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License Prices for Financially Constrained Firms

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Abstract: It is often alleged that high auction prices inhibit service deployment. We investigate this claim under the extreme case of financially constrained bidders. If demand is just slightly elastic, auctions maximize consumer surplus if consumer surplus is a convex function of quantity (a common assumption), or if consumer surplus is concave and the proportion of expenditure spent on deployment is greater than one over the elasticity of demand. The latter condition appears to be true for most of the large telecom auctions in the US and Europe. Thus, even if high auction prices inhibit service deployment, auctions appear to be optimal from the consumers' point of view.

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"It is a victory for consumerism and common sense."

Etain Doyle, Director of Ireland's regulator ComReg (then Telecommunications Regulation), commenting the decision to award 3G Irish licenses through a "beauty contest".[1]

1. Introduction

It is often alleged that high auction prices for spectrum licenses have inhibited the deployment of the related services, to the detriment of consumers. For example, telecom specialist John Tennant said that the bids on the third generation, or 3G, licenses increased the cost of debt service to the point that the companies could not borrow for infrastructure development, and ultimately accounts for the dramatic drop in share prices of the telecom sector. (McClelland, 2003). Similarly, an EC report studying 3G services identifies the €110 billion paid for licenses as a major constraint on investment. Several nations, including Finland, France, Ireland, Spain and Sweden, awarded 3G licenses for low prices not set by auction, using what are often called "beauty contests," ostensibly because this would lead to a faster deployment of services (Commission of the EC, 2002). Dr Keiji Tachikawa, president of Japan's largest cellular company (NTT DoCoMo), agrees:

"Operators will have to pass on the added cost [of auctioned licences] to consumers. This could be a hurdle for the spread of 3G."[2]

We investigate the properties of auctions, from a perspective of consumer welfare, under the extreme assumption that the bidders face salient financing constraints. This makes it theoretically possible for the critics to be right, that financing

constraints inhibit the deployment of services. We show that auctions still maximize

consumer surplus, including license revenue, provided the elasticity of demand is above

certain threshold.

Textbook economic analysis suggests that license prices are sunk costs by the

time investment decisions are made, and thus should have no effect on the deployment

of services. Moreover, if profitability of deployment varies across countries, one might

expect the high profitability countries to attract both high auction prices and rapid and

extensive deployment to capture the high profitability, inducing a positive correlation

between auction prices and service deployment. Even if profitability is constant, the

fallacy of sunk costs suggests, in addition, that psychologically the managers should

want to invest more in the regions with high-priced licenses, not less.

On the other hand, starting with Michael Jensen's 1986 free cash flow concepts,

modern corporate finance emphasizes the importance of restraining managers by

limiting their ability to invest. Moral hazard, in the form of career concerns or limited

liability, can induce managers to take excess risks. The natural response to such

managerial problems is to limit the ability of the manager to make bad choices, either by

imposing a budget constraint on the manager, or requiring the manager to use a much

higher discount factor than the actual average cost of capital for a project under

consideration. Even if budgets are "soft," in the sense that there is always more money

possible, individual executives may bear a career cost of asking for more money,

perhaps because they are seen as having mis-estimated the costs, making them

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[1] 3G Auctions: A License To Bill? *by* Nicki Hayes,
http://www.wirelessdevnet.com/channels/wireless/features/licensetobill.html, last visited May 22nd, 2006.
[2] *Business Week*, October 16, 2000, p.25.

hesitate to request more money unless the gains are very large. Such a situation mirrors a financing constraint, at least for some realizations of costs.

