

## **Economic Growth with Bubbles**

**Alberto Martín  
Jaume Ventura**

**July 2011**

*Barcelona GSE Working Paper Series*

*Working Paper n° 445*

# Economic Growth with Bubbles

Alberto Martin and Jaume Ventura\*

July 2011

## Abstract

We develop a stylized model of economic growth with bubbles. In this model, changes in investor sentiment lead to the appearance and collapse of macroeconomic bubbles or pyramid schemes. We show how these bubbles mitigate the effects of financial frictions. During bubbly episodes, unproductive investors demand bubbles while productive investors supply them. These transfers of resources improve the efficiency at which the economy operates, expanding consumption, the capital stock and output. When bubbly episodes end, these transfers stop and consumption, the capital stock and output contract. We characterize the stochastic equilibria of the model and argue that they provide a natural way of introducing shocks to asset prices into business cycle models.

**JEL classification:** E32, E44, O40

**Keywords:** bubbles, dynamic inefficiency, economic growth, financial frictions, pyramid schemes

---

\*Martin: CREI and Universitat Pompeu Fabra, amartin@crei.cat. Ventura: CREI and Universitat Pompeu Fabra, jventura@crei.cat. CREI, Universitat Pompeu Fabra, Ramon Trias Fargas 25-27, 08005-Barcelona, Spain. We thank Vasco Carvalho, Dirk Krueger and one anonymous referee for very insightful comments. We acknowledge support from the Spanish Ministry of Science and Innovation (grants ECO2008-01666 and CSD2006-00016), the Generalitat de Catalunya-DIUE (grant 2009SGR1157), and the Barcelona GSE Research Network. In addition, Ventura acknowledges support from the ERC (Advanced Grant FP7-249588), and Martin from the Spanish Ministry of Science and Innovation (grant Ramon y Cajal RYC-2009-04624).

# 1 Introduction

One of the most striking macroeconomic developments of recent decades has been the large swings in asset price movements relative to economic activity. Figure 1 illustrates this by plotting the ratio of US household net worth to GDP between 1990 and 2010. This ratio largely mirrors the behavior of real estate and stock prices.<sup>1</sup>

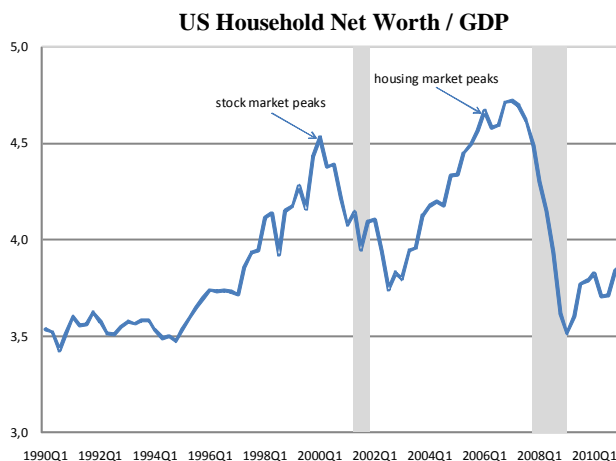


Figure 1

Figure 1 tells a fascinating story. From 1990 until 1995, and also towards the end of the sample, household net worth was approximately 3.5 times GDP. Between 1995 and 2009, however, it twice experienced episodes of substantial growth followed by quick collapses. In each of these episodes, household net worth grew and fell by many trillions of dollars in a few years, gaining a year's worth of GDP before quickly shedding it again. Moreover, each of these episodes closely tracked asset prices. The first one, starting in 1995 and peaking in early 2000, coincides exactly with the rise and fall of the stock market. During this period, the Dow Jones industrial average more than tripled before peaking in January of 2000 and starting to fall: by late 2002, the index had lost 43% of its value and, sure enough, household net worth had also lost most of the ground gained relative to GDP. Roughly around the time of this stock market collapse, though, home prices began to rise at unprecedented rates and this eventually gave rise to a second episode of spectacular growth in household net worth. Between September of 2000 and the first quarter of 2006, real home prices

---

<sup>1</sup>Real estate and holdings of corporate equity (direct and indirect) make up roughly 70% of household net worth. Other important categories of household assets are durable goods, cash and deposits, credit market instruments and other financial assets.

increased by 57,4%: this rise, eventually coupled with a rebound in stock prices, boosted household net worth once more to record levels. From there onwards home prices began to fall, slowly at first and much more rapidly after June of 2007. In March of 2009, they had reverted to their 2000 levels and this, together with the fall in stock prices, brought household net worth back to its initial value as well.

Besides their direct effect on household wealth, these movements in asset prices were associated with substantial changes in macroeconomic aggregates. As Figure 2 shows, the growth rates of output, consumption and the capital stock essentially tracked asset prices and net worth during both episodes, accelerating as asset prices increased and slowing down when asset prices fell.<sup>2</sup> Indeed, both episodes ended in economic recessions as depicted by the shaded bars in Figure 1.

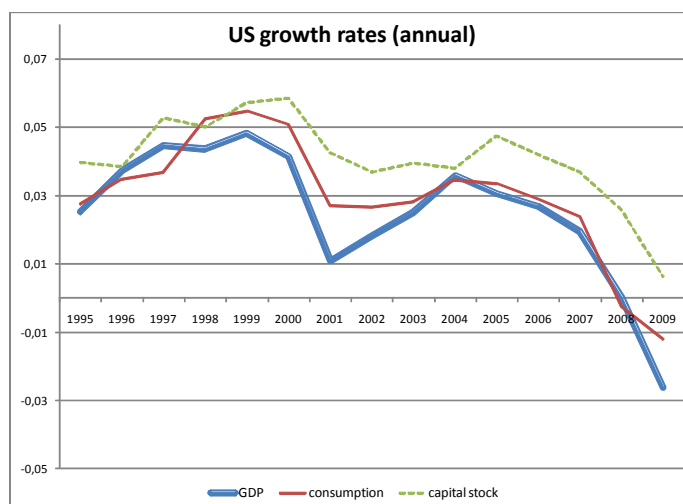


Figure 2

Remarkably, it has proved exceedingly difficult to attribute either of these episodes of assets booms and busts to changes in economic fundamentals.<sup>3</sup> Thus, they are commonly referred to as

<sup>2</sup>Real GDP and consumption data are obtained directly from the BEA. To construct the capital stock data, we use investment data from the PWT and follow Caselli (2005).

<sup>3</sup>In an extensive study on the rise of stock prices between 1995 and 2000, LeRoy (2004) documented that the value of equity far exceeded the growth of corporate earnings and of dividends throughout. He also considered and discarded other popular explanations for the rise in equity prices, most notably those based on demographics and on the valuation of intangible capital. Many of these stories have the shortcoming that, once they are invoked to account for the rapid increase in equity prices between 1995 and 1999, they are unable to account for the rapid collapse that followed. Something similar can be said of the run-up in home prices: during the boom years, people tried to account for it by invoking the evolution of demographics, the interest rate and construction costs. Already then, Shiller (2005) showed that these explanations were inadequate. Today, they seem even less plausible because they should also be consistent with the ensuing collapse in home prices.

“bubbles” or “bubbly episodes”. But what are these bubbles? What is their origin? How do they affect output, consumption, and the capital stock? To address these questions, we need a theory of economic growth with bubbles.

The goal of this paper is to provide such a theory. We consider two idealized asset classes: productive assets or “capital” and pyramid schemes or “bubbles”. Both assets are used as a store of value or savings vehicle, but they have different characteristics. Capital is costly to produce but it is then useful in production. Bubbles play no role in production, but initiating them is costless.<sup>4</sup> We consider environments with rational, informed and risk neutral investors that hold only those assets that offer the highest expected return. The theoretical challenge is to identify situations in which these investors optimally choose to hold bubbles in their portfolios and then characterize the macroeconomic consequences of their choice.

Our approach builds on the seminal papers of Samuelson (1958) and Tirole (1985) who developed a theory of rational bubbles as a remedy to the problem of dynamic inefficiency.<sup>5</sup> Their argument is based on the dual role of capital as a productive asset and a store of value. To satisfy the need for a store of value, economies sometimes accumulate so much capital that the investment required to sustain it exceeds the income that it produces. This investment is inefficient and lowers the resources available for consumption. In this situation, bubbles can be both attractive to investors and feasible from a macroeconomic perspective. For instance, a pyramid scheme that absorbs all inefficient investments in each period is feasible and its return exceeds that of the investments it replaces.

The Samuelson-Tirole model provides an elegant and powerful framework to think about bubbles. However, the picture that emerges from this theory is hard to reconcile with the episodes in Figure 1. First, the model features deterministic bubbles that exist from the very beginning of time and never burst. This is contrary to the observation that, in these episodes, bubbles seem to pop up and burst. We therefore need a model in which bubbles are transient, that is, a model of bubbly episodes. Second, and most importantly, in the Samuelson-Tirole model bubbles raise consumption

---

<sup>4</sup>It is difficult to find these idealized asset classes in financial markets, of course, as existing assets bundle or package together capital and bubbles. Yet much can be learned by working with these basic assets. To provide an obvious analogy, we have surely learned much by studying theoretical economies with a full set of Arrow-Debreu securities even though only a few bundles or packages of these basic securities are traded in the real world.

<sup>5</sup>Our research is also indebted to previous work on bubbles and economic growth. Saint-Paul (1992), Grossman and Yanagawa (1993), and King and Ferguson (1993) extend the Samuelson-Tirole model to economies with endogenous growth due to externalities in capital accumulation. In their models, bubbles slow down the growth rate of the economy. Olivier (2000) uses a similar model to show how, if tied to R&D firms, bubbles might foster technological progress and growth.

by reducing inefficient investments. As a result, bubbles slow down capital accumulation and lower output. Figure 2 shows that bubbly episodes are associated with consumption booms indeed. But they are also associated with rapid expansions in the capital stock and output. A successful model of bubbles must come to grips with these correlations.

We overcome these two shortcomings of the Samuelson-Tirole model by introducing investor sentiment shocks and imperfect financial markets into the theory of rational bubbles. Since bubbles have no intrinsic value, their current size depends on market expectations regarding their future size, i.e. on “investor sentiment”. Introducing shocks to investor sentiment is therefore crucial to generate realistic bubble dynamics in the model.<sup>6</sup> Introducing financial frictions is also crucial because these create rate-of-return differentials and allow efficient and inefficient investments to coexist. Our key observation is then quite simple: bubbles not only reduce inefficient investments, but also increase efficient ones. In our model, bubbly episodes are booms in consumption and efficient investments financed by a reduction in inefficient investments. If efficient investments increase enough, bubbly episodes expand the capital stock and output. This turns out to be the case under a wide range of parameter values.<sup>7</sup>

To understand these effects of bubbly episodes, it is useful to analyze the set of transfers that bubbles implement. Remember that a bubble is nothing but a pyramid scheme by which the buyer surrenders resources today expecting that future buyers will surrender resources to him/her. The economy enters each period with an initial distribution of bubble owners. Some of these owners bought their bubbles in earlier periods, while others just created them. When the market for bubbles opens, on the demand side we find investors who cannot obtain a return to investment above that of bubbles; while on the supply side we find consumers and investors who can obtain a return to investment above that of bubbles. When the market for bubbles clears, resources have been transferred from inefficient investors to consumers and efficient investors.

