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Dividends and Weighted Values in Games with Externalities*

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Abstract

We consider cooperative environments with externalities (games in partition function form) and provide a recursive definition of dividends for each coalition and any partition of the players it belongs to. We show that with this definition and equal sharing of these dividends the averaged sum of dividends for each player, over all the coalitions that contain the player, coincides with the corresponding average value of the player. We then construct weighted Shapley values by departing from equal division of dividends and finally, for each such value, provide a bidding mechanism implementing it.

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1 Introduction

In recent years there has been increasing interest in the construction and implementation of sharing rules for environments with externalities. Such environments encompass a large array of economic scenarios and arise whenever the benefits or costs to a group of agents depend on the coalitions formed by agents outside the group. Prominent examples are a market with several competing firms, countries negotiating on trade agreements and union-firm bargaining in a given industry.

One fruitful approach has been to proceed axiomatically and propose sharing methods (known as values) based on a collection of desirable properties. Some of the proposals extend the Shapley value (Shapley, 1953b), which is defined for games with no externalities.[1] This is the direction taken in the papers by Myerson (1977), Bolger (1989), Feldman (1996), Albizuri, Arin and Rubio (2005), Pham Do and Norde (2007), and Macho-Stadler, Pérez-Castrillo and Wettstein (2007) (that we will refer to as MPW). These proposals satisfy the properties (axioms) of efficiency, anonymity (symmetry), linearity, and the "null" player property that states that players which have no effect on the outcome should neither receive nor pay anything. The particular definitions of these properties, at times together with some additional axioms, lead to different extensions of the Shapley value.

In MPW we strengthen the symmetry property by proposing the "strong symmetry axiom" capturing the idea that players with "identical power" should receive the same outcome. We show that this axiom is equivalent to adopting an "average approach" to the problem of sharing. This approach is quite intuitive: it yields to a player in a game with externalities the Shapley value of an average game with no externalities. The average game is obtained from the original game by assigning to each coalition its (weighted) average payoff. The average approach generates an attractive family of sharing methods (see Section 2 for further details). For example, by including an additional property MPW (2007) derived a unique sharing method belonging to the family of averaging methods. This value corresponds to the one proposed, but not axiomatized, by Feldman (1996).

[1] Similarly, de Clippel and Serrano (forthcoming), rather than extend the Shapley value, focus on the implications of the marginality axiom underlying the Shapley value, for games with externalities. McQuillin (2006) also uses axioms to characterize a value that simultaneously extends the Shapley value to games with externalities and to situations where there is a prior coalition structure.

In games without externalities, another useful approach to find sharing methods has been to take a constructive point of view. Hart and Mas-Colell (1989) showed how the Shapley value can be derived through the potential approach.[2] Harsanyi (1959) used the notion of dividends accruing to a player from the various coalitions he could participate in and showed that their sum yields the Shapley value. Maschler (1982) generalized this approach to a procedure through which the payoffs accruing to agents are still given by the Shapley value.

In this paper we adopt the constructive approach both to justify the family of average values and to generate new sharing methods. We first define the dividends for all possible partition-coalition pairs (in contrast to the no-externalities case the dividends of a coalition may depend on the partition it belongs to) and then proceed to allocate them among the players. We show that the sum of dividends a player gets yields the average value defined in MPW.

This approach can be easily adopted to generate non-symmetric values, possibly capturing players's characteristics external to the environment. Hence we provide a new family of weighted Shapley value for games with externalities and show that each one coincides with the weighted Shapley value of the average game.

We then augment the cooperative approach by implementing the weighted values for games with externalities via a bidding mechanism that is based on the mechanisms appearing in Pérez-Castrillo and Wettstein (2001) and Macho-Stadler, Pérez-Castrillo and Wettstein (2006) (we will refer to this paper as $MPWb$).

In the next section, we present the environment and the average value. In Section 3, we analyze the case of dividends and in Section 4 we define the weighted Shapley value. In Section 5, we propose a mechanism to implement the weighted Shapley value.

2 The environment and the average approach

Environments with externalities are best described as games in partition function form that were first introduced by Thrall and Lucas (1963). We denote by $N = \{1, ..., n\}$ the

[2] Dutta, Ehlers and Kar (2008) adopted the potential approach to study and provide values for games with externalities.

set of players. An *embedded coalition* is a pair (S, P), where $S \subseteq N$ is a coalition and $P \ni S$ is a partition of N. An embedded coalition specifies the coalition as well as the organization of the rest of players. Also, we denote by $\mathcal{P}$ the set of all partitions of N and by $ECL = \{(S, P) \mid S \in P,\ P \in \mathcal{P}\}$ the set of embedded coalitions.

