Group strategy-proofness in private good economies without money: matching, division and house allocation

Salvador Barberà
Dolors Berga
Bernardo Moreno

February 2014

Group strategy-proofness in private good economies without money: matching, division and house allocation

by

Salvador Barberà[*], Dolors Berga[†], and Bernardo Moreno[‡]

February 2014

<u>Abstract</u>: We observe that three salient solutions to matching, division and house allocation problems are not only (partially) strategy-proof, but (partially) group strategy-proof as well, in appropriate domains of definition. That is the case for the Gale-Shapley mechanism, the uniform rule and the top trading cycle solution, respectively. We embed these three types of problems into a general framework. We then notice that the three rules, as well as many others, do share a common set of properties, which together imply their (partial) group strategy-proofness. This proves that the equivalence between individual and group strategy-proofness in all these cases is not a fortuitous event, but results from the structure of the functions under consideration.

[*]MOVE, Universitat Autònoma de Barcelona, and Barcelona GSE. Mailing address: Departament d'Economia i d'Història Econòmica,, Edifici B, 08193 Bellaterra, Spain. E-mail: salvador.barbera@uab.cat

[†]Departament d'Economia, C/ Universitat de Girona, 10; Universitat de Girona, 17071 Girona, Spain. E-mail: dolors.berga@udg.edu

[‡]Departamento de Teoría e Historia Económica, Facultad de Ciencias Económicas y Empresariales, Campus de El Ejido, 29071 Málaga, Spain. E-mail: bernardo@uma.es

1   Introduction

In many contexts where satisfactory strategy-proof mechanisms do not exist, asking for more becomes redundant. But in domains where that basic incentive property can be met, it becomes natural to investigate whether there exist mechanisms that are not only immune to manipulation by individuals, but can also resist manipulation by groups of coordinated agents.[1] This is particularly relevant because individual strategy-proofness per se is a rather fragile property, unless one can also preclude manipulations of the social outcome by potential coalitions, especially if these may be sufficiently small and easy to coordinate.

In this paper we study the characteristics of group strategy-proof mechanisms to allocate private goods among selfish agents, when no compensatory transfers are possible.

In the context of these allocation problems, it is possible to speak about the partial compliance of incentive properties: a mechanism may be individually or group strategy-proof for some of the agents. And indeed, this possibility is heavily exploited by the literature on matching, where some of the leading mechanisms, including the canonical one proposed by Gale and Shapley, only satisfy strategy-proofness for one side of the market. This introduces an apparent asymmetry between the treatment of matching and that of other similar allocation problems.

We provide a general framework that includes, as special cases, one-to-many matching, where partial notions of incentive properties prevail, as well as division and house allocation problems, where all agents abide to the same incentive constraints. In that context, we identify properties that are sufficient to precipitate the group strategy-proofness of those social choice functions that are individually strategy-proof. Hence, we do not only clarify the intriguing connection between the two properties, but also prove their link to be quite independent of whether individual and group strategy-proofness hold for all agents or only for some.

Although our framework is much more general, we use as leading examples the three allocation problems that we mentioned, and three well-known allocation mechanisms that stand out as reference points, thanks to their ability to fulfill different lists of attractive requirements. In the case of matching, we think of the Gale-Shapley mechanism (Gale and Shapley, 1962, see also Roth and Sotomayor, 1990). In the case of division, the uniform rule stands out (Sprumont, 1991). For house allocation, it is the top trading cycle (Shapley and Scarf, 1974).

---

[1] We study that question for economies with public goods in Barberà, Berga and Moreno (2011).

1

Within the context of each of these particular models and under each of the three reference rules, there are agents for whom revealing their true preferences is always a dominant strategy. For these agents we can say that the rule is strategy-proof. In the case of the Gale-Shapley mechanism, strategy-proofness only holds for one side of the market, while it applies to all agents in the case of the uniform rule or the top trading cycle.

Our research starts from the observation that, in addition to being strategy-proof for some set of agents, all three mechanisms are also group strategy-proof for those agents who cannot manipulate individually. They cannot form any coalition among themselves only and strictly gain by jointly deviating from truth-telling. Prompted by this remark, we want to take a deep look at the reasons why, in those and possibly in other cases, some mechanisms become group strategy-proof as soon as they satisfy individual strategy-proofness. In order to do that, we first provide a general framework that encompasses these three possible types of models, but also others. Then we identify some general conditions that all of our three reference models do satisfy, but that can also be asked from many other rules. Finally, we prove that any individually strategy-proof rule satisfying the specific conditions that we have unearthed must also be group strategy-proof.

Our main result shows that there is a deep link between these two incentive properties, beyond the fact that individual strategy-proofness is obviously the weaker of the two. The fact that both hold in our three reference cases is not an accident: rather, it is because all three mechanisms share the same characteristics, within their respective specifications. In fact, they share these properties with many other possible mechanisms that one can think of, and all of those strategy-proof mechanism satisfying our properties will be group strategy-proof as well.

The paper proceeds as follows. We first provide a general framework allowing us to identify the social choice functions that result from applying an allocation mechanism in our environments, and see that the three types of problems we mention do fit into it (Section 2). Then we define the specific rules that we use as reference points, and identify several properties that they all share (Section 3). Then we prove (Section 4) our main result regarding the relationship between individual and group strategy-proofness in our model, and show that the conditions involved are independent and non-redundant. A set of comments and conclusions (Section 5) close the paper.

