Manipulation Games in Economics with Indivisible Goods

Carmen Beviá

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Carmen Beviá

Departament d'Economia i d'Història Econòmica and CODE

Universitat Autònoma de Barcelona

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Abstract

In this paper we study the strategic aspects of the No-Envy solution for the problem of allocating a finite set of indivisible goods among a group of agents when monetary compensations are possible. In the first part of the paper we consider the case where each agent receives, at most, one indivisible good. We prove that the set of equilibrium allocations of any direct revelation game associated with a subsolution of the No-Envy solution coincides with the set of envy-free allocations for the true preferences. Under manipulation all the subsolutions of the No-Envy solution are equivalent. In the second part of the paper, we allow each agent to receive more than one indivisible good. In this situation the above characterization does not hold any more. We prove that any Equal Income Walrasian allocation for the true preferences can be supported as an equilibrium allocation of any direct revelation game associated with subsolutions of the No-Envy solution, but also non-efficient allocations can be supported.

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1. Introduction

An allocation mechanism can be understood as a mapping from the set of preferences to the set of feasible allocations. The question which often comes up is whether or not a mechanism is incentive compatible. It is well-known that many interesting mechanisms (those which select allocations that are efficient and satisfy some distributional requirements) are manipulable. (See for example Hurwicz (1972), Thomson (1987), Zhou (1991a), Barberá and Jackson (1995)).

The next natural question is what allocations are obtained when agents behave strategically, and what is the relationship between these allocations and those which were intended.

The first answer to this question was provided by Hurwicz (1979) who studied the manipulability of the Walrasian mechanism in economies with two goods and two agents. He proved that the equilibrium allocations of the Walrasian manipulation game are in the interior of the lens bounded by the true offer curves of the two agents. A generalization of this result for economies with l goods and two agents was given by Otani and Sicilian (1982). A counterpart of these results for the Lindahl manipulation game was established by Thomson (1979). All these results show that many equilibrium allocations are not efficient with respect to the true preferences.

In the marriage problem, Roth (1984) show that, even though agents reveal their preferences strategically, the Gale-Shapley algorithm yields stable matching as equilibrium allocations with respect to the true preferences. Zhou (1991b) completed this result by proving that any stable matching with respect to the true preferences can be supported by an equilibrium.

In economies with one indivisible good where monetary compensations are possible, Tadenuma and Thomson (1995) proved that the set of equilibrium allocations of direct revelation games associated with subsolutions of the No-Envy solution coincides with the set of envy-free allocations for the true preferences. This last result shows that the main properties of the solution are preserved under manipulation. Fujinaka and Sakai (2007) have generalized Tadenuma and Thomson's by showing that the set of equilibrium allocations of the manipulation game associated with subsolutions of the Identical Preferences Lower Bound solution coincide with the set of envy-free allocations for the true preferences.

The purpose of this paper is to continue this line of research by studying the strategic aspects of the No-Envy solution for the problem of allocating a finite set of indivisible goods among a group of agents when monetary compensations are possible.

We start analyzing the case where, even though there are more than one indivisible good, each agent can receive, at most, one object. We call this case the one object per person case. Tadenuma and Thomson's framework is a special case of this one where there is only one indivisible good. We prove that, under manipulation, all the subsolutions of the No-Envy solution are equivalent. That is, any envy-free allocation for the true preferences can be supported by an equilibrium of any direct revelation game associated with subsolutions of the No-Envy solution. And, conversely, any equilibrium of such games yields envy-free allocations for the true preferences. The characterization still holds if strong equilibrium is used. This result generalizes Tadenuma and Thomson's result. In this same framework Āzacis (2008) has proposed a simple mechanism that implements both in Nash and strong Nash equilibrium the set of true envy-free allocations. This is the closest result to ours. He concentrates in a mechanism that selects a single-valued outcome while we consider any possible subsolution of the No-Envy solution (single value or not).

In the second part of the paper, the same problem is addressed but relaxing the assumption of one agent one object. In this setting, not all envy-free allocations are efficient (as in the one object per person case). In order to keep the fairness and efficient properties of the starting solution, we consider the No-Envy and Pareto solution. The result we obtain is different from the previous one. Although we do not provide a complete characterization of the set of equilibrium, we prove that any Equal Income Walrasian allocation (an efficient and envy-free allocation) can be supported by an equilibrium of any direct revelation game associated with subsolutions of the No-Envy and Pareto solution. However, not all equilibria maintain the initial properties of the solution. In particular, we show that non-efficient allocations can arise as equilibrium allocations of the direct revelation game associated to the Equal Income Walrasian solution.

2. The one object per person case

An **economy** is a list $e = (Q, \Omega, M; R)$, where Q is a finite set of agents, Ω is a finite set of objects, $M \in \mathbb{R}$ is an amount of money, and $R = (R_i)_{i \in Q}$ is a list of preference relations defined over Ωx$\mathbb{R}$. Let P_i denote the strict preference relation associated with R_i, and I_i the indifference relation. Each preference relation is continuous, increasing in money and quasi-linear, that is, if $(\alpha, m)I_i(\beta, m')$, for all $t \in \mathbb{R}$ $(\alpha, m+t)I_i(\beta, m'+t)$. Also it is assumed that no object is infinitely desirable or undesirable when compared with another object. Thus, for any bundle $(\alpha, m) \in \Omega x \mathbb{R}$, any $i \in Q$, and any $\beta \in \Omega$, there is an amount of money m' such that $(\alpha, m)I_i(\beta, m')$. We assume that[1] $|\Omega| = |Q|$. Let $\mathcal{R}$ be the class of all such preference relations. We assume that Ω and M are known and fixed, therefore an economy is completely described by a list $R \in \mathcal{R}^q$ where $q = |Q|$.

A feasible Allocation is a pair $z = (\sigma, m)$, where $\sigma : Q \longrightarrow \Omega$ is a bijection that assigns agents to objects, and $m = (m_{\sigma(i)})_{i \in Q}$ is such that $\sum_{i \in Q} m_{\sigma(i)} = M$. For each $i \in Q$, let $z_i = (\sigma(i), m_{\sigma(i)})$ be the consumption of agent i.