Let (N, v) be a game in partition function form, where the characteristic function $v : ECL \to \mathbb{R}$ associates a real number with each embedded coalition. For each $(S, P) \in ECL$, we interpret $v(S, P)$ as the worth of coalition S when the players are organized according to the partition P. The partition P is always taken to include the empty set $\varnothing$ and, for notational convenience, when describing a partition we only list the non-empty coalitions. The characteristic function satisfies $v(\varnothing, P) = 0$.

A sharing method, or *a value*, is a mapping φ which associates with every game (N, v) a vector in $\mathbb{R}^n$ that satisfies $\sum_{i \in N} \varphi_i(N, v) = v(N, \{N\})$. A value determines the payoffs for every player in the game and, by definition, it is always *efficient* since the value of the grand coalition is shared among the players. In all the paper, it is assumed that forming the grand coalition is the most efficient way of organizing the society and thus all the players end up together. Formally, $v(N, \{N\}) \geq \sum_{S \in P} v(S, P)$ for every partition $P \in \mathcal{P}$.

To introduce the family of values proposed in MPW using the average approach we present first the axioms that characterize it.

We say that player $i \in N$ is a *null player* in the game (N, v) if $v(S, P) = v(S', P')$ for every $(S, P) \in ECL$ and for any embedded coalition (S', P') that can be obtained from (S, P) by changing the affiliation of player i.

The *addition* of two games (N, v) and (N, v') is defined as the game $(N, v + v')$ where $(v + v')(S, P) \equiv v(S, P) + v'(S, P)$ for all $(S, P) \in ECL$. Similarly, given the game (N, v) and the scalar $\lambda \in \mathbb{R}$, the game $(N, \lambda v)$ is defined by $(\lambda v)(S, P) \equiv \lambda v(S, P)$ for all $(S, P) \in ECL$.

For any permutation σ of N, the σ *permutation* of the game (N, v), denoted by $(N, \sigma v)$ is defined by $(\sigma v)(S, P) \equiv v(\sigma S, \sigma P)$ for all $(S, P) \in ECL$.

The extension of Shapley (1953b)'s basic axioms proposed in MPW for values in environments with externalities are:

1. *Linearity*: A value φ satisfies the linearity axiom if:

1.1. For any two games (N, v) and (N, v'), $\varphi(N, v + v') = \varphi(N, v) + \varphi(N, v')$.

1.2. For any game (N, v) and any scalar $\lambda \in \mathbb{R}$, $\varphi(N, \lambda v) = \lambda\varphi(N, v)$.

2. *Symmetry*: A value φ satisfies the symmetry axiom if for any permutation σ of N, $\varphi(N, \sigma v) = \sigma\varphi(N, v)$.

3. *Null player*: A value φ satisfies the null player axiom if for any player i which is a null player in the game (N, v), $\varphi_i(N, v) = 0$.

In games with no externalities where the worth of any coalition S does not depend on the organization of the other players, i.e., $v(S, P) = v(S, P')$ for every $S \subseteq N$ and $(S, P), (S, P') \in ECL$, these three basic axioms characterize a unique value (Shapley, 1953b). For expositional clarity, we denote by $(N, \widehat{v})$ a game with no externalities, where $\widehat{v} : 2^N \to \mathbb{R}$ is a function that gives the worth of each coalition. The Shapley value ϕ can be written as:

$$\phi_i(N, \widehat{v}) = \sum_{S \subseteq N} \beta_i(S, n)\widehat{v}(S) \text{ for all } i \in N, \tag{1}$$

where we have denoted by $\beta_i(S, n)$ the following numbers:

$$\beta_i(S, n) = \begin{cases} \frac{(|S|-1)!(n-|S|)!}{n!} \text{ for all } S \subseteq N, \text{ if } i \in S \\ -\frac{|S|!(n-|S|-1)!}{n!} \text{ for all } S \subseteq N, \text{ if } i \in N \backslash S. \end{cases}$$

In games with externalities, the previous axioms are satisfied by a large set of values. In MPW we propose an extension of the notion of symmetry also to the externalities created. As required by the symmetry axiom, symmetry is a property of anonymity: the payoff of a player is only derived from his influence on the worth of the coalitions, it does not depend on his "name". The strong symmetry axiom strengthens the symmetry axiom by requiring that exchanging the names of the players inducing the same externality should also not affect the payoff of any player. To formally state the axiom, we denote by $\sigma_{S,P}P$, with $P \ni S$, a new partition with $S \in \sigma_{S,P}P$ resulting from a permutation of the set $N \backslash S$. Given an embedded coalition (S, P), the $\sigma_{S,P}$ permutation of the game (N, v) denoted by $(N, \sigma_{S,P}v)$ is defined by $(\sigma_{S,P}v)(S, P) = v(S, \sigma_{S,P}P)$, $(\sigma_{S,P}v)(S, \sigma_{S,P}P) = v(S, P)$, and $(\sigma_{S,P}v)(R, Q) = v(R, Q)$ for all $(R, Q) \in ECL\backslash \{(S, P), (S, \sigma_{S,P}P)\}$.

