Approximate Knowledge of Rationality and Correlated Equilibria

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Approximate Knowledge of Rationality and Correlated Equilibria*

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Abstract

We extend Aumann's [3] theorem, deriving correlated equilibria as a consequence of common priors and common knowledge of rationality, by explicitly allowing for non-rational behavior. We replace the assumption of common knowledge of rationality with a substantially weaker one, *joint* $\boldsymbol{p}$*-belief of rationality*, where agents believe the other agents are rational with probabilities $\boldsymbol{p} = (\boldsymbol{p_i})_{i\in I}$ or more. We show that behavior in this case constitutes a type of correlated equilibrium satisfying certain $\boldsymbol{p}$-belief constraints, and that it varies continuously in the parameters $\boldsymbol{p}$ and, for $\boldsymbol{p}$ sufficiently close to one, with high probability is supported on strategies that survive the iterated elimination of strictly dominated strategies. Finally, we extend the analysis to characterizing rational expectations of interim types, to games of incomplete information, as well as to the case of non-common priors.

Keywords: Approximate common knowledge, bounded rationality, p-rational belief system, correlated equilibrium, incomplete information, non-cooperative game. *JEL Classification*: C72, D82, D83.

"Errare humanum est, perseverare diabolicum."

1 Introduction

Rationality, understood as consistency of behavior with stated objectives, information, and strategies available, naturally lies at the heart of game theory. In a celebrated paper, Aumann [3] takes as point of departure the rationality of all agents in any state of the world, knowledge thereof, as well as the existence of a common prior, to show that players' behavior will necessarily conform to a correlated equilibrium.

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Given that objectives, strategies, and information structure are part of the description of the game (or game situation) at hand, they are relatively flexible restrictions that should be adaptable to the actually given game (or game situation). At the same time it is both possible and plausible that agents may deviate from complete consistency of behavior given the assumed restrictions. This can happen in many ways, as agents can make mistakes at any stage of participating in the game:[1] they can make a mistake in perceiving or interpreting their objective (e.g., a trader may apply a wrong set of exchange exchange rates when deciding a transaction), they may make a mistake in carrying out the selected strategy (e.g., a soccer player may trip or make a wrong step when shooting a penalty kick), they may make a mistake in processing their available information (e.g., a judge may forget to consult a crucial, requested and readily available, technical report when deciding on a specific court case) and so on. It is clear that departures from full consistency or rationality not only occur, occur often, but also occur in innumerable ways.

In this paper, we build on the (ex ante) approach of Aumann [3] when formalizing the overall strategic interaction, but depart from it by allowing agents to entertain the possibility that other agents may *not* always be consistent. Specifically, we assume that agents believe that the other agents are consistent or rational with probability p_i, and that, with the remaining probability ($1 - p_i$ or less), they are *not* rational. In doing so we put as little structure as possible on what it means to be *not* rational, except that the rules of the game require agents to select *some* action from the specified set. This is in line with our belief that when modeling "social behavior" very broadly defined, mistakes or inconsistencies not only occur but also occur in innumerable ways. Therefore, without imposing assumptions on the kind of deviations from rational play that players may make, except that its probability of occurrence is limited by some vector $\boldsymbol{p}$, and maintaining the common prior assumption (common to all types of agents), we explore the consequences of what we call *joint $\boldsymbol{p}$-belief of rationality*. The two assumptions together, common prior and joint $\boldsymbol{p}$-belief of rationality, are what we call $\boldsymbol{p}$-rationality, where $\boldsymbol{p} \in [0,1]^I$.

The main result of the paper, Theorem 1, then characterizes strategic behavior in games where players are $\boldsymbol{p}$-rational. This provides a generalization of [3], since our $\boldsymbol{p}$-rational outcomes collapse to the correlated equilibria when $\boldsymbol{p} = 1$. Our second result, Theorem 2, shows basic topological properties of the $\boldsymbol{p}$-rational outcomes, in particular convexity, compactness but, more importantly, also that the set varies continuously in the underlining parameters $\boldsymbol{p}$. We also show that when $p \equiv \min \boldsymbol{p}$ is sufficiently close to 1, then strategy profiles involving strategies that do not survive the iterated elimination of strictly dominated strategies get probability at most $1 - p$ under the common prior. Following Aumann and Dreze [4], we also ask what payoffs agents should expect under $\boldsymbol{p}$-rationality, and thus characterize in Theorem 3, what we call $\boldsymbol{p}$-rational expectations of interim types; Theorem 4 shows that our main characterization result extends directly to the case of games of incomplete information. Finally, we also discuss non-common priors and show that joint $\boldsymbol{p}$-belief of rationality puts no restriction on behavior in this case.

Very closely related is the characterization of behavior under common $\boldsymbol{p}$-belief of rationality. For $\boldsymbol{p} = \mathbf{1}$

[1] We refer to [9] and [19] for surveys on bounded rationality, and to [8] for a survey of behavioral game theory.

and without requiring common priors this has been done, on one hand by Bernheim [5], Pearce [18], and Tan and Werlang [20], leading to rationalizability; and on the other by Aumann [2], leading to subjectively correlated equilibria. Though conceptually distinct, the two approaches are closely related as shown in Brandenburger and Dekel [7], and we further relate them to our approach in Section 6 where we discuss non-common priors. Also closely related is Hu [15], who studies behavior under common $\boldsymbol{p}$-belief of rationality at the interim stage (using hierarchies of beliefs and hence without any priors) and who provides, for general $\boldsymbol{p}$, a characterization in terms of what he calls $\boldsymbol{p}$-rationalizability. We complement his analysis by showing that within our finite model, assuming common $\boldsymbol{p}$-belief of rationality and a common prior at an ex ante stage, amounts to the same as assuming common knowledge of rationality; while at the interim stage with non-common priors and weakening common $\boldsymbol{p}$-belief of rationality to joint $\boldsymbol{p}$-belief of rationality puts *no* restriction on behavior. Finally, Börgers [6] also considers behavior under common $\boldsymbol{p}$-belief of rationality but adds a perfection-type assumption, which is also taken up in Hu [15].

Overall, the paper is structured as follows. Section 2 defines the $(X, \boldsymbol{p})$-correlated equilibria which are the basic building block for our epistemic characterizations. Section 3 contains the epistemic concepts and the main results characterizing the $\boldsymbol{p}$-rational outcomes and expectations. Section 4 provides some examples, Section 5 considers games of incomplete information, and Section 6 some concluding remarks. All proofs are relegated to the Appendix.

2 $(X, \boldsymbol{p})$-Correlated Equilibria and n-Games

We begin with the concept of $(X, \boldsymbol{p})$-correlated equilibrium that will play a key role in the characterization of behavior under approximate knowledge of rationality of the sections below. The concept extends the notion of correlated equilibrium by allowing agents in equilibrium to take actions that with a certain probability do not necessarily satisfy the usual incentive constraints. This allows to model – applying otherwise standard game theoretic analysis – agents that follow a mediator's advice without questioning the "rationality" of doing so (this could be in the sense of a social norm; see, e.g., [12]), or also agents that simply act irrationally in the sense of making mistakes (for whatever reason and in whichever way).

Throughout the paper, we denote by $G = \left\langle I, (A_i)_{i\in I}, (h_i)_{i\in I} \right\rangle$ a finite *game* in strategic form, where I is a finite set of players, and, for each $i \in I$, A_i is a finite set of actions and $h_i : \prod_{i\in I} A_i \to \mathbb{R}$ a payoff function; $A = \prod_{i\in I} A_i$ is the set of action profiles. The following definition generalizes that of correlated equilibrium.

Definition 1 ($(X, \boldsymbol{p})$-Correlated Equilibrium) *Let G be a game, $X = \prod_{i\in I} X_i \subseteq A$, and $\boldsymbol{p} \in [0,1]^I$. Then, $\pi \in \Delta(A)$ is a $(X, \boldsymbol{p})$-*correlated equilibrium *of G if and only if, for any $i \in I$,*

- *For any $a_i' \in X_i$, the following* incentive constraints *are satisfied,*

$$\sum_{a_{-i}\in A_{-i}} \pi\left(a_{-i}; a_i'\right)\left[h_i\left(a_{-i}; a_i'\right) - h_i\left(a_{-i}; a_i\right)\right] \geq 0, \text{ for any } a_i \in A_i,$$

- *For any $a_i \in A_i$ the following p_i-*belief constraint *is satisfied,*

$$\sum_{a'_{-i} \in X_{-i}} \pi\left(a'_{-i}; a_i\right) \geq p_i \sum_{a_{-i} \in A_{-i}} \pi\left(a_{-i}; a_i\right).$$

We denote by $(X, \boldsymbol{p})$-$CE\,(G)$ the set of $(X, \boldsymbol{p})$-correlated equilibria of G.

In other words, we fix for each player, $i \in I$, pure actions $X_i \subseteq A_i$ and a probability $p_i \in [0,1]$, such that the distribution on the overall set of action profiles A satisfies, for each $i \in I$, (i) standard *incentive constraints* for all actions in X_i, and, (ii) p_i-*belief constraints*, meaning each player assigns probability at least p_i to the *other* players all choosing action profiles from $X_{-i} = \prod_{j \neq i} X_i$. In words, the $(X, \boldsymbol{p})$-correlated equilibria allow to *relax* incentive constraints on actions not in the X_i's, while restricting the probability with which this occurs. Note that $CE(G) \subseteq (X, \boldsymbol{p})$-$CE(G)$ for any $X \subseteq A$ and $\boldsymbol{p} \in [0,1]^I$, and that, if $X = A$ or $\boldsymbol{p} = \mathbf{1}$, then $CE\,(G) = (X, \boldsymbol{p})$-$CE\,(G)$.

To illustrate, consider the following prisoner's dilemma game with distributions $\pi_{\mathbf{0.5}}, \pi_{\mathbf{0.95}} \in \Delta(A)$:

$$G \equiv \begin{array}{c|c|c|} & L & R \\ \hline T & 1,1 & 4,0 \\ \hline B & 0,4 & 3,3 \\ \hline \end{array}\ , \qquad \pi_{\mathbf{0.5}} \equiv \begin{array}{c|c|c|} & L & R \\ \hline T & 0.25 & 0.25 \\ \hline B & 0.25 & 0.25 \\ \hline \end{array}\ , \qquad \pi_{\mathbf{0.95}} \equiv \begin{array}{c|c|c|} & L & R \\ \hline T & 0.950 & 0.024 \\ \hline B & 0.024 & 0.002 \\ \hline \end{array}\ .$$

Taking X to be the dominant actions, $X = X_1 \times X_2 = \{T\} \times \{L\}$, allows to view the distributions $\pi_{\mathbf{0.5}}$ and $\pi_{\mathbf{0.95}}$ as respectively $(X, \mathbf{0.5})$- and $(X, \mathbf{0.95})$-correlated equilibria. In particular, while the unique correlated equilibrium, putting mass one on the dominant profile (T, L), is always an $(X, \boldsymbol{p})$-correlated equilibrium for any X and $\boldsymbol{p}$, there are also equilibria where players can play their cooperative (strictly dominated) action with positive probability.[2] That is, certain proportions of players are following the recommendations of the mediator even if this leads to strictly inferior payoffs. While the actions of the above game can be naturally divided into "rational" and "irrational" ones (thus suggesting a possibly natural choice of X), this is not always as straightforward in general games. In fact, as will be clearer later, even "irrational" types may accidentally play a "rational" action, and hence it becomes useful, if not necessary, to distinguish for every action two (or more) types of agents. This leads to what we will call doubled games, and the definition of $(X, \boldsymbol{p})$-correlated equilibrium will be applied to such enlarged games.