Note that $m_{\sigma(i)}$ can be either positive, negative or zero. Let **Z** be the set of feasible allocations.

A solution or a mechanism is a correspondence $\varphi : \mathcal{R}^q \longrightarrow Z$ which associates each $R \in \mathcal{R}^q$ with a subset of Z.

An important example of solution is the Pareto solution.

The Pareto solution, P: given $R \in \mathcal{R}^q$, $P(R) = \{z \in Z$ / there is no $z' \in Z$ s.t. $z'_i R_i z_i$ for all $i \in Q$ and $z'_j P_j z_j$ for some $j \in Q\}$.

One of the main fairness solution is the No-Envy solution (Foley 1967).

The No-Envy solution, N: given $R \in \mathcal{R}^q$, $N(R) = \{z \in Z$ / for all $i, j \in Q$, $z_i R_i z_j\}$.

[1] We can easily accommodate the case where $|\Omega| < |Q|$ by introducing "null objects" without loosing any of the properties of the No-Envy solution. However, although the case $|\Omega| > |Q|$ can also be considered by introducing fictitious agents who value only money, the efficiency property of envy-free allocations will be lost (among other properties). We prefer to avoid this case at the moment. (See for example Thomson (2007) for a discussion on this issue).

Through the paper, we will use the relation between the No-Envy solution and the Identical Preferences Lower Bound solution which can be described as follows.

Given $R \in \mathcal{R}^q$, for each $i \in Q$ let $R^i \in \mathcal{R}^q$ be such that $R^i_j = R_i$ for all $j \in Q$. The economy R^i is obtained from R imagining that all the agents have the same preferences as agent i. For such an economy we define,

$$E(R^i) = \{z^i \in P(R^i) \ / \ \text{for all } j, k \in Q,\ z^i_j I_i z^i_k\}.$$

The set $E(R^i)$ is the set of Pareto efficient allocations at which each agent is indifferent between what he receives and what the others receive. Since $R^i_j = R_i$, it is easy to check that $E(R^i)$ is essentially single-valued, since for all z^i, $\bar{z}^i \in E(R^i)$, z^i, $\bar{z}^i$ are Pareto indifferent. Using this set, we define the Identical Preferences Lower Bound solution.

The Identical Preferences Lower Bound solution, E : given $R \in \mathcal{R}^q$,

$$E(R) = \{z \in Z / \text{for all } i \in Q, \text{for all } z^i \in E(R^i), z_i R_i z^i_i\}.$$

Under our assumptions on preferences, given $z^i, \bar{z}^i \in E(R^i)$, if $z^i = (\sigma, m)$, and $\bar{z}^i = (\tau, \bar{m})$, there exists a permutation π of Q such that $\sigma(i) = \tau(\pi(i))$, and $m_{\sigma(i)} = \bar{m}_{\tau(\pi(i))}$. Therefore, a generic element in $E(R^i)$ can be represented as $z^i = ((\alpha, m_\alpha(R_i)))_{\alpha \in \Omega}$ such that,

(i) $\sum_{\alpha \in \Omega} m_\alpha(R_i) = M$.

(ii) $(\alpha, m_\alpha(R_i)) I_i (\beta, m_\beta(R_i))$ for all $\alpha, \beta \in \Omega, \alpha \neq \beta$.

This solution is related to the No-Envy solution by inclusion (Moulin (1990), Beviá (1996)). That is, for any economy $R \in \mathcal{R}^q$, $N(R) \subseteq E(R)$. Therefore, given $R \in \mathcal{R}^q$ and given $z \in N(R)$, $z_i R_i (\alpha, m_\alpha(R_i))$ for all $\alpha \in \Omega$.

The purpose of this paper is to evaluate the consequences of the strategic behavior of agents when the prime fairness concept is no-envy. As shown by Alkan, Demange and Gale (1991), any subsolutions of the No-Envy solution is manipulable. Thus, we would like to know which properties the allocations obtained under manipulation have according to the true economy.

In order to study such properties, we use a *direct revelation game* in which the strategy space is the space of preferences, and the outcome correspondence is a subsolution of the No-Envy solution.

Given a mechanism φ, a **direct revelation game associated with** φ is a pair $(\mathcal{R}^q, \varphi)$ such that $\mathcal{R}^q$ is the strategy space, and $\varphi : \mathcal{R}^q \longrightarrow Z$ is the outcome correspondence.

We deal with the multi-valuation of the outcome correspondence by using definitions of equilibrium which generalizes the concept of Nash equilibrium. Thomson (1984) discusses various possible generalizations of such concept. One of them reflects the idea that agents are pessimistic in the sense that they are not going to change their strategy unless the change causes an improvement in all possible outcomes. The other one is the optimistic version; agents will change their strategy whenever the change causes an improvement in some of the possible outcomes. We will refer to the first concept as a *weak-equilibrium* and to the second as an *equilibrium*

From now on, we will denote the true preferences by $R^0 = (R_1^0, ..., R_q^0)$.
A pair (R, z) is a **weak-equilibrium** of $(\mathcal{R}^q, \varphi)$ played in R^0 if $z \in \varphi(R)$ and for all $i \in Q$, for all $R'_i \neq R_i$ there is $z' \in \varphi(R'_i, R_{-i})$ such that $z_i R_i^0 z'_i$.
According to this definition R'_i is a profitable deviation for agent i if for all $z' \in \varphi(R'_i, R_{-i})$, $z'_i P_i^0 z_i$.
A pair (R, z) is an **equilibrium** of $(\mathcal{R}^q, \varphi)$ played in R^0 if $z \in \varphi(R)$ and for all $i \in Q$, for all $R'_i \neq R_i$ for all $z' \in \varphi(R'_i, R_{-i})$, $z_i R_i^0 z'_i$.
According to this definition R'_i is a profitable deviation for agent i if there is $z' \in \varphi(R'_i, R_{-i})$ such that $z'_i P_i^0 z_i$.
The set of weak-equilibria (rep. equilibria) of the game $(\mathcal{R}^q, \varphi)$ played in R^0 is denoted by $WE(\mathcal{R}^q, \varphi; R^0)$ (resp. $E(\mathcal{R}^q, \varphi; R^0)$), and the set of weak-equilibrium allocations (resp. equilibrium allocations) by $WEA(\mathcal{R}^q, \varphi; R^0)$ (resp. $EA(\mathcal{R}^q, \varphi; R^0)$).
Notice that any equilibrium is a weak-equilibrium, and therefore, $EA(\mathcal{R}^q, \varphi; R^0) \subseteq WEA(\mathcal{R}^q, \varphi; R^0)$. Furthermore, as we show in Theorem 2.1, in this model both concepts are equivalent.