2   The model and some leading interpretations

We look for a common framework encompassing different specific models where private goods must be allocated and compensatory transfers are not possible. Before providing general definitions, let us briefly discuss several models that will fit into our framework, where we distinguish among alternatives, consequences and preference domains.

Matching: A set W of workers must be allocated to some firm in the set F of potential employers, each one able to hire a given amount of workers (say quota q). Each possible outcome for this allocation problem is a matching, indicating which workers go into what firms, and those who remain jobless. The consequence of a matching for each agent is the firm where he is attached to, in the case of workers, or else staying unemployed; and in the case of a firm, it is the (possibly empty and no larger than its capacity) set of workers that are made available to it.[2] Individuals have selfish preferences over matchings: they rank equally all those that have the same consequences for them. In the case of firms, we assume that when ranking sets of workers, their preference relation respects a condition of responsiveness.[3]

The matching literature usually describes allocation rules by means of algorithms, which compute a single matching for each given specification of the preferences of workers over firms (and unemployment), and of the preferences of firms over sets of workers.

In the general language that we are going to use, borrowed from social choice theory, agents are all the workers and all the firms involved. Alternatives are matchings. Consequences for an agent are what the agent gets from some matching. Preferences are defined over consequences, but extend to matchings (alternatives) in a natural way. Admissible profiles are lists of preferences over alternatives, one for each agent. In our applications to matching we assume that all rankings of singletons are possible, and that rankings of sets are only restricted by the requirement of responsiveness. These admissible profiles constitute the domain of a social choice function that chooses one alternative at each profile.

---

[2] A special case of the many-to-one model we have described is that where firms have only one opening. It corresponds to the case where matchings are one-to-one, usually called the marriage market.

[3] In our setting, where agents have preferences over sets, responsiveness requires the following. Assume that worker s is preferred to worker t, when comparing them as singletons. Then, for any two sets sharing the same workers, except that one contains s and the other contains t, the former is preferred to the latter. We concentrate on that domain of preferences because, as we shall see, it guarantees that the Gale-Shapley algorithm, originally designed for one-to-one matchings, can be used in that more general case and provide us with a rule satisfying the good properties we are after.

Division. A set N of n individuals must share a task. An allocation is a vector of n nonnegative real numbers, adding up to one, that indicates what proportion of the total task is assigned to the individuals. The consequence of the overall allocation for agent i is just the i-th component of that vector. Individuals have preferences over consequences: there is a share s^* that they prefer most, and they hold any possible single-peaked preferences around that maximal element.[4]

Hence, in that case alternatives are vectors of shares, consequences are individual shares, and the domain of our social choice functions will be the set of selfish preferences over alternatives associated to all single-peaked preferences over individual shares.

House allocation. A set N of individuals must be allocated a maximum of one house each, out of a set of houses.[5] An allocation just designates who gets what house, if any. The consequence of an allocation for an agent is whether or not she gets a house, and which one. Individuals have preferences over houses, they typically prefer any house to none, and are otherwise unrestricted. Houses are not agents, as they are not endowed with preferences. So now individuals are agents, alternatives are allocations of at most one house to each agent, and social choice functions are defined over the domain of unrestricted rankings of houses.

The general model

Let $N = \{1, 2, ..., n\}$ be a finite set of *agents* with $n \geqslant 2$. Let B_i be the set of possible consequences for i, $i \in N$. Let $A \subseteq B_1 \times ... \times B_n$ be the set of feasible combinations of consequences for agents and $a = (a_1, ..., a_n) \in A$. A is our set of alternatives.

Each agent i has preferences denoted by R_i on B_i. As usual, we denote by P_i and I_i the strict and the indifference part of R_i, respectively. For any $a \in A$ and $R_i \in \mathcal{R}_i$, the *strict lower contour set of R_i at a_i* is $\overline{L}(R_i, a_i) = \{b_i \in B_i : a_i P_i b_i\}$ and the *strict upper contour set of R_i at a_i* is $\overline{U}(R_i, a_i) = \{b_i \in A : b_i P_i a_i\}$.

Let $\widetilde{\mathcal{R}}_i$ be the set of complete, reflexive, and transitive orderings on B_i. From preferences on B_i we can induce preferences on A as follows: For any $a, b \in A$, $a R_i b$ if and only if $a_i R_i b_i$. That is, we assume that, when evaluating different alternatives, agents are selfish. Note that, abusing notation we use the same symbol R_i to denote preferences on A and on B_i.

---

[4]That is, given any share s different than s^*, they strictly prefer any s' lying in the interval $[s, s^*]$ to the extreme s.

[5]Since we are using these models for motivational purposes, we stick to the simplest version of the house allocation model. More complex cases allow for more than one house to be allocated to the same agents, the existence of property rights and other possible variations.

Let $\mathcal{R}_i \subseteq \widetilde{\mathcal{R}}_i$ be *the set of admissible preferences for agent $i \in N$*. A *preference profile*, denoted by $R_N = (R_1, ..., R_n)$, is an element of $\times_{i \in N} \mathcal{R}_i$. We will write $R_N = (R_C, R_{N \setminus C}) \in \times_{i \in N} \mathcal{R}_i$ when we want to stress the role of coalition C in N.