The following definition generalizes the idea of *doubled game* in Aumann and Dreze [4]. For any $n \in \mathbb{N}$ we denote $N = \{0, 1 \ldots, n-1\}$; as we will see, it is the cases $n = 2$ and $n = 3$ that play a role in our epistemic

[2]Notice that distributions of the form $\tilde{\pi}_{\mathbf{0.5}} = \begin{array}{c|c|c|} & L & R \\ \hline T & 0.5 & 0 \\ \hline B & 0 & 0.5 \\ \hline \end{array}$ and $\tilde{\pi}_{\mathbf{0.95}} = \begin{array}{c|c|c|} & L & R \\ \hline T & 0.95 & 0 \\ \hline B & 0 & 0.05 \\ \hline \end{array}$ cannot be supported as respectively $(X, \mathbf{0.5})$- and $(X, \mathbf{0.95})$-correlated equilibria. This follows from the $\boldsymbol{p}$-belief constraints and the fact that we require them to be satisfied for all actions in A. Another reasonable alternative would be to require them to be satisfied only for actions in X. As we will see this is related to what we assume about the beliefs of different types of agents. Our approach is in line with the interpretation that irrationalities are due to mistakes. We come back to this issue in Remark 1 of Section 6.

characterization results.

Definition 2 ($\boldsymbol{n}$-Game) *Let* $G = \left\langle I, (A_i)_{i\in I}, (h_i)_{i\in I}\right\rangle$ *be a game, then the* n-game *is the tuple* $nG = \left\langle I, (nA_i)_{i\in I}, (h_{n,i})_{i\in I}\right\rangle$, *where for each player* $i \in I$ *we have,*

- $nA_i = A_i \times N$ *is player* i*'s set of pure actions; we denote a generic element of* $nA = \prod_{i\in I} nA_i$ *by* (a, ν), *where* $\nu \in N^I$ *specifies which copy of* A_i *in* nA_i *each player* i*'s pure action belongs to.*
- $h_{n,i}$ *is* i*'s payoff function, where for each* $(a, \nu) \in nA$, $h_{n,i}(a, \nu) = h_i(a)$.

In this context when writing the action spaces of the game nG as $nA_i = A_i \times N$ we mean that for each player there are n *copies* of the original action space A_i, which we denote by $A_i \times \{k\}$ for $k \in N$. Note that any distribution on the action profiles of nG, $\hat{\pi} \in \Delta(nA)$, induces a distribution on the action profiles of G in a natural way by taking the marginal on A, that is, $\pi = \operatorname{marg}_A \hat{\pi}$. For any subset $Y \subseteq \Delta(nA)$ we denote,

$$\mathbf{marg}_A[Y] = \left\{\operatorname{marg}_A \hat{\pi} \text{ where } \hat{\pi} \in Y\right\}.$$

Next we turn to the epistemic characterization of behavior under approximate knowledge of rationality and relating it to the $(X, \boldsymbol{p})$-correlated equilibria of the doubled game $2G$ of the underlying game G.

3 $\boldsymbol{p}$-Rational Belief Systems and $\boldsymbol{p}$-Rational Outcomes

Our main object of study are what we define as $\boldsymbol{p}$-rational belief systems. These generalize the rational belief systems of [3, 4, 13], allowing us to formalize a situation where there is no common knowledge of rationality, but where nonetheless players believe that the other players are rational with probabilities equal to or above the levels $\boldsymbol{p} = (p_i)_{i\in I} \in [0,1]^I$.

3.1 Belief Systems

Following [3] we first define general belief systems, which are more basic than the rational or $\boldsymbol{p}$-rational belief systems that follow, and which play a central role throughout the paper. They encapsulate what players believe about the game and about their opponents.

Definition 3 (Belief System) *A* belief system *for game* G *is a tuple* $B = \left\langle \Omega, (\Pi_i)_{i\in I}, (\alpha_i)_{i\in I}, (\mu_i)_{i\in I}\right\rangle$ *such that,*

- Ω *is a finite* set of sates of the world,

and for each player $i \in I$,

- Π_i *is a partition of* Ω, *where* $\Pi_i(\omega)$ *denotes the element of* Π_i *containing* ω, *for any state* $\omega \in \Omega$,

- $\alpha_i : \Omega \to A_i$ *is a* strategy map *measurable w.r.t.* Π*, and,*

- $\mu_i \in \Delta(\Omega)$ *is a* prior belief *with full support on* Ω*.*

By an *event* we mean any subset $E \subseteq \Omega$. For each player $i \in I$ and each state $\omega \in \Omega$, $\Pi_i(\omega)$ represents i's differential information, that is, the states she considers possible at state ω. We say that player $i \in I$ *knows* event $E \subseteq \Omega$ in state $\omega \in \Omega$ if $\Pi_i(\omega) \subseteq E$. Consequently, the measurability of the strategy maps w.r.t. their corresponding partitions implies that each player knows at every state what action she chooses. Additionally, priors and partitions induce *interim beliefs*, conditional on differential information:

$$\mu_i(E \mid \Pi_i(\omega)) = \frac{\mu_i(E \cap \Pi_i(\omega))}{\mu_i(\Pi_i(\omega))}, \text{ for any event } E \subseteq \Omega.$$

We say that B satisfies *the common prior assumption*, CP, if $\mu_i = \mu_j$ for all $i, j \in I$, and, in this case, denote the priors by just μ and refer to them as the *common prior*; correspondengly we write B as $\left\langle \Omega, (\Pi_i)_{i \in I}, (\alpha_i)_{i \in I}, p \right\rangle$.

For any state $\omega \in \Omega$, we say that *player i is rational at ω*, if

$$\alpha_i(\omega) \in \operatorname*{argmax}_{a_i \in A_i} \mathbb{E}_B \left(h(a_{-i}; a_i) \,|\Pi_i(\omega) \right),$$

where $\mathbb{E}_B(h(a_{-i}; a_i)\,|\Pi_i(\omega)) = \sum_{a_{-i} \in A_{-i}} \mu_i\left(\alpha_{-i}^{-1}(a_{-i}) \mid \Pi_i(\omega)\right) h_i(a_{-i}; a_i)$. We denote by R_i the set of states in which player i is rational. The set of the states in which every player but i is rational (without excluding i's possible rationality), $\bigcap_{j \in I \setminus \{i\}} R_j$, will be denoted by R_{-i}. By R, we will denote the set of states in which every player is rational. Note that a player knows at every state whether she is rational or not;[3] this follows from the measurability of each α_i w.r.t. Π_i and the fact that posterior beliefs are the same for every two states in the same element of the partition. We will say that a belief system B satisfies *common knowledge of rationality*, CKR, if every player is rational at every state, that is, if $R = \Omega$.[4]

Finally, we say that a belief system is *rational* if it satisfies both CP and CKR. It is well known from [3] that the outcome distributions over the set of strategies of a finite game in strategic form G played under some rational belief system B, are precisely the correlated equilibria of game G, which we denote by $CE(G)$.

3.2 p-Belief Operators and p-Rationality

Recall that, following Monderer and Samet [17], for any given $p_i \in [0, 1]$, and any event $E \subseteq \Omega$, we can define player i's *p_i-belief operator* as,

$$B_i^{p_i}(E) = \{\omega \in \Omega \mid \mu_i(E \mid \Pi_i(\omega)) \geq p_i\}.$$

[3] Formally this menas that for any $i \in I$, and any $\omega \in \Omega$, $\Pi_i(\omega) \subseteq R_i$, or $\Pi_i(\omega) \subseteq \Omega \setminus R_i$.

[4] As is shown in [17], p.174, this does indeed imply common knowledge of rationality. In this case, Ω is clearly such an *evident knowledge*, and, hence, also R.

For any $\omega \in \Omega$, any event E, and any $p_i \in [0,1]$, we say that *player i p_i-believes E in ω*, if $\omega \in B_i^{p_i}(E)$. For $\boldsymbol{p} = (p_i)_{i \in I} \in [0,1]^I$, we say that an event E is *$\boldsymbol{p}$-evident* if $E \subseteq \bigcap_{i \in I} B_i^{p_i}(E)$, and given any state ω, we say that an event C is *common $\boldsymbol{p}$-belief in ω* if there exists some $\boldsymbol{p}$-evident event E such that $\omega \in E \subseteq \bigcap_{i \in I} B_i^{p_i}(C)$. We denote the event that C is common $\boldsymbol{p}$-belief by $CB^{\boldsymbol{p}}(C)$.

We say that a belief system B satisfies *common $\boldsymbol{p}$-belief of rationality* if common $\boldsymbol{p}$-belief of rationality is held at every state, that is, if $CB^{\boldsymbol{p}}(R) = \Omega$, thus defining a property of the belief system, rather than one concerning a state and an event. As Ω is finite, it is easy to check that a belief system satisfies **1**-belief of rationality if and only if it satisfies common knowledge of rationality.[5] For reasons explained below, the following notion of approximate knowledge of rationality replaces the notion of common knowledge of rationality assumed in [3] and elsewhere.

Definition 4 (Joint $\boldsymbol{p}$-Belief of Rationality) *A belief system B satisfies* joint $\boldsymbol{p}$-belief of rationality, *$J\boldsymbol{p}BR$, if $\bigcap_{i \in I} B_i^{p_i}(R_{-i}) = \Omega$.*

The following are basic results regarding relations between common knowledge of rationality, common $\boldsymbol{p}$-belief of rationality and $J\boldsymbol{p}BR$.

Lemma 1 *Let G be a game and B a belief system for G, then:*

(a) For any $\boldsymbol{p} \in (0,1]^I$, B satisfies common knowledge of rationality if and only if it satisfies common $\boldsymbol{p}$-belief of rationality.

(b) For any $\boldsymbol{p} \in (0,1]^I$, if B satisfies both CP and joint $\boldsymbol{p}$-belief of rationality, then, for $p = \min \boldsymbol{p}$, $\mu(CB^{\boldsymbol{p}}(R)) \geq \mu(R) \geq p^2$.

(c) For $\boldsymbol{p} = 1$, if B satisfies CP, then B satisfies common $\boldsymbol{p}$-belief of rationality if and only if it satisfies joint $\boldsymbol{p}$-belief of rationality.

Hence, if our aim is to replace the CKR assumption by a less restrictive one involving some $\boldsymbol{p}$-belief of rationality, the lemma above suggests that $J\boldsymbol{p}BR$ could be a sensible choice, since:

- *common* $\boldsymbol{p}$-belief of rationality would lead to a model analogous to the one already considered before (or without) introducing $\boldsymbol{p}$-beliefs (this holds true with or without a common prior),
- although joint $\boldsymbol{p}$-belief of rationality is defined as a *joint* and not necessarily *common* belief, it nonetheless implies common $\boldsymbol{p}$-belief of rationality with probability $(\min \boldsymbol{p})^2$, and,
- as every p_i converges to 1, joint $\boldsymbol{p}$-belief of rationality converges to *common* **1**-belief of rationality (or *common certainty* of rationality) and therefore, to common knowledge of rationality.

[5] Hence, for any belief system B and any $\boldsymbol{p} \in [0,1]^I$, satisfying common knowledge of rationality always implies satisfying common $\boldsymbol{p}$-belief of rationality.