2'. A value φ satisfies the *strong symmetry* axiom if:

$2'.1.$ for any permutation σ of N, $\varphi(N, \sigma v) = \sigma\varphi(N, v)$,

$2'.2.$ for any $(S, P) \in ECL$ and for any permutation $\sigma_{S,P}$, $\varphi(N, \sigma_{S,P} v) = \varphi(N, v)$.

As proven in MPW any value φ satisfying linearity and null player axioms also satisfies the strong symmetry axiom if and only if it can be constructed through the "*average approach*".

The average approach consists of, first constructing an average game $(N, \widehat{v}^{\alpha})$ associated with the partition function game (N, v), by assigning to each coalition $S \subseteq N$ the average worth $\widehat{v}^{\alpha}(S) \equiv \sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P) v(S, P)$, with $\sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P) = 1$. We refer to $\alpha(S, P)$ as the "coefficient" of the partition P in the computation of the value of coalition $S \in P$.[3] Second, the average approach constructs a value φ for the Partition Function Game (N, v) by taking the Shapley value of the game with no externalities $(N, \widehat{v}^{\alpha})$. Therefore, if a value φ is obtained through the average approach then, for all $i \in N$,

$$\varphi_i(N, v) = \sum_{S \subseteq N} \beta_i(S, n) \widehat{v}^{\alpha}(S) = \sum_{(S,P) \in ECL} \alpha(S, P) \beta_i(S, n) v(S, P). \quad (2)$$

Moreover, due to the symmetry and null player axioms the coefficients must be symmetric (i.e., $\alpha(S, P)$ only depends on the sizes of the coalitions in P) and satisfy the following condition:

$$\alpha(S, P) = \sum_{R \in P \backslash S} \alpha(S \backslash \{i\}, (P \backslash (R, S)) \cup (R \cup \{i\}, S \backslash \{i\})), \quad (3)$$

for all $i \in S$ and for all $(S, P) \in ECL$ with $|S| > 1$.

There are still a variety of values satisfying the requirements of linearity, strong symmetry, and null player, each value corresponding to a different averaging method. In MPW we add a "similar influence axiom" to characterize a value with coefficients $\alpha(S, P) = \frac{\prod_{T \in P \backslash S} (|T|-1)!}{(n-|S|)!}$ (see also Feldman, 1996). Other recent proposals also satisfy the average approach. The externality-free value studied and characterized by Pham Do and

[3] In MPW we refer to $\alpha(S, P)$ as the "weight" of the partition P in the computation of the value of coalition $S \in P$. For the sake of clarity when presenting the weighted Shapley values, we will use here the term "coefficient" instead of "weight".

Norde (2007) and de Clippel and Serrano (forthcoming) corresponds to the vector of coefficients $\alpha(S,P) = 1$ if $P = \left\{S, \{j\}_{j \in N \setminus S}\right\}$ and $\alpha(S,P) = 0$ otherwise; that is, only the partition where players outside S form singleton coalitions is taken into account.[4] On the other extreme, in his extension of the Shapley value McQuillin (2006) ends up proposing a value that corresponds to $\alpha(S,P) = 1$ if $P = \{S, N \setminus S\}$ and $\alpha(S,P) = 0$ otherwise; that is, the only important partition is the one where players outside S form a single coalition. Finally, the proposal by Albizuri *et al.* (2005) corresponds to the Shapley value associated with the simple average: $\alpha(S,P) = \frac{1}{P(S)}$ where $P(S) = |\{(S,Q)/(S,Q) \in ECL\}|$.[5]

We denote by φ^α the value constructed through the average approach with a vector of coefficients α. The analysis developed in the next sections can be applied to any value φ^α, in particular to those previously discussed.

3 Dividends in games with externalities

In this section we interpret any value φ^α in terms of sharing of dividends, in the spirit of the Harsanyi (1959) dividends for games in characteristic function form.