A *social choice function* (or a rule) is a function $f : \times_{i \in N} \mathcal{R}_i \to A$.

We now turn to conditions regarding the incentives of individuals to reveal their true preferences, under a given social choice rule.

Definition 1 *A social choice function f on $\times_{i \in N} \mathcal{R}_i$ is manipulable at $R_N \in \times_{i \in N} \mathcal{R}_i$ by coalition $C \subseteq N$ if there exists $R'_C \in \times_{i \in C} \mathcal{R}_i$ ($R'_i \neq R_i$ for any $i \in C$) such that $f(R'_C, R_{N \setminus C}) P_i f(R_N)$ for all $i \in C$. A social choice function is H-group strategy-proof if it is not manipulable by any coalition $C \subseteq H$, and it is H-individually strategy-proof if it is not manipulable by any singleton $\{i\} \subseteq H$.[6]*

The requirement of individual strategy-proofness demands that revealing one's preferences is a dominant strategy for all agents in H. Likewise, group strategy-proofness requires that no subset of agents in H can coordinate their actions and obtain a better result than by just declaring their preferences. The standard definition of individual and group strategy-proofness refer to the case $H = N$.

Since the Gale-Shapley mechanism only provides one side of the market with dominant strategies to reveal the truth, we relax the standard definitions in order to also allow for partial notions of manipulation and of strategy-proofness.

Notice that the three types of allocation problems we have described before do fit well in that general framework, though they clearly differ in the exact form of the alternatives, and also in the domains of preferences over which the different social choice rules are defined.

In the case of matching, the set of agents consists of two disjoint sets W and F. The set of consequences for individuals are as follows: For each $i \in W$, $B_i = F \cup \{i\}$ and for each $j \in F$, $B_j = W \cup \{j\}$ in the case of one-to-one matching (also known as the *marriage model*) while $B_j = 2^W \cup \{j\}$ in the case of many-to-one (also known as the *college admissions problem*). The assignment problem consists of matching each agent i in W with at most one agent j in F and each j with eventually many agents in W. The set of alternatives A is given in this case by the set of all admissible matchings[7].

---

[6]Notice that our definition requires that in order to jointly manipulate, all agents in a group must derive a strict gain from their participation. A weaker version of the condition would allow for some of them to help others, as long as not making losses.

[7]A *matching* $a \in A$ is a mapping from the set N into the set of all subsets of N such that for all $i \in W$, $a_i \in B_i$ and $j \in F$, $a_j \in B_j$ and $a_i = \{j\}$ if and only if $i \in a_j$.

In the case of division, $B_i = [0,1]$, and $A = \{a \in B_1 \times ... \times B_n : \sum_{i=1}^{n} a_i = 1\}$, for each agent $i \in N$.

In the case of house allocation, B_i is a set of individual objects or houses, such that $B_i = B_j$ for each $i, j \in N$ and $\#B_i = \#N$. The set of alternatives A is given in this case by the set of possible assignments of houses to agents such that each agent receives a different house.

In the next section we define additional conditions that social choice functions may or may not satisfy, and see that, in fact, these general conditions are met by the best studied rules that apply to each context.

3   Three representative mechanisms and their common properties

In this Section, we describe with some detail three allocation rules that play a leading role in the study of matching, division and house allocation and we identify conditions that they share.

We begin by the Gale-Shapley mechanism and its associated social choice function. Given a many-to-one matching problem, one can calculate, for each preference profile, a unique matching, according to the following algorithm, that we first describe for the one-to-one case. That is, for the case where each firm can only accept one worker at most.

The algorithm starts with all workers applying to their preferred firms, and firms tentatively accepting the one worker they prefer among those that applied for it. If that leaves some workers unmatched, these are then asked to apply for their second best firms. Once their applications get in, firms may accept these new applicants if they are better for them than the ones they retained in the first round, and reject the previously accepted ones. That leads to a new matching and to a new list of unmatched workers. Again, if some workers remain unmatched, they may now re-apply to those firms that are the best among those they did not yet apply to in preceding rounds. The process continues until no further changes can occur. In the one-to-one case this method always leads to a unique alternative.[8]

---

[8]The matching thus obtained is stable: that is, it is individually rational and no two individuals, one from each side of the market, can improve upon it by forming a new pair and leaving their present match. Discussing the stability of matchings is the main concern of that literature, but since we concentrate here on incentive properties, we shall not insist on that point.

Now, to extend the use of the Gale-Shapley mechanism to our many-to-one case, simply re-define a new set of firms, so that each original one, with capacity for q workers, becomes one of q different firms with capacity one. Let all these firms still have the same preferences over singletons as before. Define the preferences of workers in such a way that they still preserve the ranking among firms that come from different original ones, and let all of them rank the small firms coming from the same one in an arbitrary, common order. Then, run the Gale-Shapley mechanism for this well-defined one-to-one matching problem, and finally assign to the "real" initial firms all the workers that are matched with any of its small, instrumental firms into which it was divided.

The use of the Gale-Shapley mechanism at each preference profile generates a social choice function. We will abuse terminology and call it the Gale-Shapley social choice function from now on.

Turn next to the uniform rule for the division problem. Again, given a profile of single-peaked preferences over shares of the job, the rule determines a unique alternative, that is, a vector of shares, as follows.