In view of this, the following generalization of rational belief systems plays a key role in our analysis.

Definition 5 ($\boldsymbol{p}$-Rational Belief System) *A belief system is* $\boldsymbol{p}$-rational *if it satisfies both* CP *and* $J\boldsymbol{p}BR$.

Our aim is to characterize the distributions over outcomes or strategy profiles that can arise under $\boldsymbol{p}$-rational belief systems, as is done for rational belief systems in [3].[6]

3.3 $\boldsymbol{p}$-Rational Outcomes

The following definition formalizes the notion of distributions over outcomes that satisfy CP and $J\boldsymbol{p}BR$.

Definition 6 ($\boldsymbol{p}$-Rational Outcome) *Let G be a game and $\boldsymbol{p} \in [0,1]^I$, then $\pi \in \Delta(A)$ is a* $\boldsymbol{p}$-rational outcome *of G if there exists a $\boldsymbol{p}$-rational belief system B such that,*

$$\pi(a) = \mu\left(\bigcap_{i\in I} \alpha_i^{-1}(a_i)\right) \text{ for any } a \in A.$$

We denote the outcome distribution over the set of strategies of G, induced by the $\boldsymbol{p}$-rational belief system B, by π_B, and the set of $\boldsymbol{p}$-rational outcomes of G, by $\boldsymbol{p}$-$RO(G)$.

We can now characterize the set $\boldsymbol{p}$-$RO(G)$ in terms of $(X,\boldsymbol{p})$-correlated equilibria of the game $2G$.

Theorem 1 *Let G be a game and $\boldsymbol{p} \in [0,1]^I$. Then the $\boldsymbol{p}$-rational outcomes of G are the marginal on A of the $(X,\boldsymbol{p})$-correlated equilibria of the doubled game $2G$, where $X = \prod_{i\in I}(A_i \times \{0\})$ is a copy of the original action space of G. Formally,*

$$\boldsymbol{p}\text{-}RO(G) = \mathbf{marg}_A\left[(X,\boldsymbol{p})\text{-}CE(2G)\right],$$

where $X = \prod_{i\in I}(A_i \times \{0\})$.

The intuition is as follows. A doubled game can be seen as splitting players' actions into ones chosen by the rational type (in $A_i \times \{0\}$) and by the irrational type (in $A_i \times \{1\}$). The $(X,\boldsymbol{p})$-correlated equilibria with $X = \prod_{i\in I}(A_i \times \{0\})$ are distributions on $\Delta(2A)$ that by definition satisfy the incentive constraints just for the rational types, and where the $\boldsymbol{p}$-belief constraints ensure that all players believe the others play rationally with probabilities $\boldsymbol{p}$ or more. Taking marginals, finally, ensures that the distributions are on $\Delta(A)$.[7]

The next two results further characterize the structure and nature of the set of $\boldsymbol{p}$-rational outcomes.

Theorem 2 *Let G be a game and $\boldsymbol{p} \in [0,1]^I$. Then the set of $\boldsymbol{p}$-rational outcomes of the game G is a nonempty, convex, compact set that varies continuously in $\boldsymbol{p}$.*[8] *Moreover, for $\boldsymbol{p} = \mathbf{0}$, we have $\mathbf{0}$-$RO(G) = \Delta(A)$, for $\boldsymbol{p} = \mathbf{1}$, we have $\mathbf{1}$-$RO(G) = CE(G)$, and for any $\boldsymbol{p} \in [0,1)^I$, we have $dim[\boldsymbol{p}\text{-}RO(G)] = dim[\Delta(A)]$.*

[6] Another natural alternative, suggested to us by Dov Samet, would be to consider belief systems that satisfy $\mu(CB^{\boldsymbol{p}}(R)) \geq 1-\epsilon$ for some $\epsilon > 0$. This is a weakening of our notion of $J_{\boldsymbol{p}}BR$ in that it imposes less structure on players' beliefs, nonetheless it appears to be less tractable; we return to this later; see Remark 1 in Section 6.

[7] Clearly, due to the symmetric role of the different copies of the action spaces of $2G$, the theorem would also hold for $X = \prod_{i\in I}(A_i \times \{1\})$, whereby only one of the two copies of players' actions satisfies the incentive constraints.

[8] A correspondence is *continuous* if it is both upper- and lower hemicontinuous; see, e.g., Ch. 17 in [1] for further details and related definitions.

Monderer and Samet [17] show (using common $\boldsymbol{p}$-beliefs) that the $\boldsymbol{p}$-rational outcomes for $\boldsymbol{p} = \mathbf{1}$ are the correlated equilibria. The above result strengthens this by showing that as $\boldsymbol{p}$ converges to 1 the $\boldsymbol{p}$-rational outcomes converge to the set of correlated equilibria. But more generally it also shows that the $\boldsymbol{p}$-rational outcomes *always* vary continuously in $\boldsymbol{p}$, at any $\boldsymbol{p} \in [0,1]^I$; and go from being the entire set $\Delta(A)$ when $\boldsymbol{p} = \mathbf{0}$ to the set of correlated equilibria when $\boldsymbol{p} = \mathbf{1}$.

The very last statement further shows that *all* strategies can be in the support of $\boldsymbol{p}$-rational outcomes whenever $\boldsymbol{p} < \mathbf{1}$. The next result qualifies this by showing that if $p = \min \boldsymbol{p}$ is close enough 1, then strategy profiles involving strategies that do not survive the iterated elimination of strictly dominated strategies get a total weight of at most $1 - p$. This can be interpreted as the $\boldsymbol{p}$-rationality counterpart of the fact that strategies that do not survive the iterated elimination of strictly dominated strategies are not in the support of correlated equilibria. In what follows we denote by A^∞ the set of all strategy profiles that survive the iterated elimination of strictly dominated strategies and denote its complement in A by $(A^\infty)^c = A \setminus A^\infty$.

Proposition 1 *Let G be a game. Then there exists $\bar{p} \in (0,1)$ such that, for any $\boldsymbol{p} \in [0,1]^I$ with $\min \boldsymbol{p} = p \in [\bar{p}, 1]$, we have that, if $\pi \in \boldsymbol{p}$-$RO(G)$, then $\pi((A^\infty)^c) \leq 1 - p$*

The above results confirm in a precise sense the robustness of the correlated equilibrium benchmark when weakening the underlying assumption of common knowledge of rationality to joint $\boldsymbol{p}$-belief of rationality.

3.4 $\boldsymbol{p}$-Rational Expectations

Following Aumann and Dreze [4], we can analyze expected payoffs or *expectations in a game* from the point of view of a fixed player who already gained differential information and has therefore, *interim* beliefs.

Definition 7 ($\boldsymbol{p}$-Rational Expectation) *Let G be a game, $\boldsymbol{p} \in [0,1]^I$, and B a $\boldsymbol{p}$-rational belief system for G. Then, a* $\boldsymbol{p}$-rational expectation *in G is an interim expected payoff of some player. We denote the set of all such $\boldsymbol{p}$-rational expectations of G, by $\boldsymbol{p}$-$RE(G)$.*

It is then easy to characterize:

Theorem 3 *Let G be a game, $X = \prod_{i \in I} (A_i \times \{0\})$, and $\boldsymbol{p} \in [0,1]^I$. Then the $\boldsymbol{p}$-rational expectations in G are the expected payoffs of the $(X, \boldsymbol{p})$-correlated equilibria of the tripled game $3G$, conditional on playing an action in $3A_i$. Moreover, the $\boldsymbol{p}$-rational expectations of players acting rationally at the interim stage are the expected payoffs of the $(X, \boldsymbol{p})$-correlated equilibria of the tripled game $3G$, conditional on playing an action in $A_i \times \{0\}$.*

This provides the joint $\boldsymbol{p}$-belief of rationality counterpart of Aumann and Dreze's characterization in [4].

4 Examples

The following examples illustrate the $\boldsymbol{p}$-rational outcomes for some 2×2 games.

Example 1 (Dominance Solvable Game) Consider the following game G_D, solvable by strict dominance with corresponding augmented game $2G_D$,

$$G_D \equiv \begin{array}{c|c|c|} & L & R \\ \hline T & 2,2 & 1,1 \\ \hline B & 1,1 & 0,0 \\ \hline \end{array}, \qquad 2G_D \equiv \begin{array}{c|c|c|c|c|} & (L,0) & (R,0) & (L,1) & (R,1) \\ \hline (T,0) & 2,2 & 1,1 & 2,2 & 1,1 \\ \hline (B,0) & 1,1 & 0,0 & 1,1 & 0,0 \\ \hline (T,1) & 2,2 & 1,1 & 2,2 & 1,1 \\ \hline (B,1) & 1,1 & 0,0 & 1,1 & 0,0 \\ \hline \end{array}.$$

To compute the $\boldsymbol{p}$-$RO(G_D)$ we compute the $(X,\boldsymbol{p})$-$CE(2G_D)$, where $X = \prod_{i=1,2}(A_i \times \{0\})$. For this notice that the strategies $(B,0)$ and $(T,1)$ of the row player and $(R,0)$ and $(L,1)$ of the column player are strictly dominated, so that the remaining constraints that need to be satisfied are the $\boldsymbol{p}$-belief constraints, and one obtains,

$$\boldsymbol{p}\text{-}RO(G_D) = \left\{ \pi \in \Delta(A) \;\middle|\; \begin{array}{c} \pi_{TL} \geq p_1(\pi_{TL}+\pi_{TR}), \pi_{BL} \geq p_1(\pi_{BL}+\pi_{BR}) \\ \pi_{TL} \geq p_2(\pi_{TL}+\pi_{BL}), \pi_{TR} \geq p_2(\pi_{TR}+\pi_{BR}) \end{array} \right\}.$$

Figures 1 and 2 show the set $\boldsymbol{p}$-$RO(G_D)$ for $\boldsymbol{p} = (0.95, 0.95)$ together with respectively the ϵ-neighborhood of the set of correlated equilibria of G_D, $N_\epsilon(CE(G_D))$, and the set of ϵ-correlated equilibria, ϵ-$CE(G_D)$,[9] both with $\epsilon = 0.20$. Clearly the three sets are all distinct.

Example 2 (Matching Pennies Game) Consider the following version G_{MP} of matching pennies, with corresponding doubled game $2G_{MP}$,

$$G_{MP} \equiv \begin{array}{c|c|c|} & L & R \\ \hline T & 1,0 & 0,1 \\ \hline B & 0,1 & 1,0 \\ \hline \end{array}, \qquad 2G_{MP} \equiv \begin{array}{c|c|c|c|c|} & (L,0) & (R,0) & (L,1) & (R,1) \\ \hline (T,0) & 1,0 & 0,1 & 1,0 & 0,1 \\ \hline (B,0) & 0,1 & 1,0 & 0,1 & 1,0 \\ \hline (T,1) & 1,0 & 0,1 & 1,0 & 0,1 \\ \hline (B,1) & 0,1 & 1,0 & 0,1 & 1,0 \\ \hline \end{array}.$$

The set $\boldsymbol{p}$-$RO(G_{MP})$ is now somewhat more tedious to characterize, nonetheless we know it is a compact, convex polyhedron around $\bar{\pi} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, which converges to $\bar{\pi}$ as $\boldsymbol{p}$ converges to 1. In particular it contains profiles that do not yield the agents their value of the game, but rather something in a neighborhood thereof.