A key aspect of the theory is how the distribution of bubble owners is determined. As in the Samuelson-Tirole model, our economy is populated by overlapping generations that live for two

---

<sup>6</sup>To the best of our knowledge, Weil (1987) was the first to consider stochastic bubbles in general equilibrium.

<sup>7</sup>The introduction of financial frictions also solves an empirical problem of the theory of rational bubbles. Abel et al. (1989) examined a group of developed economies and found that, in all of them, investment falls short of capital income. This finding, which means that the average investment is dynamically efficient, has often been used to argue that in real economies the conditions for the existence of rational bubbles are not satisfied. But this argument is not quite right. Even if the average investment is dynamically efficient, the economy might contain some dynamically inefficient investments that could support a bubble. Moreover, it is also possible that an expansionary bubble, by lowering the return to investment, creates itself the dynamically inefficient investments that support it. Woodford (1990) and Azariadis and Smith (1993) were, to the best of our knowledge, the first to show that financial frictions could relax the conditions for the existence of rational bubbles.

periods. The young invest and the old consume. The economy enters each period with two types of bubble owners: the old who acquired bubbles during their youth, and the young who are lucky enough to create new bubbles. Since the old only consume, bubble creation by efficient young investors plays a crucial role in our model: it allows them to finance additional investment by selling bubbles.

But where is the market for bubbles in real economies? We show that no formal changes to the theory are required if bubbles are attached to specific assets like firms or real estate. Bubbly episodes are then characterized by high and rising firm and real estate prices. Young individuals are nonetheless willing to purchase firms and real state in the expectation that their price will remain high in the future. Some of these individuals are even lucky enough to create new bubbles, i.e. to increase the size of bubbles attached to existing assets. If some of these young individuals are productive enough to obtain a return above that of bubbles, they can effectively sell them by borrowing against them in the credit market. Thus, nothing changes when we replace the abstract market for bubbles with stock, real state and credits markets, except that the transfer of resources towards efficient investment is now carried out in these markets and not in the market for bubbles.

There has been quite a bit of theoretical interest on bubbles recently. Like our paper, most of this work has been motivated by recent events in the United States and similar ones elsewhere<sup>8</sup>, and it has focused on the effects of bubbles in the presence of financial frictions: (i) Caballero and Krishnamurthy (2006) and Farhi and Tirole (2011) show that bubbles can be a useful source of liquidity;<sup>9</sup> (ii) Kocherlakota (2009) and Martin and Ventura (2011) show that bubbles can also raise collateral or net worth;<sup>10</sup> and (iii) Ventura (2011) shows that bubbles can lower the cost of capital.<sup>11</sup> Unlike these papers, and due to the simplicity of our setup, we are able to provide a full

---

<sup>8</sup>Japan experienced a similar event in the late 1980s, which created and destroyed substantial amounts of wealth in a few years. The stock market nearly doubled between 1987 and 1990, while land prices increased until the market value of all the land in Japan exceeded four times the land value of the United States. This boom was followed by a bust that, by 1993, had completely undone the increase in the stock market and halved the price of land. Despite the magnitude of this episode, it is hard to attribute it to changes in economic fundamentals (French and Poterba (1991)).

<sup>9</sup>There is, of course, a long tradition of papers that view fiat money as a bubble. Indeed, Samuelson (1958) adopted this interpretation. For a recent paper that also emphasizes the liquidity-enhancing role of fiat money in the presence of financial frictions, see Kiyotaki and Moore (2008).

<sup>10</sup>Giglio and Severo (2011) study an environment in which physical capital can be used as collateral whereas intangible capital cannot. They show that this can lead to excessive investment in physical capital, which may make rational bubbles possible.

<sup>11</sup>This paper is the closest in spirit to ours. Ventura (2011) models a multi-country world in which financial frictions impede capital flows. In this model, there are many markets for country bubbles. When this market is active, the capital stock falls in the country, but this lowers the price of investment goods and raises the capital stock in the rest of the world. The paper then uses a few examples to study how shocks are transmitted across countries.

characterization of all the stochastic equilibria of the model and to show that they provide a natural way of introducing asset-price shocks into business-cycle models. Finally, there are two papers that have used rational bubbles to interpret recent macroeconomic developments more directly: Kraay and Ventura (2007) use a model of bubbles and capital flows to study the origin and effects of global imbalances, while Martin and Ventura (2011) use a model of bubbles and the financial accelerator to interpret the 2007-08 financial crisis and its effects.

There has also been a growing empirical interest on the general relationship between asset prices and the macroeconomy. This interest has been partly motivated by some of the events outlined above but also by the development of macroeconomic models with financial frictions, which give asset prices an important role in determining the level of financial intermediation and economic activity. Our theory is consistent with the main findings of this literature. First, large movements in asset prices are fairly common in industrialized economies. An extensive IMF (2003) study on asset prices in these economies found that, during the postwar period, equity price busts occurred on average once every 13 years whereas housing busts occurred on average every 20 years. Both equity and housing price busts entailed significant average price declines, of 45 and 30 percent respectively. Second, there is ample evidence that equity and housing price changes are closely correlated with – and tend to lead – output growth.<sup>12</sup> In industrialized economies, the average equity bust of the postwar period has been associated with GDP losses of about 4 percent whereas the average housing bust has been associated with GDP losses of about 8 percent (IMF 2003). Third, there is mounting evidence based on firm-level data that asset prices have a direct and independent effect on investment decisions.<sup>13</sup> Gan (2007) analyzed firm- and loan-level data corresponding to the late 1990s in Japan in order to quantify the impact of a large decline in asset markets on firms' investment decisions. Based on a sample containing all publicly traded manufacturing firms, he found that the collapse of land prices had a significant and negative effect on corporate investment.<sup>14</sup> More recently, Chaney et al. (2008) have documented similar results for the US economy during the 1993-2007 period.<sup>15</sup>

---

<sup>12</sup>See IMF (2000) for a review of the literature that documents these correlations.

<sup>13</sup>Our theory would also predict that, through this effect on investment, asset prices should affect the misallocation of resources and hence the dispersion of productivity. Although there is some firm-level evidence indicating that misallocation indeed increases during recessions (Eisfeldt and Rampini (2006), Kehrig (2011), and Sandleris and Wright (2010) in the context of the Argentine crisis), we are not aware of any evidence that relates this misallocation to asset prices.

<sup>14</sup>Specifically, he found a reduction in the investment rate of 0.8% for every 10% decline in land value. In a related study, Goyal and Yamada (2004) found that the evolution of stock prices in Japan during the late 1980s and early 1990s also had a significant effect on corporate investment.

<sup>15</sup>Using firm-level data, they found that a one dollar increase in the value of its real estate leads the average US



## 2 The Model

This section develops a model that builds on the seminal contributions of Samuelson (1958), Diamond (1965) and Tirole (1985). It introduces two new elements which turn out to be crucial for the analysis. The first one is random creation and destruction of bubbles. The second one is financial frictions. None of these two pieces is new. But their combination creates a novel and quite suggestive view of the origins and effects of bubbly episodes in real economies.

### 2.1 Setup

Consider an economy inhabited by overlapping generations of young and old. Time starts at  $t = 0$  and then goes on forever. Define  $h_t$  as the vector of all shocks in period  $t$ . As we shall see, this economy does not experience technology or preference shocks. However, this economy displays stochastic equilibria with bubble or investor sentiment shocks. We define  $h^t$  as a history of shocks until period  $t$ , i.e.  $h^t = \{h_0, h_1, \dots, h_t\}$ ; and we let  $H_t$  be the set of all possible histories, i.e.  $h^t \in H_t$ . All variables should be indexed by period and history, i.e.  $t$  and  $h^t$ . To simplify notation, we omit the history sub-index since this is not confusing. We assume that all individuals know the history of shocks  $h^t$  when they compute expectations in period  $t$ .

Each generation contains a continuum of individuals of size one, indexed by  $i \in I_t$ . Individuals maximize expected old-age consumption, i.e.  $U_{it} = E_t \{c_{it+1}\}$ ; where  $U_{it}$  and  $c_{it+1}$  are the utility and the old-age consumption of individual  $i$  of generation  $t$ . Individuals supply one unit of labor when young. Since they care only about old age consumption, individuals save their entire labor income. Since they are risk-neutral, individuals choose the portfolio that maximizes the expected return to their savings.

The output of the economy is given by a Cobb-Douglas production function:  $F(l_t, k_t) = l_t^{1-\alpha} \cdot k_t^\alpha$  with  $\alpha \in (0, 1)$ , where  $l_t$  and  $k_t$  are the labor force and capital stock, respectively. Since the young have one unit of labor,  $l_t = 1$ . Markets are competitive and factors of production are paid the value of their marginal product:

$$w_t = (1 - \alpha) \cdot k_t^\alpha \quad \text{and} \quad r_t = \alpha \cdot k_t^{\alpha-1}, \quad (1)$$

---

corporation to raise its investment by 6 cents. This implies that a drop in real estate prices of 35%, like the one that has happened in the US since 2006, depresses aggregate investment by more than 5% purely because of financial frictions.

where  $w_t$  and  $r_t$  are the wage and the rental rate, respectively.

The stock of capital in period  $t + 1$  depends on the investment made by generation  $t$  during its youth.<sup>16</sup> In particular, we have that:

$$k_{t+1} = A_t \cdot s_t \cdot k_t^\alpha, \quad (2)$$

where  $s_t$  is the investment rate, i.e. the fraction of output that is devoted to capital formation; and  $A_t$  is the investment efficiency, i.e. the units of capital that are created for each unit of output that is devoted to capital formation.

To solve the model, we need to find the investment rate and its efficiency. At this point, it is customary to assume that the young use all their savings to build capital. This means that the investment rate equals the savings of the young. Since the latter equal labor income, which is a constant fraction  $1 - \alpha$  of output, the investment rate is constant as in the classic Solow model:

$$s_t = 1 - \alpha \equiv s. \quad (3)$$

How efficient are the young at building capital? We now introduce a key feature of the model, namely, a friction in financial markets that lowers investment efficiency. We start by creating gains from financial trade by assuming that some individuals are better at investing than others. In particular, a fraction  $\varepsilon \in [0, 1]$  of the young can produce one unit of capital with one unit of output, while the rest only have access to an inferior technology that produces  $\delta < 1$  units of capital with one unit of output. We refer to these two types as “productive” and “unproductive” investors, respectively. If financial markets worked well, productive investors would borrow from unproductive ones and invest on everyone’s behalf. The aggregate investment efficiency would be one. We assume however a friction in financial markets that prevents this borrowing altogether and, as a result, unproductive investors are forced to make their own investments.<sup>17</sup> This means that the average investment efficiency is given by:

$$A_t = \varepsilon + (1 - \varepsilon) \cdot \delta \equiv A. \quad (4)$$

---

<sup>16</sup>We assume that (i) capital fully depreciates in production; and (ii) the first generation found some positive amount of capital to work with, i.e.  $k_0 > 0$ .

<sup>17</sup>To fix ideas, assume that individuals cannot commit to making any future payments due to weak enforcement institutions. This effectively prevents them from issuing any contingent or non-contingent debts. In section 4, we shall discuss further the origins and effects of this financial friction.

Since all individuals invest the same amount, the average efficiency of investment is determined by the population weights of both types of investors.