Ask each agent for its preferred share of the job. If the sum of the desired shares exceeds one, find a number λ having the following property. If all agents who demand less than λ are allowed to have their preferred share, and all others are required to accept λ, then the total assignment adds up to one. If the sum of the desired shares is short of one, find a number λ' having the following property. If all agents who demand more than λ' are allowed to have their preferred share, and all others are required to accept λ', then the total assignment adds up to one.

These values for λ or λ' always exist, and thus the rule based on them determines an assignment of shares that is always feasible. It therefore defines a social choice function on the set of admissible profiles.

Finally, for the housing problems we have described before, the top trading cycle mechanism also determines a unique alternative, this time an allocation of houses, according to the following rule. Ask agents to point at their preferred house. There will always be some set of agents (maybe several) whose demands form a cycle.[9] Give those agents in the cycle their preferred houses and remove them. Now ask the remaining set of agents for their preferred houses over the remaining ones, and proceed likewise until all houses are assigned.

Again, this procedure leads to a well defined alternative for each preference profile, and

---

[9]Agents who point at their own house, form a cycle by themselves.

thus gives rise to what we call, abusing terminology, the top trading cycle social choice function.

Notice that all three methods, each one applying to a different world, do generate specific social choice functions. Because of that, we can investigate the common traits of these different functions, if any.

The first remarkable coincidence is that all three rules are not only individually but also group strategy-proof. But there are also other common characteristics, that we emphasize in turn.

We start by the domains of definition. They are clearly different, and it is not even the case that the restrictions that make sense when alternatives are matchings would also be applicable to the case of division, or vice-versa. Yet, here are general requirements on the profiles of preferences over consequences that all three domains we consider do satisfy.

Definition 2 *A set of individual preferences $\mathcal{R}_i$ is **top rich** if for any $R_i \in \mathcal{R}_i$*
(1) there exists a unique $\tau(R_i) \in B_i$ such that for all $x_i \in B_i$, $\tau(R_i) P_i x_i$.[10] and
(2) for any $a_i, b_i \in B_i$ such that $b_i P_i a_i$, there exists R_i' such that $\overline{U}(R_i, a_i) = \overline{U}(R_i', a_i)$, $\overline{L}(R_i, a_i) = \overline{L}(R_i', a_i)$ and $b_i = \tau(R_i')$.

Definition 3 *Let $H \subseteq N$. A domain of preferences $\times_{i \in N} \mathcal{R}_i$ is H-**top rich** if for any $i \in H$, $\mathcal{R}_i$ is top rich.*

We can argue that the standard domains of definition for our three rules satisfy H-top rich for some H. What are these domains?

In housing problems, no restrictions are imposed on the agent's preferences over houses, and therefore the admissible preferences over consequences are top rich for all individuals in N.

In many-to-one matching models, no restrictions are imposed on the preferences of the set W of workers other than they are linear orders. Hence, the family of admissible preferences is also trivially top rich. As for firms, it is usually assumed in the literature that their preferences over sets of workers satisfy responsiveness. The family of such preferences

---

[10]Notice that the fact that there is a unique top in the preferences over consequences does not imply that individual preferences over alternatives have a unique top, because of the selfishness assumption.

is not top rich.[11] But we can still say that the typical domain for many-to-one matching problems is W-top rich.

As for division problems, the family of all single-peaked preferences is again top rich because given any $a_i, b_i \in B_i$ such that $b_i P_i a_i$, one can trivially find another single-peaked preference $R_i' \in \mathcal{R}_i$ with $b_i = \tau(R_i')$ and such that $\overline{U}(R_i, a_i) = \overline{U}(R_i', a_i)$, $\overline{L}(R_i, a_i) = \overline{L}(R_i', a_i)$.

Next we identify two more conditions on social choice functions that are satisfied by our leading examples.

The first one is a restricted version of non-bossiness. That condition was proposed by Satterthwaite and Sonnenschein (1981) and requires that no agent can change the consequences for another agent unless the consequences for her also change. In our case this condition would be unnecessarily strong, because the Gale-Shapley social choice function does not satisfy it, not even for agents in one side of the market.[12]

The following concept of H-respectfulness is a mild version of non-bossiness, where the requirement is limited to changes in consequences induced by a limited type of preference changes, and is only predicated for some of the agents.

Definition 4 *Let $H \subseteq N$ be a subset of agents. A social choice function f on $\times_{i \in N} \mathcal{R}_i$ is H-respectful if*

$$f_i(R_N) = f_i(R_i', R_{N \setminus \{i\}}) \text{ implies } f_j(R_N) = f_j(R_i', R_{N \setminus \{i\}}), \ \forall j \in H \setminus \{i\},$$

for each $i \in H$, $R_N \in \times_{i \in N} \mathcal{R}_i$, and $R_i' \in \mathcal{R}_i$ such that $\overline{U}(R_i, f_i(R_N)) = \overline{U}(R_i', f_i(R_N))$ and $\overline{L}(R_i, f_i(R_N)) = \overline{L}(R_i', f_i(R_N))$.

The weakening of standard non-bossiness comes from two sources. In the first place, the definition only imposes requirements on some subset H of agents. More important is the fact that the lack of ability to affect the consequences on others when not affecting one's own is restricted to special and limited cases of preference changes.

