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A COMPARISON OF INPUT-OUTPUT MODELS:
GHOSH REDUCES TO LEONTIEF
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Abstract

Ghosh's model is discussed in this paper under two alternative scenarios. In an open version we compare it with Leontief's model and prove that they reduce to each other under some specific productive conditions. We then move onto reconsidering Ghosh's model alleged implausibility and we do so reformulating the model to incorporate a closure rule. The closure solves, to some extent, the implausibility problem very clearly put out by Oosterhaven for then value-added is correctly computed and responsive to allocation changes resulting from supply shocks.

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1. INTRODUCTION

The debate on the validity and plausibility of the so called ‘supply-driven’ input-output model of Ghosh (1958) seems to keep resurfacing every so often. The difficulty to interpret Ghosh’s model within conventional production theory has led to numerous interpretations and assertions that, periodically, put into question the structure and meaning of the model. Giarratani (1980), for instance, discussed the lack of well understood economic behaviour behind it. Oosterhaven (1988, 1989), in turn, called the attention over the ‘implausibility’ of a model that allocates output in response to changes in value-added in a given sector without those changes in output translating into further changes in value-added. In whatever way output turns out to be produced and allocated among sectors it surely makes little sense that value-added is not responsive to a general system reallocation. Dietzenbacher (1997) ‘vindicates’ Ghosh by way or reinterpreting it as a price model, which then happens to be fully formally equivalent to Leontief’s price model and we are back to the well-known and standard interindustry model. More recently de Mesnard (2009) has claimed the model to be ‘uninteresting’ since it is implausible as an output model, unnecessary as a price model and less informative than Leontief’s dual quantity and price models. More in-depth discussion and details can be found in the references provided by these authors but the essence of the problematic issues about Ghosh’s model has been sufficiently laid out.

We will organize our discussion here distinguishing two aspects. First, on the ‘validity’ side, we will analyze Ghosh’s model vis a vis Leontief’s and we will do so under the standard open model configurations. We show that both models, even though they share some basic common linearity assumptions, are in fact incompatible except when quantities are held constant. In this case we show that they reduce to each other and Dietzenbacher’s (1997) proposal is in fact reinforced. Second, we address the

‘plausibility’ debate regarding unresponsive value-added that was pointed out by Oosterhaven’s (1989, 1989). His criticism still stands since unresponsive value-added to output changes is a hard to sell economic fact. Under Ghosh’s model conditions, suppose that value-added in sector j , say, increases. When this shock is subsequently absorbed by the output allocation system, we observe an increase in the output of sector i ($i \neq j$), as a result of the endogenous output reallocation, but at the same time value-added in sector i is surprisingly unaffected. Needless to say this seems to violate common sense as well as some version of Debreu’s axiom on the impossibility of the Land of Cockaigne (Debreu, 1959, chapter 3). Let us consider for a moment Leontief’s open quantity model. When autonomous final demand for good j increases, the system generates increases in output and value-added in all sectors. There is more value-added around but this does not have, however, any effect whatsoever in final demand for other goods ($i \neq j$). This is also somewhat surprising as far as economic logic goes. How can it be that consumption behaves in an unresponsive way to the new additional income? There are at least two ways out of this situation. The first one is to close Leontief’s open model and make consumption endogenous using linearity assumptions. The second one is to move up from the input-output model towards general equilibrium models where consumption is endogenous and price and income responsive. If Ghosh’s model is not ‘plausible’ because value-added is unresponsive to output reallocations, then a similar case could be made for Leontief’s model being somewhat ‘implausible’ too because of the fact that consumption is unresponsive to income generation. This having been said, perhaps the road to endow Ghosh’s model with a bit more plausibility is formally similar to the road taken with Leontief’s model: close it with an additional layer of endogeneity. If for Leontief we make the ‘driving demand’ force (consumption) endogenous, then for Ghosh we may attempt to make the ‘driving supply’ force (value-

added) endogenous. Since Ghosh's model shares the basic mathematical linearity of the standard interindustry model, closing it may follow the same formal logic. We first need a rule stipulating a relationship between value-added and some output measure and, secondly, we need an instrument that reflects and captures external supply shocks that are subsequently incorporated into the allocation system.

The paper is organized as follows. In Section 2 we compare both models and examine their relationship under their open versions, we summarize the main conclusions drawn by previous researchers and we supply a new proof regarding their formal connection. In Section 3 we extend Ghosh's model by formulating a possible closure rule. We verify its consequences for a correct accounting of output changes as well as value-added changes. We then illustrate the results with some numerical examples in Section 4. Section 5 concludes.