Our next result proves that under manipulation, all the subsolutions of the No-Envy solution satisfying *complete indifference* are equivalent. *Complete indifference* is a mild condition met by all the solutions in this setting that have been discussed in the literature.

A solution φ satisfies **complete indifference** if for all $z \in Z$ such that for all $i, j \in Q$ $z_i I_i z_j$, then $z \in \varphi(R)$.

Theorem 2.1. *Let $\varphi : \mathcal{R}^q \longrightarrow Z$ be such that for all $R \in \mathcal{R}^q$, $\varphi(R) \subseteq N(R)$ and satisfies complete indifference. Then*

$$N(R^0) = WEA(\mathcal{R}^q, \varphi; R^0) = EA(\mathcal{R}^q, \varphi; R^0).$$

Proof. Step 1. $N(R^0) \subseteq EA(\mathcal{R}^q, \varphi; R^0)$.

Let $z \in N(R^0)$, and let $R \in \mathcal{R}^q$ be such that $z_i I_i z_j$ for all $i, j \in Q$. By complete indifference $z \in \varphi(R)$. It remains to prove that (R, z) is an equilibrium of $(\mathcal{R}^q, \varphi)$ played in R^0. Let $i \in Q$ be given, we will show that no $R'_i \neq R_i$ can be a profitable deviation for agent i. Let $z' = (\tau, m') \in \varphi(R'_i, R_{-i})$ be given. Since $\varphi(R'_i, R_{-i}) \subseteq N(R'_i, R_{-i})$ and $N(R'_i, R_{-i}) \subseteq E(R'_i, R_{-i})$, it should be the case that $m'_{\tau(j)} \geq m_{\tau(j)}(R_j)$ for all $j \neq i$ and $m'_{\tau(i)} \geq m_{\tau(i)}(R'_i)$. By the choice of R, $m_{\tau(j)}(R_j) = m_{\tau(j)}(R_i) = m_{\tau(j)}$ for all $j \in Q$. By feasibility $m'_{\tau(i)} \leq m_{\tau(i)}(R_i)$. Since $z \in N(R^0)$, $z_i R^0_i z_j$ for all $j \in Q$, in particular, $z_i R^0_i (\tau(i), m_{\tau(i)}(R_i)) R^0_i (\tau(i), m'_{\tau(i)})$. Then, for all $z' \in \varphi(R'_i, R_{-i})$, $z_i R^0_i z'_i$. Thus, (R, z) is an equilibrium of $(\mathcal{R}^q, \varphi)$ played in R^0.

Step 2. $WEA(\mathcal{R}^q, \varphi; R^0) \subseteq N(R^0)$.

Let (R, z) be a weak-equilibrium of $(\mathcal{R}^q, \varphi)$ played in R^0. Suppose that $z \notin N(R^0)$. Then, there is at least one agent $i \in Q$ who envies somebody. For each $\alpha \in \Omega$ let m^i_α be such that $(\sigma(i), m_{\sigma(i)}) I^0_i (\alpha, m^i_\alpha)$.

Case 1. Suppose that $\sum_{\alpha \in \Omega} m^i_\alpha < M$.

For each $\alpha \in \Omega$, let $m_\alpha(R^0_i)$ be such that $(\alpha,\, m_\alpha(R^0_i)) I^0_i (\beta,\, m_\beta(R^0_i))$ and $\sum_{\alpha \in \Omega} m_\alpha(R^0_i) = M$. By monotonicity of preferences in money, $m^i_\alpha < m_\alpha(R^0_i)$ for all $\alpha \in \Omega$. If agent i switches from R_i to R^0_i, for all $z' = (\tau, m') \in \varphi(R^0_i, R_{-i}) \subseteq N(R^0_i, R_{-i})$, since $N(R^0_i, R_{-i}) \subseteq E(R^0_i, R_{-i})$, we obtain that $m'_{\tau(i)} \geq m_{\tau(i)}(R^0_i) > m^i_{\tau(i)}$. Then,

$$(\tau(i), m'_{\tau(i)}) R^0_i (\tau(i), m_{\tau(i)}(R^0_i)) P^0_i (\tau(i), m^i_{\tau(i)}) I^0_i (\sigma(i), m_{\sigma(i)}).$$

Therefore, R_i^0 is a profitable deviation for agent i, in contradiction with the fact that (R, z) is an equilibrium of $(\mathcal{R}^q, \varphi)$ played in R^0.

Case 2. Suppose that $\sum_{\alpha\in\Omega} m_\alpha^i \geq M$.

Let $Q' \subset Q\backslash\{i\}$ be such that $j \in Q'$, $z_j P_i^0 z_i$. Let $q' = |Q'|$, and $R_i' \neq R_i$ be such that

(i) $m_{\sigma(i)}(R_i') = m_{\sigma(i)} + \varepsilon$,

(ii) $m_{\sigma(j)}(R_i') = m_{\sigma(j)} - k$ for all $j \in Q'$,

(iii) $m_{\sigma(l)}(R_i') = m_{\sigma(l)} + \frac{q'k-\varepsilon}{q-q'-1}$ for all $l \notin Q' \cup \{i\}$,

with k and ε such that:

(1) $\varepsilon, k > 0$,

(2) $\frac{(q-q')\varepsilon}{q'} < k < m_{\sigma(j)} - m_{\sigma(j)}^i$ for all $j \in Q'$.

We claim that for all $z' \in \varphi(R_i', R_{-i})$, $z_i' P_i^0 z_i$.