Figures 3 and 4 show the set $\boldsymbol{p}$-$RO(G_{MP})$ for $\boldsymbol{p} = (0.95, 0.95)$ together with the ϵ-neighbourhood of the set of correlated equilibria, $N_\epsilon(CE(G_{MP}))$, and the set of ϵ-correlated equilibria, ϵ-$CE(G_{MP})$, both with $\epsilon = 0.10$, respectively. Again, the sets $\boldsymbol{p}$-$RO(G_{MP})$ and $N_\epsilon(CE(G_{MP}))$ and ϵ-$CE(\Gamma_{MP})$ are visibly distinct.

The following example further illustrates the relationship with the ϵ equilibria.

[9] In general, this is the set of probability distributions $\pi \in \Delta(A)$ that satisfy the incentive constraints for correlated equilibria with a slack of ϵ, analagous to Radner's ϵ-Nash equilibria, formally, π is an *ϵ-correlated equilibrium* (ϵ-CE) if for any $i \in I$,

$$\sum_{a_i \in A_i} \max_{a'_i \in A_i} \sum_{a_{-i} \in A_{-i}} \pi(a_i, a_{-i}) \left(h_i(a'_i, a_{-i}) - h_i(a_i, a_{-i})\right) \leq \epsilon.$$

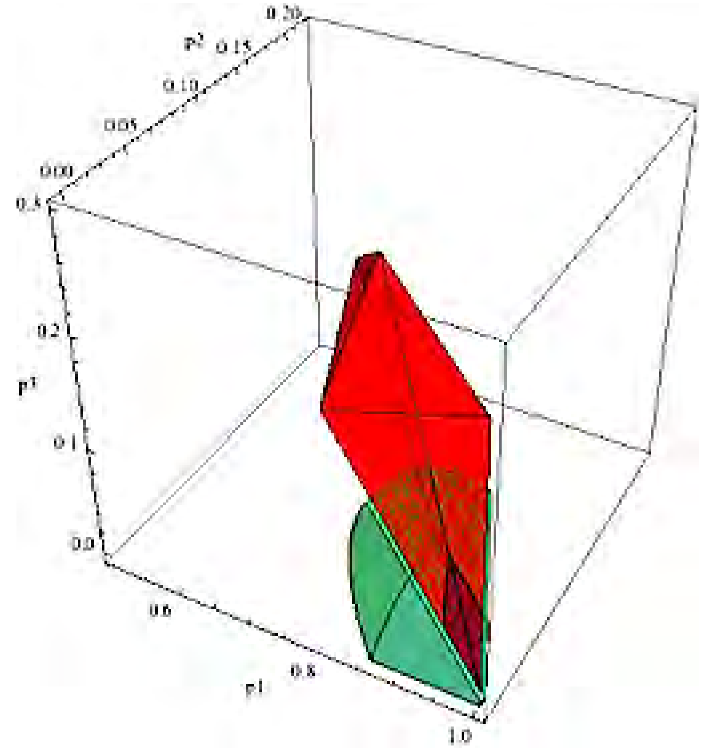

Figure 1: 0.95-$RO(\Gamma_D)$ (blue), 0.80-$RO(\Gamma_D)$ (red), $N_{0.10}CE(\Gamma_D)$ (green)

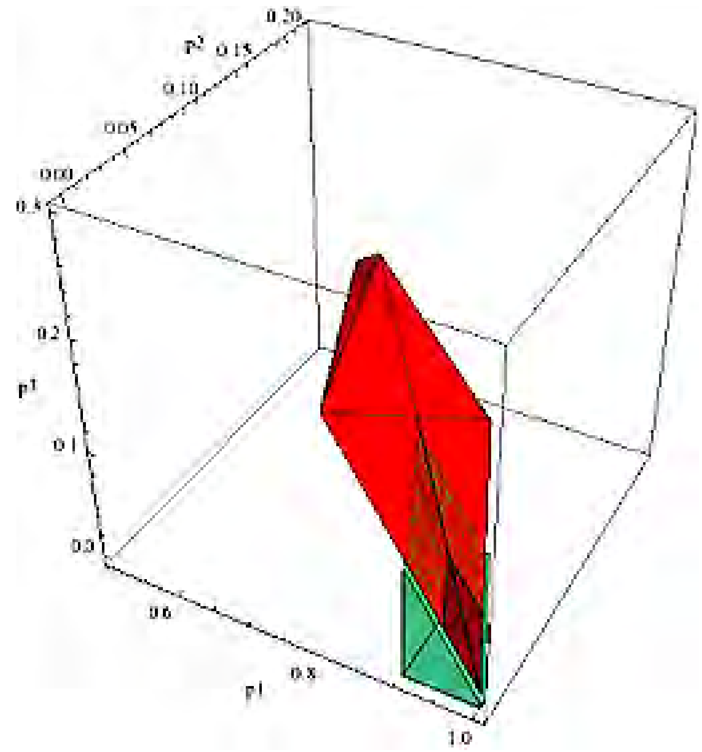

Figure 2: 0.95-$RO(G_D)$ (blue), 0.80-$RO(G_D)$ (red), 0.10-$CE(G_D)$ (green)

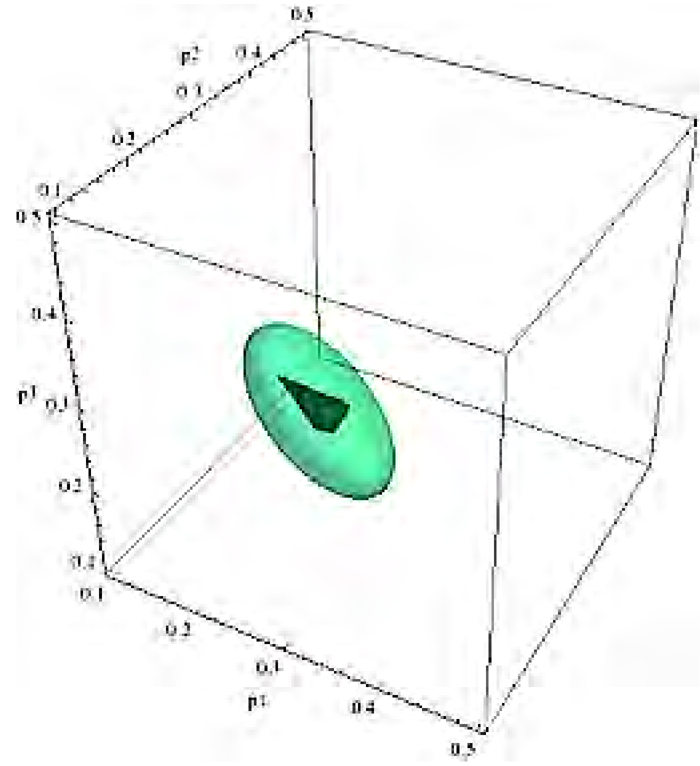

Figure 3: 0.95-$RO(G_{MP})$ (blue), $N_{0.10}CE(G_{MP})$ (green)

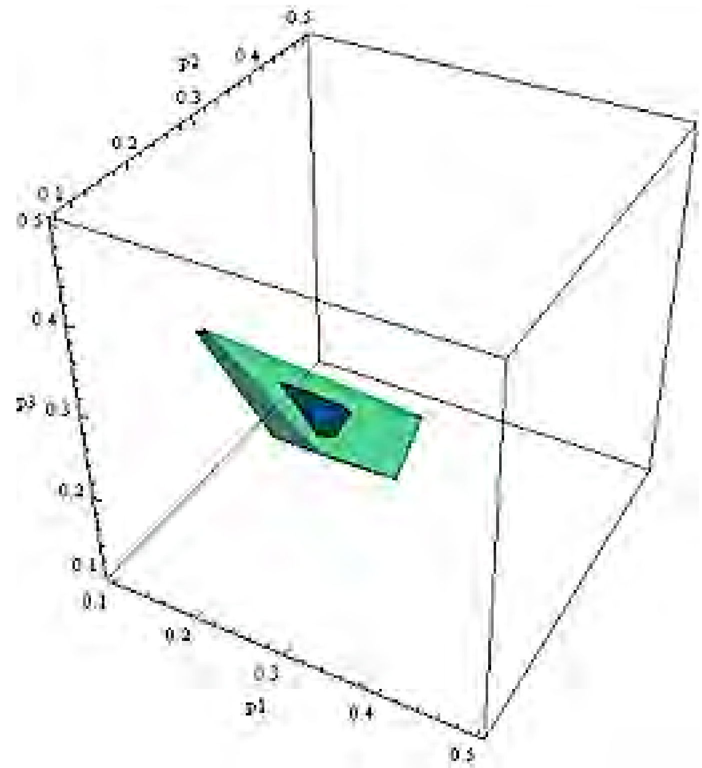

Figure 4: 0.95-$RO(G_{MP})$ (blue), 0.10-$CE(G_{MP})$ (green)

Example 3 (Prisoner's Dilemma Game) Consider the following game G_{PD} with corresponding probability of play π_{PD},

$$G_{PD} \equiv \begin{array}{c|c|c|} & L & R \\ \hline T & 1,1 & 4,0.99 \\ \hline B & 0.99,4 & 3.99,3.99 \\ \hline \end{array}\,, \qquad \pi_{PD} \equiv \begin{array}{c|c|c|} & L & R \\ \hline T & 0 & 0 \\ \hline B & 0 & 1 \\ \hline \end{array}\,.$$

It is clear that π_{PD} constitutes both an ϵ-Nash and an ϵ-correlated equilibrium, for any $\epsilon > 0.01$, yet $\pi_{PD} \in \boldsymbol{p}$-$RO(G_{PD})$ if and only if $\boldsymbol{p} = 0$.

5 Games with Incomplete Information

Next, we extend the characterization of $\boldsymbol{p}$-rational behavior to standard games of incomplete information. Following Forges [11], we define such a game in the usual way by a tuple $G = \langle I, (A_i)_{i\in I}, (T_i)_{i\in I}, f, (h_i)_{i\in I}\rangle$, where I is a finite set of players, and, for each $i \in I$, A_i is a finite set of actions, T_i a finite *set of types* with generic element t_i, and $h_i : \prod_{i\in I} T_i \times A \to \mathbb{R}$ a payoff function; $T = \prod_{i\in I} T_i$ is the set of type profiles. Finally, f is a *common prior*, a distribution on T with marginal full support on each T_i that represents players' *ex ante* beliefs on the set of types, and which induces interim beliefs by conditioning:

$$f_i(t_i)(E) = \frac{f(E \cap (T_{-i} \times \{t_i\}))}{f(T_{-i} \times \{t_i\})}, \text{ for any } i \in I,\ t_i \in T_i \text{ and } E \subseteq T.$$

5.1 Belief Systems on Games with Incomplete Information

In this set-up, a belief system for G must take into consideration players' types in the description of the states of the world. This is achieved by defining $B = \langle \Omega, (\Pi_i)_{i\in I}, (\alpha_i)_{i\in I}, (\tau_i)_{i\in I}, \mu\rangle$, where everything is analogous to the definition of belief systems for games with complete information as in Section 3, except that now, for each $i \in I$, we have $\tau_i : \Omega \to T_i$, a *type map* measurable *w.r.t.* Π_i. For consistency of beliefs, following Forges [10, 11], the following are required:

- $\mu\left(\bigcap_{i\in I} \tau_i^{-1}(t_i)\right) = f(t)$, for any $t \in T$, and,
- $\mu\left(\bigcap_{j\neq i} \tau_j^{-1}(t_j) \mid \tau_i^{-1}(t_i)\right) = \mu\left(\bigcap_{j\neq i} \tau_j^{-1}(t_j) \mid \Pi_i(\omega)\right)$, for any $i \in I$, $t \in T$, and $\omega \in \tau_i^{-1}(t_i)$.