The recognition that agency problems -- either moral hazard or asymmetric infomation or both -- might have an impact on corporate financing and investment probably begins with Stiglitz and Weiss (1981), which argues that asymmetric information can impede credit markets, and Greenwald, Stiglitz and Weiss (1984), which argues that equity financing does not cure the agency problem created by asymmetric information. Another important paper was Myers and Majluf's 1984 analysis that asymmetric information can drive a wedge between the interests of new investors and creditors, thereby creating an agency problem distinct from that identified by the Stiglitz and Weiss. Lewis and Sappington (1989a, 1989b), and Greenwood and McAfee (1991) show that asymmetric information can lead to inflexible rules, and in particular may fix capital investment at a level unaffected by the state of the world unless the state is very extreme. These rules work precisely like a financing constraint provided to a manager. Hart and Moore (1995) develop Jensen's free cash flow concept, and show that debt seniority can be used as a versatile instrument to induce more efficient project selection. In particular, a mix of "hard" debt, which cannot be postponed, and soft debt create a limit on the ability to raise future capital, thus inducing future financing constraints. Clementi and Hopenhayn (2006) develop a dynamic model in which borrowing constraints arise endogenously and relax as the value of the prospects of the firm improves. Overall, the thrust of the theoretical literature is that budget or financing constraints imposed on firm managers play an important role in ameliorating incentive problems.

Many empirical tests corroborate the view that firms are financially-constrained to some degree, by showing that internal and external financing are not perfect substitutes. The theme of the empirical studies is that investment decisions are affected by the amount of cash on hand in the firm. Fazzari and Athey (1987) show that the availability of internal financing affects investment. Fazzari, Hubbard and Petersen (1988) emphasize financing constraints in their study of the determinants of investment. Whited (1992) corroborates the existence of financing constraints. Fazzari and Petersen (1993) use working capital as a way of controlling for errors in measured variables that might create the spurious appearance of financial constraints, and finds evidence that previous studies had in fact *underestimated* the importance of financing constraints. More recently, Love (2003) estimates the effects of financing constraints across many nations, and finds that strong capital markets in developed nations reduce, but don't eliminate, the significance of financing constraints.[3] If these studies are relevant to the telecom firms, then the cost of spectrum licenses could have an impact on investment in deployment of services.

The behavior of the telecom industry during the 1990s reinforces the importance of managerial incentive problems. Some companies bid in excess of the maximum values suggested by their own analyses. Stefan Zehle describes a 3G bidder in the U.K. who bid £5 billion for a license that the company estimated was worth £1 billion. He also describes an executive who called the auctions a "spectrum landgrab" and that the bidders should not worry whether the prices made business sense (McClelland,

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[3] The empirical literature is also discussed in Clementi and Hopenhayn (2006). Although larger, mature firms are typically subject to less serious financing constraints, Whited (1992) concluded that financial constraints are still important for large firms. Moreover, even for these firms, asymmetric information is the element that is stressed, and it should be important for firms in a new market, like 3G services.

2003).[4] One author (McAfee) was repeatedly asked by spectrum bidders for auction-theoretic reasons for bidding in excess of the net present value. The managers were very disappointed to hear about the winner's curse, which goes the other direction.[5] In addition, many of the bidders believed that other bidders faced financing constraints and consulted with economists in an attempt to estimate just what those constraints might be.

In a world of possibly financially constrained firms, do auctions maximize consumer surplus, counting the revenue raised by the auction as part of consumer surplus?[6] We characterize conditions under which auctions yield an optimal price from the perspective of consumers, in spite of the presence of binding financing constraints.

Rather than explicitly consider auctions, we consider a posted price, which simplifies the analysis. This price may range from zero to a maximum where the firms earn zero profits. Generally the zero profit point is the price that would emerge from an auction among symmetric firms; financing constraints are salient if they strictly bind at this price.

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[4] For an opposing view of the European 3G auctions, see Klemperer (2002).

[5] The likely reason for this tendency to bid in excess of net present value was the 1980s experience. The actual number of U.S. cell phone users in 1990 was ten times the expectation projected in 1980 for 1990, and cellular profits represented a large fraction of total telecom profits, mostly because there were only two firms in most regions and limited capacity. This dramatic underestimate of the value of wireless fueled an unjustified optimism.

[6] Auctions with budget constraints have been examined by Pitchik and Schotter (1986, 1988), Che and Gale (1988), and Benoit and Krishna (2001). The focus of these papers is on the firms' ability to bid in subsequent auctions, given the prices paid in earlier auctions, and on the proposition that bidders might artificially inflate the price of earlier sales as a means of reducing the ability of the winners to pay for later items. In contrast, we examine the ability of firms to deploy a service after the sale. Haan and Toolsema (2003) introduce credit rationing due to market uncertainty and limited liability in a model of license allocation, providing an alternative source of "budget constraints". In their model, higher nominal interest rates (debt burden) induce firms to take a more aggressive behavior in the market, due to the assumption that uncertainty is resolved after market competition.