In games with no externalities, the Shapley value can be characterized as the value that distributes the so-called *Harsanyi dividends* of the game equally among the players in the corresponding coalitions. Harsanyi (1959) assumed that every coalition would negotiate a vector of "dividends" such that the sum of all coalitions's dividends vectors would be a feasible allocation for the grand coalition. Therefore, the dividends of a coalition S are what is left after all proper subcoalitions of S have received their corresponding dividends. More formally, the dividends $\Delta_{\widehat{v}}(S)$ of a game in characteristic form $(N, \widehat{v})$ are defined recursively: $\Delta_{\widehat{v}}(\varnothing) = 0$ and $\Delta_{\widehat{v}}(S) = \widehat{v}(S) - \sum_{R \subsetneq S} \Delta_{\widehat{v}}(R)$ if $S \neq \varnothing$. Harsanyi (1959) showed that all the equal shares of the dividends a player is entitled to sum up to his

[4]The main contribution of de Clippel and Serrano (forthcoming) is the analysis of the marginality principle in games with externalities.

[5]The values proposed by Myerson (1977) and Bolger (1989) can not be constructed through the average approach. Some of the values proposed by Dutta, Ehlers and Kar (2008) can also be constructed to the average approach while others can not.

Shapley value:

$$\phi_i(N, \widehat{v}) = \sum_{S \ni i} \frac{1}{|S|} \Delta_{\widehat{v}}(S) \text{ for all } i \in N.$$

In a similar spirit, consider a game with externalities (N, v) and a system of coefficients α. We can obtain the dividends of a coalition S when it is part of the partition P taking into account that the dividends of subcoalitions of S should matter only in proportion to its corresponding coefficients. Formally, we can also define the dividends inductively as follows:

$$\Delta_v^\alpha(\varnothing, P) = 0$$
$$\Delta_v^\alpha(S, P) = v(S, P) - \sum_{\substack{(R,Q) \in ECL \\ R \subsetneq S}} \alpha(R, Q) \Delta_v^\alpha(R, Q) \text{ if } S \neq \varnothing.$$

Theorem 1 provides an interpretation of any value φ^α in terms of dividends. Therefore, it is a result that can be applied, in particular, to the values studied by Feldman (1996) and MPW (2007); Albizuri *et al.* (2005); Pham Do and Norde (2007) and de Clippel and Serrano (forthcoming); and McQuillin (2006).

Theorem 1 *The value φ^α can be written as follows:*

$$\varphi_i^\alpha(N, v) = \sum_{\substack{(S,P) \in ECL \\ S \ni i}} \frac{1}{|S|} \alpha(S, P) \Delta_v^\alpha(S, P).$$

Proof. We denote by $\Delta_{\widehat{v}^\alpha}(S)$ the dividends of the average game $(N, \widehat{v}^\alpha)$ associated with the (N, v). We prove by induction over the number of elements of S that

$$\Delta_{\widehat{v}^\alpha}(S) = \sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \alpha(S, P) \Delta_v^\alpha(S, P).$$

The proof is immediate when $S = \varnothing$ (and it is also immediate when $|S| = 1$). Assuming that the expression holds for any coalition R with $|R| < |S|$, we can write

$$\sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \alpha(S, P) \Delta_v^\alpha(S, P) = \sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \alpha(S, P) \left[v(S, P) - \sum_{\substack{(R,Q) \in ECL \\ R \subsetneq S}} \alpha(R, Q) \Delta_v^\alpha(R, Q) \right]$$
$$= \sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \alpha(S, P) v(S, P) - \sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \alpha(S, P) \sum_{R \subsetneq S} \Delta_{\widehat{v}^\alpha}(R) = \widehat{v}^\alpha(S) - \sum_{R \subsetneq S} \Delta_{\widehat{v}^\alpha}(R) = \Delta_{\widehat{v}^\alpha}(S).$$

Since ϕ is the Shapley value of the (characteristic form game) $\widehat{v}^{\alpha}$, we can write $\phi_i(N, v)$ in terms of sum of dividends:

$$\phi_i(N, \widehat{v}^{\alpha}) = \sum_{S \ni i} \frac{1}{|S|} \Delta_{\widehat{v}^{\alpha}}(S),$$

therefore,

$$\varphi_i^{\alpha}(N, v) = \phi_i(N, \widehat{v}^{\alpha}) = \sum_{S \ni i} \frac{1}{|S|} \sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \alpha(S, P) \Delta_v^{\alpha}(S, P) = \sum_{\substack{(S,P) \in ECL \\ S \ni i}} \frac{1}{|S|} \alpha(S, P) \Delta_v^{\alpha}(S, P).$$

■

Maschler (1982) showed that the sharing method based on dividends was a special case of a procedure where a given sequence of coalitions sequentially claim their worth. At each step the worth of one coalition is equally shared by its members. The process then generates a new game where the worth of each coalition containing the sharing coalition is reduced by the worth of the sharing coalition, whereas the worth of other coalitions is not altered. The next coalition in the sequence then proceeds to share its worth. This process ends when all coalitions arrive at zero worth. It was then shown that this process must end, and the sum of payoffs accruing to each player is his Shapley value. The value φ^{α} can be arrived at via these more general procedures as well if the proportion $\alpha(S, P)$ of the dividend associated with each embedded coalition (S, P) is assigned to each player $i \in S$ and the coefficient $\alpha(S, P)$ is also used to reduce the worth of each coalition containing S.