In the present framework, for any $i \in I$, $\omega \in \Omega$, and $a_i \in A_i$, we denote the interim expected payoffs of playing a given action by

$$\mathbb{E}_B[h_i(a_{-i}; a_i) | \Pi_i(\omega)] = \sum_{t_{-i}\in T_{-i}} \sum_{a_{-i}\in A_{-i}} \mu\left(\alpha_{-i}^{-1}(a_{-i}) \cap \tau_{-i}^{-1}(t_{-i}) \mid \Pi_i(\omega)\right) h_i((t_{-i}; \tau_i(\omega)), (a_{-i}; a_i)),$$

so all definitions in Section 3 (except Definition 3) can be easily adapted.

As with the case of games with complete information, we are interested in the study of the distributions over action profiles induced by $\boldsymbol{p}$-rational belief systems for each type profile. These are defined as follows.

Definition 8 ($\boldsymbol{p}$-Rational Bayesian Outcome) *Let G be a game and $\boldsymbol{p} \in [0,1]^I$, then $(\pi_t)_{t\in T} \subseteq \Delta(A)$ is a* $\boldsymbol{p}$-Bayesian rational outcome *of G if there exists a $\boldsymbol{p}$-rational belief system B such that for any $t \in T$,*

$$\pi_t(a) = \mu\left(\bigcap_{i\in I} \alpha_i^{-1}(a_i) \,\middle|\, \bigcap_{i\in I} \tau_i^{-1}(t_i)\right) \text{ for any } a \in A.$$

We denote the set of $\boldsymbol{p}$-rational Bayesian outcomes of G, by $\boldsymbol{p}$-RBO (G).

Notice that if G is a game with incomplete information such that its set of types consists of a single element, then G is indeed a game with complete information. It is straightforward to check that in such a case, all the theory presented above corresponds to the one in Section 2 and in consequence, for $\boldsymbol{p} \in [0,1]^I$, we have $\boldsymbol{p}$-*RBO* $(G) = \boldsymbol{p}$-*RO* (G) (see the proof of Theorem 1, in the Appendix for more details). Additionally, in the case that $\boldsymbol{p} = \mathbf{1}$, the set of interim expected payoffs induced by some $\mathbf{1}$-rational belief system are precisely the belief invariant Bayesian solutions of G, as defined in [10].

5.2 Characterization of the Set of $\boldsymbol{p}$-Rational Bayesian Outcomes

The following is parallel to the definition in the case of complete information.

Definition 9 ($(\boldsymbol{X},\boldsymbol{p})$-Bayesian Correlated Equilibrium) *Let G be a game with incomplete information, $X = \prod_{i\in I} X_i \subseteq A$, and $\boldsymbol{p} \in [0,1]^I$. Then, $(\pi(t))_{t\in T} \subseteq \Delta(A)$ is a* $(X,\boldsymbol{p})$-Bayesian correlated equilibrium *of G, if and only if for any $i \in I$:*

- *For any $t_i \in T_i$ and any $a_i' \in X_i$, the following* incentive constraints *are satisfied,*

$$\sum_{t_{-i}\in T_{-i}} \sum_{a_{-i}\in A_{-i}} f_i(t_i)(t_{-i})\, \pi_{(t_{-i};t_i)}(a_{-i};a_i')\left[h_i((t_{-i};t_i),(a_{-i};a_i')) - h_i((t_{-i};t_i),(a_{-i};a_i))\right] \geq 0,$$

  for any $a_i \in A_i$.

- *For any $t_i \in T_i$ and any $a_i \in A_i$, the following* p_i-belief constraint *is satisfied,*

$$\sum_{t_{-i}\in T_{-i}} f_i(t_i)(t_{-i})\, \pi_{(t_{-i};t_i)}(X_{-i}\times\{a_i\}) \geq p_i \sum_{t_{-i}\in T_{-i}} f_i(t_i)(t_{-i})\, \pi_{(t_{-i};t_i)}(A_{-i}\times\{a_i\}).$$

We denote by $(X,\boldsymbol{p})$-BCE (G) the set of $(X,\boldsymbol{p})$-Bayesian correlated equilibrium of G.

Again, when G is a game of incomplete information with degenerate or single element type sets, it can easily be checked that for any $\boldsymbol{p} \in [0,1]^I$, we have $\boldsymbol{p}$-*BCE* $(G) = \boldsymbol{p}$-*CE* (G) (see the proof of Theorem 1, in the Appendix). It is also easy to check that in the case $X = A$, or $\boldsymbol{p} = \mathbf{1}$, the set of expected payoffs conditional on playing a certain action of the $(X, \boldsymbol{p})$-Bayesian correlated equilibria coincides with the belief invariant Bayesian solutions of G.

The generalization of the n-Game concept to games with incomplete information is straightforward. Recalling that $n \in \mathbb{N}$, $N = \{0, 1, \ldots, n-1\}$, we can state the following.

Definition 10 ($\boldsymbol{n}$-Game for Games with Incomplete Information) *Let G be a game with incomplete information and $n \in \mathbb{N}$. The* n-game *of G is a tuple $nG = \left\langle I, (nA_i)_{i\in I}, (T_i)_{i\in I}, f, (h_{n,i})_{i\in I} \right\rangle$ where for any $i \in I$:*

- *$nA_i = A_i \times N$ is player i's set of pure actions; we denote a generic element of $nA = \prod_{i\in I} nA_i$ by (a, ν), where $\nu \in N^I$ specifies which copy of A_i in nA_i each player i's pure action belongs to.*
- *$h_{n,i}(t, (a, \nu)) = h_i(t, a)$, for any $t \in T$, and any $(a, \nu) \in nA$.*

That is, we enlargen the underlying incomplete information game, analogous to Section 2, but without modifying the belief structure. Any element $(\hat{\pi}_t)_{t\in T} \in (\Delta(nA))^T$ induces some $(\pi_t)_{t\in T} \in (\Delta(A))^T$ by taking marginals on A, i.e., $\pi_t = \mathrm{marg}_A \hat{\pi}_t$ for any $t \in T$. For any subset $Y \subseteq (\Delta(A))^T$ we denote:

$$\mathbf{marg}_A[Y] = \left\{ (\mathrm{marg}_A \hat{\pi}_t)_{t\in T} \text{ where } (\hat{\pi}_t)_{t\in T} \in Y \right\}.$$

The following characterization directly extends the one under complete information of Theorem 1.

Theorem 4 *Let $G = \left\langle I, (A_i)_{i\in I}, (T_i)_{i\in I}, f, (h_i)_{i\in I} \right\rangle$ be a game with incomplete information and $\boldsymbol{p} \in [0,1]^I$. Then the $\boldsymbol{p}$-rational Bayesian outcomes of G are the marginals on A of the $(X, \boldsymbol{p})$-Bayesian correlated equilibria of the doubled game $2G$, where $X = \prod_{i\in I}(A_i \times \{0\})$ is a copy of the original action space of G. Formally,*

$$\boldsymbol{p}\text{-}RBO\,(G) = \mathbf{marg}_A\left[(X, \boldsymbol{p})\text{-}BCE\,(2G)\right],$$

where $X = \prod_{i\in I}(A_i \times \{0\})$.

As with the complete information counterpart, we can write the set $\boldsymbol{p}$-$RBO(G)$ in terms of $(X, \boldsymbol{p})$-Bayesian correlated equilibria of the game $2G$, carrying over the robustness of the correlated equilibrium concept to the present incomplete information case.

6 Some Remarks

We conclude with a few remarks.

Remark 1 ($\boldsymbol{P}$-Rational Outcomes and Expectations) An important objective of the paper was to put as few restrictions on non-rational behavior as possible, so as to cover all sorts of departures from rationality.

However, throughout the paper we implicitly assumed – as part of the notion of $J\boldsymbol{p}BR$ – that players always believe the other players are rational with probability p_i or more; whether or not they are rational at the given state. Indirectly, this restricts behavior of non-rational types. An alternative benchmark – in line with our motivation – is to drop any restriction on the non-rational types and allow them to have any kind of beliefs about others. This can be formalized by assuming a larger vector $\boldsymbol{P} = (\boldsymbol{p}, \boldsymbol{0}) \in [0,1]^{2I}$, where the components associated to states in which the agents are rational give the usual probabilities $\boldsymbol{p}$, while the components associated to the non-rational states are all zero. This leads to $\boldsymbol{P}$*-rational outcomes* of G. These are again marginals of $(A \times \{0\}, \boldsymbol{P})$-correlated equilibria of $2G$, in that they are distributions satisfying the same conditions as the $\left(\prod_{i\in I}(A_i \times \{0\}), \boldsymbol{p}\right)$-$CE(2G)$ except that the $\boldsymbol{p}$-belief constraints for the non-rational types are dropped. In other words, $\left(\prod_{i\in I}(A_i \times \{0\}), \boldsymbol{p}\right)$-$CE(2G) \subset \left(\prod_{i\in I}(A_i \times \{0\}), \boldsymbol{P}\right)$-$CE(2G)$ and therefore also $\boldsymbol{p}$-$RO(G) \subset \boldsymbol{P}$-$RO(G)$.

While the correspondence $\boldsymbol{P}$-$RO(G)$ maintains the basic topological properties of the correspondence $\boldsymbol{p}$-$RO(G)$, it need not converge to the set of correlated equilibria of G as $\boldsymbol{P} \to (\boldsymbol{1}, \boldsymbol{0})$, but does so if one also requires $\boldsymbol{P} \to (\boldsymbol{1}, \boldsymbol{1})$. This can be seen already in Example 1. A $(\boldsymbol{1}, \boldsymbol{0})$-rational belief system can be very far from a $(\boldsymbol{1}, \boldsymbol{1})$-rational belief system, in that the former need not put any restriction on the total mass of states where all players are rational $\mu(R)$.[10]

The alternative notion of approximate knowledge of rationality requiring $\mu(CB^{\boldsymbol{p}}(R)) > 1 - \epsilon$, for $\epsilon > 0$, (instead of $J_{\boldsymbol{p}}BR$), is more flexible with respect to the players' beliefs in that it only restricts the total mass of common $\boldsymbol{p}$-belief and hence does not specify directly what beliefs individual players and types have. A characterization of $\boldsymbol{p}$-rational outcomes with this definition is possible along the lines of our Theorem 1, but involves more complicated incentive and $\boldsymbol{p}$-belief constraints to be imposed over all possible subsets and permutations of players.