If the elasticity of demand, ε, exceeds $1 + 1/n$ evaluated at the (Cournot) unconstrained output, where n is the number of licenses, then the price where financing constraints just bind falls short of the consumer surplus-maximizing price. In fact, if the consumer surplus is convex in output (a case which arises with linear demand) then the consumer-surplus maximizing price is always the auction price. If consumer surplus is concave (e.g. constant elasticity of demand), then ε exceeding $1 + 1/n$ is a necessary condition for auctions to be optimal. A sufficient condition in this instance is that no more than $(\varepsilon\text{-}1)/\varepsilon$ of the budget is spent on the licenses.

The formulas we derive are well rooted in economic analysis, e.g. they are the natural counterpart of Ramsey pricing in the regulation of natural monopoly. Also, they have the virtue of being simple and readily checkable. In particular, in the United States PCS auctions, licenses costs were estimated to be about 40% of the costs of deploying a PCS service. Even if the firms were financially constrained, and even if consumer surplus was concave, an auction was consumer surplus-maximizing provided the elasticity of demand for PCS services exceeded 1.66. We argue in Section 5 that the demand for new wireless services is elastic.[7] Under that assumption, even if the critics are right that auctions curtail output (deployment) relative to beauty contests, auctions nonetheless are in the best interest of consumers.

Auctions have an important advantage that is not considered in this paper – auctions tend to pick the most able companies. This advantage is set aside to provide a stronger case for using beauty contests, since reducing the advantages of auctions makes our

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[7] Over the rollout of PCS services, prices have dropped by 50% or so, and the number of customers has grown by at least several hundred percent, suggesting elasticities over 4. However, the technology has changed substantially as well, with smaller phones with many more features like cameras and instant messaging, which may account for some of the increased sales.

conclusion that auctions are nonetheless optimal even stronger. Other simplifying assumptions have this same effect of making the case stronger for beauty contests. We comment on some of them in Section 4 . In general, the model is stylized for clarity but we anticipate the effects identified to carry over to more elaborate settings, although of course new phenomena may crop up in such settings.

The paper proceeds as follows. The second section develops the basic Cournot theory and the third section proves the main results. The fourth section considers whether auctions produce the zero profit price, when this price is unique, and shows that the theory is robust to asymmetries in the budget. The fifth section considers the application of the theory to the European experience with 3G services. The sixth section concludes.

2. The Model

There are *n* licenses, and at least *n*+1 identical firms. A license is a right to compete in a symmetric Cournot industry. [8]   We discuss below alternative assumptions (e.g., price competition). If industry output is Q, then the realized price is *p*(Q). The elasticity of demand is

(1) $\quad \varepsilon(Q) = \dfrac{-p(Q)}{Q p'(Q)}.$

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[8] We assume the Cournot model not because it is necessarily the best model of any specific industry but primarily for its tractability. However, using the homogeneous-good, Cournot model makes our analysis comparable to many other regulation studies, and has the added advantage that the effects of budget constraints have a natural interpretation in the Cournot model. In contrast, in a differentiated products model, the effects of budget constraints could be to limit capacity but could also affect the dissimilarity of the products. In addition, differentiated product models are notoriously challenging to analyze. However, the analysis of such models represents the natural next step. Competition among cellular telephone companies has both a quantity and a differentiated product aspect. For part of their history, the cell companies have been capacity constrained, and these capacity constraints are alleviated by the denser deployment of towers. With respect to 3G services, however, differentiation is an important aspect of competition, although even there, investments needed to deploy any 3G services were slow in coming.

Where the risk of confusion is minimal, we will suppress the dependence of ε on Q. We assume that for all Q,

(2) $2p'(Q) + Qp''(Q) < 0$.

Inequality (2) is the condition that marginal revenue is downward sloping, and insures that the second order conditions hold globally for Cournot equilibrium.