Substituting Equations (3) and (4) into Equation (2), we find the dynamics of the capital stock:

$$k_{t+1} = A \cdot s \cdot k_t^\alpha. \quad (5)$$

Equation (2) constitutes a very stylized version of a workhorse model of modern macroeconomics. This model can be extended by adding more sophisticated formulations of preferences and technology, various types of shocks, a few market imperfections, and a role for money. Instead, we follow Samuelson (1958) and Tirole (1985) and open a market for bubbles or pyramid schemes.

## 2.2 Equilibria with bubbles

We introduce now a market for bubbles. Unlike capital, bubbles start randomly and without cost, they do not produce any output and the only reason to purchase them is to resell them later. Bubbles are akin to pyramid schemes. In a pyramid scheme, any contribution is voluntary and entitles the contributor to receive next period's contribution. There are two aspects to such a scheme that are of interest to us. The creator of the pyramid scheme obtains a windfall, since he/she receives the first contribution without having contributed before. Participants in the scheme (other than the creator) effectively purchase the right to the next contribution with their own contribution. These two aspects also define bubbles. The creator of a bubble obtains a windfall equal to its initial market price, while individuals that purchase a fraction of this bubble obtain a pro-rata share of its price next period. A key feature of bubbles is that they do not constitute a promise by the seller to deliver future payments. Thus, bubbles might be traded even in situations such as the one considered here in which borrowing is not possible at all.

What does it mean to purchase a bubble? One could think of bubbles as being attached to specific objects and, in this case, bubble purchases would entail the physical transfer of these objects. One could alternatively think of bubbles as a collective memory of past contributions to pyramid schemes and, in this case, bubble purchases would amount to registering new entries in this collective memory. Without imposing further structure, however, the theory presented here is silent on these issues and has implications only for the following aggregates: (i)  $b_t$  which is the market price of the portfolio that contains *all* old bubbles, i.e. already existing before period  $t$  or created by earlier generations; and (ii)  $b_t^{NP}$  and  $b_t^{NU}$  which are the market prices of the portfolios

that contain *all* new bubbles created by productive and unproductive investors respectively, i.e. bubbles added in period  $t$  or created by generation  $t$ .<sup>18</sup> To keep the formal analysis as general as possible, we first develop theoretical implications for these aggregates only. In section 4, we impose further structure and re-interpret bubbles in terms of stock, real estate prices and credit.

Let us describe how the market for bubbles works.<sup>19</sup> On the supply side, there are two types of bubble owners: the old who acquired bubbles during their youth and the young who are lucky enough to create new ones. On the demand side, there can only be the young since the old do not save. Throughout, we assume that there is free disposal of bubbles. This implies that  $b_t \geq 0$ ,  $b_t^{NP} \geq 0$  and  $b_t^{NU} \geq 0$ . Then, the following conditions must hold in all dates and histories in which  $b_t + b_t^{NP} + b_t^{NU} > 0$ :

$$E_t \left\{ \frac{b_{t+1}}{b_t + b_t^{NP} + b_t^{NU}} \right\} \begin{cases} = \delta \cdot \alpha \cdot k_{t+1}^{\alpha-1} & \text{if } \frac{b_t + b_t^{NP}}{(1-\varepsilon) \cdot s \cdot k_t^\alpha} < 1 \\ \in [\delta \cdot \alpha \cdot k_{t+1}^{\alpha-1}, \alpha \cdot k_{t+1}^{\alpha-1}] & \text{if } \frac{b_t + b_t^{NP}}{(1-\varepsilon) \cdot s \cdot k_t^\alpha} = 1 \\ = \alpha \cdot k_{t+1}^{\alpha-1} & \text{if } \frac{b_t + b_t^{NP}}{(1-\varepsilon) \cdot s \cdot k_t^\alpha} > 1 \end{cases}, \quad (6)$$

$$0 \leq b_t \leq s \cdot k_t^\alpha. \quad (7)$$

Equation (6) follows from the first-order conditions of the portfolio allocation problem of individuals. For bubbles to be attractive to a particular investor, they must deliver at least the same return as capital. The return to holding the bubble consists of its growth over the holding period. The purchase price of the bubble is  $b_t + b_t^{NP} + b_t^{NU}$ , and the selling price is  $b_{t+1}$ . The return to investing on capital equals the rental rate divided by the cost of capital, which is one for productive investors and  $\delta^{-1}$  for unproductive ones. Equation (6) then recognizes that the marginal buyer of the bubble changes as the bubble grows. If the bubble is small, the marginal buyer is an unproductive investor and the expected return to the bubble must equal the return to unproductive investments. If the bubble is large, the marginal buyer is a productive investor and the expected

<sup>18</sup>Throughout, we assume that there is free disposal of bubbles. This implies that  $b_t \geq 0$ ,  $b_t^{NP} \geq 0$  and  $b_t^{NU} \geq 0$ .

<sup>19</sup>Let  $b_{it}$  and  $b_{it}^N$  denote the bubble demanded and created by individual  $i \in I_t$  in period  $t$ , respectively. We can write the intertemporal budget constraint of this individual as follows:

$$c_{it+1} = r_{t+1} \cdot A_i \cdot (w_t + b_{it}^N - b_{it}) + \left( \frac{b_{t+1}}{b_t + b_t^{NP} + b_t^{NU}} \right) \cdot b_{it},$$

where  $A_i = 1$  if individual  $i$  is productive and  $A_i = \delta$  otherwise and  $\frac{b_{t+1}}{b_t + b_t^{NP} + b_t^{NU}}$  is the return to holding bubbles. Naturally,  $b_t = \int_{i \in I_t} b_{it}$ ,  $b_t^{NP} = \int_{i \in I_t^P} b_{it}^N$  and  $b_t^{NU} = \int_{i \in I_t^U} b_{it}^N$ ; where  $I_t^P$  and  $I_t^U$  are the sets of productive and unproductive investors in generation  $t$ .

return to the bubble must be the return to productive investments.<sup>20</sup> Equation (7) imposes the non-negativity constraints on both bubbles and capital. That bubbles must be positive follows from our free-disposal assumption. That bubbles cannot exceed the savings of the young, i.e.  $s \cdot k_t^\alpha + b_t^{NP} + b_t^{NU} \geq b_t + b_t^{NP} + b_t^{NU}$ , follows from the non-negativity constraint on the capital stock.

One can summarize this discussion by saying that the theory imposes two restrictions on the type of bubbles that can exist. On the one hand, bubbles must grow fast enough or otherwise the young will not be willing to purchase them. This restriction is embedded in Equation (6). On the other hand, the aggregate bubble cannot grow too fast or otherwise the young will not be able to purchase them. This restriction is embedded in Equation (7). The tension between these two restrictions is what determines the set of equilibrium bubbles, as we show in section 3.

The presence of a market for bubbles has potentially important macroeconomic effects that work through capital accumulation. To see this, we first derive the dynamics of the capital stock in the presence of bubbles:

$$k_{t+1} = \begin{cases} A \cdot s \cdot k_t^\alpha + (1 - \delta) \cdot b_t^{NP} - \delta \cdot b_t & \text{if } \frac{b_t + b_t^{NP}}{(1 - \varepsilon) \cdot s \cdot k_t^\alpha} < 1 \\ s \cdot k_t^\alpha - b_t & \text{if } \frac{b_t + b_t^{NP}}{(1 - \varepsilon) \cdot s \cdot k_t^\alpha} \geq 1 \end{cases}, \quad (8)$$

Equation (8) has two steps that depend on who is the marginal buyer of the bubble. When the bubble is small, the marginal buyer is an unproductive investor. In this case, capital accumulation equals the savings of the productive investors times their efficiency (which is one), i.e.  $\varepsilon \cdot s \cdot k_t^\alpha + b_t^{NP}$ ; plus the savings of the unproductive investors minus the value of the bubbles they purchase times their efficiency (which is  $\delta$ ), i.e.  $\delta \cdot [(1 - \varepsilon) \cdot s \cdot k_t^\alpha + b_t^{NU} - b_t - b_t^{NP} - b_t^{NU}]$ . When the bubble is large, the marginal buyer is a productive investor. In this case, unproductive investors do not build capital and capital accumulation equals the savings of the productive investors i.e.  $\varepsilon \cdot s \cdot k_t^\alpha + b_t^{NP}$ ; minus the bubbles they purchase, i.e.  $b_t + b_t^{NP} + b_t^{NU} - (1 - \varepsilon) \cdot s \cdot k_t^\alpha - b_t^{NU}$ .

Equation (8) nicely illustrates the two macroeconomic effects of bubbles. The first one is the classic *crowding-out* effect: when the old sell bubbles to the young, consumption grows and investment falls. This is why  $b_t$  slows down capital accumulation. Interestingly, the bubble crowds out first unproductive investments. It is only when there are no unproductive investments left that

---

<sup>20</sup>Bubbles cannot deliver a higher return than productive investments. Assume this were the case. Then, nobody would invest and the return to investment would be infinite. But this means that the bubble would be growing at an infinite rate and this is not possible.

the bubble starts to crowd out productive investments. This ability of the bubble to eliminate the ‘right’ investments raises average investment efficiency and minimizes this crowding-out effect. The second macroeconomic effect of bubbles is a new *reallocation* effect: when the productive young sell bubbles to the unproductive young, productive investments replace unproductive ones. This effect further raises average investment efficiency and explains why  $b_t^{NP}$  speeds up capital accumulation. The relative magnitudes of these two effects is unclear at this point since we do not know the relative size of  $b_t$  and  $b_t^{NP}$ . We return to this issue in section 3 when we discuss the equilibria of this economy.<sup>21</sup>

We now provide a formal definition of a competitive equilibrium for this economy. For a given initial capital stock and bubble,  $k_0 > 0$  and  $b_0 \geq 0$ , a competitive equilibrium is a non-negative sequence  $\{k_t, b_t, b_t^{NP}, b_t^{NU}\}_{t=0}^{\infty}$  satisfying Equations (6), (7) and (8). Closing down the market for bubbles is equivalent to adding the additional equilibrium restriction that  $b_t + b_t^{NP} + b_t^{NU} = 0$  for all periods and histories. This restriction cannot be justified on ‘a priori’ grounds, but we note that there always exists one equilibrium in which it is satisfied. This “fundamental” equilibrium, as described by Equation (5), is the one macroeconomics has focused on almost exclusively.

Before analyzing the equilibria of this model, we explain how it differs from (and what it adds to) the original models of Samuelson (1958) and Tirole (1985). Unlike us, both Samuelson and Tirole restricted their analysis to the subset of equilibria that are deterministic and do not involve bubble creation or destruction. That is, they imposed the additional restrictions that  $E_t b_{t+1} = b_{t+1}$  and  $b_t^{NP} = b_t^{NU} = 0$  for all  $t$ . With these restrictions, any bubble must have existed from the very beginning of time and it can never burst, i.e. its value can never be zero. This makes their models unsuitable to study the type of episodes that interest us. We therefore relax these restrictions here and allow for stochastic equilibria with bubble creation. Unlike us, both Samuelson and Tirole assumed that financial markets are frictionless. Since this allows productive investors to invest on behalf of unproductive ones, this is akin to imposing the additional restriction that  $\delta = 1$ . With this restriction, bubbles only have crowding-out effects and slow down capital accumulation. This makes their models inconsistent with the empirical evidence that bubbly episodes tend to speed up capital accumulation. We therefore introduce financial frictions and allow for the possibility that

---

<sup>21</sup>For simplicity, we have assumed a financial friction that prevents financial intermediation altogether. The same effects would be present if, as in Martin and Ventura (2011), financial frictions constrained intermediation without totally eliminating it. As long as the return to investment differs across investors, there are incentives for unproductive investors to demand bubbles and for productive investors to supply them. Of course, the type and size of bubbles that are possible in equilibrium will depend on the severity of the financial friction (see footnote 26).

bubbles be expansionary.

