would set a vector of monopoly prices,  $\mathbf{p}_i^m = \operatorname{argmax}_{\mathbf{p}_i} \pi_i(\mathbf{p}_i)$ , with  $u_i^m = S(\mathbf{p}_i^m, \mathbf{q}_i^*(\mathbf{p}_i^m))$  and  $\pi_i(u_i^m) \equiv \pi_i(\mathbf{p}_i^m)$ . Hence, for Assumption U to hold with  $u_a^m = u_b^m$ , we restrict attention to cases with  $\pi_a(u) = \pi_b(u) = \pi(u)$ .

Under suitable demand assumptions, there can exist a unique efficient price vector

that maximizes a firm's profits subject to the constraint of supplying a given utility draw  $u$ , such that  $\mathbf{p}^*(u) = \operatorname{argmax}_{\mathbf{p}} \pi(\mathbf{p})$  subject to  $S(\mathbf{p}, \mathbf{q}^*(\mathbf{p})) = u$ , with resulting profits per consumer,  $\pi(u) \equiv \pi(\mathbf{p}^*(\mathbf{u}))$ .<sup>24</sup> It then follows that  $\bar{\Pi}_i = \theta_i \pi(\mathbf{p}^m) + \frac{x_i}{1-x_i} A_i$ ,  $\alpha_i = 1 - \frac{A_j}{x_i(1-\theta)\pi(\mathbf{p}^m)}$ , and  $F_i(u) = \frac{x_i \theta_j [\pi(\mathbf{p}^m) - \pi(\mathbf{p}^*(\mathbf{u}))] + A_j}{x_i(1-\theta)\pi(\mathbf{p}^*(\mathbf{u}))}$ , where  $\bar{\mathbf{p}} = \mathbf{p}^*(u^m) = \mathbf{p}^m$  and  $\underline{\mathbf{p}} = \mathbf{p}^*(\bar{u})$ , and where  $x_i$  and  $\bar{u}$  follow from amended versions of (5) and (6).

To consider how our framework then reproduces the standard clearinghouse equilibrium, suppose that the market is symmetric. It then follows that  $x_i = 0.5$ . Further, let  $K = 1$  such that  $\mathbf{p} \equiv p$ ,  $\bar{\Pi} = \frac{\theta}{2} \pi(p^m) + A$ ,  $\alpha = 1 - \left( \frac{2A}{(1-\theta)\pi(p^m)} \right)$ , and  $\pi(\bar{u}) = \left( \frac{\theta\pi(p^m) + 4A}{(2-\theta)} \right)$ . We can then use  $F(p) = 1 - F(u)$  to find the price distribution (conditional on advertising)  $F_A(p) \equiv \frac{1-F(u)}{\alpha}$  which equals  $\frac{1}{\alpha} \left[ 1 - \left( \frac{\theta[\pi(p^m) - \pi(p)] + 4A}{2(1-\theta)\pi(p)} \right) \right]$  with  $\underline{p} = \pi^{-1} \left( \frac{\theta\pi(p^m) + 4A}{(2-\theta)} \right)$  and  $\bar{p} = p^m$ . Finally, to consider how our framework reproduces the equilibrium of Simester (1997), suppose the market is symmetric with  $K \geq 1$  and  $A \rightarrow 0$ , and let all marginal costs equal zero. One can then replicate the equilibrium using  $x_i = 0.5$  under the additional restriction that non-shoppers and shoppers share common demand functions.

### B3. Equilibrium with Two-Part Tariffs

Consider the previous analysis of downward-sloping demand. While symmetric profit functions are no longer required, we keep this assumption for exposition with  $K$  products, demand functions,  $\mathbf{q}$ , and marginal costs,  $\mathbf{c}$ . However, now let each firm  $i$  set a  $K$ -dimensional vector of marginal prices (per unit of consumption),  $\mathbf{p}_i$ , and a single fixed fee,  $f_i \geq 0$ . It then follows that  $\pi(\mathbf{p}_i, f_i) = \mathbf{q}(\mathbf{p}_i)'(\mathbf{p}_i - \mathbf{c}) + f_i$  and  $u_i = S(\mathbf{p}_i) - f_i$  where  $S(\mathbf{p}_i)$  denotes a consumer's surplus at firm  $i$  gross of firm  $i$ 's fixed fee. To generate any utility,  $u'$ , firm  $i$  will choose  $\mathbf{p}_i$  and  $f_i$  to maximize  $\pi(\mathbf{p}_i, f_i)$  subject to  $S(\mathbf{p}_i) - f_i = u'$ . This implies marginal cost pricing,  $\mathbf{p}_i = \mathbf{c}$ , with optimal fixed fee,  $f_i = S(\mathbf{c}) - u'$ . The full equilibrium can then be derived using  $\pi(u) = S(\mathbf{c}) - u$ ,  $u^m = 0$  and  $\pi(u^m) = S(\mathbf{c})$  and shown to exhibit the features listed within the text.