4 Weighted Shapley values in games with externalities

In some economic applications, the symmetry axiom can be challenged. Players may have different bargaining powers or may have invested different levels of capital or effort essential to the generation of the surplus to be shared. These differences which are not reflected in the characteristic function should possibly play a role in the sharing of the surplus, thus leading to a violation of the symmetry property. Shapley (1953a) addressed this issue by proposing a family of weighted (Shapley) values. The weights are used to determine the proportions in which the players share the surplus in (characteristic function) unanimity games. Adopting the dividends approach these weighted Shapley values correspond to allocations where the dividends of a coalition are distributed among players according to the proportions implied by the weights.

Formally, for a given vector of weights $w \in \mathbb{R}^n$, with $w_i > 0$ for $i \in N$, the corresponding *weighted Shapley value* (Shapley, 1953a) ϕ^w associates with every game $(N, \widehat{v})$ a vector in $\mathbb{R}^n$ that satisfies:

$$\phi_i^w(N, \widehat{v}) = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{w_i}{w_S} \Delta_{\widehat{v}}(S) \text{ for all } i \in N,$$

where $w_S \equiv \sum_{i \in S} w_i$. The Shapley value $\phi_i(N, \widehat{v})$ is arrived at when all the weights w_i are equal.

We can proceed in a similar way to obtain non-symmetric values for games with externalities. We define, for a given vector of weights $w \in \mathbb{R}^n$, with $w_i > 0$ for $i \in N$, (and for each vector of factors α) a *weighted Shapley value for games with externalities*:

$$\varphi_i^{\alpha w}(N, v) = \sum_{\substack{(S,P) \in ECL \\ S \ni i}} \frac{w_i}{w_S} \alpha(S, P) \Delta_v^\alpha(S, P).$$

It follows easily that there is a relationship between the weighted Shapley value for games with externalities that we have defined through the dividends and the weighted Shapley value of the corresponding average game $(N, \widehat{v}^\alpha)$. We state this result in the next proposition:

Proposition 1 *The value $\varphi^{\alpha w}$ can be written as follows:*

$$\varphi_i^{\alpha w}(N, v) = \phi_i^w(N, \widehat{v}^\alpha).$$

Proof. We write the value ϕ^w:

$$\phi_i^w(N, \widehat{v}^\alpha) = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{w_i}{w_S} \Delta_{\widehat{v}^\alpha}(S) = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{w_i}{w_S} \sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \alpha(S, P) \Delta_v^\alpha(S, P),$$

following the proof of Theorem 1. Therefore,

$$\phi_i^w(N, \widehat{v}^\alpha) = \sum_{\substack{S \subseteq N \\ S \ni i}} \sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \frac{w_i}{w_S} \alpha(S, P) \Delta_v^\alpha(S, P) = \sum_{\substack{(S,P) \in ECL \\ S \ni i}} \frac{w_i}{w_S} \alpha(S, P) \Delta_v^\alpha(S, P) = \varphi_i^{\alpha w}(N, v)$$

■

As Owen (1968) has shown it might be the case that increasing a player's weight may decrease the value of the player. Haeringer (2006) addressed this issue and suggested a different family of weighted Shapley values, where the weights satisfy the property that an increase in the player's weight does not decrease the player's value (hence might justifiably reflect a measure of power). To that effect he used two systems of weights. One system was used to allocate the dividend of a coalition among its members in the case where the dividend was positive and another (derived from the first one by a strictly decreasing function, for instance taking the reciprocals of the weights) for the case of a negative dividend. Taking the same route for games with externalities we can use two systems of weights, to share $\Delta_v^\alpha(S,P)$, for any (S,P) in ECL, depending on its sign.

5 Implementation of weighted Shapley values in games with externalities

The value $\varphi^{\alpha w}(N,v)$ can be implemented via bidding mechanisms in the same spirit as the ones that appear in Pérez-Castrillo and Wettstein (2001) and $MPWb$), for any system of non-negative coefficients, i.e., $a(S,P) \geq 0$ for all $(S,P) \in ECL$ that satisfy (3). The implementation is carried out for environments that can be characterized either via negative or via positive externalities, and slightly different mechanisms are used for each environment. Positive and negative externalities are defined as follows.