### 3 Bubbly episodes and their macroeconomic effects

An important payoff of analyzing stochastic equilibria with bubble creation and destruction is that this allows us to rigorously capture the notion of a bubbly episode. Within a given history, the economy generically fluctuates between periods in which  $b_t + b_t^{NP} + b_t^{NU} = 0$  and periods in which  $b_t + b_t^{NP} + b_t^{NU} > 0$ . We say that the economy is in the *fundamental state* if  $b_t + b_t^{NP} + b_t^{NU} = 0$ . We say instead that the economy is experiencing a *bubbly episode* if  $b_t + b_t^{NP} + b_t^{NU} > 0$ . A bubbly episode starts when the economy leaves the fundamental state and ends when the economy returns to the fundamental state. We study next the nature of bubbly episodes and their macroeconomic effects.

#### 3.1 Existence of bubbly episodes

To study the types of bubbly episodes that can occur in equilibrium, it is useful to exploit a trick that makes the model recursive. Let  $x_t$ ,  $x_t^{NP}$  and  $x_t^{NU}$  be the stock of old and new bubbles as a share of the savings of the young or wealth of the economy, i.e.  $x_t \equiv \frac{b_t}{s \cdot k_t^\alpha}$ ,  $x_t^{NP} \equiv \frac{b_t^{NP}}{s \cdot k_t^\alpha}$  and  $x_t^{NU} \equiv \frac{b_t^{NU}}{s \cdot k_t^\alpha}$ . Then, we can rewrite Equations (6) and (7) as saying that, if  $x_t + x_t^{NP} + x_t^{NU} > 0$ , then

$$E_t x_{t+1} \begin{cases} = \frac{\alpha}{s} \cdot \frac{\delta \cdot (x_t + x_t^{NP} + x_t^{NU})}{A + (1 - \delta) \cdot x_t^{NP} - \delta \cdot x_t} & \text{if } \frac{x_t + x_t^{NP}}{1 - \varepsilon} < 1 \\ \in \left[ \frac{\alpha}{s} \cdot \frac{\delta \cdot (x_t + x_t^{NP} + x_t^{NU})}{A + (1 - \delta) \cdot x_t^{NP} - \delta \cdot x_t}, \frac{\alpha}{s} \cdot \frac{x_t + x_t^{NU} + x_t^{NP}}{1 - x_t} \right] & \text{if } \frac{x_t + x_t^{NP}}{1 - \varepsilon} = 1 \\ = \frac{\alpha}{s} \cdot \frac{x_t + x_t^{NP} + x_t^{NU}}{1 - x_t} & \text{if } \frac{x_t + x_t^{NP}}{1 - \varepsilon} > 1 \end{cases} \quad (9)$$

$$0 \leq x_t \leq 1. \quad (10)$$

Equations (9) and (10) fully describe the bubble dynamics that can take place in our economy. There are two sources of randomness in these dynamics: shocks to bubble creation, i.e.  $x_t^{NP}$  and  $x_t^{NU}$ ; and shocks to the value of the existing bubble, i.e.  $x_t$ . Any admissible stochastic process for  $\{x_t, x_t^{NP}, x_t^{NU}\}$  satisfying Equations (9) and (10) describes an equilibrium of the model. By admissible, we mean that this stochastic process must ensure that  $x_t^{NP} \geq 0$  and  $x_t^{NU} \geq 0$  for all  $t$ . Conversely, any equilibrium of the model can be expressed as an admissible stochastic sequence

for  $\{x_t, x_t^{NP}, x_t^{NU}\}$ .

The following proposition provides the conditions for the existence of bubbly episodes:

**Proposition 1** *Bubbly episodes are possible iff*  $\alpha < \begin{cases} s \cdot \frac{A}{\delta} & \text{if } A > 1 - \varepsilon \\ s \cdot \frac{A}{\delta} \cdot \max \left\{ 1, \frac{1}{4 \cdot (1 - \varepsilon) \cdot A} \right\} & \text{if } A \leq 1 - \varepsilon \end{cases}$ .

To prove Proposition 1 we ask if, among all admissible stochastic processes for  $\{x_t, x_t^{NP}, x_t^{NU}\}$  that satisfy Equation (9), there is at least one that also satisfies Equation (10). Consider first the case in which there is no bubble creation after a bubbly episode starts. Figure 3 plots  $E_t x_{t+1}$  against  $x_t$ , using Equation (9) with  $x_t^{NP} = x_{t_0}^{NP}$ ,  $x_t^{NU} = x_{t_0}^{NU}$  and  $x_t^{NP} = x_t^{NU} = 0$  for all  $t > t_0$ , where  $t_0$  is the period in which the episode starts. The left panel shows the case in which  $\alpha \geq s \cdot \frac{A}{\delta}$  and the slope of  $E_t x_{t+1}$  at the origin is greater than or equal to one. This means that any initial bubble would be demanded only if it were expected to continuously grow as a share of labor income, i.e. if it violates Equation (10), and this can be ruled out. The right panel of Figure 3 shows the case in which  $\alpha < s \cdot \frac{A}{\delta}$ . Now the slope of  $E_t x_{t+1}$  at the origin is less than one and, as a result,  $E_t x_{t+1}$  must cross the 45 degree line once and only once. Let  $x^*$  be the value of  $x_t$  at that point. Any initial bubble  $x_{t_0+1} > x^*$  can be ruled out. But any initial bubble  $x_{t_0}^N \leq x^*$  can be part of an equilibrium since it is possible to find a stochastic process for  $x_t$  that satisfies Equations (9) and (10).

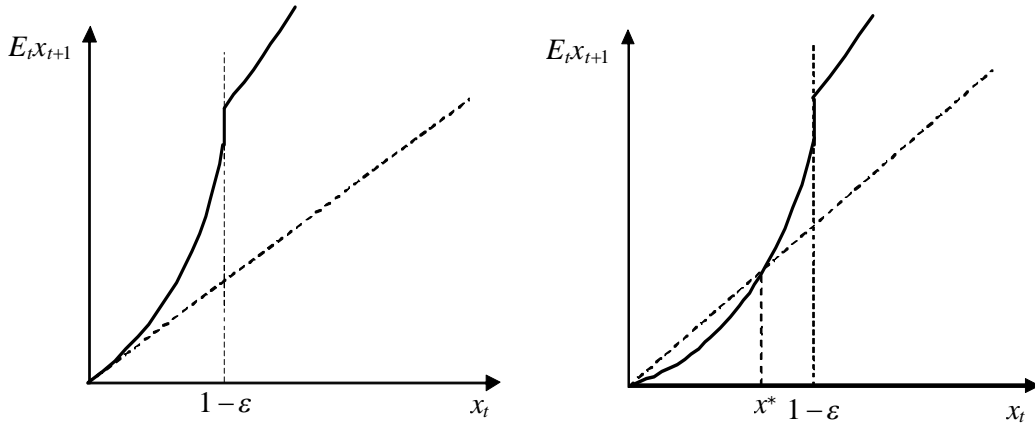


Figure 3

Is it possible that bubble creation relaxes the conditions for the existence of bubbly episodes? The answer is negative if we consider bubble creation by unproductive investors, i.e.  $x_t^{NU}$ . To see



this, note that bubble creation shifts upwards the schedule  $E_t x_{t+1}$  in Figure 3. The intuition is clear: new bubbles compete with old bubbles for the income of next period's young, reducing their return and making them less attractive.

Consider next bubble creation by productive investors, i.e.  $x_t^{NP}$ . Equation (9) shows that this type of bubble creation shifts the schedule  $E_t x_{t+1}$  upwards if  $x_t \in (0, A] \cup (1 - \varepsilon, 1]$ , but it shifts it downwards if  $x_t \in (A, 1 - \varepsilon]$ . To understand this result, it is important to recognize the double role played by bubble creation by productive investors. On the one hand, new bubbles compete with old ones for the income of next period's young. This effect reduces the demand for old bubbles and shifts the schedule  $E_t x_{t+1}$  upwards. On the other hand, productive investors sell new bubbles to unproductive investors and use the proceeds to invest, raising average investment efficiency and the income of next period's young. This effect increases the demand for old bubbles and shifts the schedule  $E_t x_{t+1}$  downwards. This second effect operates whenever  $x_t \leq 1 - \varepsilon$ , and it dominates the first effect only if  $x_t \geq A$ . Hence, if  $A > 1 - \varepsilon$ , bubble creation by productive investors cannot relax the condition for the existence of bubbly episodes.

If  $A \leq 1 - \varepsilon$ , bubble creation does relax the condition for the existence of bubbles. Namely, this condition becomes  $\alpha < s \cdot \frac{A}{\delta} \cdot \max \left\{ 1, \frac{1}{4 \cdot (1 - \varepsilon) \cdot A} \right\}$ . Figure 4 provides some intuition for this result by plotting  $E_t x_{t+1}$  against  $x_t$ , using Equation (9) and assuming that  $x_t^{NU} = 0$  and

$$x_t^{NP} = \begin{cases} 0 & \text{if } x_t \in (0, A] \cup (1 - \varepsilon, 1] \\ 1 - \varepsilon - x_t & \text{if } x_t \in (A, 1 - \varepsilon] \end{cases},$$

for all  $t > t_0$ . In both panels, this bubble creation by productive investors shifts the schedule  $E_t x_{t+1}$  downward. The left panel shows the case in which this does not affect the conditions for the existence of bubbly episodes, i.e.  $4 \cdot (1 - \varepsilon) \cdot A > 1$ . The right panel shows instead the case in which bubble creation by productive investors weakens the conditions for the existence of bubbly episodes, i.e.  $4 \cdot (1 - \varepsilon) \cdot A < 1$ . This completes the proof of Proposition 1.