Indeed, two of our reference methods are quite trivially N-respectful, because the uniform rule and the top trading cycle solution are in fact non-bossy. But notice that the

---

[11] The following example shows that the set of responsive preferences is not top rich. Let $W = \{1, 2, 3, 4\}$ and $f \in F$ with capacity 2. Let R_f be a responsive preference such that $\{i\} P_f \{f\}$, for any $i \in W$ and $\{2\} P_f \{3, 4\}$. Note that for each responsive R_f' such that $2 = \tau(R_f')$, $\overline{L}(R_f, \{3, 4\}) \neq \overline{L}(R_f', \{3, 4\})$ because $\{f\} P_f' \{i\}$, for each $i \in W \setminus \{2\}$.

[12] Kojima (2010) proves that stability is incompatible with non-bossiness. Since the Gale-Shapley mechanism produces stable outcomes it must be bossy.

Gale-Shapley social choice function is bossy, even if we restrict attention to changes in the side of the workers/proposers. This is clear from the following example.

Example 1 *The Gale-Shapley social choice function is bossy. Let $W = \{1, 2, 3, 4\}$ and $F = \{a, b, c, d\}$. For each $i \in W$, $B_i = F \cup \{i\}$ and for each $j \in F$, $B_j = W \cup \{j\}$ (one-to-one matching). The preferences of the agents in W, P_W, on B_i are given in the following table:*

R_1	R_2	R_3	R_4
c	d	b	b
a	c	d	d
b	b	c	a
d	a	a	c
1	2	3	4

and the preferences of the agents in F, P_F, on B_j are given in the following table:

R_a	R_b	R_c	R_d
2	2	3	4
4	4	1	3
3	3	2	2
1	1	4	1
a	b	c	d

Let $H = W$. The alternative selected by the Gale-Shapley social choice function for R_N when the workers are the proposers is $(f_1(R_N), f_2(R_N), f_3(R_N), f_4(R_N)) = (a, b, c, d)$. Let R'_1 be as follows: $aP'_1cP'_1bP'_1dP'_1 1$. Note that P'_1 is such that $\overline{U}(R_1, f_1(R_N)) \neq \overline{U}(R'_1, f_1(R_N))$ and $\overline{L}(R_1, f_1(R_N)) \neq \overline{L}(R'_1, f_1(R_N))$. Let $R'_N = (R'_1, R_{-1})$. The alternative selected by the Gale-Shapley social choice function for R'_N when the workers are the proposers is $(f_1(R'_N), f_2(R'_N), f_3(R'_N), f_4(R'_N)) = (a, c, d, b)$. Since $f_1(R'_N) = f_1(R_N)$, but $f_i(R'_N) \neq f_i(R_N)$ for any $i \in F \backslash \{1\}$ we have that the Gale-Shapley social choice function is bossy.

Yet, even if bossy, the Gale-Shapley social choice function is W-respectful. To check that, consider two preference profiles R_N and R'_N such that, for each agent i, the upper and lower contour sets of $f_i(R_N)$ coincide in R_N and R'_N. Then, $f(R_N)$ will be stable under profile R'_N, and $f(R'_N)$ will be stable under profile R_N. Since preferences are strict, and the choice of f is optimal for the workers among all stable outcomes, it must be that $f(R_N) = f(R'_N)$.

The second condition is a limited form of monotonicity, that we call H-top monotonicity. The condition needs to only hold for some group H, and then requires the following. Take a preference profile, and a subset of agents in H. Suppose that the preferences of all agents in the subset do change, and that the consequences they were getting under the previous profile become now their best choice. Then, these agents should retain the same outcome at the new profile.

Formally,

Definition 5 *Let $H \subseteq N$ be a subset of agents. A social choice function f satisfies H-top monotonicity on $\times_{i \in N} \mathcal{R}_i$ if for any $R_N \in \times_{i \in N} \mathcal{R}_i$, $C \subseteq H$, and $R'_C \in \times_{i \in C} \mathcal{R}_i$ such that $\tau(R'_i) = f_i(R_N)$ for all $i \in C$, then $f_i(R'_C, R_{N \setminus C}) = f_i(R_N)$ for all $i \in C$.*

Let us check that this property is also satisfied by the social choice functions generated through our mechanisms. In the case of the top trading cycle this is clear, since agents endowed with their very best house will just retain it. In the division problem, agents who did not get their best amount at a certain profile and now request that same amount as being their best will not change the outcome of the uniform rule. Hence, N-monotonicity is satisfied. As for the Gale-Shapley social choice function, it will satisfy W-monotonicity, by the following reasoning. Start from a profile R_N, and consider another profile R'_N where all agents in a subset C of W consider that $f_i(R_N)$ is their best alternative in R'_N. Notice that $f(R_N)$ will also be stable at profile R'_N. Moreover, since the Gale-Shapley social choice function selects the W-best stable matching, $f(R_N)$ has this property in R_N, and clearly it will also have it in R'_N. Thus, $f(R_N) = f(R'_N)$.

4 Individual versus group strategy-proofness

We have observed that, in spite of several important formal differences regarding the space of alternatives and the domains of preferences, our three reference social choice functions share a number of properties. In particular, all of them are group strategy-proof. The following theorem proves that the equivalence between that property and individual strategy-proofness is not a lucky coincidence, but a result of the fact that any strategy-proof social choice function satisfying the remaining common requisites that we just exhibited, for individuals in a set H must also be immune to manipulation by subsets of H.