Let $z' = (\tau, m') \in \varphi(R_i', R_{-i})$. Since $\varphi(R_i', R_{-i}) \subseteq N(R_i^0, R_{-i}) \subseteq E(R_i^0, R_{-i})$, if $\tau(i) = \sigma(i)$, clearly $z_i' P_i^0 z_i$. If $\tau(i) = \sigma(j)$ for $j \in Q'$, since $k < m_{\sigma(j)} - m_{\sigma(j)}^i$, then $z_i' P_i^0 z_i$. If $\tau(i) = \sigma(l)$ for $l \notin Q' \cup \{i\}$ and $m_{\sigma(l)} + \frac{q'k-\varepsilon}{q-q'-1} > m_{\sigma(l)}^i$, then $z_i' P_i^0 z_i$.

Let $Q^* = \{l \notin Q' \cup \{i\}$ such that $m_{\sigma(l)} + \frac{q'k-\varepsilon}{q-q'-1} < m_{\sigma(l)}^i\}$. To complete the proof we only need to show that for all $l \in Q^*$, $z_i' \neq (\sigma(l), m_i')$ with $m_i' \in [m_{\sigma(l)} + \frac{q'k-\varepsilon}{q-q'-1}, m_{\sigma(l)}^i]$.

Suppose that there is $z' = (\tau, m') \in \varphi(R_i', R_{-i})$ such that $\tau(i) = \sigma(l)$ for some $l \in Q^*$ and $m_i' \in [m_{\sigma(l)} + \frac{q'k-\varepsilon}{q-q'-1}, m_{\sigma(l)}^i]$.

Claim 1. For all $l \in Q^*$, $\tau(l) \neq \sigma(i)$.

Suppose that $\tau(l) = \sigma(i)$. Let $m^*, \bar{m} \in \mathbb{R}$ be such that $(\sigma(l), m_i') I_l (\sigma(i), m^*)$, and $(\sigma(l), m_i') I_i' (\sigma(i), \bar{m})$. Since $z' \in \varphi(R_i', R_{-i}) \subseteq N(R_i', R_{-i})$, $\bar{m} \geq m_{\tau(l)} \geq m^*$. But, let us see that this is not possible because m^* is bigger than $\bar{m}$.

Let $m_{\sigma(i)}^l$ be such that

$$(\sigma(l), m_{\sigma(l)} + \frac{q'k-\varepsilon}{q-q'-1}) I_l (\sigma(i), m_{\sigma(i)}^l).$$

Since $\frac{(q-q')\varepsilon}{q'} < k$, we know that $\frac{q'k-\varepsilon}{q-q'-1} > \varepsilon$. By monotonicity of preferences in money, $(\sigma(l), m_{\sigma(l)} + \frac{q'k-\varepsilon}{q-q'-1}) P_l (\sigma(l), m_{\sigma(l)} + \varepsilon)$. Since $z \in \varphi(R) \subseteq N(R)$, $(\sigma(l), m_{\sigma(l)}) R_l (\sigma(i), m_{\sigma(i)})$, by quasi-linearity, $(\sigma(l), m_{\sigma(l)} + \varepsilon) R_l (\sigma(i), m_{\sigma(i)} + \varepsilon)$. By monotonicity of preferences in money, $m_{\sigma(i)}^l > m_{\sigma(i)} + \varepsilon$.

Thus, by quasi-linearity $m^* > \bar{m}$.

Claim 2. For all $l \in Q^*$, and for $j \in Q'$, $\tau(l) \neq \sigma(j)$.

Suppose that there is $l \in Q^*$ such that $\tau(l) = \sigma(j)$ for some $j \in Q'$. Let $m^*, \bar{m} \in \mathbb{R}$ be such that $(\sigma(l), m_i')I_l(\sigma(j), m^*)$, and $(\sigma(l), m_i')I_i'(\sigma(j), \bar{m})$. Since $z' \in \varphi(R_i', R_{-i}) \subseteq N(R_i', R_{-i})$, $\bar{m} \geq m_{\tau(l)} \geq m^*$. But, let us see that this is not possible because m^* is bigger than $\bar{m}$.

Let $m_{\sigma(j)}^l$ be such that

$$(\sigma(l), m_{\sigma(l)} + \frac{q'k - \varepsilon}{q - q' - 1})I_l(\sigma(j), m_{\sigma(j)}^l).$$

Since $z \in \varphi(R) \subseteq N(R)$, $(\sigma(l), m_{\sigma(l)})R_l(\sigma(j), m_{\sigma(j)})$. Since $\frac{q'k-\varepsilon}{q-q'-1} > 0$,

$$(\sigma(l), m_{\sigma(l)} + \frac{q'k - \varepsilon}{q - q' - 1})P_l(\sigma(j), m_{\sigma(j)}).$$

Therefore, $m_{\sigma(j)}^l > m_{\sigma(j)}$. Thus, by quasi-linearity $m^* > \bar{m}$.

Claim 3. For all $l \in Q^*$, and for all $n \notin Q' \cup \{i\}$, $n \neq l$, $\tau(l) \neq \sigma(n)$.

Suppose that $\tau(l) = \sigma(n)$ for some $n \neq l, n \notin Q' \cup \{i\}$. Let $m^*, \bar{m} \in \mathbb{R}$ be such that $(\sigma(l), m_i')I_l(\sigma(n), m^*)$, and $(\sigma(l), m_i')I_i'(\sigma(n), \bar{m})$. Since $z' \in \varphi(R_i', R_{-i}) \subseteq N(R_i', R_{-i})$, $\bar{m} \geq m_{\tau(l)} \geq m^*$.

Let $m_{\sigma(n)}^l$ be such that

$$(\sigma(l), m_{\sigma(l)} + \frac{q'k - \varepsilon}{q - q' - 1})I_l(\sigma(n), m_{\sigma(n)}^l).$$

Since $z \in \varphi(R) \subseteq N(R)$, $(\sigma(l), m_{\sigma(l)})R_l(\sigma(n), m_{\sigma(n)})$, by quasi-linearity

$$(\sigma(l), m_{\sigma(l)} + \frac{q'k - \varepsilon}{q - q' - 1})R_l(\sigma(n), m_{\sigma(n)} + \frac{q'k - \varepsilon}{q - q' - 1}).$$

By monotonicity of preferences in money $m_{\sigma(n)}^l \geq m_{\sigma(n)} + \frac{q'k-\varepsilon}{q-q'-1}$.