2. THE OPEN VERSION OF GHOSH'S MODEL:

IF PLAUSIBLE, IT REDUCES TO LEONTIEF PRICE MODEL.

Without a doubt, one of Leontief's most enduring contributions was to provide analysts with an empirically applicable general equilibrium system. Ignoring the difference between industry and commodity, in Leontief's system each intermediate input is only a function of total output:

$$x_{ij} = \Theta_{ij}(X_j) \tag{1}$$

In Leontief's model, inputs are used in fixed proportions and there is a unique technique for each sector. Since input substitution is not allowed, changes in relative prices have no influence on technical coefficients. For an economy with n goods and industries, the production function for this non-substitution case can be written as:

$$X_j = \text{Min} \left\{ \frac{x_{1j}}{a_{1j}}, \dots, \frac{x_{nj}}{a_{nj}} \right\} \quad \forall j = 1, \dots, n \quad (2)$$

where a_{ij} indicates the (minimal) amount of good i required to produce a single unit of output j , and the technology represented in (2) corresponds to the inverse function of (1), which refers to the conditional input demand functions. According to the assumptions in Leontief's model, the supply-demand balance equations, that state how total production is distributed among intermediate and final uses, can be presented as a system of n equations with n unknowns X_1, X_2, \dots, X_n :

$$X_j = \sum_{i=1}^n a_{ij} \cdot X_i + f_j \quad (3)$$

where f_j is final demand for good j . In simpler matrix notation, the system in (3) can be written as:

$$\mathbf{X} = \mathbf{A} \cdot \mathbf{X} + \mathbf{f} \quad (4)$$

Provided some technicalities that are associated to matrix \mathbf{A} are satisfied¹, Expression (4) can be solved having a non-negative solution:

$$\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{f} \quad (5)$$

with $(\mathbf{I} - \mathbf{A})^{-1}$ being the so-called Leontief inverse. This inverse can also be expressed as the sum of direct and indirect effects from unitary changes in the exogenous vector \mathbf{f} , culminating in a matrix of multipliers:

$$(\mathbf{I} - \mathbf{A})^{-1} = (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \dots) = \sum_{s=0}^{\infty} \mathbf{A}^s \quad (6)$$

Combining (5) and (6) we observe:

¹ \mathbf{A} is non-negative, productive and $(\mathbf{I} - \mathbf{A})$ is singular. See Waugh (1950).

$$\mathbf{X} = \mathbf{f} + \mathbf{A} \cdot \mathbf{f} + \mathbf{A}^2 \cdot \mathbf{f} + \mathbf{A}^3 \cdot \mathbf{f} + \dots + \mathbf{A}^k \cdot \mathbf{f} + \dots \quad (7)$$

These multipliers stem from the existing input demand inter-industrial linkages. Equation (7) indicates how the level of output of each sector is generated “round” by “round”, capturing how industrial interdependencies take place in the economy “down the stream” and showing the consolidated structure of the economy’s production chains.

Leontief’s original formulation assumes that all price effects are neglected. This has led to a consideration of the Leontief model as a “quantity model”, i.e. each of the equations of the system (3) links the physical inputs requirements of the *i*-th sector to the output level of the *j*-th sector. This does not imply that changes in prices cannot be reflected when using Leontief’s’ approach. These variations in prices can be evaluated but changes in values and quantities are not simultaneous within this model. This is in fact one of the distinctive characteristics of input-output analysis and Leontief’s theoretical approach. It closely corresponds to the case of a competitive equilibrium where the supply curve is perfectly elastic and thus the overall situation, in terms of equilibrium quantities, is determined by the demand side.

Ghosh (1958) considered Leontief’s model appropriate for describing this economic reality. Ghosh pointed out, however, that in certain situations that depart from the perfectly competitive scenario with excess capacity allocation considerations rather than technological ones should play a major role. In a formally similar way to Leontief’s model, in the Ghosh system intermediate input purchases are also a function of total output:

$$x_{ij} = g_{ij}(X_i) \quad (8)$$

but now X_i refers to the sum of rows in the input-output table, that is, total commodity sales delivered by the *i*-th sector. The inverse function of (8), however, does not

correspond to any production function (Oosterhaven, 1988) but rather to an allocation or distribution function of the form:

$$X_i = \text{Max} \left\{ \frac{x_{i1}}{a_{i1}^*}, \dots, \frac{x_{in}}{a_{in}^*} \right\} \quad (9)$$

where a_{ij}^* represents now the so-called allocation or supply coefficients, which are assumed to be fixed. Any increase in the production of sector i will therefore be allocated in a fixed proportion to all the recipient sectors. Differently to Ghosh's model, in Leontief's model, these supply coefficients do not remain fixed when there is an exogenous change in final demand. In other words, under Leontief's model assumptions, the *ex-ante* and *ex-post* allocation coefficients differ when there is an exogenous shock in final demand while input coefficients remain constant. The situation reverses when using the Ghoshian approach; when there is an exogenous change in value-added, the *ex-ante* and *ex-post* input coefficients differ while allocation coefficients remain constant.