### B4. Equilibrium with Non-Price Sales

For brevity, we consider non-price sales in a symmetric market with single products and unit demand. However, more complex settings can also be considered. Following the motivation in the text, suppose that each firm's price is fixed at  $p > 0$ , and that each

---

<sup>24</sup>This constrained pricing decision can be thought of as a Ramsey problem. Individual prices are hard to fully characterize, but with additional restrictions, firms can be shown to optimally use lower prices on products that are more price-elastic and complementary to other products. See Armstrong and Vickers (2001) and Simester (1997) for more discussion.



firm chooses some other strategic sales variable  $z_i \in [\underline{z}, \bar{z}]$ . To avoid any unnecessary complications, we ensure unique correspondences between  $z_i$ ,  $u_i$ , and  $\pi(u_i)$  by making two assumptions. First, let both the consumers' willingness to pay for firm  $i$ 's product,  $V(z_i)$ , and firm  $i$ 's marginal (per unit) cost,  $c(z_i)$ , be strictly increasing in  $z_i$ , such that  $u(z_i) = V(z_i) - p$  is strictly increasing in  $z_i$ , while  $\pi(z_i) = p - c(z_i)$  is strictly decreasing in  $z_i$ . Second, let  $u(\underline{z}) = V(\underline{z}) - p \geq 0$  and  $\pi(\bar{z}) = p - c(\bar{z}) > 0$ . Because profits and utilities are monotone in  $z$ , we have  $z = V^{-1}(p + u)$ . We can then derive  $\pi(u) = p - c(V^{-1}(p + u))$  and  $u^m = V(\underline{z}) - p$ . To ensure the equilibrium exists, we verify that  $\pi(u^m) = p - c(\underline{z}) > 0$  and that  $\pi'(u) < 0$ . The full equilibrium can then be explicitly derived and shown to exhibit the features listed within the text.

## Appendix C: Relaxing the Single Visit Assumption

In this appendix, we provide further details on how the model can be generalized to allow the shoppers to sequentially visit multiple firms. We focus on the duopoly model - similar (but more tedious) arguments can also be made for the  $n$ -firm model. Suppose that the cost of visiting any first firm is  $s_1$  and the cost of visiting any second firm is  $s_2$ . The main model implicitly assumed  $s_1 = 0$  and  $s_2 = \infty$ . However, we now use some arguments related to the Diamond paradox (Diamond, 1971) to show that our equilibrium results remain under sequential shopper visits for any  $s_2 > 0$  provided that i) the costs of any first visit are not too large,  $s_1 \in [0, u^m)$ , and ii) shoppers can only purchase from a single firm. The latter 'one-stop shopping' assumption is frequently assumed in consumer search models and the wider literature on price discrimination.

First, suppose  $s_1 \in [0, u^m)$  but maintain  $s_2 = \infty$ . Beyond  $s_1 = 0$ , this can now permit cases where the first visit cost is strictly positive provided  $u^m > 0$  as consistent with downward-sloping demand and linear prices. In particular, provided  $u^m > s_1 \geq 0$ , shoppers will still be willing to make a first visit and the equilibrium will remain unchanged.

Second, suppose  $s_1 \in [0, u^m)$  but allow for any  $s_2 > 0$  subject to a persistent assumption of one-stop shopping, such that a shopper cannot combine utility offers by buying from more than one firm. By definition, the behavior of the non-shoppers will remain unchanged. Therefore, to demonstrate that our equilibrium remains robust, we need to show that shoppers will endogenously refrain from making a second visit. Initially suppose that the firms keep playing their original equilibrium strategies and that a given shopper receives  $h \in \{0, 1, 2\}$  adverts. Given  $s_2 > 0$  and one-stop shopping, the gains

from any second visit will always be strictly negative for all  $h$ . In particular, if  $h \geq 1$ , then a shopper will first visit the firm with the highest advertised utility,  $u^* > u^m$ , and any offer from a second visit will necessarily provide  $u < u^*$ . Alternatively, if  $h = 0$ , then both firms will offer  $u^m$ , such that any second visit would be sub-optimal. Now suppose that the firms can deviate from their original equilibrium strategies. To see that the logic still holds, note that only the behavior of any non-advertising firms is relevant and that such firms are unable to influence any second visit decisions due to their inability to communicate or commit to any  $u < u^m$ . Hence, firms' advertising and utility incentives remain unchanged and the original equilibrium still applies.