Definition 1 *The game (N,v) has negative externalities if $v(S,P) \geq v(S,P')$ for every P,P', when each element in P' is given by a union of elements in P.*

Definition 2 *The game (N,v) has positive externalities if $v(S,P) \leq v(S,P')$ for every P,P', when each element in P' is given by a union of elements in P.*

In both types of environments it must be the case that the departure of a single player from a coalition results in efficiency losses. This is a mild requirement usually referred to as zero-monotonicity in games without externalities. It can be extended in several ways to games with externalities, one extension appropriate for the case of negative externalities is given by:

Definition 3 *The game* (N, v) *strictly zero-monotonic-A if*

$$v(S, P) > v(S\backslash\{i\}, (P\backslash S) \cup (S\backslash\{i\}, \{i\})) + v(\{i\}, (P\backslash S) \cup (S\backslash\{i\}, \{i\}))\}$$

for every $(S, P) \in ECL$ *and every* $i \in S$ *.*

Zero-monotonicity-A requires that the addition of a singleton player to a coalition is always beneficial, considering that the organization of the other players does not change.

For environments with negative externalities, we implement the value $\varphi^{\alpha w}(N, v)$ via the mechanism $M^{-}(\alpha, w)$ which modifies the $M^{-}(\alpha)$ mechanism appearing in $MPWb$. Environments with positive externalities can be similarly analyzed by modifying the mechanism $M^{+}(\alpha)$ in the same way.

The mechanism $M^{-}(\alpha, w)$ can be informally described as follows:

At each round, there is a set of "insiders" S and a set of "outsiders" $N\backslash S$. At the first round, the set of insiders is N. Each round is composed of two stages; the first stage is played among the insiders, the second one, if reached, is played among the outsiders. In the *insiders stage*, the players in S select a proposer among themselves through a multibidding procedure. Each player's bid is weighted according to the vector w so that the influence of a player's bid in the multibidding procedure is proportional to the player's weight. Once a proposer is chosen, he pays the bids he made and then makes a proposal to the other members in S on the sharing of the (expected) benefits if they stay together (i.e., if they form the coalition S). If the proposal is rejected, the proposer joins the set of outsiders and the remaining insiders go to the next round. If the proposal is accepted, S forms and the organization of the outsiders is determined in the outsiders stage. Note that in the case where $S = N$ the outsiders' stage is redundant.

At the *outsiders stage*, the set of players is $N\backslash S$, those agents whose proposals have been rejected at previous rounds of the mechanism. First, a "candidate partition" of N, including S, is randomly selected, where the probability of selecting a particular partition is the coefficient $\alpha(S, P)$ associated with this partition by the averaging system α. Second, the members of each coalition in such a partition, other than S, play a game that determines whether the candidate partition (or some finer partition) is the final organization. This phase is constructed to encourage the proposer in the insiders stage to make acceptable proposals. The formal description of the mechanism $M^{-}(\alpha, w)$ follows.

The mechanism $M^{-}(\alpha, w)$

The mechanism $M^{-}(\alpha, w)$ proceeds in rounds. Each round is characterized by a coalition $S \subseteq N, S \neq \varnothing$. At the first round of the mechanism, $S = N$. For each round, in the case where $s = |S| > 1$, we move to $I(S)$, the insiders' stage, otherwise we move to $O(S)$, the outsiders' stage.

$I(S)$: *Insiders' stage*

$I(S)$.1: Each agent $i \in S$ makes bids $b_j^i \in \mathbb{R}$, for every $j \in S \backslash \{i\}$. Agents' bids are simultaneous.

Define the aggregate net bid to each player $i \in S$ by $B_i = \sum_{j \in S \backslash \{i\}} w_i b_j^i - \sum_{j \in S \backslash \{i\}} w_j b_i^j$. Let $\gamma_s = \text{argmax}_i (B_i)$ where an arbitrary tie-breaking rule is used in the case of a non-unique maximizer. The proposer is then taken to be γ_s and prior to moving to the next stage pays every player $i \in S \backslash \{\gamma_s\}$ the amount $b_i^{\gamma_s}$.

$I(S)$.2: The proposer γ_s makes a proposal $x_i^{\gamma_s} \in \mathbb{R}$ to every $i \in S \backslash \{\gamma_s\}$.

$I(S)$.3: The agents in $S \backslash \{\gamma_s\}$, sequentially, either accept or reject the offer. If an agent rejects it, then the offer is rejected and the game moves to the next round of the insider's stage characterized by the coalition $S \backslash \{\gamma_s\}$ (hence, γ_s becomes an ousider). Otherwise, the offer is accepted, agent γ_s pays $x_i^{\gamma_s}$ to each agent $i \in S \backslash \{\gamma_s\}$, and then the final outcome is given after $O(S)$.