Remark 2 (Non-Common Priors) Throughout the paper we assumed the existence of a common prior (CP). This together with the notion of joint $\boldsymbol{p}$-belief of rationality allowed us to derive relatively stringent restrictions on behavior. Clearly, it is very natural to analyze what happens when the common prior assumption is relaxed. As it turns out, under *subjective* or *non-common* priors, joint $\boldsymbol{p}$-belief of rationality puts *no* restrictions on possible behavior – even when $\boldsymbol{p} = \boldsymbol{1}$. This provides a stark contrast with the cases of common knowledge of rationality and also common p-belief of rationality, as studied respectively in

[10]To see this, let $\boldsymbol{P} = (\boldsymbol{p}, \boldsymbol{q}) \in [0,1]^{2I}$, where $\boldsymbol{p}$, $\boldsymbol{q}$ are the probabilities for the rational and non-rational types respectively. To see that in a $(\boldsymbol{1}, \boldsymbol{0})$-rational belief system the total mass of states where are non-rational is unrestricted, take the game in Example 1 and consider the belief system $B = \left\langle \Omega, (\Pi_i)_{i\in I}, (\alpha_i)_{i\in I}, \mu \right\rangle$, where $\Omega = A$, $s_i(a_{-i}; a_i) = a_i$, for all $(a_{-i}; a_i) \in A_i$, $i \in I$, and where $\mu \in \Delta(A)$ is given by $\mu_{TL} = \mu_{TR} = \mu_{BL} = 0$ and $\mu_{BR} = 1$. It can be checked that it is $(\boldsymbol{1}, \boldsymbol{0})$-rational and clearly $\mu(R) = 0$. At the same time, in a $(\boldsymbol{p}, \boldsymbol{q})$-rational belief system it is always the case that, for any $i \in I, \omega \in \Omega$, $\mu\left(R_{-i} \mid \Pi_i(\omega)\right) \geq q_i$, hence

$$\mu(R_{-i} \cap \Pi_i(\omega)) \geq q_i \mu(\Pi_i(\omega)) \Longrightarrow \sum_{\Pi_i(\omega)\in\Pi_i} \mu(R_{-i} \cap \Pi_i(\omega)) = q_i \sum_{\Pi_i(\omega)\in\Pi_i} \mu(\Pi_i(\omega)) \Longrightarrow \mu(R_{-i}) \geq q_i,$$

which besides confirming the expected convergence to the correlated equilibria as $(\boldsymbol{p}, \boldsymbol{q}) \to (\boldsymbol{1}, \boldsymbol{1})$, also shows that positive q_i's do put restrictions on the total mass of states where agents are rational $\mu(R)$.

[2, 5, 7, 18, 20] and [6, 15], and in a sense further highlights the stringency of the common prior assumption.[11]

To see the non-common prior case, define belief system $B = \left\langle \Omega, (\Pi_i)_{i\in I}, (\alpha_i)_{i\in I}, (\mu_i)_{i\in I} \right\rangle$ to be *subjectively* $\boldsymbol{p}$*-rational* if $\Omega = \bigcap_{i\in I} B_i^{p_i}(R_{-i})$. Given a finite game in strategic form G with set of players I and set of action profiles A, and given $\boldsymbol{p} \in [0,1]^I$, we say that a family of distributions $(\pi_i)_{i\in I} \in (\Delta(A))^I$ is a $\boldsymbol{p}$*-subjectively rational outcome* of G ($\boldsymbol{p}$-$SRO(G)$) if there exists some subjectively $\boldsymbol{p}$-rational belief system $B = B = \left\langle \Omega, (\Pi_i)_{i\in I}, (\alpha_i)_{i\in I}, (\mu_i)_{i\in I} \right\rangle$ for G such that for any $i \in I$, we have $\pi_i = \mu_i \circ \alpha^{-1}$. As shown in the Appendix, it is easy to see that, for any $\boldsymbol{p} \in [0,1]^I$, the whole space is obtained, namely,

$$\boldsymbol{p}\text{-}SRO(G) = (\Delta(A))^I .$$

In particular, an agent that is certain that all the other agents are rational, given his priors, may still select a non-rational action given. Hence, *any* pure strategy profile in A is consistent with subjective $\boldsymbol{p}$-rationality, even when $\boldsymbol{p} = \mathbf{1}$.

Remark 3 (Comparison with Further Solution Concepts) The sets of $\boldsymbol{p}$-rational outcomes define sets of probability distributions of play that are broader than the correlated equilibria that follow their own logic. As the examples show, they are distinct from ϵ-neighborhoods of the correlated equilibria, thus putting further structure on the types of deviations from the set $CE(G)$ that occur as $\boldsymbol{p}$ departs from $\mathbf{1}$. At the same time, they are distinct from the ϵ-correlated equilibria, reflecting the fact that they impose no constraints on the *type* of departure from rationality assumed – unlike with the ϵ-correlated equilibria, which assume the agents are ϵ-optimizers. A similar remark applies to the quantal response equilibria of McKelvey and Palfrey [16] or other models such as the level-k reasoning models (e.g, [8]) that put specific restrictions on how players can deviate from rationality. Also related are the rationalizable and the $\boldsymbol{p}$-rationalizable strategy profiles (see respectively [5, 18] and [15]), which are derived at the interim stage and without appealing to priors. Unlike the $\boldsymbol{p}$-rational outcomes, whose set of distributions is fully supported on A, whenever $\boldsymbol{p} < \mathbf{1}$, both the rationalizable and the $\boldsymbol{p}$-rationalizable profiles may be strict subsets of A. It remains an empirical question to what extent the $\boldsymbol{p}$-rational outcomes bound observed behavior in a robust and useful manner.

Remark 4 (Learning to Play $\boldsymbol{p}$-Rational Outcomes) Clearly, all learning dynamics that lead to correlated equilibria (see e.g., [14]) will also lead to $\boldsymbol{p}$-rational outcomes. The question arises as to what further dynamics (not necessarily converging to correlated equilibria) may converge to $\boldsymbol{p}$-rational outcomes and whether they include interesting dynamics that for example allow for faster or more robust convergence.

[11] Recall that the result of part (a) of Lemma 1 also holds with non-common priors.

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APPENDIX

A Proof of Lemma 1

(a) By definition, $CB^{\boldsymbol{p}}(R) \subseteq \bigcap_{i\in I} B_i^{p_i}(R)$, and therefore, for any $i \in I$, $CB^{\boldsymbol{p}}(R) \subseteq B_i^{p_i}(R_i)$. But

$$\omega \in B_i^{p_i}(R_i) \iff \mu_i(R_i \mid \Pi_i(\omega)) = \frac{\mu_i(R_i \cap \Pi_i(\omega))}{\mu_i(\Pi_i(\omega))} \geq p_i.$$

So, as $p_i > 0$, $R_i \cap \Pi_i(\omega) \neq \emptyset$, and therefore, $\Pi_i(\omega) \subseteq R_i$ and in particular, $\omega \in R_i$.

Thus, $B_i^{p_i}(R_i) \subseteq R_i$.

(b) First, if $\bigcap_{i\in I} B_i^p(R_{-i}) = \Omega$, then

$$R = \Omega \cap R = \bigcap_{i\in I}(B_i^p(R_{-i}) \cap R_i) = \bigcap_{i\in I} B_i^{p_i}(R) \subseteq CB^p(R),$$

and therefore, $R \subseteq CB^p(R)$. Now, again, since $\bigcap_{i\in I} B_i^p(R_{-i}) = \Omega$, we have both

$$\mu(R_{-i}) \geq p, \text{ and } \mu(R_{-i} \cap R_i) \geq p\mu(R_i).$$

The fact that, for any $j \neq i$, $\mu(R) = f(R_{-i}|R_i)\,\mu(R_i) \geq p\mu(R_i) \geq p\mu(R_{-j}) \geq p^2$ completes the proof.

(c) It is easy to check that

$$CB^{\mathbf{1}}(R) = \Omega \iff \bigcap_{i\in I} B_i^{\mathbf{1}}(R) = \Omega,$$

so it suffices to prove that

$$\bigcap_{i\in I} B_i^{\mathbf{1}}(R) = \Omega \iff \bigcap_{i\in I} B_i^{\mathbf{1}}(R_{-i}) = \Omega.$$

The right implication is immediate. For the proof of the left one, from part (a) of the lemma, it is enough to check that

$$\bigcap_{i\in I} B_i^{\mathbf{1}}\left(R_{-i}\right) = \Omega \implies R = \Omega.$$

But this is immediate. For, let $i, j \in I$, $i \neq j$, then $B_j^{p_j}\left(R_{-j}\right) \subseteq B_j^{p_j}\left(R_i\right)$, and therefore, $B_j^{p_j}\left(R_i\right) = \Omega$. Hence

$$\mu\left(R_i\right) = \sum_{\omega\in\Omega} \mu\left(R_i \cap \Pi_j\left(\omega\right)\right) = \sum_{\omega\in\Omega} \mu\left(\Pi_j\left(\omega\right)\right) = 1.$$

Since μ has full support on Ω, the latter implies that $R_i = \Omega$. As the proof applies for any $i \in I$, we can infer $R = \Omega$.

B Proofs of the Characterization Results

In this section we prove Theorems 1, 3, and 4. We first show a lemma that is used in the proof of each of the three theorems. Next, we prove Theorem 4, then Theorem 1 as a special case of Theorem 4, and finally Theorem 3. The auxiliary lemma is as follows:

Lemma 2 *Let G a game with incomplete information, $\boldsymbol{p} \in [0,1]^I$, $n \in \mathbb{N}$, and $\left(\hat{\pi}_t\right)_{t\in T} \in (X, \boldsymbol{p})$-$BCE\,(nG)$, where $X = \prod_{i\in I}\left(A_i \times \{k\}\right)$ with $k \leq n$. Additionally, let $B = \left\langle \Omega, \left(\Pi_i\right)_{i\in I}, \left(\alpha_i\right)_{i\in I}, \left(\tau_i\right)_{i\in I}, \mu\right\rangle$, where,*

- $\Omega = \left\{(a,\nu,t) \in A \times N^I \times T \text{ such that } t \in \operatorname{supp} f \text{ and } (a,\nu) \in \operatorname{supp} \hat{\pi}_t\right\}$,

and, for any $(a,\nu,t) \in \Omega$,

- $\Pi_i\left(a,\nu,t\right) = \left(A_{-i} \times \{a_i\}\right) \times \left(N^{I\setminus\{i\}} \times \{\nu_i\}\right) \times \left(T_{-i} \times \{t_i\}\right)$, $i \in I$,
- $\alpha_i\left(a,\nu,t\right) = a_i$, $i \in I$,
- $\tau_i\left(a,\nu,t\right) = t_i$, $i \in I$, *and,*
- $\mu\left(a,\nu,t\right) = f\left(t\right)\hat{\pi}_t\left(a,\nu\right)$.

Then, B is a $\boldsymbol{p}$-rational belief system for G such that the family of distributions induced on A is $\left(\operatorname{marg}_A \hat{\pi}_t\right)_{t\in T}$.