## References

- ALLEN, J., R. CLARK, AND J.-F. HOUDE (2014): "The Effect of Mergers in Search Markets: Evidence from the Canadian Mortgage Industry," *American Economic Review*, 104, 3365–3396.
- AN, Y., M. R. BAYE, Y. HU, J. MORGAN, AND M. SHUM (2015): "Identification and Estimation of Online Price Competition With an Unknown Number of Firms," *Journal of Applied Econometrics*, forthcoming.
- ANDERSON, S., A. BAIK, AND N. LARSON (2015): "Personalized Pricing and Advertising: An Asymmetric Equilibrium Analysis," *Games and Economic Behavior*, 92, 53–73.
- ARMSTRONG, M. (2015): "Search and Ripoff Externalities," *Review of Industrial Organization*, 47, 273–302.
- ARMSTRONG, M. AND J. VICKERS (2001): "Competitive Price Discrimination," *RAND Journal of Economics*, 32, 579–605.
- ARNOLD, M., C. LI, C. SALIBA, AND L. ZHANG (2011): "Asymmetric Market Shares, Advertising and Pricing: Equilibrium With an Information Gatekeeper," *Journal of Industrial Economics*, 59, 63–84.
- ARNOLD, M. A. AND L. ZHANG (2014): "The Unique Equilibrium in a Model of Sales with Costly Advertising," *Economics Letters*, 124, 457–460.

- BARRON, J. M., B. A. TAYLOR, AND J. R. UMBECK (2004): “Number of Sellers, Average Prices, and Price Dispersion,” *International Journal of Industrial Organization*, 22, 1041–1066.
- BAYE, M. R., D. KOVENOCK, AND C. G. D. VRIES (1992): “It Takes Two to Tango: Equilibria in a Model of Sales,” *Games and Economic Behavior*, 4, 493–510.
- BAYE, M. R. AND J. MORGAN (2001): “Information Gatekeepers on the Internet and the Competitiveness of Homogeneous Product Markets,” *American Economic Review*, 91, 454–474.
- (2009): “Brand and Price Advertising in Online Markets,” *Management Science*, 55, 1139–1151.
- BAYE, M. R., J. MORGAN, AND P. SCHOLTEN (2004a): “Price Dispersion in the Small and in the Large: Evidence from an Internet Price Comparison Site,” *Journal of Industrial Economics*, 52, 463–496.
- (2004b): “Temporal Price Dispersion: Evidence from an Online Consumer Electronics Market,” *Journal of Interactive Marketing*, 18, 101–115.
- (2006): “Information, Search, and Price Dispersion,” in *Handbook on Economics and Information Systems*, ed. by T. Hendershott, Amsterdam: Elsevier, vol. 1, chap. 6, 323–375.
- BROWN, J. R. AND A. GOOLSBEE (2002): “Does the Internet Make Markets More Competitive? Evidence from the Life Insurance Industry,” *Journal of Political Economy*, 110, 481–507.
- BURDETT, K. AND K. L. JUDD (1983): “Equilibrium Price Dispersion,” *Econometrica*, 51, 955–969.
- CHANDRA, A. AND M. TAPPATA (2011): “Consumer Search and Dynamic Price Dispersion: An Application to Gasoline Markets,” *The RAND Journal of Economics*, 42, 681–704.
- CHEN, H., H. MARMORSTEIN, M. TSIROS, AND A. R. RAO (2012): “When More is Less: The Impact of Base Value Neglect on Consumer Preferences for Bonus Packs over Price Discounts,” *Journal of Marketing*, 76, 64–77.