According to (5), the Leontief system may also be rewritten in matrix notation as a vector function ϕ that relates the structural matrix \mathbf{A} and the final demand vector \mathbf{f} to total output:

$$\mathbf{X} = \phi(\mathbf{A}, \mathbf{f}) = (\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{f} \quad (10)$$

with $[(\mathbf{I} - \mathbf{A})^{-1}]_{ij} = [\alpha]_{ij} = \alpha_{ij}$. From here:

$$X_i = \sum_{j=1}^n \alpha_{ij} \cdot f_j \Rightarrow \alpha_{ij} = \frac{\partial \phi_i}{\partial f_j} \quad (11)$$

Notice that the coefficients α_{ij} can be interpreted as the partial derivative of total output of the i -th sector with respect to the final demand of the j -th good. This implies that if each sector's final demand marginally changes by one unit, *caeteris paribus*, the

total effect on the production level of sector j is captured by the expression $\sum_{i=1}^n \alpha_{ij}$, which is commonly and widely interpreted as the backward linkage of this sector. Nevertheless, as indicated in the introduction, it might be also subjected to some criticism since in equilibrium final demand changes in a sector are unresponsive to other sectoral final demand flows.

Under Ghosh's model conditions total output is a vector function φ of the matrix of allocation coefficients, \mathbf{A}^* , and net national income or value added, \mathbf{VA} , generated in each sector:

$$\mathbf{X}' = \varphi(\mathbf{A}^*, \mathbf{VA}) = \mathbf{VA}' \cdot (\mathbf{I} - \mathbf{A}^*)^{-1} \quad (12)$$

with $\left[(\mathbf{I} - \mathbf{A}^*)^{-1} \right]_{ij} = \alpha_{ij}^*$ and \mathbf{VA}' and \mathbf{X}' standing now for row vectors.

Similarly as in (11), we find:

$$X_j = \sum_{i=1}^n \alpha_{ij}^* \cdot VA_i \Rightarrow \alpha_{ij}^* = \frac{\partial \varphi_j}{\partial VA_i} \quad (13)$$

The allocation coefficients α_{ij}^* can therefore be considered too as the partial derivative of the total output of a sector j -th with respect to value added in the i -th sector (Oosterhaven, 1988). Notice however that an increase in primary inputs in one sector would be unconceivable without a further simultaneous increase in the primary input requirements of the remaining sectors in the economy. Thus the assumption of *caeteris paribus* does not hold any more in the Ghoshian approach. This is the argument backing the implausibility of the Ghoshian model as a quantity model claimed by Oosterhaven (1988, 1989) and that has been widely accepted by researchers (Dietzenbacher, 1993, de Mesnard, 2009).

Once we have described the mechanisms and assumptions governing both models, we now proceed to prove that even though both models are in fact theoretically incompatible in their assumptions, if quantities are held constant, the Ghoshian approach is just a reformulation of Leontief's price model.

Proposition: *If input-output ratios and output ratios remain constant (i.e. no changes in quantities take place) then Ghosh's model reduces to Leontief's.*

Proof: Let \mathbf{Z} be the matrix of intermediate deliveries among sectors, $[\mathbf{Z}]_{ij} = x_{ij}$ and let $\hat{\mathbf{X}}$ be the diagonal matrix representation of vector \mathbf{X} . The matrix \mathbf{A} of technical coefficients is obtained as:

$$\mathbf{A} = \mathbf{Z} \cdot \hat{\mathbf{X}}^{-1} \quad (14)$$

whereas the matrix \mathbf{A}^* of allocation coefficients is computed as:

$$\mathbf{A}^* = \hat{\mathbf{X}}^{-1} \cdot \mathbf{Z} \quad (15)$$

Combining Expressions (14) and (15) we find:

$$\mathbf{Z} = \mathbf{A} \cdot \hat{\mathbf{X}}' = \hat{\mathbf{X}} \cdot \mathbf{A}^* \quad (16)$$