Proof. First, it is immediate, when Ω is finite and μ has full support on Ω, that, for any $i \in I$: (i) Π_i is a partition of Ω, and (ii) both α_i and τ_i are measurable w.r.t. Π_i. B is therefore a well defined belief system for G which by construction, satisfies the common prior assumption. Since $\left(\hat{\pi}_t\right)_{t\in T}$ is a $(X, \boldsymbol{p})$-Bayesian correlated equilibrium of nG, we have:

- From the incentive constraints, we know that for any $j \in I$, any $(a,\nu,t) \in \Omega$ such that $\nu_j = k$, $(a,\nu,t) \in R_j$, so that, for any $i \in I$ and any $(a,\nu,t) \in \Omega$,

$$\mu\left(R_{-i} \cap \Pi_i\left(a,\nu,t\right)\right) \geq \mu\left(\Omega \cap \left(A_{-i} \times \{\boldsymbol{k}_{-i}\} \times T_{-i} \times \{(a_i,\nu_i,t_i)\}\right)\right).$$

- From the $\boldsymbol{p}$-belief constraints we know that for any $i \in I$, any $((a_{-i};\nu_{-i}),(a_i;\nu_i)) \in nA$ and any $(t_{-i};t_i) \in T$:

$$\sum_{t'_{-i}\in T_{-i}} f_i\left(t'_{-i};t_i\right)\hat{\pi}_{\left(t'_{-i};t_i\right)}\left(\left(A_{-i}\times\{\boldsymbol{k}_{-i}\}\right)\times\{(a,\nu_i)\}\right)\geq$$

$$\geq p_i\sum_{t'_{-i}\in T_{-i}} f_i\left(t'_{-i};t_i\right)\hat{\pi}_{\left(t'_{-i};t_i\right)}\left(nA_{-i}\times\{(a_i,\nu_i)\}\right),$$

and so,

$$\mu\left(\Omega\cap\left(A_{-i}\times\{\boldsymbol{k}_{-i}\}\times T_{-i}\times\{(a_i,\nu_i,t_i)\}\right)\right)\geq p_i\mu\left(\Pi_i\left((a_{-i};a_i),(\nu_{-i},\nu_i),(t_{-i};t_i)\right)\right).$$

Thus, we obtain that, for any $(a,\nu,t)\in\Omega$, $\mu\left(R_{-i}\cap\Pi_i(a,\nu,t)\right)\geq p_i\mu\left(\Pi_i(a,\nu,t)\right)$, and therefore that B is $\boldsymbol{p}$-rational.

Finally, it is straightforward to check that, for any $t\in T$ and any $a\in A$,

$$\mu\left(\bigcap_{i\in I}\alpha_i^{-1}(a_i)\,\middle|\,\bigcap_{i\in I}\tau_i^{-1}(t_i)\right)=\mathrm{marg}_A\hat{\pi}_t(a)\;.$$

This ends the proof of the lemma. ■

B.1 Proof of Theorem 4

$\subseteq$

Let $B=\left\langle\Omega,(\Pi_i)_{i\in I},(\alpha_i)_{i\in I},(\tau_i)_{i\in I},\mu\right\rangle$ a $\boldsymbol{p}$-rational belief system for G. For each player $i\in I$ and each $t_i\in T_i$ we can define

$$\begin{array}{rccc}\beta_{t_i}: & \tau_i^{-1}(t_i) & \longrightarrow & 2A_i\\ & \omega\in R_i & \to & (\alpha_i(\omega),0)\\ & \omega\notin R_i & \to & (\alpha_i(\omega),1)\end{array}$$

Now, for any $t\in\mathrm{supp}f$, let $\hat{\pi}_t(a,\nu)=\mu\left(\bigcap_{i\in I}\beta_{t_i}^{-1}(a_i,\nu_i)\mid\bigcap_{i\in I}\tau_i^{-1}(t_i)\right)$ for any $(a,\nu)\in 2A$. For $t\notin\mathrm{supp}f$ any distribution satisfies what is required. Hence we have,

- For $t_i\in T_i$, $(a_i,0)\in 2A_i$ such that $\beta_{t_i}^{-1}(a_i,0)\neq\emptyset$, and $(\bar{a}_i,\nu_i)\in 2A_i$ we have,[12]

$$\sum_{t_{-i}\in T_{-i}}\sum_{(a_{-i},\nu_i)\in 2A_{-i}} f_i(t_i)(t_{-i})\,\pi_{(t_{-i};t_i)}\left((a_{-i},\nu_{-i});(a_i,0)\right)h_{2,i}\left((t_{-i}:t_i),((a_{-i},\nu_{-i});(\bar{a},\nu_i))\right)=$$

$$=\sum_{\omega\in\beta_{t_i}^{-1}(a_i,0)}\mu\left(\omega\,|\,\tau_i^{-1}(t_i)\right)\mathbb{E}_B\left[h_i(a_{-i};\bar{a}_i)\,|\Pi_i(\omega)\right].$$

[12]There is a slight abuse of notation in that we are considering $\hat{\pi}_t$ which are not defined if $t\notin\mathrm{supp}f$. But this is automatically resolved since in such a case, $f_i(t_i)(t_{-i})=0$.

Since by construction, $\beta_{t_i}^{-1}(a_i, 0) \subseteq R_i$, $(a_i, 0)$ is maximizer of the above.

- For $(a_i, \nu_i) \in 2A_i$ and $t_i \in T_i$ we have,

$$\begin{aligned}
&\sum_{t_{-i} \in T_{-i}} f_i(t_i)(t_{-i})\, \hat{\pi}_{(t_{-i};t_i)} \left(\prod_{j \neq i} (A_j \times \{0\}) \times \{(a_i, \nu_i)\} \right) \\
&\quad \geq \sum_{t_{-i} \in T_{-i}} f_i(t_i)(t_{-i})\, \mu \left(R_{-i} \cap \beta_{t_i}^{-1}(a_i, \nu_i) \,\middle|\, \bigcap_{i \in I} \tau_i^{-1}(t_i) \right) \\
&\quad \geq p_i \sum_{t_{-i} \in T_{-i}} f_i(t_i)\, \mu \left(\beta_{t_i}^{-1}(a_i, \nu_i) \,\middle|\, \bigcap_{i \in I} \tau_i^{-1}(t_i) \right) \\
&\quad = p_i \sum_{t_{-i} \in T_{-i}} f_i(t_i)(t_{-i})\, \hat{\pi}_{(t_{-i};t_i)} \left(2A_{-i} \times \{(a_i, \nu_i)\} \right).
\end{aligned}$$

Therefore, $(\hat{\pi}_t)_{t \in T} \in (X, \boldsymbol{p})\text{-}BCE(G)$ for $X = \prod_{i \in I}(A_i \times \{0\})$. Finally, note that for any $t \in \mathrm{supp} f$ we have, for any $a \in A$,

$$\mathrm{marg}_A \hat{\pi}_t(a) = \mu \left(\bigcap_{i \in I} \alpha_i^{-1}(a_i) \,\middle|\, \bigcap_{i \in I} \tau_i^{-1}(t_i) \right).$$

$\supseteq$

Just apply Lemma 2 to $n = 2$ and $k = 0$.

B.2 Proof of Theorem 1

This theorem can be seen as a corollary of Theorem 4. To see this, note that if G is a game with complete information, we can define a game with incomplete information $G' = \langle I, (A_i)_{i \in I}, (T_i)_{i \in I}, f, (h'_i)_{i \in I} \rangle$, where (i) I is the same set of players of game G, and for each $i \in I$ (ii) A_i is the same set of actions as in game G, (iii) $T_i = \{t_i\}$, and (iv) $h'_i(t, a) = h_i(a)$, where h_i is i's payoff function in G, for any $a \in A$. Obviously, it follows that $f = 1_{\{t\}}$, where t is the only element in T. It is immediate that, for any $X = \prod_{i \in I} X_i \subseteq A$, and any $\boldsymbol{p} \in [0, 1]^I$,

$$(X, \boldsymbol{p})\text{-}CE(G) = (X, \boldsymbol{p})\text{-}BCE(G').$$

But note also, that for any tuple $B' = \langle \Omega, (\Pi_i)_{i \in I}, (\alpha_i)_{i \in I}, (\tau_i)_{i \in I}, \mu \rangle$, which is a candidate to be a belief system for B', for any $i \in I$, forcefully $\tau_i = 1_{\{t_i\}}$, and therefore, B' is a belief system for G' if and only if $B = \langle \Omega, (\Pi_i)_{i \in I}, (\alpha_i)_{i \in I}, \mu \rangle$ is a belief system for G. Thus, it is immediate that, for any $\boldsymbol{p} \in [0, 1]^I$,

$$\boldsymbol{p}\text{-}RO(G) = \boldsymbol{p}\text{-}RBO(G').$$

So, let G be a game, and $\boldsymbol{p} \in [0, 1]^I$. Then, we just checked above that both $\boldsymbol{p}\text{-}RO(G) = \boldsymbol{p}\text{-}RBO(G')$ and

$(X,\boldsymbol{p})\text{-}CE\,(2G) = (X,\boldsymbol{p})\text{-}BCE\,(2G')$, where $X = \prod_{i\in I}(A_i \times \{0\})$, hold, so as a consequence of Theorem 4,

$$\boldsymbol{p}\text{-}RO\,(G) = (X,\boldsymbol{p})\text{-}CE\,(2G)\ ,$$

where $X = \prod_{i\in I}(A_i \times \{0\})$.

B.3 Proof of Theorem 3

We prove the first statement; the next one concerning the $\boldsymbol{p}$-rational expectations of rational types then follows directly. We suppose we are taking some player i_0's expectation.

$\subseteq$

Let $B = \left\langle \Omega, (\Pi_i)_{i\in I}, (\alpha_i)_{i\in I}, \mu \right\rangle$ be a $\boldsymbol{p}$-rational belief system for G, $i_0 \in I$, and $\omega_0 \in \Omega$. For any $i \in I \setminus \{i_0\}$, we define,

$$\begin{array}{cccc} \beta_i : & \Omega & \longrightarrow & 3A_i \\ & \omega \in R_i & \rightarrow & (\alpha_i(\omega), 0) \\ & \omega \notin R_i & \rightarrow & (\alpha_i(\omega), 1) \end{array}$$

and for i_0 we define,

$$\begin{array}{cccc} \beta_{i_0} : & \Omega & \longrightarrow & 3A_{i_0} \\ & \omega \in R_{i_0} \setminus \Pi_{i_0}(\omega_0) & \rightarrow & (\alpha_{i_0}(\omega), 0) \\ & \omega \notin R_{i_0} \cup \Pi_{i_0}(\omega_0) & \rightarrow & (\alpha_{i_0}(\omega), 1) \\ & \omega \in \Pi_{i_0}(\omega_0) & \rightarrow & (\alpha_{i_0}(\omega), 2) \end{array}$$

By an argument similar to the one in the first part of the proof of Theorem 4, reduced to the degenerate case where $|T| = 1$, we can conclude that

$$\hat{\pi}(\,\cdot\,) = \mu\left(\bigcap_{i\in I} \beta_i^{-1}(\,\cdot\,)\right)$$

is a $(X,\boldsymbol{p})$-correlated equilibrium of G, where $X = \prod_{i\in I}(A_i \times \{0\})$; moreover, it is immediate that player i_0's expectation conditional on playing $(\alpha_{i_0}(\omega), 2)$ induced by $\hat{\pi}$ is exactly $\mathbb{E}_B\left[h_{i_0}(a_{-i_0}; \alpha_{i_0}(\omega))\,|\Pi_{i_0}(\omega)\right]$.

$\supseteq$

It is again a reduction of Lemma 2, this time to the case of $n = 3$, $k = 0$, and $|T| = 1$.