- CHIOVEANU, I. (2008): “Advertising, Brand Loyalty and Pricing,” *Games and Economic Behavior*, 64, 68–80.
- DIAMOND, P. A. (1971): “A Model of Price Adjustment,” *Journal of Economic Theory*, 3, 156–168.
- DUBOVIK, A. AND M. C. JANSSEN (2012): “Oligopolistic Competition in Price and Quality,” *Games and Economic Behavior*, 75, 120 – 138.
- ELICKSON, P. B. AND S. MISRA (2008): “Supermarket Pricing Strategies,” *Marketing Science*, 27, 811–828.
- GIULIETTI, M., M. WATERSON, AND M. WILDENBEEST (2014): “Estimation of Search Frictions in the British Electricity Market,” *The Journal of Industrial Economics*, 62, 555–590.
- GU, Y. AND T. WENZEL (2014): “Strategic Obfuscation and Consumer Protection Policy,” *The Journal of Industrial Economics*, 62, 632–660.
- HENDEL, I., A. LIZZERI, AND N. ROKETSKIY (2014): “Nonlinear pricing of storable goods,” *American Economic Journal: Microeconomics*, 6, 1–34.
- HOSKEN, D. AND D. REIFFEN (2004): “Patterns of Retail Price Variation,” *RAND Journal of Economics*, 35, 128–146.
- (2007): “Pricing Behavior of Multiproduct Retailers,” *The B.E. Journal of Theoretical Economics*, 7, 1354–1354.
- IRELAND, N. J. (1993): “The Provision of Information in a Bertrand Oligopoly,” *The Journal of Industrial Economics*, 41, 61–76.
- JANSSEN, M. C. AND J. L. MORAGA-GONZÁLEZ (2004): “Strategic Pricing, Consumer Search and the Number of Firms,” *The Review of Economic Studies*, 71, 1089–1118.
- JING, B. (2007): “Product Differentiation under Imperfect Information: When Does Offering a Lower Quality Pay?” *Quantitative Marketing and Economics*, 5, 35–61.
- KAPLAN, G. AND G. MENZIO (2016): “Shopping Externalities and Self-Fulfilling Unemployment Fluctuations,” *Journal of Political Economy*, 124, 771–825.
- KOCAS, C. AND T. KIYAK (2006): “Theory and Evidence on Pricing by Asymmetric Oligopolies,” *International Journal of Industrial Organization*, 24, 83–105.

- LACH, S. (2002): “Existence and Persistence of Price Dispersion: An Empirical Analysis,” *The Review of Economics and Statistics*, 84, 433–444.
- LEWIS, M. (2008): “Price Dispersion and Competition with Differentiated Sellers,” *The Journal of Industrial Economics*, 56, 654–678.
- MORAGA-GONZÁLEZ, J. L., Z. SÁNDOR, AND M. R. WILDENBEEST (2013): “Semi-Nonparametric Estimation Of Consumer Search Costs,” *Journal of Applied Econometrics*, 28, 1205–1223.
- MORAGA-GONZÁLEZ, J. L. AND M. R. WILDENBEEST (2012): “Comparison Sites,” *Handbook of the Digital Economy M. Peitz and J. Waldfogel (eds)*, 224–53.
- NAKAMURA, E. AND J. STEINSSON (2008): “Five Facts about Prices: A Reevaluation of Menu Cost Models,” *The Quarterly Journal of Economics*, 123, 1415–1464.
- NARASIMHAN, C. (1988): “Competitive Promotional Strategies,” *The Journal of Business*, 61, 427–449.
- PALAZON, M. AND E. DELGADO-BALLESTER (2009): “Effectiveness of Price Discounts and Premium Promotions,” *Psychology & Marketing*, 26, 1108–1129.
- PENNERSTORFER, D., P. SCHMIDT-DENGLER, N. SCHUTZ, C. WEISS, AND B. YONTCHEVA (2015): “Information and Price Dispersion: Theory and Evidence,” *Working paper*.
- ROBERT, J. AND D. O. STAHL (1993): “Informative Price Advertising in a Sequential Search Model,” *Econometrica*, 61, 657–86.
- ROY, S. (2000): “Strategic Segmentation of a Market,” *International Journal of Industrial Organization*, 18, 1279–1290.
- SHANKAR, V. AND R. N. BOLTON (2004): “An Empirical Analysis of Determinants of Retailer Pricing Strategy,” *Marketing Science*, 23, 28–49.
- SHELEGIA, S. (2012): “Asymmetric Marginal Costs in Search Models,” *Economics Letters*, 116, 551–553.
- SHERMAN, J. AND A. WEISS (2015): “Search and Price Dispersion at an Open-Air Market,” *Working paper*.

- SIMESTER, D. (1997): “Optimal Promotion Strategies: A Demand-Sided Characterization,” *Management Science*, 43, pp. 251–256.
- SORENSEN, A. T. (2000): “Equilibrium Price Dispersion in Retail Markets for Prescription Drugs,” *Journal of Political Economy*, 108, 833–862.
- SPIEGLER, R. (2015): “Choice Complexity and Market Competition,” *Annual Review of Economics*.
- STAHL, D. O. (1989): “Oligopolistic Pricing with Sequential Consumer Search,” *American Economic Review*, 79, 700–712.
- STEENKAMP, J.-B. E., V. R. NIJS, D. M. HANSENS, AND M. G. DEKIMPE (2005): “Competitive Reactions to Advertising and Promotion Attacks,” *Marketing Science*, 24, 35–54.
- VARIAN, H. R. (1980): “A Model of Sales,” *American Economic Review*, 70, 651–659.
- WILDENBEEST, M. R. (2011): “An Empirical Model of Search with Vertically Differentiated Products,” *The RAND Journal of Economics*, 42, 729–757.