Suppose that $m_{\sigma(n)}^l > m_{\sigma(n)} + \frac{q'k-\varepsilon}{q-q'-1}$. Then, in this case and by quasi-linearity, $m^* > \bar{m}$, in contradiction with the fact that $z' \in \varphi(R_i', R_{-i}) \subseteq N(R_i', R_{-i})$. Thus, if $\tau(l) = \sigma(n)$ for some $n \neq l, n \notin Q' \cup \{i\}$, $m_{\sigma(n)}^l = m_{\sigma(n)} + \frac{q'k-\varepsilon}{q-q'-1}$. In this case agent n should receives $\sigma(i)$, or $\sigma(j)$ for some $j \in Q'$ or $\sigma(s)$ for some $s \notin Q' \cup \{i\}$, $s \neq l, n$. But in any of those cases, we can repeat the above arguments. Thus, an agent $s \notin Q' \cup \{i\}$, $s \neq l, n$ should receive $\sigma(i)$ or $\sigma(j)$ for some $j \in Q'$, but this is not possible by claims 1 and 2.

Therefore, for all $z' \in \varphi(R_i', R_{-i})$, and for all $l \in Q^*$, $z_i' \neq (\sigma(l), m_i')$ with $m_i' \in [m_{\sigma(l)} +$

$\frac{q'k-\varepsilon}{q-q'-1}, m^i_{\sigma(l)}]$.
Thus, for all $z' \in \varphi(R'_i, R_{-i})$, $z'_i P^0_i z_i$, which is a contradiction with the fact that (R, z) is a weak-equilibrium of $(\mathcal{R}^q, \varphi)$ for R^0. ■

Notice that since $N(R^0) \neq \emptyset$ for all $R^0 \in \mathcal{R}^q$, the first part of the proof also shows the existence of equilibria of $(\mathcal{R}^q, \varphi)$ for R^0. Furthermore, since all the envy-free allocations are efficient (Svenson (1983)), the result also shows that efficiency is not destroyed under manipulation.

In the next result we show that the above characterization also holds if a refinement of the equilibrium is used.

A pair (R, z) is a **strong equilibrium** of $(\mathcal{R}^q, \varphi)$ for R^0 if $z \in \varphi(R)$, and for all $S \subseteq Q$, $S \neq \emptyset$, for all $R'_S = (R_i)_{i \in S}$ with $R'_S \neq R_S$, for all $z' \in \varphi(R'_S, R_{-S})$, $z_i R^0_i z'_i$ for all $i \in S$.

Let $SE(\mathcal{R}^q, \varphi; R^0)$ be the set of strong equilibria of $(\mathcal{R}^q, \varphi)$ for R^0, and $SEA(\mathcal{R}^q, \varphi; R^0)$ be the set of strong equilibrium allocations of $(\mathcal{R}^q, \varphi)$ for R^0.

Theorem 2.2. *Let $\varphi : \mathcal{R}^q \longrightarrow Z$ be such that for all $R \in \mathcal{R}^q$, $\varphi(R) \subseteq N(R)$ and satisfies complete indifference. Then*

$$N(R^0) = SEA(\mathcal{R}^q, \varphi; R^0).$$

Proof. Let $z \in N(R^0)$, and let $R \in \mathcal{R}^q$ be such that $z_i I_i z_j$ for all $i, j \in Q$. By complete indifference $z \in \varphi(R)$. It remains to prove that (R, z) is a strong equilibrium of $(\mathcal{R}^q, \varphi)$ for R^0. Suppose that there is $S \subseteq Q$, $|S| \geq 2$ such that for some $R'_S = (R'_i)_{i \in S}$ with $R'_S \neq R_S$, there is $z' = (\tau, m') \in \varphi(R'_S, R_{-S})$ such that for all $j \in S$, $z'_j P^0_j z_j$. For each $j \in S$ and for each $\alpha \in \Omega$, let m^j_α be such that $(\alpha, m^j_\alpha) I^0_j (\sigma(j), m_{\sigma(j)})$. Since $z \in N(R^0)$, $m^j_{\sigma(i)} \geq m_{\sigma(i)}$ for all $i \in Q$. Since for all $j \in S$, $z'_j P^0_j z_j$, $m'_{\tau(j)} > m^j_{\tau(j)} \geq m_{\tau(j)}$ By feasibility, for some $k \in Q \backslash S$, $m'_{\tau(k)} < m_{\tau(k)}$. But $z' \in \varphi(R'_S, R_{-S}) \subseteq E(R'_S, R_{-S})$ and $(\sigma(k), m_{\sigma(k)}) I_k (\sigma(i), m_{\sigma(i)})$ for all $i \in Q$. Therefore, $m'_{\tau(k)} \geq m_{\tau(k)}$ in contradiction with feasibility. ■

3. The general model with indivisible goods

In this section we study the strategic aspects of the No-Envy solution in economies with indivisible goods in which each agent is allowed to consume more that one indivisible good.

An **economy** is a list $e = (Q, \Omega, M; R)$, where Q is a finite set of agents, Ω is a finite set of objects, $M \in \mathbb{R}$ is an amount of money, and $R = (R_i)_{i \in Q}$ is a list of preference relations defined over $\mathcal{P}(\Omega)$x$\mathbb{R}$, where $\mathcal{P}(\Omega)$ denotes the power set of Ω. Let P_i denote the strict relation associated with R_i, and I_i the indifference relation. Each preference relation is continuous, increasing in money and quasi-linear, that is, if $(A, m)I_i(B, m')$ with $A, B \in \mathcal{P}(\Omega)$, for all $t \in \mathbb{R}$, $(A, m+t)I_i(B, m'+t)$. We assume that no set of objects is infinitely desirable or undesirable when compared with another set. Thus, for any bundle $(A, m) \in \mathcal{P}(\Omega)x\mathbb{R}$, any $i \in Q$, and any $B \in \mathcal{P}(\Omega)$, there is an amount of money m' such that $(A, m)I_i(B, m')$. Let $\mathcal{R}$ be the class of all such preference relations. We assume that Ω, and M are known and fixed, therefore an economy is completely described by a list $R \in \mathcal{R}^q$ where $q = |Q|$.