Hence the allocation matrix can be written as:

$$\mathbf{A}^* = \hat{\mathbf{X}}^{-1} \cdot \mathbf{A} \cdot \hat{\mathbf{X}} \quad (17)$$

Rewrite (17) as:

$$\mathbf{I} - \mathbf{A}^* = \mathbf{I} - \hat{\mathbf{X}}^{-1} \cdot \mathbf{A} \cdot \hat{\mathbf{X}} \quad (18)$$

Using the inverse of a sum of matrices², we obtain:

$$(\mathbf{I} - \mathbf{A}^*)^{-1} = (\mathbf{I} - \hat{\mathbf{X}}^{-1} \cdot \mathbf{A} \cdot \hat{\mathbf{X}})^{-1} = \mathbf{I} + \hat{\mathbf{X}}^{-1} \cdot \mathbf{A} \cdot (\mathbf{I} - \mathbf{A})^{-1} \cdot \hat{\mathbf{X}} \quad (19)$$

² See H.V. Henderson and S.R. Searle (1981) for the derivation of the inverse of a sum of matrices.

Expression (19) allows us to relate Ghosh's inverse to Leontief's technical coefficient matrix and output ratios. Next we need to show under which conditions exogenous variations in value-added lead to equivalent endogenous effects under both models. The balance accounting equation in value terms in Ghosh's model is given by:

$$\mathbf{X}' = \mathbf{VA}' \cdot (\mathbf{I} - \mathbf{A}^*)^{-1} \quad (20)$$

Substituting now Expression (19) into Expression (20) we can rewrite it as:

$$\mathbf{X}' = \mathbf{VA}' \cdot (\mathbf{I} + \hat{\mathbf{X}}^{-1} \cdot \mathbf{A} \cdot (\mathbf{I} - \mathbf{A})^{-1} \cdot \hat{\mathbf{X}}) \quad (21)$$

Let us translate Expressions (20) and (21) into algebraic notation:

$$\begin{aligned} X_j &= \sum_{i=1}^n VA_i \cdot \alpha_{ij}^* = \sum_{i=1}^n VA_i \cdot \left[\left(\sum_{k=1}^n a_{ik} \cdot \alpha_{kj} \right) \cdot \frac{X_j}{X_i} \right] \quad i \neq j \\ X_j &= \sum_{i=1}^n VA_i \cdot \alpha_{ij}^* = VA_j + \sum_{i=1}^n VA_i \cdot \left[\left(\sum_{k=1}^n a_{ik} \cdot \alpha_{kj} \right) \cdot \frac{X_j}{X_i} \right] \quad i = j \end{aligned} \quad (22)$$

If there is any exogenous change in the value-added flow of sector i -th, and according to (22), the partial endogenous change in the j -th sector is given by:

$$\begin{aligned} \frac{\partial X_j}{\partial VA_i} &= \alpha_{ij}^* = \left(\sum_{k=1}^n a_{ik} \cdot \alpha_{kj} \right) \cdot \frac{X_j}{X_i} \quad (i \neq j) \\ \frac{\partial X_j}{\partial VA_i} &= \alpha_{ij}^* = 1 + \left(\sum_{k=1}^n a_{ik} \cdot \alpha_{kj} \right) \cdot \frac{X_j}{X_i} \quad (i = j) \end{aligned} \quad (23)$$

This concludes the proof. Notice that, according to Expression (23), both the Leontief and the Ghosh models are simultaneously equivalent in their partial effects provided the output ratios (X_j / X_i) remain constant. But this is only possible if there are not any changes in quantities, neither endogenous nor exogenous (the partial derivatives in (23) refer to partial variations in value terms and in quantities), or the changes are proportional everywhere. In this case the own output ratio would remain constant which is incompatible with Ghoshian endogenous quantity impacts, for when $i=j$ the

quantity effects in (23) would then be equal to zero. If quantities remain constant, the only difference between Leontief's and Ghosh's model is the way endogenous value impacts are expressed. Under the standard Leontief's price approach endogenous changes refer to value changes per unit of output while under the "reinterpretation" of the Ghoshian model, price changes are expressed in absolute terms (Dietzenbacher, 1997).