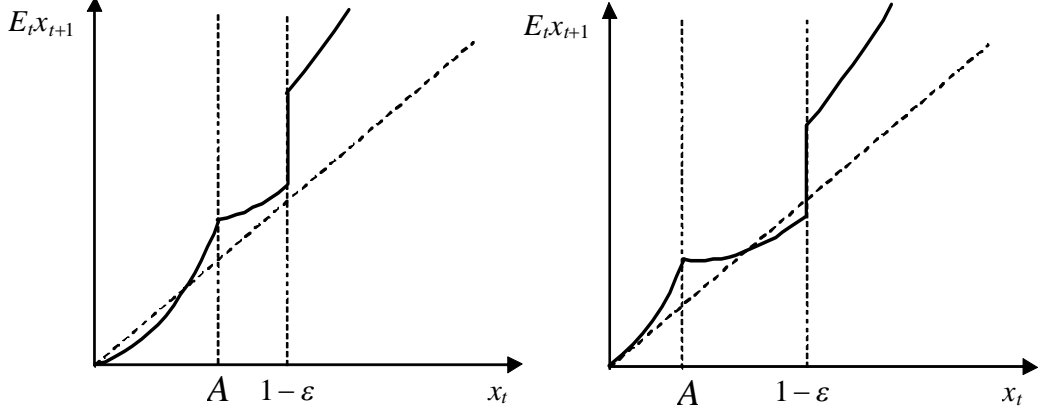


Figure 4

Proposition 1 provides the condition for existence of bubbly episodes of any sort. It is also useful to describe the conditions for the existence of bubbles according to their effects on capital accumulation. Recall that these effects depend on whether  $x_t^{NP}$  is smaller or greater than  $x_t \cdot \frac{\delta}{1-\delta}$ . We label a bubbly episode as contractionary if  $x_t^{NP} < x_t \cdot \frac{\delta}{1-\delta}$  throughout its duration. We similarly label a bubbly episode as expansionary if  $x_t^{NP} > x_t \cdot \frac{\delta}{1-\delta}$  throughout its duration.<sup>22</sup> With these definitions at hand, we can state the following proposition:

**Proposition 2** *Contractionary bubbly episodes are possible iff  $\alpha < \alpha_C \equiv s \cdot \frac{A}{\delta}$ . Expansionary bubbly episodes are possible iff  $\alpha < \alpha_E \equiv s \cdot \frac{A}{\delta} \cdot \begin{cases} (1-\delta) & \text{if } A > 0.5 \\ \frac{1}{4 \cdot (1-\epsilon) \cdot A} & \text{if } A \leq 0.5 \end{cases}$ .*

The proof of Proposition 2 follows almost immediately from the proof of Proposition 1 and we omit it here. Instead, we summarize the content of Proposition 2 with the help of Figure 5.<sup>23</sup>

<sup>22</sup>Formally, the existence of a contractionary (expansionary) bubbly episode requires the existence of an admissible stochastic process  $\{x_t, x_t^{NP}, x_t^{NU}\}$  satisfying  $E_t x_{t+1} < x_t$  and  $x_t \cdot \frac{\delta}{1-\delta} < x_t^{NP}$  ( $x_t \cdot \frac{\delta}{1-\delta} > x_t^{NP}$ ) for all  $t$ , where  $E_t x_{t+1}$  is as in Equation (9). Some bubble episodes are neither contractionary nor expansionary according to these definitions since their effects on the capital stock and output vary through time within a given history.

<sup>23</sup>Figure 5 has been drawn for a fixed value of  $\epsilon < 0.5$ . This guarantees that Region IV exists.

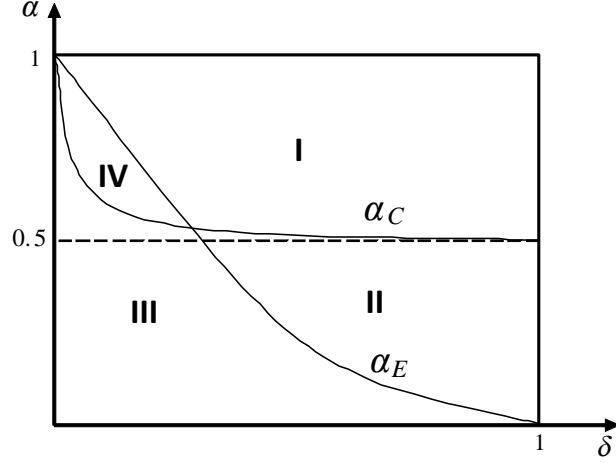


Figure 5

Bubbly episodes are possible in Regions II-IV, but not in Region I. In Regions II and III,  $\alpha < \alpha_C$  and contractionary episodes are possible. In Region III and IV,  $\alpha < \alpha_E$  and expansionary episodes are possible. In the limiting case  $\delta \rightarrow 1$ , only contractionary episodes are possible. As  $\delta$  declines, the value of  $\alpha$  required for the existence of both types of bubbly episodes declines. In the limiting case  $\delta \rightarrow 0$ , both types of bubbly episodes are possible regardless of  $\alpha$ .

### 3.2 Bubbles and dynamic inefficiency

During a bubbly episode, the young reduce their investments and purchase bubbles. They do so voluntarily in the expectation that the revenues from selling these bubbles will exceed the foregone investment income. These revenues are nothing but the reduction in the investments of the next generation of young minus the value of any new bubbles. Thus, bubbly episodes are possible if there exists a chain of investments that is expected to absorb resources, that is, a chain whose cost is expected to exceed the income it produces in all periods. More formally, let  $I_t$  be a chain of investments and let  $D_t$  be the resources it absorbs. This chain is expected to absorb resources in all periods if, for all  $t$ ,

$$E_t \{I_{t+1} - R_{t+1} \cdot I_t\} = E_t \{D_{t+1}\} \geq 0, \quad (11)$$

where  $R_{t+1}$  is the equilibrium return to the investments in the chain. We say that a chain is “dynamically inefficient” if it satisfies Equation (11). We provide next the economic intuition behind Propositions 1 and 2 by showing that bubbles can exist if the chains of investments they replace are dynamically inefficient *at the equilibrium return to investment*. Intuitively, investors are

happy not to make these investments and instead purchase bubbles or pyramid schemes. The latter can offer as much as  $E_t \{I_{t+1}\}$ , while the investments can only offer  $E_t \{R_{t+1} \cdot I_t\}$ . We emphasize that Condition (11) must be evaluated at the equilibrium rate of return because this observation plays a subtle but crucial role in what follows: a chain of investments might not be dynamically inefficient in the fundamental or other equilibria, and yet this same chain might be dynamically inefficient in the equilibrium in which the bubble replaces it. It is the latter that is required for a bubbly episode to exist.

In the proof of Proposition 1, we began by considering bubbly episodes in which there is no bubble creation after their start. To determine whether these episodes are possible, we must simply check whether there exist dynamically inefficient chains of investments to be replaced, i.e. satisfying Condition (11) for some  $D_t \geq 0$ . Since it is easier to construct dynamically inefficient chains of unproductive investments, we take a chain of such investments  $I_t = x_t \cdot s \cdot k_t^\alpha$ . Since the equilibrium rate of return to unproductive investments is  $R_t = \delta \cdot \alpha \cdot k_t^{\alpha-1}$ , this chain satisfies Condition (11) if and only if

$$E_t x_{t+1} \cdot s \cdot k_{t+1}^\alpha \geq \delta \cdot \alpha \cdot k_{t+1}^{\alpha-1} \cdot x_t \cdot s \cdot k_t^\alpha = \frac{x_t \cdot \delta}{A - \delta \cdot x_t} \cdot \alpha \cdot k_{t+1}^\alpha \geq 0, \quad (12)$$

for all  $t$ .<sup>24</sup> The LHS of Condition (12) is  $E_t \{I_{t+1}\}$  while the RHS is  $E_t \{R_{t+1} \cdot I_t\}$ . A chain of investments can satisfy Condition (12) if and only if

$$\alpha < s \cdot \frac{A}{\delta}.$$

Otherwise  $x_t$  would have to grow continuously and eventually exceed one, which is not possible. But this is the condition for the existence of bubbly episodes without bubble creation that we found in the proof of Proposition 1. Since these episodes are all contractionary, this is also the condition for being in regions II and III of Figure 3.

We then asked whether bubble creation could relax the conditions for bubbly episodes to exist. At first sight, one might be tempted to dismiss this possibility at once. With bubble creation, bubbles replace chains of investment that absorb a strictly positive amount of resources, i.e.  $D_t > 0$ ; and this seems to make Condition (11) more stringent. But this reasoning is incomplete because it fails to recognize that Condition (11) must be evaluated at the equilibrium rate of return, which

---

<sup>24</sup>Here we have used Equation (8) and the definition of  $x_t$  to eliminate  $k_t$ .

might be lower in the equilibrium with bubble creation. Thus, take again a chain of unproductive investments  $I_t = (x_t + x_t^{NP} + x_t^{NU}) \cdot s \cdot k_t^\alpha$  that absorbs resources  $D_t = (x_t^{NP} + x_t^{NU}) \cdot s \cdot k_t^\alpha$ . Since the equilibrium rate of return to unproductive investments is  $R_t = \delta \cdot \alpha \cdot k_t^{\alpha-1}$ , this chain satisfies Condition (11) if and only if

$$E_t x_{t+1} \cdot s \cdot k_{t+1}^\alpha \geq \delta \cdot \alpha \cdot k_{t+1}^{\alpha-1} \cdot (x_t + x_t^{NP} + x_t^{NU}) \cdot s \cdot k_t^\alpha = \frac{(x_t + x_t^{NP} + x_t^{NU}) \cdot \delta}{A + (1 - \delta) \cdot x_t^{NP} - \delta \cdot x_t} \cdot \alpha \cdot k_{t+1}^\alpha, \quad (13)$$

for all  $t$ . The LHS of Condition (13) is  $E_t \{I_{t+1} - D_{t+1}\}$  while the RHS is  $E_t \{R_{t+1} \cdot I_t\}$ . A chain of investments can satisfy this condition if and only if

$$\alpha < \begin{cases} s \cdot \frac{A}{\delta} & \text{if } A > 1 - \varepsilon \\ s \cdot \frac{A}{\delta} \cdot \max \left\{ 1, \frac{1}{4 \cdot (1 - \varepsilon) \cdot A} \right\} & \text{if } A \leq 1 - \varepsilon \end{cases}.$$

Otherwise  $x_t$  would have to grow continuously and eventually exceed one, which is not possible. But this is the condition for the existence of bubbles in Proposition 1. It is also the condition for being in regions II-IV of Figure 3. Bubble creation thus makes the bubbly episodes of region IV possible. In these episodes, bubbles lower the rate of return making the chains of investments they replace dynamically inefficient.

We end this discussion by noting that the special case in which investments are homogenous and financial frictions are irrelevant, i.e.  $\delta = 1$ , exhibits an interesting property: if there exists a dynamically inefficient chain of investments, then the chain of all investments must also be dynamically inefficient. This is because all investments are homogenous.<sup>25</sup> Thus, the condition for bubbly episodes to exist implies that aggregate investment exceeds capital income, i.e.  $\alpha < s$ . Abel et al. (1989) used this result to call into doubt the existence of rational bubbly episodes in industrial economies, since in all of them aggregate investment falls short of capital income.

Our analysis shows that this reasoning is based on the dubious assumption that financial frictions do not matter in industrial economies. If  $\delta < 1$ , we have shown that rational bubbly episodes are possible even if  $\alpha > s$ . This is for two reasons: (i) if  $s < \alpha < s \cdot \frac{A}{\delta}$  (regions II and III), in the fundamental state there are dynamically inefficient chains of investments; and (ii) if  $s \cdot \frac{A}{\delta} \leq \alpha < s \cdot \frac{A}{\delta} \cdot \frac{1}{4 \cdot (1 - \varepsilon) \cdot A}$  (region IV), there are no dynamically inefficient chains of investment in the

---

<sup>25</sup>Once again, we note that the chain that contains all investments in the economy is dynamically inefficient in equilibria in which bubbles *do not* replace all investments. There exists no equilibrium in which all investments are replaced by bubbles.

fundamental state but expansionary bubbly episodes that lower the return to investment would create such chains themselves.<sup>26</sup>

### 3.3 Shocks to investor sentiment as a source of macroeconomic fluctuations

We study next the macroeconomic effects of bubbly episodes. To do this, rewrite the law of motion of the capital stock using the definition of  $x_t$  and  $x_t^{NP}$ :

$$k_{t+1} = \begin{cases} [A + (1 - \delta) \cdot x_t^{NP} - \delta \cdot x_t] \cdot s \cdot k_t^\alpha & \text{if } \frac{x_t + x_t^{NP}}{1 - \varepsilon} < 1 \\ (1 - x_t) \cdot s \cdot k_t^\alpha & \text{if } \frac{x_t + x_t^{NP}}{1 - \varepsilon} \geq 1 \end{cases}. \quad (14)$$

Equation (14) describes the dynamics of the capital stock for any admissible stochastic process for the bubble  $\{x_t, x_t^{NP}, x_t^{NU}\}$  satisfying Equations (9) and (10). Interestingly, bubbly episodes can be literally interpreted as shocks to the law of motion of the capital stock. These shocks do not reflect any fundamental change in preferences and technology. Instead, they can aptly be described as shocks to investor sentiment.