Let $\mathbb{P}(\Omega, q)$ be the set of all the partitions of Ω with q elements. That is, $\mathbb{P}(\Omega, q) = \{P = \{A_1, .., A_q\} / A_i \in \mathcal{P}(\Omega), \cup_{i \in Q} A_i = \Omega,\ A_i \cap A_j = \emptyset \text{ for all } i \neq j\}$.

A feasible allocation is a triple $z = (P, \sigma, m)$, where $P \in \mathbb{P}(\Omega, q)$, $\sigma : Q \longrightarrow P$ is a bijection, and $m = (m_{\sigma(i)})_{i \in Q}$ is such that $\sum_{i \in Q} m_{\sigma(i)} = M$. The bijection σ is called an assignment. For each $i \in Q$, let $z_i = (\sigma(i), m_{\sigma(i)})$ be the consumption of agent i. Note that $m_{\sigma(i)}$ can be either positive, negative or zero. Let Z be the set of feasible allocations.

The Pareto solution, the No-Envy solution and the Identical Preferences Lower Bound solution are defined as in Section 2.

Given an efficient allocation, (P, σ, m), we will refer to σ as the efficient assignment, and P as the efficient partition. Given an economy $R \in \mathcal{R}^q$, let $\Sigma(R)$ be the set of efficient assignments.

If agents can receive more than one object, not all envy-free allocations are efficient (Beviá (1998)). In order to keep efficiency we consider only subsolutions of the No-Envy and the Pareto solution. We denote the intersection of the No-Envy solution and the Pareto solution by NP.

In the one object per person case, the No-Envy solution also coincide with the Equal Income Walrasian solution (with divisible goods, this is the Walrasian solution from an equal distribution of the resources, we properly define the solution in the case of indivisible goods below). Thus, Theorem 2.1 also tell us that the set of equilibrium allocations of the game associated with any subsolution of the Equal Income Walrasian solution is the set of Equal Income Walrasian allocations for the true preferences. This result contrasts with the one obtained by Otani and Sicilian (1982) in the case of divisible goods where, even though the Walrasian allocations for the true preferences are in the set of equilibrium allocations , non-efficient allocations are also in that set. When we allow agents to consume more than one object, the results becomes closer to the ones with divisible goods. Although we do not have a complete characterization of the set of equilibrium allocations, we show that the Equal Income Walrasian allocation for the true preferences can be supported as an equilibrium allocation of the direct revelation game associated to any subsolution of the No-Envy and the Pareto solution. This is good news, but we give an example where some other non-efficient allocations can also be supported as equilibrium.

We start with the formal definition of the Equal Income Walrasian solution.

Let $p = (p_\alpha)_{\alpha\in\Omega} \in \mathbb{R}^{|\Omega|}$ denote prices for the indivisible goods (prices are expressed in units of money). If agent i buys the set A of objects he will pay $p_A = \sum_{\alpha\in A} p_\alpha$.

The Equal Income Walrasian solution, WI: given $R \in \mathcal{R}^q$, $z = (P, \sigma, m) \in WI(R)$ if $z \in Z$, and there exists a price vector $p \in \mathbb{R}^{|\Omega|}$ such that,

(i) if $(A, m_A)P_i(\sigma(i), m_{\sigma(i)})$, then $p_A + m_A > p_{\sigma(i)} + m_{\sigma(i)}$.

(ii) $p_{\sigma(i)} + m_{\sigma(i)} = p_{\sigma(j)} + m_{\sigma(j)}$ for all $i, j \in Q$.

The inclusion of the No-Envy solution in the Identical Lower Bound solution was essential in the proof of Theorem 2.1. This relation does not hold in the general model (Beviá (1998)). But still, we can apply the following lemma.

Lemma 1. *Let $R \in \mathcal{R}^q$, and let $z = (P, \sigma, m) \in N(R)$. For each $i \in Q$, let $(\sigma(j), m_{\sigma(j)}(R_i))_{j\in Q}$ be a feasible allocation such that $(\sigma(j), m_{\sigma(j)}(R_i))I_i(\sigma(k), m_{\sigma(k)}(R_i))$ for all $j, k \in Q$. Then, $z_iR_i(\sigma(j), m_{\sigma(j)}(R_i))$ for all $j \in Q$.*

Proof. Suppose that for all $j \in Q$, $(\sigma(j), m_{\sigma(j)}(R_i))P_i z_i$. Since z is envy free, $z_i R_i z_j$ for all $j \in Q$. Thus, $m_{\sigma(j)}(R_i) > m_j$ for all $j \in Q$, in contradiction with feasibility. ■

In Section 2 complete indifference was required in order to obtain the characterization of the equilibrium allocations. In the present model, this condition is not satisfied by the Pareto solution. For our result, we will use a weaker condition which will be satisfied by all the Pareto solutions that are presented in the paper. In the one object per person case, both conditions are equivalent.

A subsolution of the Pareto solution, φ, satisfies **P-complete indifference** if for all $R \in \mathcal{R}^q$, for all $z \in P(R)$ such that for all $i, j \in Q$, $z_i I_i z_j$, then $z \in \varphi(R)$.

Under this condition we are ready to establish our first result.

Theorem 3.1. *Let $\varphi : \mathcal{R}^q \longrightarrow Z$ be such that for all $R \in \mathcal{R}^q$, $\varphi(R) \subseteq NP(R)$ and satisfies P-complete indifference. Then, $WI(R^0) \subseteq EA(\mathcal{R}^q, \varphi; R^0)$.*

In order to prove the theorem we need the following lemmas, which give some properties of the Equal Income Walrasian solution.