Let λ be the price of a license, and *B* the budget of each firm. We model the financing constraint as a "hard" budget constraint, primarily to favor the case that financing constraints might interfere with subsequent production. That is, "soft" budget or financing constraints are generally going to have less of an effect than "hard" financing constraints. Also, as we will show in the next section, if firms are asymmetrically constrained the case for auctions is actually stronger. If a firm chooses to produce the quantity *q*, the constraint becomes

(3) $cq + \lambda \le B$,

where *c* is the marginal cost of output.[9] We assume that *c* is below the demand price $p(Q)$ for some positive quantity Q.

We look for a symmetric equilibrium in output. Suppose the symmetric equilibrium quantity choice is q^*. Each firm's profits are

(4) $\pi_i = \max_{cq + \lambda \le B} p(q + (n-1)q^*)q - cq - \lambda$.

We first consider the quantity choices of the firms, which are characterized in Lemma 1.

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[9] We do not include retained earnings as part of the budget constraint because borrowing against the prospect of future earnings for investment is notoriously difficult. New and speculative investments are rarely funded by borrowing. It would clearly be an improvement to model the dynamic process of deployment with investment and debt.

Lemma 1: *There is a unique Cournot equilibrium, and it is symmetric and has industry output Q satisfying:*

$$(5) \qquad p(Q)\left(1-\frac{1}{n\varepsilon(Q)}\right)-c\begin{cases}=0 & \text{if } cQ/n+\lambda < B \\ \\ \geq 0 & \text{if } cQ/n+\lambda = B\end{cases}$$

All proofs are contained in the appendix.

When the financing constraint does not bind, condition (5) can be expressed as

$$(6) \qquad p(Q)=\frac{n\varepsilon}{n\varepsilon-1}c.$$

The solution to (6) is the unconstrained Cournot outcome. Let Q_c be this solution. The associated profits are

$$(7) \qquad \pi = p(Q_c)\frac{Q_c}{n}-c\frac{Q_c}{n}-\lambda=\left(\frac{n\varepsilon}{n\varepsilon-1}-1\right)c\frac{Q_c}{n}-\lambda=\frac{1}{n}\frac{cQ_c}{n\varepsilon-1}-\lambda.$$

If the financing constraint binds,

$$(8) \qquad p\left(n\frac{B-\lambda}{c}\right)\geq\frac{n\varepsilon}{n\varepsilon-1}c,$$

and profits are

$$(9) \qquad \pi = p\left(n\frac{B-\lambda}{c}\right)\frac{B-\lambda}{c}-c\frac{B-\lambda}{c}-\lambda=p\left(n\frac{B-\lambda}{c}\right)\frac{B-\lambda}{c}-B.$$

We assume that license prices will have to satisfy voluntary participation of firms. The maximum that any firm would pay for a license is a level leading to zero profits. We denote the zero-profits license price by λ_0:

$$(10) \qquad \lambda_0 = (p(Q)-c)\frac{Q}{n}$$

We will discuss in Section 4 the existence and uniqueness of this price. For the moment, assume it is indeed unique.

3. *Optimal prices*

What is the optimal license price λ? We will consider two scenarios, depending on whether the firm profits are counted as part of welfare. If the firms are local firms, and their profits are fully counted as part of the local welfare, then it is appropriate to maximize the total gains from trade. In this case, giving the licenses away (which relaxes the firms' financing constraints) maximizes welfare. If, in contrast, the profits of the firms are not part of the objective of the licensor, the results are more interesting. For sufficiently elastic demand, an auction is optimal, and we will derive a sharp characterization of how elastic demand must be.

If the licensing authority counts firm profits in welfare, the welfare measure, as a function of output, is composed of three terms, the consumer surplus, the firm profits, and the license revenue. If Q is the quantity produced by the industry, welfare is:

$$(11) \quad W = \int_0^Q (p(x) - p(Q))dx + n\left(\frac{Q}{n}p(Q) - c\frac{Q}{n} - \lambda\right) + n\lambda = \int_0^Q p(x)dx - cQ = \int_0^Q (p(x) - c)dx.$$

In this case, the closer is output to the level where price equals marginal cost, the more efficient is the solution. Since price exceeds marginal cost at the unconstrained solution Q_c, and price is increasing (weakly if the financing constraint does not bind) in λ, W is maximized over non-negative λ at $\lambda=0$, or giving the licenses away free. More generally, any λ small enough that the financing constraint does not bind maximizes welfare. If the financing constraint binds strictly, the value of λ is too high to maximize

welfare. Note that this could entail a negative value of λ if the financing constraint binds at $\lambda=0$.