This is in fact what justifies Dietzenbacher's results whereby the Leontief and Ghosh models led to exactly the same endogenous changes as far as these changes relate to value (Ghoshian approach) or price changes (Leontief approach). In proving the *Proposition*, however, we have shown that it is only possible to formally draw these conclusions when the Ghosh model reduces to the Leontief approach, or the other way around. This suggests that the interpretation of the Ghosh model as a price model formulated by Dietzenbacher (1997) should rather be considered as a reformulation of the Leontief model itself under the assumption that quantities are held constant and only value effects are allowed to occur in the economic system. This observation was first suggested by de Mesnard (2009a). Additionally, we have also proven that both models are simultaneously incompatible when analysing quantity changes. This has also an important implication for the validity of the so-called "mixed-model" that merges aspects of both input-output models. Since the allocation and production mechanisms presented in Expressions (2) and (9) are simultaneously incompatible, mixed-models lose their theoretical appeal. The conclusion is that they should not be used in a combined way to measure inter-industrial linkages as some authors have done in the past to measure push-effects using Ghosh's model (Mesnard, 2009b).

3. CLOSING GHOSH'S MODEL.

The previous results show why Ghosh's model reduces to Leontief's and provides additional support for Dietzenbacher's reinterpretation. But can Ghosh's model itself be reclaimed from its implausibility clause? Are there any socio-economic rules for which Ghosh's vision of the economic game make still some sense? We will approach this issue considering a non-market economy where decisions on output allocation are taken by a benevolent central planner whose mission is to enhance the collective good and guarantee a viable distribution of goods. This alleged economy comprises three productive units and distinguishes a private agent (citizens) and a public one (the planner). The private agent provides labour services to all sectors and in exchange receives income (value-added) that is used to finance his consumption needs and his contribution to the sustainment of the collective. From this contribution the planner provides infrastructure services that are used in the allocation process. These services also provide value to the collective, which is in turn used by the public agent to facilitate goods to society in the form of public goods. The aggregate level of these public goods is of course constrained by the overall contributions to the collective.

Let us begin considering a reference or benchmark allocation table for this 3 good-3 sector, economy. The reference data in value flows for such an economy is represented in Table 1. Data in the Table represent an economic arrangement that is allocation feasible and budget feasible for all agents involved, and all magnitudes are value magnitudes.

Table 1: Benchmark allocation data

	Sector 1	Sector 2	Sector 3	Private Agent	Collective	Total
Sector 1	x_{11}	x_{12}	x_{13}	f_1	c_1	X_1
Sector 2	x_{21}	x_{22}	x_{23}	f_2	c_2	X_2
Sector 3	x_{31}	x_{32}	x_{33}	f_3	c_3	X_3
Value-added	VA_1	VA_2	VA_3			
Collective	t_1	t_2	t_3			
Total	X_1	X_2	X_3			

Because of viability the following accounting identities hold true:

$$\sum_{j=1}^3 x_{ij} + f_i + c_i = X_i \quad (i = 1, 2, 3) \quad (24)$$

$$\sum_{i=1}^3 x_{ij} + VA_j + t_j = X_j \quad (j = 1, 2, 3) \quad (25)$$

In Expressions (24) and (25) we have that x_{ij} is the amount of good i flowing to sector j , f_i is the consumption of good i by the private agent, c_i is collective consumption of good i , VA_j is income accruing to the private agent in sector j whereas t_j is the materialization of the contribution to the collective. The identity in Expression (24) shows, by rows, the ‘output’ distribution for each of the goods. Using columns, identity (25) shows the ‘input’ repercussions of the said output allocations that are budget feasible. Because of a ‘Walras-like’ aggregate feasibility constraint (24) and (25) imply:

$$\sum_{i=1}^3 f_i + \sum_{i=1}^3 c_i = \sum_{j=1}^3 VA_j + \sum_{j=1}^3 t_j \quad (26)$$

The left-hand side of (26) can be interpreted as national output as calculated from the expenditure side. The right-hand side, in turn, is national output as obtained from the income side. Alternatively, if the private and public agents behave so as to satisfy some sort of disciplined budget constraint, such as:

$$\sum_{i=1}^3 f_i = \sum_{j=1}^3 VA_j \quad (27)$$

$$\sum_{i=1}^3 c_i = \sum_{j=1}^3 t_j$$

then the national output accounting identity (26) follows from aggregation of the budget constraints in (27).