These shocks also have independent effects on consumption and therefore welfare, which in this model happen to be exactly the same:

$$c_t = (\alpha + x_t \cdot s) \cdot k_t^\alpha. \quad (15)$$

Equation (15) shows how bubbles affect consumption through two channels. First, contemporary bubbles increase consumption by raising the share of output in the hands of the old. This first effect is the same for all bubbly episodes, regardless of their type. Second, past bubbles affect consumption through their effect on the contemporary capital stock. This second effect clearly depends on the type of bubbly episode. In contractionary bubbly episodes, it lowers the capital stock and consumption. In expansionary bubbly episodes, it raises the capital stock and consumption.

To illustrate the potential of investor sentiment shocks for business cycle theory, we use next a simple example. Let  $z_t \in \{F, B\}$  be the random variable that determines whether the economy is in a fundamental state or in a bubbly episode, where  $\Pr[z_{t+1} = B/z_t = F] = q$  and

---

<sup>26</sup>This discussion provides a sense of how the financial friction relaxes the condition for the existence of bubbly episodes. The constraint that the financial friction imposes on the reallocation of resources must be tight enough to (i) make the unproductive investments inefficient in the fundamental state, or to; (ii) make the gains from reallocation sufficiently high, and hence bubble creation sufficiently expansionary, to render the unproductive investments inefficient in the bubbly state.

$\Pr [z_{t+1} = F/z_t = B] = p$  for all  $t$ . That is, the economy switches between the fundamental state and bubbly episodes with transition probabilities  $q$  and  $p$ . When the economy is in the fundamental state, i.e.  $z_t = F$ , we have that  $x_t^{NP} = x_t^{NU} = x_t = 0$ . When the economy is in a bubbly episode, i.e.  $z_t = B$ , we have that:

$$x_t^{NP} = \eta_0 + \eta_1 \cdot x_t, \quad x_t^{NU} = 0, \quad \text{and} \quad x_t = \frac{\alpha}{s \cdot (1-p)} \cdot \frac{\delta \cdot [(1 + \eta_1) \cdot x_t + \eta_0]}{A + (1 - \delta) \cdot \eta_0 + [(1 - \delta) \cdot \eta_1 - \delta] \cdot x_t} + u_t \quad (16)$$

where  $u_t = \begin{cases} \sigma & \text{with prob. } 0.5 \\ -\sigma & \text{with prob. } 0.5 \end{cases}$ . Thus,  $E_t u_{t+1} = 0$  and  $E_t u_{t+1}^2 = \sigma^2$ . In this specific example, the vector of all shocks in period  $t$  is  $h_t = \{z_t, u_t\}$ . These two shocks can be interpreted as changes in investor sentiment. While  $u_t$  embodies small changes in investor sentiment that lead to fluctuations in the bubble within a bubbly episode,  $z_t$  embodies drastic changes in investor sentiment that start and end bubbly episodes. In applications, one would choose the process for the vector of shocks  $h_t$  that best describes the empirical behavior of the bubble, just as we typically choose processes for productivity shocks that best describe the empirical behavior of total factor productivity.<sup>27</sup>

Figure 6 shows the result of simulating the economy using this example.<sup>28</sup> The figure plots output ( $k_t^\alpha$ ), consumption ( $c_t$ ) and the bubble ( $b_t + b_t^{NP} + b_t^{NU}$ ) in each period. Initially, the economy is in the fundamental steady state. In period 15, there is a shock to investor sentiment that fuels a bubbly episode and raises the efficiency of investment. Consequently, the law of motion of the capital stock shifts upwards and the economy starts transitioning towards a higher, bubbly, steady state. Output and consumption increase, although they fluctuate throughout the bubbly episode along with the bubble. In period 35, a shock to investor sentiment ends this first episode and the economy suffers a sharp contraction. Output and consumption collapse and they stabilize around the fundamental steady state. Only 6 periods later, there is another shock to investor sentiment that starts a second bubbly episode and the economy expands again. This economy therefore experiences a macroeconomic fluctuations that are driven solely by investor sentiment shocks. Despite its simplicity, this example shows that introducing these shocks into quantitative

---

<sup>27</sup>Indeed, there is a strong parallelism between technology shocks and investor sentiment shocks, since their data counterparts are both obtained as residuals. Growth in total factor productivity is the difference between growth in output and growth in factor usage. Growth in the bubble is the difference between the growth of asset prices and growth in the net present value of their payoffs. See Carvalho et al. (2011).

<sup>28</sup>To produce this figure, we assume that  $p = 0.11$ ,  $q = 0.11$ ,  $\eta_0 = 0.15$ ,  $\eta_1 = 0.18$ ,  $\delta = 0.1$ ,  $\sigma = 0.035$ ,  $\varepsilon = 0.02$  and  $\alpha = 0.4$ . These parameters ensure that bubbly episodes never exceed the savings of unproductive investors, i.e.  $\frac{x_t + x_t^{NP}}{1 - \varepsilon} < 1$ ; and bubbly episodes are always expansionary, i.e.  $x_t^{NP} > x_t \cdot \frac{\delta}{1 - \delta}$ .

business-cycle models is a promising strategy to account for the type of episodes mentioned in the introduction.

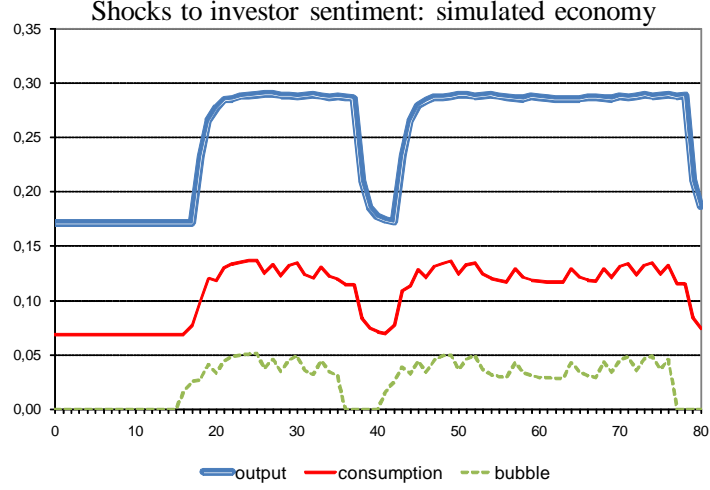


Figure 6

In Figure 6, the effects of bubbly episodes are transitory because the economy is stationary. This is due to diminishing returns, i.e. capital accumulation makes capital abundant and lowers its average product. Bubbly episodes would have long-lasting effects, however, if the economy were non-stationary. To illustrate this, in the appendix we generalize the production structure of the economy and allow for constant or increasing returns to capital accumulation. In particular, we assume that the final good is produced by assembling a continuum of intermediate inputs. The presence of fixed costs then creates a market-size effect, i.e. capital accumulation increases the number of intermediate inputs and this raises the average product of capital. We find that output is  $y_t = k_t^{\alpha \cdot \mu}$ ; where the parameters  $\alpha < 1$  and  $\mu > 1$  reflect these two opposing forces. If diminishing returns are strong and market-size effects are weak, i.e.  $\alpha \cdot \mu < 1$ , capital accumulation lowers the average product of capital and the economy is stationary. If instead diminishing returns are weak and market-size effects are strong, i.e.  $\alpha \cdot \mu \geq 1$ , capital accumulation raises the average product of capital and the economy is non-stationary. Interestingly, this generalization does not affect the conditions for the existence of bubbly episodes in Propositions 1 and 2.<sup>29</sup> It does however make it possible for transitory bubbly episodes to have permanent effects. We illustrate this in the appendix

<sup>29</sup>Moreover, it has only minor effects on the formal structure of the model: Equations (9) and (10) remain the same while, in Equation (14), the exponents of  $k_t$  become  $\alpha \cdot \mu$  instead of  $\alpha$ .



with the help of an example in which a bubbly episode takes the economy out of a negative-growth trap. Even though the bubbly episode ends, the path of the economy has changed forever.

## 4 Where is the market for bubbles?

We have developed a stylized model of economic growth with bubbles. In this model, a financial friction impedes productive investors to borrow from unproductive ones. When the market for bubbles is closed, each investor is forced to invest his/her own resources. As a result, the average efficiency of investment, the capital stock and output are all low. When the market for bubbles opens, productive investors sell bubbles to unproductive ones. This reallocation of resources raises the average efficiency of investment, the capital stock and output. Thus, fluctuations in activity in the market for bubbles create macroeconomic fluctuations even in the absence of fundamental shocks to preferences and technology. But, where is the market for bubbles in real economies? We show next that the transactions performed in the market for bubbles can be replicated with the help of stock and credit markets.

Consider an economy with the same preferences and technology as the benchmark economy of sections two and three. In this modified economy there is no market for bubbles and instead we have stock and credit markets. In particular, we make the following assumptions:

1. Production and investment must take place within firms that are owned and managed by entrepreneurs. The young can become entrepreneurs by purchasing existing firms in the stock market or by creating new ones at zero cost. Let  $v_t$  denote the value of the stock market, i.e. the price of all firms.
2. The productivity of a firm is that of its entrepreneur. In particular, firms owned and managed by productive entrepreneurs have an investment efficiency of one, while firms owned and managed by unproductive ones have an investment efficiency of  $\delta$ . Let  $v_t^P$  and  $v_t^U$  be the price of all productive and unproductive firms, respectively. Naturally,  $v_t^P + v_t^U = v_t$ .
3. Entrepreneurs can obtain credit to purchase their firms and/or to invest in them. But entrepreneurs cannot pledge the output of their firms to their creditors.<sup>30</sup> Since capital fully

---

<sup>30</sup>That is, we assume that old entrepreneurs can appropriate the firm's entire output net of wage payments. Utility maximization ensures that they will indeed choose to do so regardless of whatever they might have promised in their youth.

depreciates in production, entrepreneurs can only pledge an empty firm as collateral to their creditors. Credit contracts last for one period and specify an ex-post payment in each possible history. These payments cannot exceed the price of the entrepreneur's empty firm.