Now consider the consumer surplus, which doesn't count firm profits, but does count license revenues; we'll refer to values of λ that maximize this measure as optimal. Consumer surplus (plus license revenue) is probably a more realistic objective function for regulators in general.[10] More relevant to our particular problem, this is an objective function that better fits declared intentions of governments when assigning telecommunication license.[11] The objective function for the government is in this case

$$(12) \quad CS + R = \int_0^Q (p(x) - p(Q))dx + n\lambda .$$

CS (and then CS+R) is a convex function of output when

$$(13) \quad p'(Q) + Qp''(Q) < 0 .$$

Note that (13) guarantees the second order condition (2). CS is convex whenever a tax on a monopoly is only partially passed on to consumers, which is a common assumption. The condition (13) is satisfied by concave, and in particular linear demand.

Denote the level that leads to the financing constraint just binding by λ_B. Note that λ_B is defined by

$$(14) \quad n\frac{B - \lambda_B}{c} = Q_c .$$

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[10] We assume the shadow cost of public funds to be 1. In fact, our results would be strengthened under the more plausible, standard hypothesis that this cost is above 1.

[11] For instance, for the British government, the main goal was the "efficient utilisation of the spectrum and the enhancement of competition between operators to the benefit of consumers." (page 6 of The Auction of Radio Spectrum for the Third Generation of Mobile Telephones, Report by the Comptroller and Auditor General, October 1991 http://www.nao.org.uk/intosai/wgap/0102233.pdf). Revenue, although less openly recognized, was also welcomed.

For values of $\lambda < \lambda_B$, CS is (locally) independent of λ. Indeed, firms are effectively unconstrained, and their output will be equal to the Cournot output. For $\lambda > \lambda_B$, (14) is the expression of output as a function of λ, for general values of Q and λ. Thus,

$$(15) \quad \frac{d(CS+R)}{d\lambda} = \begin{cases} \frac{n}{c}\left(Q\frac{dp(Q)}{dQ} + c\right) = \frac{n}{c}\left(c - \frac{p(Q)}{\varepsilon(Q)}\right) & \text{if } \lambda > \lambda_B \\ \\ n & \text{if } \lambda < \lambda_B \end{cases}$$

and CS+R is increasing for $\lambda < \lambda_B$. Thus, a sufficient condition for optimal prices to be strictly larger than λ_B is that $c - \frac{p}{\varepsilon} > 0$ at Q_c. But at that level of output, (5) is just satisfied with equality and any reduction in quantity would violate (5). Therefore,

Lemma 2: *The value of λ that maximizes CS+R is at least λ_B. If $\varepsilon(Q_c) > \frac{n+1}{n}$, then the CS+R-maximizing value of λ exceeds λ_B.*

For sufficiently elastic demand (evaluated at the Cournot quantity), an optimal license price should cause the financing constraints to bind, if at that price firms earn positive profits. If $\lambda_B > \lambda_0$, an auction maximizes consumer surplus, since the financing constraint is not salient (all prices firms voluntarily agree to pay are not financing constrained.) The interesting case is when $\lambda_B < \lambda_0$. In this case, should the government attempt to extract the highest possible price for the licenses?

CS+R is convex in λ for $\lambda > \lambda_B$ if CS is convex in output (because (3) holds with equality). Thus, in this case we can conclude immediately from Lemma 2 that λ_0 is indeed the optimal price if the conditions of that lemma are satisfied. If CS+R is increasing at λ_B it is also increasing at any $\lambda > \lambda_B$. This demonstrates:

Theorem 1: *If CS is convex and* $\varepsilon(Q_c) > \dfrac{n+1}{n}$, *then* λ_0 *is the optimal price.*

The interpretation of Theorem 1 is that if demand is just a little elastic and satisfies the usual condition that a monopolist absorbs part of a per unit tax, then auction prices are *not* too high to maximize consumer surplus, even if financing constraints strictly bind. That is, even if the critics of auctions are right that auctions interfere with service deployment, if demand is slightly elastic, consumers prefer the auction revenue to wider deployment and less revenue.