$O(S)$: *Outsiders stage*

A partition P, with $S \in P$, is chosen with probability $\alpha(S, P)$. Denote by T_{s+1} the coalition in P containing the last rejected proposer, γ_{s+1}.[6] A proposer $\beta(T)$ is randomly chosen for every $T \in P \backslash S$ with $|T| > 1$, the only restriction is that $\beta(T_{s+1}) \neq \gamma_{s+1}$, when $|T_{s+1}| > 1$. The agents in each such coalition T play the game $G(T)$, described below.

$G(T)$.1: Player $\beta(T)$ makes a proposal $x_i^{\beta(T)} \in \mathbb{R}$ to every $i \in T \backslash \beta(T)$.

$G(T)$.2: The agents in $T \backslash \beta(T)$, sequentially, either accept or reject the proposal. When an agent $\delta(T)$ rejects it, then the proposal is rejected. In this case, all the players in $T \backslash \delta(T)$ play the game $G(T \backslash \delta(T))$ with $\beta(T)$ as the proposer and player $\delta(T)$ stays as a

[6]If $S = N$, then there is no γ_{s+1}, and the grand coalition N is chosen with probability 1.

singleton. Otherwise, the proposal is accepted, the coalition T is formed, and $\beta(T)$ pays $x_i^{\beta(T)}$ to every $i \in T \backslash \beta(T)$.

Following these games we obtain a partition $P(S)$ consisting of S, the coalitions resulting from the $G(T)$ games, and the singleton coalitions in P.

Outcome.

We denote by S^* the coalition of insiders which is formed and $P^* \equiv P(S^*)$ the final partition formed. Agent $i \in S^* \backslash \{\gamma_{s^*}\}$ obtains $x_i^{\gamma_{s^*}} + \sum_{k=s^*}^{n} b_i^{\gamma_k}$. Agent γ_{s^*} gets $v(S^*, P^*) - \sum_{i \in S^* \backslash \{\gamma_{s^*}\}} x_i^{\gamma_{s^*}} + \sum_{k=s^*+1}^{n} b_{\gamma_{s^*}}^{\gamma_k} - \sum_{i \in S^* \backslash \{\gamma_{s^*}\}} b_i^{\gamma_{s^*}}$.[7]

The outcomes for the set of outsiders, $N \backslash S^* = \{\gamma_m\}_{m=s^*+1,\ldots,n}$, are given as follows: The final outcome of player γ_m, for $m = s^* + 1, \ldots, n$, is $v(\{\gamma_m\}, P^*) + \sum_{k=m+1}^{n} b_{\gamma_m}^{\gamma_k} - \sum_{i \in S_m} b_i^{\gamma_m}$ if $\{\gamma_m\} \in P^*$, where $S_m = N \backslash \{\gamma_m, \ldots, \gamma_n\}$. Otherwise, denote by T_m the coalition in P^* containing agent γ_m and by $\beta(T_m)$ the proposer in that coalition. The final payoff of player γ_m is $x_{\gamma_m}^{\beta(T_m)} + \sum_{k=m+1}^{n} b_{\gamma_m}^{\gamma_k} - \sum_{i \in S_m} b_i^{\gamma_m}$ if $\gamma_m \neq \beta(T_m)$ and $v(T_m, P^*) - \sum_{i \in T_m \backslash \gamma_m} x_i^{\gamma_m} + \sum_{k=m+1}^{n} b_{\gamma_m}^{\gamma_k} - \sum_{i \in S_m} b_i^{\gamma_m}$ if $\gamma_m = \beta(T_m)$.

Theorem 2 characterizes the equilibrium outcome of this mechanism:

Theorem 2 *If the game (N, v) has negative externalities and it is strictly zero-monotonic-A, then the mechanism $M^-(\alpha, w)$ implements in SPE the value $\varphi^{\alpha w}(N, v)$.*

Proof. We first prove that every SPE of $M^-(\alpha, w)$ leads to a payoff vector coinciding with $\varphi^{\alpha w}(N, v)$.

Denote by $\phi_j^w(S, \widehat{v}^\alpha)$ the weighted Shapley value of player $j \in S$ in the game with no externalities $(S, \widehat{v}^\alpha)$.

We fix the size n of the set of players N and proceed by induction over the number s of insiders, for $s = 1, \ldots, n$. The induction property is the following: for any set of insiders S of size s, if the game reaches the insiders stage $I(S)$, then at any SPE of $M^-(\alpha, w)$ the coalition S indeed forms and any player j in S receives from this stage onwards (i.e., without taking into account the payments made or received before the stage $I(S)$) the payoff $\phi_j^w(S, \widehat{v}^\alpha)$.