A difference between this modified economy and the benchmark economy of sections two and three is that now individuals have access to three savings options rather than two: (i) purchasing firms in the stock market; (ii) building capital within these firms or within new firms created at zero cost; and (iii) becoming a lender in the credit market.<sup>31</sup> Since the labor market is still assumed to be competitive, the wage equals the marginal product of labor as given by Equation (1). And this means that the return to investment (option (ii)) is as follows:

$$R_{i,t+1}^K = \begin{cases} \delta \cdot \alpha \cdot k_{t+1}^{\alpha-1} & \text{if } i \notin I_t^P \\ \alpha \cdot k_{t+1}^{\alpha-1} & \text{if } i \in I_t^P \end{cases} \quad (17)$$

where  $R_{i,t+1}^K$  is the return to investment in the firm owned and managed by individual  $i$ . Consistent with our assumptions, Equation (17) says that this return varies across investors. Let  $R_{t+1}$  be the average ex-post return on credit contracts (option (iii)). Since individuals are risk neutral, all contracts must offer the same expected return. We refer to this required expected return as the interest rate. Equilibrium in the credit market requires that the interest rate satisfy the following restrictions:

$$E_t R_{t+1} = \begin{cases} \delta \cdot \alpha \cdot k_{t+1}^{\alpha-1} & \text{if } \frac{E_t v_{t+1}^P}{(1-\varepsilon) \cdot s \cdot k_t^\alpha - v_t^U} < \delta \cdot \alpha \cdot k_{t+1}^{\alpha-1} \\ \frac{E_t v_{t+1}^P}{(1-\varepsilon) \cdot s \cdot k_t^\alpha - v_t^U} & \text{if } \delta \cdot \alpha \cdot k_{t+1}^{\alpha-1} \leq \frac{E_t v_{t+1}^P}{(1-\varepsilon) \cdot s \cdot k_t^\alpha - v_t^U} \leq \alpha \cdot k_{t+1}^{\alpha-1} \\ \alpha \cdot k_{t+1}^{\alpha-1} & \text{if } \alpha \cdot k_{t+1}^{\alpha-1} < \frac{E_t v_{t+1}^P}{(1-\varepsilon) \cdot s \cdot k_t^\alpha - v_t^U} \end{cases}, \quad (18)$$

To understand Equation (18), recall first that the only collateral that productive firms can pledge in period  $t$  is the expected discounted value of their firms in period  $t+1$ , i.e.  $\frac{E_t v_{t+1}^P}{E_t R_{t+1}}$ . Notice also that, after purchasing their firms, unproductive individuals are left with an amount of funds

---

<sup>31</sup>Let  $v_{it}$  denote the value of the firms purchased or created by individual  $i \in I_t$  in period  $t$ , respectively. Let  $f_{it}$  be the credit obtained by individual  $i$ . We can write the intertemporal budget constraint of individual  $i$ :

$$c_{it+1} = r_{t+1} \cdot A_i \cdot (w_t + f_{it} - v_{it}) + v_{it+1} - R_{t+1} \cdot f_{it},$$

where  $A_i = 1$  if individual  $i$  is productive and  $A_i = \delta$  otherwise. Naturally,  $v_t = \int_{i \in I_t} v_{it}$ ,  $v_t^P = \int_{i \in I_t^P} v_{it}$  and  $v_t^U = \int_{i \in I_t^U} v_{it}$ ; where  $I_t^P$  and  $I_t^U$  are the sets of productive and unproductive entrepreneurs in generation  $t$ .

equal to  $(1 - \varepsilon) \cdot s \cdot k_t^\alpha - v_t^U$  that can be used to invest in their firms or to give credit to productive firms. With these two observations at hand, it is straightforward to see that the three segments of Equation (18) describe three different situations relating these two quantities.

Assume first that the collateral of productive firms falls short of these available funds when the interest rate equals the return to unproductive investments:  $\frac{E_t v_{t+1}^P}{(1 - \varepsilon) \cdot s \cdot k_t^\alpha - v_t^U} < \delta \cdot \alpha \cdot k_{t+1}^{\alpha-1}$ . Then, collateral is so scarce and credit constraints so tight that productive firms cannot absorb all available funds for investment. As a result, some investments take place in unproductive firms and the interest rate indeed equals the return to unproductive investments.

Assume instead that the collateral of productive firms falls short of available funds when the interest rate equals the return to productive investments, but it exceeds available funds if the interest rate equals the return to unproductive investments:  $\delta \cdot \alpha \cdot k_{t+1}^{\alpha-1} \leq \frac{E_t v_{t+1}^P}{(1 - \varepsilon) \cdot s \cdot k_t^\alpha - v_t^U} \leq \alpha \cdot k_{t+1}^{\alpha-1}$ . Then, the interest rate is such that it makes the credit constraint just binding. Collateral is sufficient to ensure that productive firms absorb all available funds for investment. But it is not sufficient to allow competition among productive firms until the interest rate equals the return to productive investments.

Assume finally that the collateral of productive firms exceeds available funds when the interest rate equals the return to productive investments:  $\alpha \cdot k_{t+1}^{\alpha-1} < \frac{E_t v_{t+1}^P}{(1 - \varepsilon) \cdot s \cdot k_t^\alpha - v_t^U}$ . Then, the interest rate equals the return to productive investments. Collateral is now enough to allow productive firms to absorb all the available funds for investment and to compete among themselves until the interest equals the return to productive investments.

We can use these results to write the law of motion of the capital stock as a function of stock prices:

$$k_{t+1} = \begin{cases} A \cdot s \cdot k_t^\alpha + (1 - \delta) \cdot \left( \frac{E_t v_{t+1}^P}{\delta \cdot \alpha \cdot k_{t+1}^{\alpha-1}} - v_t^P \right) - \delta \cdot v_t & \text{if } \frac{E_t v_{t+1}^P}{(1 - \varepsilon) \cdot s \cdot k_t^\alpha - v_t^U} \leq \delta \cdot \alpha \cdot k_{t+1}^{\alpha-1} \\ s \cdot k_t^\alpha - v_t & \text{if } \frac{E_t v_{t+1}^P}{(1 - \varepsilon) \cdot s \cdot k_t^\alpha - v_t^U} \geq \delta \cdot \alpha \cdot k_{t+1}^{\alpha-1} \end{cases}, \quad (19)$$

The formal similarity between Equations (8) and (19) is not a coincidence. Indeed, the predictions of the modified and benchmark models for quantities and welfare are seen to be equivalent if<sup>32</sup>

$$b_t = v_t, \quad b_t^{NP} = \frac{E_t v_{t+1}^P}{E_t R_{t+1}} - v_t^P \quad \text{and} \quad b_t^{NU} = \frac{E_t v_{t+1}^U}{E_t R_{t+1}} - v_t^U \quad (20)$$

---

<sup>32</sup>Note that this also ensures that  $E_t \left\{ \frac{b_{t+1}}{b_t + b_t^{NP} + b_t^{NU}} \right\} = E_t R_{t+1}$ .

This equivalence follows from the observation that firms bundle together capital and bubbles. Since the output of a firm cannot be pledged to creditors, its market price only reflects the bubbles this firm owns, i.e.  $b_t = v_t$ .<sup>33</sup> It is therefore in the stock market that the old sell their bubbles to the young. When a new entrepreneur adds additional bubbles to the firm, this is reflected in the firm's value growing (in expectation) above the interest rate, i.e.  $b_t^{NP} = \frac{E_t v_{t+1}^P}{E_t R_{t+1}} - v_t^P > 0$  and  $b_t^{NU} = \frac{E_t v_{t+1}^U}{E_t R_{t+1}} - v_t^U > 0$ . If this bubble creation takes place in an unproductive firm, the entrepreneur is happy to hold this value until old age and use it then to finance additional consumption. If this bubble creation takes place in a productive firm, the entrepreneur might want to borrow against this value and *de facto* sell the new bubbles to the firm's creditors. This would be the case if collateral is low or intermediate and the return to productive investments exceeds the interest rate. It is therefore in the credit market that the productive young sell their new bubbles to the unproductive young. All the equilibria of the benchmark model studied in section three, can now be re-interpreted as equilibria of the modified model in which the transactions in the market for bubbles take place in the stock and credit markets. It is important to stress that no new firms need to be created in these equilibria, so that changes in  $b_t = v_t$  reflect changes in the price of all firms but also in the price of the average firm.

This re-interpretation of our model connects the abstract theory presented here with the evidence reported in section one. Activity in the market for bubbles has a natural interpretation in terms of activity in the stock and credit markets.<sup>34</sup> The onset of a bubbly episode is characterized by a large and unexpected increase in stock prices. This provides productive firms with enough collateral to borrow and invest. Thus, the average efficiency of the economy increases and so does the capital stock and output. Interestingly, this happens at the same time as the interest rate declines. Throughout the bubbly episode, the stock market outgrows the interest rate and consumption and welfare are both high.<sup>35</sup> Eventually, the bubbly episode ends and stock prices collapse. This leads

---

<sup>33</sup>This amounts to assuming that credit backed by bubbles is less prone to agency costs and asymmetric information problems than credit backed by physical output or by capital. This seems like a reasonable assumption, though. In general, agency costs increase with the manager's ability to influence the firm's value and decrease with the shareholders' ability to observe the actions of the manager. The output of a firm depends on its capital stock and on how it is managed, and it is likely to be influenced by managers through a variety of channels that are difficult to observe. The value of a bubble depends instead on the expectations of a rational market, which are unlikely to be influenced by the actions of a manager unless the market decided to use the manager as a sunspot to coordinate these expectations. Even in this case, it seems unlikely the manager could exploit this ability to his/her advantage without the shareholders knowing it.

<sup>34</sup>The same could be said of real estate markets, as very little would change in the model of this section if relabeled firms as real estate.

<sup>35</sup>During the bubbly episode, the price of the average firm outgrows the interest rate due to the risk of the bubble bursting and to bubble creation.

to a collapse in collateral and intermediation. The average efficiency of investment declines and so do the capital stock and output. The result is a decline in consumption and welfare.<sup>36</sup>

This re-interpretation of our model also connects the theory with existing macroeconomic models with financial frictions, which stress the importance of asset prices in the determination of credit and macroeconomic activity. As in these models, we show how higher asset prices enable productive individuals to borrow and expand their investment thereby raising average efficiency and growth. What is new here is that asset prices can experience booms and busts even in the absence of shocks to fundamentals. We think this addition to the theory will be crucial for understanding the relationship between asset prices and the macroeconomy in the recent past. In ongoing work, Carvalho et al. (2011), we develop a quantitative model of the financial accelerator with asset bubbles. This quantitative model contains a much more sophisticated and realistic description of preferences and demography, and cannot be analyzed with the simple analytical methods we have used here. Business cycles are driven by two types of shocks: fundamental shocks that affect technology and preferences; and investor sentiment shocks that lead to the appearance and collapse of bubbles in financial markets. Our immediate goal is to calibrate the model with data from industrialized economies and use it to explore the relative importance of both types of shocks in recent macroeconomic history.

---

<sup>36</sup>Of course, although qualitatively similar, not all bubbly episodes need to be exactly alike. In a richer model with individuals and financial intermediaries, for example, the impact of a bubbly episode on macroeconomic aggregates might depend on the identity of agents holding the bubble. This possibility, which has been invoked to account for the severity of the last recession, provides an exciting avenue for future research.