In matrix terms the input-output data information in Table (1) takes this shape:

$$\mathbf{Z} \cdot \mathbf{e} + \mathbf{f} + \mathbf{c} = \mathbf{X} \quad (24')$$

$$\mathbf{e}' \cdot \mathbf{Z} + \mathbf{VA}' + \mathbf{t}' = \mathbf{X}' \quad (25')$$

where \mathbf{e} is a summation vector. The rest of the notation for matrix \mathbf{Z} and vectors \mathbf{f} , \mathbf{c} , \mathbf{VA}' and \mathbf{t}' is self-explanatory, while \mathbf{e} (\mathbf{e}') denotes a column (row) vector. Let us define now the matrix \mathbf{A}^* of 'allocation' coefficients, that is to say, the information on how output is sectorally distributed among productive agents:

$$[\mathbf{A}^*]_{ij} = \left[\hat{\mathbf{X}}^{-1} \cdot \mathbf{Z} \right]_{ij} = a_{ij}^* = \left(\frac{Z_{ij}}{X_i} \right) \quad (28)$$

The notation, $\hat{\mathbf{X}}$, as before, stands for the diagonalised version of vector \mathbf{X} while $\hat{\mathbf{X}}^{-1}$ is the inverse matrix of $\hat{\mathbf{X}}$. Solving for \mathbf{Z} in Expression (28) and substituting in identity (25') we obtain now an equation in \mathbf{X} :

$$\mathbf{e}' \cdot \mathbf{Z} + \mathbf{VA}' + \mathbf{t}' = \mathbf{e}' \cdot \hat{\mathbf{X}} \cdot \mathbf{A}^* + \mathbf{VA}' + \mathbf{t}' = \mathbf{X}' \cdot \mathbf{A}^* + \mathbf{VA}' + \mathbf{t}' = \mathbf{X}' \quad (29)$$

This equation corresponds to the familiar 'supply-driven' Ghosh's equation and allocated output can be meaningfully solved provided matrix \mathbf{A}^* satisfies the usual productivity condition:

$$\mathbf{X}' = (\mathbf{VA}' + \mathbf{t}') \cdot (\mathbf{I} - \mathbf{A}^*)^{-1} \quad (30)$$

We will now postulate a possible closing for value-added. Define the coefficient λ_i as value-added per unit of aggregate private consumption and d_j as the allocation coefficient for private consumption of good j :

$$\begin{aligned}\lambda_i &= \frac{VA_i}{\sum_{j=1}^3 f_j} \\ d_j &= \frac{f_j}{X_j}\end{aligned}\tag{31}$$

Because of (31) we find:

$$VA_i = \lambda_i \cdot \sum_{j=1}^3 d_j \cdot X_j\tag{32}$$

so that in compact matrix terms (32) becomes:

$$\mathbf{VA} = \begin{pmatrix} VA_1 \\ VA_2 \\ VA_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \cdot (d_1, d_2, d_3) \cdot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \boldsymbol{\lambda} \cdot \mathbf{d}' \cdot \mathbf{X}\tag{33}$$

where the matrix $\boldsymbol{\lambda} \cdot \mathbf{d}'$ reflects the allocation coefficients for private consumption in terms of value-added. Under this possible closed version of the Ghoshian approach, this matrix allows endogenising changes in value added \mathbf{VA} generated and accumulated by the private agent when there is an exogenous change in that part of the production value contributed to the collective, i.e. \mathbf{t} .

We now incorporate Expression (33) into (29) via its transpose:

$$\mathbf{VA}' = (\boldsymbol{\lambda} \cdot \mathbf{d}' \cdot \mathbf{X})' = \mathbf{X}' \cdot (\boldsymbol{\lambda} \cdot \mathbf{d}')' = \mathbf{X}' \cdot \mathbf{d} \cdot \boldsymbol{\lambda}'\tag{34}$$

and we then obtain:

$$\mathbf{X}' = \mathbf{X}' \cdot \mathbf{A}^* + \mathbf{VA}' + \mathbf{t}' = \mathbf{X}' \cdot \mathbf{A}^* + \mathbf{X}' \cdot \mathbf{d} \cdot \boldsymbol{\lambda}' + \mathbf{t}'\tag{35}$$

We can now solve again for \mathbf{X}' under this additional assumption to obtain:

$$\mathbf{X}' = \mathbf{t}' \cdot (\mathbf{I} - \mathbf{A}^* - \mathbf{d} \cdot \boldsymbol{\lambda}')^{-1} \quad (36)$$

The inverse matrix in Expression (36) can be interpreted as the ‘extended’ Ghosh inverse since it incorporates allocation coefficients for material flows, \mathbf{A}^* , and value-added flows, $\mathbf{d} \cdot \boldsymbol{\lambda}'$. Supply shocks are caused by the exogenous actions of the central planner as represented by changes decreed in contributions to the collective, $\Delta \mathbf{t}'$, for example. The output vector \mathbf{X}' that satisfies condition (36) can be interpreted as an ‘allocation equilibrium’ for this economy, and such an equilibrium turns out to be consistent with the allocation rules implicit in \mathbf{A}^* and $\mathbf{d} \cdot \boldsymbol{\lambda}'$ and with the value of contributions to the collective, i.e. \mathbf{t}' . Allocated output is coherently distributed among sectors while at the same time it is value feasible. The new ‘equilibrium’ can be visualized in differential terms from:

$$\Delta \mathbf{X}' = \Delta \mathbf{t}' \cdot (\mathbf{I} - \mathbf{A}^* - \mathbf{d} \cdot \boldsymbol{\lambda}')^{-1} \quad (37)$$