There is at least one commonly-assumed class of demand functions of interest that does not satisfy (13): constant-elasticity demand functions. For this class, CS is concave. Let us now assume that CS is concave in output, so that CS+R is concave in λ, for $\lambda > \lambda_B$. Concavity of CS+R guarantees that CS+R is increasing in $\lambda < \lambda_0$ if it is increasing at λ_0, which occurs when $c - \dfrac{p}{\varepsilon} > 0$ when evaluated at the quantity associated with λ_0. The zero profits condition (10), together with budget binding (3, with equality) implies that

$$(16) \quad \frac{p}{c} = \frac{B}{B-\lambda}.$$

Therefore,

Theorem 2: *If CS is concave and at the price* $\lambda_0, \varepsilon \geq \dfrac{B}{B-\lambda_0}$, *then* λ_0 *is the optimal price.*

Theorem 2 relates variables that may readily be estimated: the elasticity of demand at the quantity associated with the auction price, overall expenditures (budget) and license prices. Under the assumption of CS concavity, this allows us to check whether a realized price is excessive. When CS is concave the hypothesis of Lemma 2, that

$\varepsilon(Q_c) > \dfrac{n+1}{n}$, is not sufficient for optimality of λ_0. However one can easily check that it

is a necessary condition for λ_0 to be optimal.

4. *Auctions, budgets, and prices; some comments on robustness*

In this section we first analyze the existence and uniqueness of zero-profit prices and its

relationship with auction prices. Afterward, we relax the assumption of equal budgets

and discuss alternative market models.

Let K represent "output capacity" after any payment for a license. That is, a firm

that has budget B and pays the price λ has an output capacity equal to $K = \dfrac{B-\lambda}{c}$. Note

that once licenses have been acquired, if firms have capacity in excess of Q_c/n, then the

output in the market will be Q_c. If firms have a capacity $K < Q_c/n$, then output will be nK.

Using this notation, we can represent a firm's market profits as a function of capacity

$$
\pi_m(K) = \begin{cases} (p(nK) - c)K & \text{if } K \le \dfrac{Q_c}{n} \\ (p(Q_c) - c)\dfrac{Q_c}{n} & \text{otherwise} \end{cases}
$$

The function π_m is concave to the left of Q_c, and is constant beyond that point. It attains

a maximum at the K equal to ($1/n^{\text{th}}$ of) the monopoly output. We have represented this

function as the thick line in Figure 1.

In Figure 1 we have also represented several straight lines with slope $-c$. Take

any of these and consider its intersection with the horizontal axis. For a firm with a

budget given by c times this intersection, this linear function measures for each capacity

level the license fee that would leave the firm with that level of capacity. Then, the

intersection between the function $\pi_m(K)$ and the straight line corresponding to B defines a zero profit license fee when firms have a budget B.

Note that $\pi_m(K)$ is zero at K=0. Also, we know that the (left) derivative of $\pi_m(K)$ with respect to K at $K=Q_c/n$ is

$$p'(Q_c)Q_c + (p(Q_c) - c) = p(Q_c)\left[-\frac{1}{\varepsilon} + 1\right] - c.$$

From (6), this is equal to

$$c\left[\left(-\frac{1}{\varepsilon} + 1\right)\frac{n\varepsilon}{n\varepsilon - 1} - 1\right] = -c\frac{n-1}{n\varepsilon - 1}.$$

This value is negative but larger than $-c$ when $\varepsilon > 1$ at Q_c. Thus, since $\pi_m(K)$ is concave in K, when $\varepsilon > 1$ at Q_c there exists one and only one zero-profit capacity/output (and then, one zero-profit λ) for each value of B. Therefore, whenever the conditions of Lemma 2 or Theorems 1 and 2 hold there is no ambiguity as to what is the zero-profit license price.[12]

In this model, the relationship between the auction price and the zero-profit price when the latter is unique is trivial. Any standard (only winners pay) auction has a unique pure strategy equilibrium where all firms bid this price. Comparative statics for this price with respect to B are equally straightforward, and are illustrated by Figure 1. The auction price is (weakly) increasing in the budget.