Lemma 2. *Given $z \in WI(R^0)$, if $\tau \in \Sigma(R^0)$, and $z' = (P', \tau, m')$ is such that for all $i \in Q$, $z_i I_i^0 z_i'$, then $z' \in WI(R^0)$.*

Proof. Step 1. $z' \in Z$

Suppose that $\sum_{i \in Q} m_i' = M' < M$. Let $z'' = (P', \tau, m'')$ be such that $m_i'' = m_i' + \frac{M-M'}{q}$. Thus $z'' \in Z$ and $z_i'' P_i^0 z_i' I_i^0 z_i$ for all $i \in Q$, in contradiction with the efficiency of z.Then, $\sum_{i \in Q} m_i' = M' \geq M$. Suppose that $\sum_{i \in Q} m_i' = M' > M$. Let $z'' = (P', \tau, m'')$ be such that $m_i'' = m_i' + \frac{M-M'}{q}$. Thus $z'' \in Z$ and $z_i' I_i^0 z_i P_i^0 z_i''$ for all $i \in Q$. But this is a contradiction since z'' is Pareto efficient because is feasible and $\tau \in \Sigma(R^0)$. Therefore $\sum_{i \in Q} m_i' = M$.

Step 2. $z' \in WI(R^0)$.

Let $p \in \mathbb{R}^{|\Omega|}$ be the equilibrium prices at z. Since $z_i I_i^0 z_i'$ and z' is Pareto efficient, we claim that for all $i \in Q$, $p_{\tau(i)} + m_{\tau(i)} = p_{\sigma(i)} + m_{\sigma(i)}$. Suppose that $p_{\tau(i)} + m_{\tau(i)} < p_{\sigma(i)} + m_{\sigma(i)}$ for some $i \in Q$. Let $m'_i = m_{\tau(i)} + d$, where d is such that $p_{\tau(i)} + m_{\tau(i)} + d \leq p_{\sigma(i)} + m_{\sigma(i)}$. Then $(\tau(i), m'_i)P^0_i(\sigma(i), m_{\sigma(i)})$ which is a contradiction because $z \in WI(R^0)$. Therefore for all $i \in Q$, $p_{\tau(i)} + m_{\tau(i)} \geq p_{\sigma(i)} + m_{\sigma(i)}$. Then, $\sum_{i\in Q}(p_{\tau(i)} + m_{\tau(i)}) \geq \sum_{i\in Q}(p_{\sigma(i)} + m_{\sigma(i)})$. But $\sum_{i\in Q} m_{\tau(i)} = \sum_{i\in Q} m_{\sigma(i)}$, and $\cup_{i\in Q}\tau(i) = \cup_{i\in Q}\sigma(i)$, then $\sum_{i\in Q} p_{\tau(i)} = \sum_{i\in Q} p_{\sigma(i)}$. Therefore, $p_{\tau(i)} + m_{\tau(i)} = p_{\sigma(i)} + m_{\sigma(i)}$. ■

Lemma 3. *Given $z \in WI(R^0)$ let P be any other non-efficient partition. For any $A \in P$ and for any $i \in Q$, let m^i_A be such that $z_i I^0_i(A, m^i_A)$, and let $\bar{m}_A = \min_i m^i_A$. Then $\sum_{A\in P} \bar{m}_A \geq M$.*

Proof. For any $A \in P$ there is $i \in Q$ such that $(A, \bar{m}_A)I^0_i(\sigma(i), m_{\sigma(i)})$. Since $z \in WI(R^0)$, $p_A + \bar{m}_A \geq p_{\sigma(i)} + m_{\sigma(i)}$. Since $p_{\sigma(i)} + m_{\sigma(i)} = p_{\sigma(j)} + m_{\sigma(j)}$ for all $i, j \in Q$, we conclude that $\sum_{A\in P}(p_A + \bar{m}_A) \geq \sum_{i\in Q}(p_{\sigma(i)} + m_{\sigma(i)})$. Since $\cup_{A\in P} A = \cup_{i\in Q}\sigma(i)$, $\sum_{A\in P} p_A = \sum_{i\in Q} p_{\sigma(i)}$. Therefore, $\sum_{A\in P} \bar{m}_A \geq \sum_{i\in Q} m_{\sigma(i)} = M$. ■

Proof of Theorem 3.1. Let $z \in WI(R^0)$, and let $R \in \mathcal{R}^q$ be such that

(i) $z_i I_i z_j$ for all $i, j \in Q$.

(ii) for all $\tau \in \Sigma(R^0)$ and $z' = (P', \tau, m')$ such that for all $i \in Q$ $z_i I^0_i z'_i$, let $z_i I_i z'_i I_i z'_j$ for all $i, j \in Q$.

(iii) for any other partition P, and for any $A \in P$, let $\bar{m}_A$ be as it was described in Lemma 3. If $\sum_{A\in P} \bar{m}_A = M$, let $z_i I_i(A, \bar{m}_A)$ for all $i \in Q$, for all $A \in P$. If $\sum_{A\in P} \bar{m}_A > M$, let $m^*_A = \bar{m}_A - d$, with $d = \frac{\sum_{A\in P} \bar{m}_A - M}{q}$, and let $z_i I_i(A, m^*_A)$ for all $i \in Q$, for all $A \in P$.

By (i), (ii), (iii), $z \in P(R)$, and by P-complete indifference $z \in \varphi(R)$. It remains to prove that $(R, z) \in EA(\mathcal{R}^q, \varphi; R^0)$. Let $i \in Q$, we will show that no $R'_i \neq R_i$ can be a profitable deviation for agent i. Let $R'_i \neq R_i$, and let $z' = (P', \tau, m') \in \varphi(R'_i, R_{-i}) \subseteq NP(R'_i, R_{-i})$, by Lemma 1, $m'_{\tau(j)} \geq m_{\tau(j)}(R_j)$ for all $j \neq i$. Since $m_{\tau(j)}(R_i) = m_{\tau(j)}(R_j)$ for all i, j, and $\sum_{k\in Q} m_{\tau(k)}(R_j) = M$, by feasibility $m'_{\tau(i)} \leq m_{\tau(i)}(R_i)$. Since $z \in WI(R^0)$, by Lemmas 2 and 3, and by (i), (ii) and (iii), for $m_{\tau(i)}$ such that $z_i I^0_i(\tau(i), m_{\tau(i)})$ should be the case that $m_{\tau(i)} \geq m_{\tau(i)}(R_i)$. Therefore, by monotonicity of preferences in money, $z_i I^0_i(\tau(i), m_{\tau(i)})R^0_i(\tau(i), m_{\tau(i)}(R_i))R^0_i(\tau(i), m'_{\tau(i)})$. Consequently, $(R, z) \in EA(\mathcal{R}^q, \varphi; R^0)$. ■

The next example shows that inefficient allocations can also be supported as equilibrium of the direct revelation game associated to subsolutions of the No-Envy and Pareto solution. In particular, in the example we consider the Equal Income Walrasian solution.