C Proof of Theorem 2

Nonemptiness follows from the fact that correlated equilibria always exist for any finite game G and constitute $\boldsymbol{p}$-rational outcomes for any $\boldsymbol{p} \in [0,1]^I$. Given that the set of $\boldsymbol{p}$-rational outcomes is a projection (under

$\mathbf{marg}_A$) of the $(X, \boldsymbol{p})$-correlated equilibria of $2G$, with $X = A \times \{k\}$ a copy of the action space of the original game G, the remaining properties follow once they have been shown for the $(X, \boldsymbol{p})$-correlated equilibria of $2G$. This is what we do next. For the given game G, define the $(X, \boldsymbol{p})$-correlated equilibrium correspondence, where $X = A \times \{k\}$, $k \in \{0, 1\}$, is fixed:

$$\begin{array}{rccc} \rho: & [0,1]^I & \longrightarrow & \Delta(2A) \\ & \boldsymbol{p} & \rightarrow & (X, \boldsymbol{p})\text{-}CE\,(2G) \ . \end{array}$$

Clearly ρ is convex- and compact-valued; it remains to show that it is also continuous. We do this by showing that it is upper- and lower-hemicontinuous (respectively, *uhc* and *lhc*) as a correspondence of $\boldsymbol{p}$.

uhc

Since $2A$ is finite, $\Delta(2A)$ is compact, and hence upper-hemicontinuity is equivalent to showing that ρ has a closed graph. But this is immediate from inspection of the inequalities defining the sets $(X, \boldsymbol{p})$-$CE\,(2G)$. In particular, the inequalities are all weak inequalities, linear in $\boldsymbol{p}$. Moreover, the domain $[0,1]^I$ is compact.

lhc

Denote by $\Gamma_\rho \subset [0,1]^I \times \Delta(2A)$ the graph of the correspondence ρ. Fix $(\boldsymbol{p}, \hat{\pi}) \in \Gamma_\rho$ and let $(\boldsymbol{p}^n)_n \subset [0,1]^I$ be a sequence converging to $\boldsymbol{p}$. We need to show that there exists a sequence $((\hat{\pi})^n)_n$ converging to $\hat{\pi}$ such that $(\hat{\pi})^n \in \rho(\boldsymbol{p}^n)$ for sufficiently large n. Take the point $(\boldsymbol{p}, \hat{\pi})$. Clearly this satisfies all inequalities defining $\rho(\boldsymbol{p})$, in particular also the $\boldsymbol{p}$-rationality constraints. Consider the following sequence $(\boldsymbol{p}^n, \hat{\pi})_n \subset [0,1]^I \times \Delta(2A)$. If for sufficiently large n the elements are contained in Γ_ρ we are done. So consider the case where they are not. Consider the family of projections $\Pi_\rho : [0,1]^I \times \Delta(2A) \longrightarrow [0,1]^I \times \Delta(2A)$ that map, for fixed $\bar{\boldsymbol{p}} \in [0,1]^I$, any element $(\bar{\boldsymbol{p}}, \bar{\pi}) \in [0,1]^I \times \Delta(2A)$ to the point in the set $\{\bar{\boldsymbol{p}}\} \times \rho(\bar{\boldsymbol{p}})$ that is closest to $(\bar{\boldsymbol{p}}, \bar{\pi})$. Since the sets $\rho(\cdot)$ are always nonempty, convex, compact polyhedra, we have that $\Pi_\rho(\boldsymbol{p}^n, \hat{\pi})$ is uniquely defined and moreover, $\Pi_\rho(\boldsymbol{p}^n, \hat{\pi}) \in \Gamma_\rho$ for all points in the sequence $(\boldsymbol{p}^n, \hat{\pi})_n$. It remains to show that the sequence $(\Pi_\rho(\boldsymbol{p}^n, \hat{\pi}))_n$ converges to the point $(\boldsymbol{p}, \hat{\pi})$.

Apart from the $\boldsymbol{p}$-belief constraints all other constraints defining $\rho(\boldsymbol{p})$ are independent of $\boldsymbol{p}$. Hence, if $(\boldsymbol{p}, \hat{\pi})$ satisfies those constraints, then so must any other point in the sequence $(\boldsymbol{p}^n, \hat{\pi})_n$. Therefore the only constraints that can be violated by elements of the sequence $(\boldsymbol{p}^n, \hat{\pi})_n$ are the $\boldsymbol{p}$-belief constraints. Consequently, any point in the sequence $(\Pi_\rho(\boldsymbol{p}^n, \hat{\pi}))_n$ lies on the boundary of the polyhedra defined by the $\boldsymbol{p}$-belief constraints. As mentioned, these constraints are linear in $\boldsymbol{p}$, and since they also define nonempty, convex, compact polyhedra, the sequence $(\Pi_\rho(\boldsymbol{p}^n, \hat{\pi}))_n$ indeed converges to $(\boldsymbol{p}, \hat{\pi})$. This shows the continuity of ρ and hence also of $\boldsymbol{p}$-$RO(G)$ in $\boldsymbol{p}$.

Finally, the claims that, for $\boldsymbol{p} = 0$, we have 0-$RO(G) = \Delta(A)$, and for $\boldsymbol{p} = 1$, we have 1-$RO(G) = CE(G)$, are immediate. To see that for any $\boldsymbol{p} \in [0,1)$, we have $\dim[\boldsymbol{p}\text{-}RO(G)] = \dim[\Delta(A)]$, notice that the $(X, \boldsymbol{p})$-correlated equilibria with $X = A^1$ and $\boldsymbol{p} < 1$ entail distributions that put strictly positive weight on all strategies in A^2 as well as all convex combinations of such distributions. Projecting onto the original space $\Delta(A)$ implies distributions with strictly positive weights on all strategies in A as well as all possible convex combinations. This concludes the proof.

D Proof of Proposition 1

Fix G and let $A^n = \Pi_{i\in I} A_i^n$ denote the space of all pure strategy profiles that survive n rounds of iterated elimination of strictly dominated strategies in G, and similarly for the individual sets A_i^n. Let G^n denote the subgame of G with strategies restricted to A^n. Because G is finite, the limit sets A_i^∞, A^∞, and G^∞ are well defined (and are obtained after finitely many iterations). Also, for any subset $Y \subset A$, let $Y^c = A \setminus Y$ denote the complement of Y in A.

For any given $p \in [0,1]$, let $\boldsymbol{p} \in [0,1]^I$ be such that $\min \boldsymbol{p} \geq p$. We show that for p sufficiently close to 1, behavior is supported with high probability (p) in A^∞. Specifically, we construct a $\bar{p} < 1$ such that for any $p \in [\bar{p}, 1]$, if $\pi \in \boldsymbol{p}$-$RO(G)$, then $\pi\left((A^\infty)^c\right) \leq 1 - p$.

Consider the game $G^0 = G$ and pick some $p^1 < 1$. It immediately follows from p-rationality that for $p \in [p^1, 1]$, if $\pi \in \boldsymbol{p}$-$RO(G)$, we have $\pi\left(\left(A^1\right)^c\right) \leq 1 - p$.

Suppose now that the above statement is true for $n-1$, namely there exists $p^{n-1} < 1$ such that for $p \in [p^{n-1}, 1]$, if $\pi \in \boldsymbol{p}$-$RO(G)$, then we have $\pi\left(\left(A^{n-1}\right)^c\right) \leq 1 - p$. We show that the statement also holds for n.

Fix the game G^{n-1}. It follows from finiteness of G and continuity of the payoffs that there exists $p^n \in [p^{n-1}, 1)$ such a strategy in $A^{n-1} \setminus A^n$ that is strictly dominated in G^{n-1} (by some strategy in G^{n-1} and hence in G) is also strictly dominated in G (by the same strategy) given a $\boldsymbol{p}$-rational belief system with $\min \boldsymbol{p} \geq p$ and $p \geq p^n$; (this follows from $p^n \geq p^{n-1}$, and because $\pi \in \boldsymbol{p}$-$RO(G)$ with $p \geq p^{n-1}$ implies $\pi\left(\left(A^{n-1}\right)^c\right) \leq 1 - p$). This implies that for any $p \in [p^n, 1]$ and any $\pi \in \boldsymbol{p}$-$RO(G)$, we also have $\pi\left((A^n)^c\right) \leq 1 - p$.

Finiteness of the game implies that the process ends after finitely many steps implying that indeed there exists $p^\infty < 1$ such that for $p \in [p^\infty, 1]$ and any $\pi \in \boldsymbol{p}$-$RO(G)$, we have $\pi\left((A^\infty)^c\right) \leq 1 - p$. Taking $\bar{p} = p^\infty$ shows the claim.

E Proof of Result in Remark 2

Following Aumann [2, 3], for any $X = \prod_{i\in I} X_i \subseteq A$, we say that the family $(\pi_i)_{i\in I} \subseteq (\Delta(A))^I$ is a $(X, \boldsymbol{p})$-*subjective correlated equilibrium* of G, if, for any $i \in I$,

- For any $a_i' \in X_i$, the following incentive constraints are satisfied,

$$\sum_{a_{-i} \in A_{-i}} \pi_i \left(a_{-i}; a_i'\right) \left[h_i \left(a_{-i}; a_i'\right) - h_i \left(a_{-i}; a_i\right)\right] \geq 0, \text{ for any } a_i \in A_i.$$

- For any $a_i \in A_i$ the following p_i-belief constraint is satisfied,

$$\sum_{a_{-i}' \in X_{-i}} \pi_i \left(a_{-i}'; a_i\right) \geq p_i \sum_{a_{-i} \in A_{-i}} \pi_i \left(a_{-i}; a_i\right).$$

We denote the set of $(X, \boldsymbol{p})$-subjective correlated equilibria of game G by $(X, \boldsymbol{p})\text{-}SCE\,(G)$. Given $n \in \mathbb{N}$ and a n-game nG, we have a map $\mathbf{marg}_{A^I} : (\Delta\,(nA))^I \to (\Delta\,(A))^I$, where for any $(\hat{\pi}_i)_{i \in I} \in (\Delta\,(nA))^I$, $\mathbf{marg}_{A^I}\left((\hat{\pi}_i)_{i \in I}\right) = \left(\mathrm{marg}_A\,(\hat{\pi}_i)\right)_{i \in I}$. Then, the proof of the identity,

$$\boldsymbol{p}\text{-}SRO\,(G) = \mathbf{marg}_{A^I}\left[(X, \boldsymbol{p})\text{-}SCE\,(2G)\right],$$

where $X = \prod_{i \in I} (A_i \times \{0\})$, is the same as the one for Theorem 1 after slight modifications (just add sub-indices where needed). To see that the above marginals constitute the whole space, let $\left(a^i\right)_{i \in I} \subseteq A$, and for any $i \in I$, $\pi_i = 1_{\{a^i\}}$. Fix $k \in \{0, 1\}$, and define, for any $i \in I$, $\hat{\pi}_i = 1_{\left\{\left(\left(a_{-i}^i, k_{-i}\right); \left(a_i^i, -k\right)\right)\right\}}$. It is immediate that $\mathbf{marg}_A\left((\hat{\pi}_i)_{i \in I}\right) = (\pi_i)_{i \in I}$. Now, let $i \in I$, then the incentive constraints are trivially satisfied, since $\hat{\pi}_i\,(2A_{-i} \times (A_i \times \{0\})) = 0$. Moreover, the p_i-belief constraint is also satisfied, because regardless of i's action, the sums are on both sides 1 or 0. We conclude that, again for $X = \prod_{i \in I} (A_i \times \{0\})$, we have $\left(1_{\{a^i\}}\right)_{i \in I} \in \mathbf{marg}_A\left((X, \boldsymbol{p})\text{-}SCE\,(2G)\right)$ for any $\left(a^i\right)_{i \in I} \subseteq A$, so by convexity, $\mathbf{marg}_A\left((X, \boldsymbol{p})\text{-}SCE\,(2G)\right) = (\Delta\,(A))^I$.