## References

- Abel, A., G. Mankiw, L. Summers, and R. Zeckhauser, 1989, Assessing Dynamic Efficiency: Theory and Evidence, *Review of Economic Studies* 56, 1-19.
- Azariadis, C., and B. Smith, 1993, Adverse Selection in the Overlapping Generations Model: The Case of Pure Exchange, *Journal of Economic Theory* 60, 277–305.
- Caballero, R. and A. Krishnamurthy, 2006, Bubbles and Capital Flow Volatility: Causes and Risk Management, *Journal of Monetary Economics* 53(1), 33-53.
- Carvalho, V., A. Martin and J. Ventura, 2011, Bubbly Business Cycles, work in progress.
- Chaney T., D. Sraer, Princeton and D. Thesmar, 2008, The Collateral Channel: How Real Estate Shocks Affect Corporate Investment, NBER working paper 16060.
- Diamond, P., 1965, Government Debt in a Neoclassical Growth Model, *American Economic Review* 74, 920-30.
- Eisfeldt, A. and A. Rampini, 2006, Capital Reallocation and Liquidity, *Journal of Monetary Economics* 53, 369-99.
- Farhi, E. and J. Tirole, 2011, Bubbly Liquidity, NBER working paper 16750.
- French, K. and J. Poterba, 1991, Were Japanese stock prices too high?, *Journal of Financial Economics*, 29, 337-363.
- Gan, J., 2007, Collateral, Debt Capacity, and Corporate Investment: Evidence from a Natural Experiment, *Journal of Financial Economics*, 85, 709-734
- Giglio, G. and T. Severo, 2011, Intangible Capital, Relative Asset Shortages and Bubbles, mimeo, Harvard University.
- Goyal, V. and T. Yamada, 2004, Asset Price Shocks, Financial Constraints, and Investment: Evidence from Japan, *Journal of Business*, 77, 175-801.
- French, K. and J. Poterba, 1991, Were Japanese stock prices too high?, *Journal of Financial Economics*, 29(2), 337-363.

- Grossman, G. and N. Yanagawa, 1993, Asset Bubbles and Endogenous Growth, *Journal of Monetary Economics* 31, 3-19.
- French, K. and J. Poterba, 1991, Were Japanese stock prices too high?, *Journal of Financial Economics*, 29(2), 337-363.
- International Monetary Fund, 2000 and 2003, *World Economic Outlook*.
- Kehrig, M., 2011, The Cyclicalilty of Productivity Dispersion, mimeo, Northwestern University.
- King, I. and D. Ferguson, 1993, Dynamic Inefficiency, Endogenous Growth, and Ponzi Games, *Journal of Monetary Economics* 32, 79-104.
- Kiyotaki, N. and J. Moore, 2008, Liquidity, Business Cycles and Monetary Policy, mimeo, Princeton.
- Kocherlakota, N. 2009, Bursting Bubbles: Consequences and Cures, Minneapolis Fed.
- Kraay, A., and J. Ventura, 2007, The Dot-Com Bubble, the Bush Deficits, and the US Current Account, in *G7 Current Account Imbalances: Sustainability and Adjustment*, R. Clarida (eds.), The University of Chicago.
- LeRoy, S., 2004, Rational Exhuberance, *Journal of Economic Literature* 42, 783-804.
- Martin, A. and J. Ventura, 2011, Theoretical Notes on Bubbles and the Current Crisis, *IMF Economic Review*, forthcoming.
- Olivier, J., 2000, Growth-enhancing Bubbles, *International Economic Review* 41, 133-151.
- Saint Paul, G., 1992, Fiscal Policy in an Endogenous Growth Model, *Quarterly Journal of Economics* 107, 1243-1259.
- Samuelson, P., 1958, An Exact Consumption-loan Model of Interest with or without the Social Contrivance of Money, *Journal of Political Economy* 66, 467-482.
- Sandleris, G. and M. Wright, 2010, *The Cost of Financial Crises: Resource Misallocation, Productivity and Welfare in the 2001 Argentine Crisis*, mimeo, UCLA.
- Shiller, R. *Irrational Exuberance*, Princeton University Press 2005.

Tirole, J., 1985, Asset Bubbles and Overlapping Generations, *Econometrica* 53 (6), 1499-1528.

Ventura, J., 2011, Bubbles and Capital Flows, *Journal of Economic Theory*, forthcoming.

Ventura, J., 2005, A Global View of Economic Growth, in *Handbook of Economic Growth*, Philippe Aghion and Stephen Durlauf (eds.), 1419-1497.

Weil, P., 1987, Confidence and the Real Value of Money, *Quarterly Journal of Economics* 102, 1-22.

Woodford, M., 1990, Public Debt as Private Liquidity, *American Economic Review*, 80, 382—388.



## 5 Appendix: the model with endogenous growth

Assume that the production of the final good consists of assembling a continuum of intermediate inputs, indexed by  $m \in [0, m_t]$  according to a symmetric CES function:

$$y_t = \lambda \cdot \left( \int_0^{m_t} q_{tm}^{\frac{1}{\mu}} \cdot dm \right)^{\mu}, \quad (21)$$

where  $q_{tm}$  denotes units of the variety  $m$  of intermediate inputs and  $\mu > 1$ . The constant  $\lambda = (\mu)^{-\mu} \cdot (1 - \mu)^{\mu-1}$  is a normalization parameter. Final-good producers are competitive. Production of intermediate inputs entails variable and fixed costs:

$$q_{tm} = (l_{tm,v})^{1-\alpha} \cdot (k_{tm,v})^{\alpha}, \quad (22)$$

$$f_{tm} = \begin{cases} 1 = (l_{tm,f})^{1-\alpha} \cdot (k_{tm,f})^{\alpha} & \text{if } q_{tm} > 0 \\ 0 & \text{if } q_{tm} = 0 \end{cases}, \quad (23)$$

where  $f_{tm}$  is the fixed cost and  $l_{tm,v}$ ,  $l_{tm,f}$ ,  $k_{tm,v}$  and  $k_{tm,f}$  are the labor and capital, variable and fixed costs of producing variety  $m$ . Input varieties become obsolete in one generation and, as a result, all generations must incur the fixed costs. The production of intermediate inputs takes place under under monopolistic competition and free entry.

This production structure is a special case of that considered by Ventura (2005). He shows that, under the assumptions made, the output of the economy is given by  $y_t = k_t^{\alpha\mu}$ , whereas competition in factor markets implies that  $w_t = (1 - \alpha) \cdot k_t^{\alpha\mu}$  and  $r_t = \alpha \cdot k_t^{\alpha\mu-1}$ . We can generalize Equations (6), (7) and (8) as follows:

$$E_t \left\{ \frac{b_{t+1}}{b_t + b_t^{NP} + b_t^{NU}} \right\} \begin{cases} = \delta \cdot \alpha \cdot k_t^{\alpha\mu-1} & \text{if } \frac{b_t + b_t^{NP}}{(1 - \varepsilon) \cdot s \cdot k_t^{\alpha\mu}} < 1 \\ \in [\delta \cdot \alpha \cdot k_t^{\alpha\mu-1}, \alpha \cdot k_t^{\alpha\mu-1}] & \text{if } \frac{b_t + b_t^{NP}}{(1 - \varepsilon) \cdot s \cdot k_t^{\alpha\mu}} = 1 \\ = \alpha \cdot k_t^{\alpha\mu-1} & \text{if } \frac{b_t + b_t^{NP}}{(1 - \varepsilon) \cdot s \cdot k_t^{\alpha\mu}} > 1 \end{cases}, \quad (24)$$

$$0 \leq b_t \leq s \cdot k_t^{\alpha\mu}, \quad (25)$$

$$k_{t+1} = \begin{cases} s \cdot A \cdot k_t^{\alpha\mu} + (1 - \delta) \cdot b_t^{NP} - \delta \cdot b_t & \text{if } \frac{b_t + b_t^{NP}}{(1 - \varepsilon) \cdot s \cdot k_t^{\alpha\mu}} < 1 \\ s \cdot k_t^{\alpha\mu} - b_t & \text{if } \frac{b_t + b_t^{NP}}{(1 - \varepsilon) \cdot s \cdot k_t^{\alpha\mu}} \geq 1 \end{cases}, \quad (26)$$

The only difference with the model in the main text lies in the exponents of  $k_t$ , which are now  $\alpha \cdot \mu$  instead of  $\alpha$ . For a given initial capital stock and bubble,  $k_0 > 0$  and  $b_0 \geq 0$ , a competitive equilibrium is a sequence  $\{k_t, b_t, b_t^{NP}, b_t^{NU}\}_{t=0}^{\infty}$  satisfying Equations (24), (25) and (26). It is easy to verify that this generalization of the production structure does not affect the conditions for the existence of bubbly episodes, as Propositions 1 and 2 apply for any value of  $\mu$ . To see this, re-define  $x_t \equiv \frac{b_t}{s \cdot k_t^{\alpha \cdot \mu}}$ ,  $x_t^{NP} \equiv \frac{b_t^{NP}}{s \cdot k_t^{\alpha \cdot \mu}}$  and  $x_t^{NU} \equiv \frac{b_t^{NU}}{s \cdot k_t^{\alpha \cdot \mu}}$ , and check that Equations (9) and (10) still apply.

When  $\alpha \cdot \mu \geq 1$ , even transitory bubbly episodes can have permanent effects on the levels and growth rates of capital and output. Figure A1 illustrates this. The left panel depicts the case of an expansionary bubble. Initially, the economy is in the fundamental state and the appropriate law of motion is  $k_{t+1}^F$ . Initially, the capital stock is below the fundamental steady state, i.e.  $k_t < k^F \equiv (s \cdot A)^{\frac{1}{1-\alpha \cdot \mu}}$ , and growth is negative. This economy is caught in a “negative-growth trap”. When an expansionary bubble pops up, it reduces unproductive investments and uses part of these resources to increase productive investments. During the bubbly episode, the law of motion of capital lies above  $k_{t+1}^F$ : in the figure,  $k_{t+1}^B$  represents the initial law of motion when the episode begins. Throughout the episode,  $k_{t+1}^B$  may shift as the bubble grows or shrinks. Growth may be positive if, during the bubbly episode, the capital stock lies above its steady-state value. Eventually, the bubble bursts but the economy might keep on growing if the capital stock at the time of bursting exceeds  $k^F$ . The bubbly episode, though temporary, leads the economy out of the negative-growth trap and it has a permanent effect on long-run growth. Naturally, it is also possible for contractionary bubbles to lead the economy into a negative growth trap thereby having permanent negative effects on growth: the right panel of Figure A1 shows this possibility.

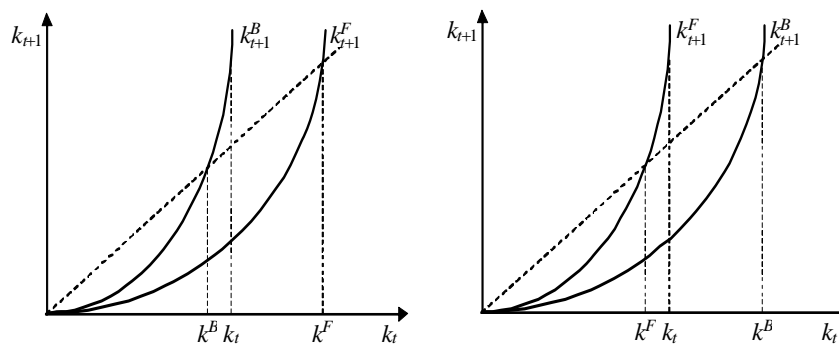


Figure A1