Theorem 1 If f is H-strategy-proof, H-top monotonic, H-respectful and defined on an H-top rich domain, then f is also H-group strategy-proof.

Proof By contradiction suppose that there exists $R_N \times_{i \in N} \mathcal{R}_i$, $C \subseteq H$, $\widetilde{R}_C \in \times_{i \in C} \mathcal{R}_i$ such that for any agent $i \in C$, $f_i(\widetilde{R}_C, R_{-C}) P_i f_i(R_N)$. Let $b = f(\widetilde{R}_C, R_{-C})$ and $a = f(R_N)$. Note that for each $i \in C$, $f_i(R_N) \neq \tau(R_i)$.

By top richness of $\mathcal{R}_i$ where $i \in C$, there exists R'_C such that $\overline{U}(R_i, a_i) = \overline{U}(R'_i, a_i)$, $\overline{L}(R_i, a_i) = \overline{L}(R'_i, a_i)$ and $b_i = \tau(R'_i)$ for each $i \in C$. Without loss of generality, assume that $C = \{1, 2, ..., c\}$. Consider the sequence of preference profiles $R^0 = R_N$, $R^1 = (R'_1, R_{-1})$, $R^2 = (R'_1, R'_2, R_{-\{1,2\}})$,..., $R^c = (R'_C, R_{-C})$. By H-strategy-proofness, $f_1(R^0) = f_1(R^1)$, and by H-respectfulness, $f_j(R^0) = f_j(R^1)$ for each $j \in H \backslash \{1\}$. Again, by H-strategy-proofness, $f_2(R^1) = f_2(R^2)$, and by H-respectfulness, $f_j(R^1) = f_j(R^2)$ for each $j \in H \backslash \{2\}$. Repeating the same argument, we obtain that $a_j = f_j(R^0) = f_j(R^c)$ for each $j \in H$.

Finally, since $R'_C \in \times_{i \in C} \mathcal{R}_i$ is such that $\tau(R'_i) = b_i$ for each $i \in C$, then $f_i(\widetilde{R}_C, R_{-C}) = f_i(R'_C, R_{-C})$ for each $i \in C$ by H-top monotonicity.

Thus, we obtain the desired contradiction since, on the one hand $f_j(R'_C, R_{-C}) = a_j$ and on the other hand $f_j(R'_C, R_{-C}) = b_j$, for each $j \in C$.

Our main purpose in this paper is achieved: we have proven that the coincidence between the two incentive properties in the apparently diverse worlds of matching, division and house allocation is the consequence of a shared structure that goes beyond the details of each particular model.

The following examples show that the result is robust in that all assumptions we use are needed.

Example 2 *Violation of Theorem 1 when the domain is not top-rich. Let $W = \{1, 2, 3\}$ and $F = \{4, 5, 6\}$. For each $j \in F$, $B_j = W \cup \{j\}$ and for agent $3 \in W$, $B_3 = F \cup \{3\}$ any linear order on B_l is admissible. For agents 2 and $3 \in W$, $B_i = F \cup \{i\}$, the set of admissible preferences are as follows:*

R_1^1	R_1^2	R_1^3	R_2^1	R_2^2	R_2^3
4	5	6	4	5	6
5	4	4	5	4	5
6	6	5	6	6	4
1	1	1	2	2	2

Define the following rule f where neither firms nor agent 3 play any role, that is:

$f(\cdot, R_3, R_F)$	R_2^1	R_2^2	R_2^3
R_1^1	$4, 5, 6, 1, 2, 3$	$4, 5, 6, 1, 2, 3$	$5, 6, 4, 3, 1, 2$
R_1^2	$5, 4, 6, 2, 1, 3$	$5, 4, 6, 2, 1, 3$	$5, 6, 4, 3, 1, 2$
R_1^3	$6, 4, 5, 2, 3, 1$	$6, 4, 5, 2, 3, 1$	$5, 4, 6, 2, 1, 3$

For $H = W$, the above rule is restricted H-respectful, H-top monotonic, H-strategy-proof, but it is not H-group strategy-proof (agents 1 and 2 deviates from (R_1^3, R_2^3) to (R_1^1, R_2^1) and they are strictly better off).

Example 3 *A social choice function defined on a top-rich domain that is H-respectful, H-top monotonic but not H-strategy-proof. The much celebrated Boston mechanism provides an example. In a first round, each student applies to his (reported) top choice and each school admits applicants one at a time according to its preferences until either capacity is exhausted or there are no more students who ranked it first. In Round k, each unmatched student applies to his kth choice and schools with remaining capacity admits applicants one at a time according to its preferences until either the remaining capacity is exhausted or there are no more students who ranked it kth.*

Example 4 *A social choice function defined on a top-rich domain that is H-top monotonic, H-strategy-proof but not H-respectful. Let $W = \{1, 2\}$ and $F = \{3, 4\}$. For each $j \in F$, $B_j = W \cup \{j\}$ and any linear order on B_j is admissible. For each $i \in W$, $B_i = F \cup \{i\}$ and any linear order on B_i is admissible. The latter is as in the following table:*

R_i^1	R_i^2	R_i^3	R_i^4	R_i^5	R_i^6
3	4	3	4	i	i
4	3	i	i	3	4
i	i	4	3	4	3

Define the following rule f where firms do not play any role, that is for any $R_F \in \mathcal{R}_F$:

$f(\cdot, R_F)$	R_2^1	R_2^2	R_2^3	R_2^4	R_2^5	R_2^6
R_1^1	$3, 4, 1, 2$	$3, 4, 1, 2$	$3, 2, 1, 4$	$3, 4, 1, 2$	$4, 2, 3, 1$	$4, 2, 3, 1$
R_1^2	$4, 3, 2, 1$	$4, 3, 2, 1$	$4, 3, 2, 1$	$4, 2, 3, 1$	$4, 2, 3, 1$	$4, 2, 3, 1$
R_1^3	$3, 4, 1, 2$	$3, 4, 1, 2$	$3, 2, 1, 4$	$3, 4, 1, 2$	$4, 2, 3, 1$	$4, 2, 3, 1$
R_1^4	$4, 3, 2, 1$	$4, 3, 2, 1$	$4, 3, 2, 1$	$4, 2, 3, 1$	$4, 2, 3, 1$	$4, 2, 3, 1$
R_1^5	$1, 3, 2, 4$	$1, 3, 2, 4$	$1, 3, 2, 4$	$1, 3, 2, 4$	$4, 3, 2, 1$	$4, 3, 2, 1$
R_1^6	$1, 3, 2, 4$	$1, 3, 2, 4$	$1, 3, 2, 4$	$1, 3, 2, 4$	$4, 3, 2, 1$	$4, 3, 2, 1$

For any $R_F \in \mathcal{R}_F$, note that $f(R_1^5, R_2^5, R_F) = (4, 3, 2, 1)$. Observe that $f_1(R_1^5, R_2^5, R_F) = f_1(R_1^3, R_2^5, R_F) = 4$, $\overline{U}(R_1^5, 4) = \overline{U}(R_1^3, 4)$, $\overline{L}(R_1^5, 4) = \overline{L}(R_1^3, 4)$ but $f_2(R_1^5, R_2^5, R_F) \neq f_2(R_1^3, R_2^5, R_F)$, violating H-respectfulness.

Example 5 *A social choice function defined on a top-rich domain that is H-respectful, H-strategy-proof but not H-top monotonic. Let $W = \{1, 2, 3\}$ and $F = \{4, 5, 6\}$. For each $i \in W$, $B_i = F \cup \{i\}$ and any linear order on B_i is admissible. For each $j \in F$, $B_j = W \cup \{j\}$ and any linear order on B_j is admissible. For any $R_1 \in \mathcal{R}_1$, let $\tau^2(R_1)$ be the most preferred alternative on $B_1 \backslash \tau(R_1)$ of R_1. Define the following rule f where neither firms nor agents in $W \backslash \{1\}$ play any role:*

	$\forall R_{-1}, f(R_N)$
If $\tau(R_1) = 5$ then	$5, 6, 4, 3, 1, 2$
If $\tau(R_1) = 6$ then	$6, 5, 4, 3, 2, 1$
If $\tau(R_1) = 1$ then	$1, 5, 4, 3, 2, 6$
If $\tau(R_1) = 4$ and $\tau^2(R_1) = 5$ then	$5, 4, 6, 2, 1, 3$
If $\tau(R_1) = 4$ and $\tau^2(R_1) = 6$ then	$6, 4, 5, 2, 3, 1$
If $\tau(R_1) = 4$ and $\tau^2(R_1) = 1$ then	$1, 4, 5, 2, 3, 6$

For $H = W$, the above rule is restricted H-respectful, H-strategy-proof, but it is not H-top monotonic. To see this consider the following preference profile:

$\widehat{R}_1$	$\widehat{R}_2$	$\widehat{R}_3$
4	5	4
1	4	5
5	6	6
6	2	3

For any $R_F \in \mathcal{R}_F$, note that $f(\widehat{R}_1, \widehat{R}_2, \widehat{R}_3, R_F) = (1, 4, 5, 2, 3, 6)$. Suppose that the preferences of agents in $H = W$ change and the consequences they have obtained are now the most preferred ones

$\widetilde{R}_1$	$\widetilde{R}_2$	$\widetilde{R}_3$
1	4	5
4	5	4
5	6	6
6	2	3

14

For any $R_F \in \mathcal{R}_F$, note that $f(\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3, R_F) = (1, 5, 4, 3, 2, 6)$. Observe that $f_1(\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3, R_F) = f_1(\widehat{R}_1, \widehat{R}_2, \widehat{R}_3, R_F) = 1$ but $f_i(\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3, R_F) \neq f_i(\widehat{R}_1, \widehat{R}_2, \widehat{R}_3, R_F)$ for $i = 2, 3$, violating H-top monotonicity.

5   Conclusions

We have proven that the coincidence between individual and group strategy-proofness in diverse worlds, where these properties may be satisfied for all agents, or only by a subset of them, is the consequence of a shared structure that goes beyond the details of each particular model.

We have motivated our choice of framework by showing that it encompasses classical mechanisms to solve a variety of allocation problems. But the result goes well beyond the three examples we have used. Even within the framework of those problems we have focused on, alternative mechanisms have been discussed that satisfy group strategy-proofness for the same reasons we just discussed. This is the case, for the division problem, of the non-anonymous and non-neutral social choice functions induced by sequential mechanisms, as described in Barberà, Jackson and Neme (1997), or in Massó and Neme (2001). Solutions to housing problems as the one discussed by Pápai (2000) are also within the scope of our results. And so are mechanisms for school choice (Abdulkadiroglu and Sönmez, 2003).