Provided the extended allocation matrix $\mathbf{A}^* + \mathbf{d} \cdot \boldsymbol{\lambda}'$ is also productive, in the sense that $(\mathbf{A}^* + \mathbf{d} \cdot \boldsymbol{\lambda}') \cdot \mathbf{X}' \leq \mathbf{X}'$ for all possible row vectors $\mathbf{t}' \geq 0$, then the new ‘equilibrium’ defined in (37) might be also rewritten as a power series of the form:

$$\Delta \mathbf{X}' = \Delta \mathbf{t}' + \Delta \mathbf{t}' \cdot (\mathbf{A}^* + \mathbf{d} \cdot \boldsymbol{\lambda}') + \Delta \mathbf{t}' \cdot (\mathbf{A}^* + \mathbf{d} \cdot \boldsymbol{\lambda}')^2 + \dots + \Delta \mathbf{t}' \cdot (\mathbf{A}^* + \mathbf{d} \cdot \boldsymbol{\lambda}')^k + \dots \quad (38)$$

Expression (37) indicates that the endogenous effect on output levels can be decomposed into the following components: the “pure” impact of the contribution to the collective that adds value to production, i.e. $\Delta \mathbf{t}'$. This “pure” impact in output should be allocated in the system in the form of intermediate and private final demand, i.e. $\Delta \mathbf{t}' \cdot (\mathbf{A}^* + \mathbf{d} \cdot \boldsymbol{\lambda}')$ generating additional multiplicative effects in output levels. This “second-round” impact further increases production in the remaining sectors round by

round according the structure of the “allocation path” defined in (37), i.e.

$$\Delta t' \cdot (\mathbf{A}^* + \mathbf{d} \cdot \lambda')^2 + \dots + \Delta t' \cdot (\mathbf{A}^* + \mathbf{d} \cdot \lambda')^k + \dots$$

4. CLOSING GHOSH’S MODEL: A NUMERICAL EXAMPLE

To illustrate the formal description and interpretation of the closed Ghosh model presented in Section 3, we use now a numerical example. We start using the 3 sector, 3 good economy whose reference data in value flows is shown in Table 2a.

TABLE 2a: Reference data: Numerical example

	Sector 1	Sector 2	Sector 3	Private Agent	Collective	Total
Sector 1	30	20	10	35	5	100
Sector 2	20	10	40	5	25	100
Sector 3	10	20	5	30	35	100
Value-added	20	10	40			
Collective	20	40	5			
Total	100	100	100			

This economic system has a benevolent central planner that decides to allocate additional resources to sector 1 in such a way that its value contribution increases by 1 units of value. As an example, these additional exogenous resources decided by the central planner would be materialized in new equipment whose services could be used in sector 1 and that increases production levels either in value or quantity terms. This refers to what we have named the “pure impact” in Expression (37), i.e. $\Delta t'$. This impact additionally boosts output levels due to the multiplicative effects generated by this supply shock in the remaining sectors according to the structure of the allocation path in (37), i.e. if additional intermediate demand is allocated to the remaining sectors, there would be endogenous supply effects coming from these sectors that further affect the output values in sector 1 increasing overall value-added in the system.

TABLE 3: Synthetic indicators after evaluating $\Delta t_i=1$ units sequentially in each sector.

Exogenous Shock	% Endogenous Changes in “key” variables				
	ΔVA	ΔX_1	ΔX_2	ΔX_3	ΔX
$\Delta t_1 = 1$ units	2.10	2.60	1.18	1.65	1.81
$\Delta t_2 = 1$ units	1.29	1.08	1.81	1.41	1.43
$\Delta t_3 = 1$ units	1.46	0.96	0.84	2.12	1.31

As it can be asserted from Table 3, the impact of an identical unitary exogenous increase in the contribution from the collective is distributed in an unequal way, picking up the distinct values of the allocation coefficients in each sector. If the flow is contributed to Sector 1, for instance, total value-added in all these sectors increases by 2.10 percent and, on average, total output increases by 1.81 percent. Differently to the open version of the Ghoshian approach, value-added changes everywhere and does so simultaneously and homogeneously. The homogeneity of the endogenous change in value-added is due to the “allocation rules” dictated by the matrix $\mathbf{d} \cdot \lambda'$. The set of Tables 2b-2d in the Annex show the readjusted allocation flows in sectoral detail. Note that each of the additional exogenous units contributed to each sector is fully and endogenously redistributed over collective consumption according to allocation rules. This is because, following the disciplined budget constraints defined in Expression (27) of this closed version of the Ghoshian approach, the exogenous unit contributed from the central planner cannot be withheld by the private agent but rather devoted to collective consumption.