$(s = 1)$ If there is one player in S, $S = \{j\}$, the rules of the mechanism $M^-(\alpha, w)$ imply that the game directly goes to the outsiders stage $O(\{j\})$, hence the coalition S

[7] If $|S^*| = 1$, then there are no payments $x_i^{\gamma_{s^*}}$ since $S^* \backslash \{\gamma_{s^*}\} = \varnothing$.

indeed forms. Since the outsiders stage here is the same as in $MPWb$, we can use Lemma 1 in that paper that states that any chosen partition selected at stage $O(S)$ actually forms. Given that the probability that partition $P \ni \{j\}$ is chosen, is $\alpha(\{j\},P)$, the expected payoff for j from this stage $(I(\{j\}))$ on is given by:

$$\sum_{P \ni \{j\}} \alpha(\{j\},P) v(\{j\},P) = \widehat{v}(\{j\}) = \phi_j^w(\{j\},\widehat{v}^\alpha).$$

We now assume the induction property holds for any set R with a number of players smaller than k and prove it also holds for any set S with k players.

$(s=k)$ We can follow similar steps as in $MPWb$ to prove Claims 1 and 2:

Claim 1. In any SPE, the proposer γ_s makes an offer $x_j^{\gamma_s} = \phi_j^w(S\backslash\{\gamma_s\},\widehat{v}^\alpha)$ to every player $j \in S\backslash\{\gamma_s\}$ and these players accept the offer.

Claim 2. In any SPE the aggregate bids are all zero, i.e., $B_i = 0$ for all $i \in S$. Moreover, any player $i \in S$ is indifferent with respect to the identity of the proposer.

Now, by Claims 1 and 2 (and Lemma 1 in $MPWb$), if player i is the proposer his final payoff from this stage onwards is $\widehat{v}^\alpha(S) - \widehat{v}^\alpha(S\backslash\{i\}) - \sum_{j\in S\backslash\{i\}} b_j^i$; while if player $j \in S\backslash\{i\}$ is the proposer, the final payoff from this stage onwards of player i is $\phi_i^w(S\backslash\{j\},\widehat{v}^\alpha) + b_i^j$. Given that player i is indifferent between all s possible proposers, we can denote what player i gets for any of the proposers by u_i. Thus we have:

$$\begin{aligned}\left(\sum_{k\in S} w_k\right) u_i &= w_i \left[\widehat{v}^\alpha(S) - \widehat{v}^\alpha(S\backslash\{i\}) - \sum_{j\in S\backslash\{i\}} b_j^i\right] + \sum_{j\in S\backslash\{i\}} w_j \left[\phi_i^w(S\backslash\{j\},\widehat{v}^\alpha) + b_i^j\right] \\ &= w_i \left[\widehat{v}^\alpha(S) - \widehat{v}^\alpha(S\backslash\{i\})\right] + \sum_{j\in S\backslash\{i\}} w_j \phi_i^w(S\backslash\{j\},\widehat{v}^\alpha)\end{aligned}$$

since $B_i = \sum_{j\in S\backslash\{i\}} w_i b_j^i - \sum_{j\in S\backslash\{i\}} w_j b_i^j = 0$.

Hence

$$u_i = \frac{1}{\sum_{k\in S} w_k}\left[w_i \left[\widehat{v}^\alpha(S) - \widehat{v}^\alpha(S\backslash\{i\})\right] + \sum_{j\in S\backslash\{i\}} w_j \phi_i^w(S\backslash\{j\},\widehat{v}^\alpha)\right] = \phi_i^w(S,\widehat{v}^\alpha).^{8}$$

Hence, the induction property holds for S.

This (taking the case where $S = N$) shows that every SPE payoff of $M^-(\alpha,w)$ is $\varphi^{\alpha w}(N,v)$. Indeed, the induction property implies that, at the SPE of the mechanism

[8] See Lemma 1 in Perez-Castrillo and Wettstein (2001).

$M^{-}(\alpha, w)$, the agents accept the offer made by the proposer at the first round, and their final equilibrium payoff is given by $\phi_j^w(S, \widehat{v}^{\alpha})$, which corresponds to the value $\varphi^{\alpha w}(N, v)$.

Similar to $MPWb$ we can explicitly construct an SPE strategy profile that yields the value as an outcome. ■

We note that the weighted Shapley value for the case of positive externalities can be implemented as well by modifying the outsiders' stage in $M^{-}(\alpha, w)$. The basic feature of the modification is that a rejection in that stage leads to a partition composing of singletons.

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