In all these cases, functions satisfying our performance and domain requirements are presented, and group strategy-proofness holds. But this is often presented as a lucky consequence of the basic quest for individual strategy-proofness, while we have emphasized here that group strategy-proofness can and should be a fundamental and attainable objective per se.

Let us also remark that the designer's choice of mechanism may be guided by other criteria than those we have emphasized here. This is particularly true in the case of matching, where the basic concern has been to find procedures that guarantee stability. Our emphasis on different forms of strategy-proofness and their relationship has sidestepped that main concern, though we have identified conditions that are mild enough to still admit the deferred acceptance procedure proposed by Gale and Shapley as part of our universe. Hence, our results also apply to possible mechanisms that do not meet the requirement of stability. For example, you may think of a segmented society with two culturally differentiated groups h and l, match their men M_h and M_l, and their W_h, W_l, according to the Gale-Shapley deferred acceptance mechanism. However, the men M_l are the proposers to W_l in group l,

while the women W_h in h are the proposers to M_h. The composed matchings for these two segments of society result in an overall matching that clearly does not guarantee overall stability, but that fits well in our context. It is easy to see that the induced social choice function will be strategy-proof relative to the union of M_l and W_h, and satisfy all of our conditions relative to that set as well.

In our search for a common ground for the classical and attractive mechanisms that we have highlighted, we have identified the condition of H-respectfulness, and seen that it is weaker than the property of non-bossiness that has been often used in the mechanism design literature since its introduction by Satterthwaite and Sonnnschein (1981). The reader familiar with the preceding literature will be aware that non-bossiness, coupled with individual strategy-proofness, precipitates a strong form of group strategy-proofness. Hence, one may have wondered why we did not take that shortcut to explain the link between these two properties. Our reasons may be apparent by now, after seeing the results, but let us insist on our logic. On the one hand, we wanted to stay in a world where comparisons between the classical solution for matching and those for other allocation problems did coexist. And that required a weakening of non-bossiness, because the Gale-Shapley social choice function is bossy. On the other hand, notice that our notion of group strategy-proofness is a mild one, where in order for a group to manipulate, all agents in it must strictly gain. Indeed, we think that this is an attractive property per se, and the natural one in our context where the set of alternatives is discrete. A stronger condition would result from allowing a weaker notion of manipulation, where manipulation by groups are considered relevant even if some agents do not gain by misrepresenting in favor of others[13]. Individual strategy-proofness and non-bossiness in fact precipitate that stronger form of manipulation, and that may exclude some attractive social choice rule that only satisfy our milder, and we think more natural condition. For all these reasons, we have departed from the strong and often ill-justified use of non-bossiness.

Our main message is a plea for consideration of group strategy-proofness as an extremely attractive property that may be attained in contexts of relevance, then avoiding the fragility of individually strategy poof rules when those are possibly manipulable by small and easy to coordinate groups.

---

[13]The distinction gets blurred in other contexts where there is a continuum of alternatives and preferences are continuous, because then both conditions can be proven to coincide, under very mild additional assumptions. This is, for example the case in exchange economies (see Barberà and Jackson, 1995).

References

[1] A. Abdulkadiroglu, and T. Sönmez. "School Choice: A Mechanism Design Approach". *American Economic Review* **93,** (3): 729-747 (2003).

[2] Barberà, S., Berga, D., Moreno, B. "Individual versus group strategy-proofness: when do they coincide?". *Journal of Economic Theory* **145**: 1648-1674 (2010).

[3] S. Barberà, and M. Jackson. "Strategy-proof Exchange". *Econometrica* **63**, (1): 51-87 (1995).

[4] S. Barberà, M. Jackson, and A. Neme. "Strategy-Proof Allotment Rules". *Games and Economic Behavior* **18**: 1-21 (1997).

[5] D. Gale, and L. Shapley. "College Admissions and the Stability of Marriage". *American Mathematical Monthly* **69**: 9-15 (1962).

[6] F. Kojima. "Impossibility of stable and nonbossy matching mechanisms". *Economics Letters* **107**, (1): 69–70 (2010).

[7] J. Massó, and A. Neme. "A Maximal Domain of Preferences for the Division Problem". *Games and Economic Behavior* **37**: 367-387 (2001).

[8] S. Pápai. "Strategyproof Assignment by Hierarchical Exchange". *Econometrica* **68**, (6): 1403–1433 (2000).

[9] A. Roth, and M. Sotomayor. *Two-sided Matching: A Study in Game-Theoretic Modeling and Analysis.* Cambridge University Press, Cambridge, England. [Econometrica Society Monographs No. 18] (1990).

[10] M. A. Satterthwaite, and H. Sonnenschein. "Strategy-Proof Allocation Mechanisms at Diferentiable Points". *Review of Economic Studies* **48**: 587-597 (1981).

[11] L. Shapley, and H. Scarf. "On Cores and Indivisibilities". *Journal of Mathematical Economics* **1**: 23-37 (1974).

[12] Sprumont, Y. "The division problem with single-peaked preferences: A characterization of the uniform allocation rule". *Econometrica* **59:** 509-519 (1991).