In order to avoid problems of existence, we restrict the domain of preferences to satisfy the following properties:

For each agent $i \in Q$, preferences are representable by a utility function such that:

(i) For each $(A, m) \in \mathcal{P}(\Omega)\mathrm{x}\mathbb{R}$, $u_i(A, m) = v_i(A) + m$, where $v_i(A)$ is the reservation value of holding the set of object A.

(ii) The reservation value functions are submodular. That is, $v_i(A) - v_i(A \backslash \alpha) \leq v_i(B) - v_i(B \backslash \alpha)$ for all $\alpha \in B \subseteq A$.

Let $\mathcal{R}_s$ be this class of quasi-linear preferences.

Although submodularity is not sufficient to imply existence of Walrasian allocations if indivisible objects are not perfect substitutes for one another (Beviá et al. (1999)), existence is guarantee with submodularity if objects are identical (Henry (1970)). In the following example we deal with identical objects.

Example 1. *Let $e = (Q, \Omega, M; R^0)$ be such that $Q = \{1, 2\}$, $\Omega = \{\alpha, \alpha, \alpha, \alpha\}$ that is, we have four units of an identical indivisible object, $M = 10$, and the preferences $R_i^0 \in \mathcal{R}_s$ of agent i holding m_i units of money and the set A of object is described by the utility function $u_i(A, m_i) = v_i(A) + m_i$ where $v_1(\alpha) = 1$; $v_1(2\alpha) = 2$; $v_1(3\alpha) = 3$; $v_1(4\alpha) = 4$; $v_2(\alpha) = 2$; $v_2(2\alpha) = 4$; $v_2(3\alpha) = 6$; $v_2(4\alpha) = 6$.*

The efficient assignment of objects in this economy requires to give one unit of α to agent one and three units of α to agent two. Notice that, in this particular example, the Equal Income Walrasian allocation is the standard Walrasian allocation obtained when the initial resources of agents are the equal division of the total resources, $((2\alpha, 5), (2\alpha, 5))$. Thus, it is not difficult to see that the unique Equal Income Walrasian allocation for this economy is $z = (z_1, z_2)$ such that $z_1 = (\alpha, 6)$; $z_2 = (3\alpha, 4)$, and $p_\alpha = 1$.

Let us consider the direct revelation game associated with WI. Notice first that, for the allocation

z, the preferences described in (i), (ii) and (iii) in Theorem 3.1 satisfy submodularity. So, we can follow the proof of Theorem 3.1 to conclude that z can be supported as an equilibrium allocation of the game $(\mathcal{R}_s^2, WI)$ played in R^0. But let us see that we can also support allocations which are not efficient for the economy e. In particular let $z^* = (z_1^*, z_2^*)$ be such that $z_1^* = (2\alpha, 5)$ and $z_2^* = (2\alpha, 5)$. Clearly, this allocation is not efficient for the initial economy. Let $R \in \mathcal{R}_s^2$ be such that

$$(2\alpha, 5)I_i(3\alpha, 5)I_i(4\alpha, 5)I_i(\alpha, 7)I_i(\varnothing, 9) \text{ for all } i \in \{1, 2\}.$$

For this new economy the unique efficient assignment requires to give each agent two units of α. Any price between cero and two can support this efficient assignment. And clearly, $z^* \in WI(R)$. It remains to prove that (R, z^*) is an equilibrium of $(\mathcal{R}_s^2, WI)$ played in R^0. Let us show first that no $R_1' \neq R_1$ can be a profitable deviation for agent 1. Let $z' = (\tau, m') \in WI(R_1', R_2)$ be given. If R_1' is a profitable deviation for agent 1, then

$$(\tau(1), m_1')P_1^0(2\alpha, 5)I_1^0(\alpha, 6)I_1^0(3\alpha, 4)I_1^0(4a, 3)I_1^0(\varnothing, 7).$$

Furthermore, since $z' \in WI(R_1', R_2)$, for agent 2 it should be the case that

$$(\tau(2), m_2')R_2(2\alpha, 5)I_2(3\alpha, 5)I_2(4\alpha, 5)I_2(\alpha, 7)I_2(\varnothing, 9).$$

Notice that there is no a feasible allocation compatible with those requirements. Finally, let us show that no $R_2' \neq R_2$ can be a profitable deviation for agent 2. Let $z'' = (\rho, m'') \in WI(R_1, R_2')$ be given. If R_2' is a profitable deviation for agent 2, then

$$(\rho(2), m_2'')P_2^0(2\alpha, 5)I_2^0(3\alpha, 3)I_2^0(4\alpha, 3)I_2^0(\alpha, 7)I_2^0(\varnothing, 9).$$

Furthermore, since $z'' \in WI(R_1, R_2')$, for agent 1 it should be the case that

$$(\rho(1), m_1'')R_1(2\alpha, 5)I_1(3\alpha, 5)I_1(4\alpha, 5)I_1(\alpha, 7)I_1(\varnothing, 9).$$

Again notice that there is no a feasible allocation compatible with those requirements. Thus $z^* = (z_1^*, z_2^*)$ is an equilibrium of the game $(\mathcal{R}_s^2, WI)$ played in R^0.

Finally, we can argue that, although z^* is not efficient, it is envy free. Thus, we can still thinking that, perhaps, all envy free allocations can be supported as an equilibrium. However, let us see that in the above example, the allocation $\bar{z} = ((\varnothing, 8), (4\alpha, 2))$, which is envy free, can not be supported as an equilibrium. Suppose that there is R^* such that $(R^*, \bar{z})$ is an equilibrium. Notice first that $(2\alpha, 5)P_2(4\alpha, 2)$. Then, agent two can guarantee himself at least the utility of the initial endowments by announcing his true preferences, because the Equal Income Walrasian allocation is always weakly prefer to $(2\alpha, 5)$. Therefore, $(R^*, \bar{z})$ can not be an equilibrium.

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