According to the simulation results presented in Table 3, if the benevolent central planner wished to maximise economy-wide effects, the contribution to the collective

should be decreed in Sector 1. This is so because the implied reallocation effects, both in value-added and in output, are higher here than those obtainable should the contribution be allotted in Sectors 2 or 3.

These conclusions are independent from the benchmark value flows of the contribution to the collective. Consequently, there is no need to perform any normalization to appraise their robustness. Notice that the exogenous shock is carried out homogeneously in all three sectors and the impacts in Table 3 depict the percentage between benchmark and simulated allocations.

These numerical examples of the closed Ghosh model outlined in Section 3 approximate better, we believe, the initial idea posed by Ghosh. In his seminal work, this author highlighted that his approach could be used, in planned economies, for the assessment of economy-wide impacts of government employment programs. The main question that Ghosh wanted to address using his modelling proposal was the following: if the labour force is forcefully allocated in a given sector, what would the economy-wide impact be according to the allocation rules that are used in planned economies?. The economy-wide output impacts of the open version of this model turn out not to be value feasible when answering this question (Oosterhaven, 1988). Our close version, however, not only makes it possible to answer this question in a more plausible way but also helps in understanding the initial purposes of A. Ghosh.

5. CONCLUSIONS

The objective of this paper is simply to recover the initial purposes of A. Ghosh while trying to contribute to the long debate in the literature around his original model published in 1958. Since then, there has been an over-use of his model by researchers that, after some time, has given rise to an over-criticism. The main target of this paper

is then to present a compromise between the two existing positions in academics around this issue.

To this end we describe a way of closing the Ghoshian approach that resolves, to some extent, the implausibility problem that afflicts its open version (Oosterhaven, 1988, 1989). Supply shocks in this modified version of the Ghosh model stem from the actions of a benevolent central planner. This central planner exogenously contributes to production which will generate value to the economic system, applying it in one or more sectors. This initial impact is spread through the economic system further boosting output levels that, differently to the open version, are accompanied by simultaneous endogenous and allocation compatible increases in value-added. Therefore, closing Ghosh's model makes it more plausible.

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ANNEX OF TABLES

TABLE 2b: Simulated percentage changes of the Reference data after evaluating an exogenous supply shock in Sector 1: $\Delta t_1 = 1$ units of value.

	Sector 1	Sector 2	Sector 3	Private Agent	Collective	Total
Sector 1	30.78	20.52	10.26	35.91	5.13	102.60
Sector 2	20.24	10.12	40.47	5.06	25.29	101.18
Sector 3	10.17	20.331	5.08	30.49	35.58	101.65
Value-added	20.42	10.21	40.84			
Collective	21.00	40.00	5.00			
Total	102.60	101.18	101.65			

TABLE 2c: Simulated percentage changes of the Reference data after evaluating an exogenous supply shock in Sector 2: $\Delta t_2 = 1$ units of value

	Sector 1	Sector 2	Sector 3	Private Agent	Collective	Total
Sector 1	30.33	20.22	10.11	35.38	5.05	101.08
Sector 2	20.36	10.18	40.72	5.09	25.45	101.81
Sector 3	10.14	20.28	5.07	30.42	35.49	101.41
Value-added	20.26	10.13	40.51			
Collective	2.00	41.00	5.00			
Total	101.08	101.81	101.41			

TABLE 2.d: Simulated percentage changes of the Reference data after evaluating an exogenous supply shock in Sector 3: $\Delta t_3 = 1$ units of value.

	Sector 1	Sector 2	Sector 3	Private Agent	Collective	Total
Sector 1	30.29	20.19	10.10	35.34	5.05	100.96
Sector 2	20.17	10.08	40.34	5.04	25.21	100.84
Sector 3	10.21	20.42	5.11	30.64	35.74	102.12
Value-added	20.29	10.15	40.58			
Collective	20.00	40.00	6.00			
Total	100.96	100.84	102.12			