Next, we discuss asymmetries in budgets. We will refer to the firms by their ranking, from higher to lower, in terms of budgets, and let B_i denote firm i's budget. It is

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[12] When $\varepsilon < 1$ at Q_c there may be multiple (up to three) zero-profit prices λ for some values of B. Intuitively, when license price rise, the budget constraints cause the quantity to fall, and inelastic demand implies that revenues rise. This entails revenues being an increasing function of the license prices, which can offset the increased license price.

an expositional simplification to assume that no two firms have the same budget. Also, assume that any two firms that obtain a license pay the same price.

In the appendix (Lemma 3) we show that given the license price and any set of winners, only the firms with smallest budgets will be constrained and only if the smallest budget is insufficient to meet the license price and the Cournot quantity. Firms with larger budgets all produce the same, larger output. Since the market price is common to all firms, this means that firms with larger budgets will have larger profits. The immediate implication is that if a firm j is willing to pay some given price for the license in order to compete with some set of $n-1$ other firms in the market, then any other firm $i<j$ should be also willing to pay that price for the license.

A price λ will clear the market if it is below the profits of firm n and above the profits of firm $n+1$, both computed in competition with firms 1 through $n-1$. There is an interval of such prices.[13] Define λ_0 as the largest of all these prices, and λ_B as the price at which the financing constraint of firm n just binds, when competing against firms 1 through $n-1$.

Assuming that firms 1 through n win a license, we can replicate the major results of Section 3. Indeed, for any price lower than λ_B, CS+R is locally independent of λ, under this construction. Also (15) defines now a lower bound for the slope of CS+R with respect to the license price. To see this, note that the slope of revenues with respect to λ is not affected, but the slope of Q with respect to λ is lower: When not all firms are financially constrained by the license price, then an increase in this price reduces Q by

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[13] An oral auction, or a sealed bid auction with full information, would tend to pick the low end of this interval.

less than $\frac{n}{c}$. Therefore whenever (15) is positive, at some value of Q (and P), the slope of CS+R is positive at that value. It follows that Lemma 2 and Theorems 1 and 2 still hold with asymmetric financial constraints. [14]

We have modeled the market for telecommunications as a simple Cournot market. It may be argued that telecommunications are characterized by network externalities, which may render reductions in output more costly. For instance, one could postulate, as in Amir and Lazzati (2007), a demand function $P(Q;Z)$, where Z is the expected size of total output (network), [15] and assume that $P_1<0$, and $P_2>0$, where P_i represents the partial derivative of P with respect to the ith argument.[16] In equilibrium, $Q=Z$.

Let $A = \frac{P_1}{P_1+P_2} > 1$. Then, the condition of Lemma 2 becomes $\varepsilon(Q_c) > \dfrac{\frac{n}{A}+1}{n}$,

where now $\varepsilon(Q) = \dfrac{-p(Q;Z)}{Q} \dfrac{1}{P_1}\bigg|_{Q=Z}$, i.e., the elasticity of demand for a fixed expected total output. [17] Therefore, the sufficient conditions for auctions to be optimal (when the consumer surplus is convex) are even weaker, when we consider the effects of externalities.

Finally, we should note that assuming price competition would not have an effect on the qualitative results, as is some times the case in oligopoly models. Indeed, as

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[14] However, it may be in the interest of the government to assign a number of licenses smaller than the technically feasible, selling fewer licenses at higher prices. With symmetric firms, more licenses are generally preferable if more licenses produce a more competitive outcome.
[15] Z may be a measure of network quality, when it is the size of total industry investment what matters for consumer, and not individual firm investment.
[16] Amir and Lazzati (2007) show that under mild conditions a Cournot equilibrium exists, although some additional assumptions are required for uniqueness.

long as firms are constrained and have a total capacity $nk=Q$ below the competitive quantity, in a Bertrand equilibrium all firms set prices equal to $P(Q)$. Then, equation (15) still holds, except that λ_B should now be defined as the license price that leave firms with just enough resources to produce the competitive quantity, instead of the Cournot quantity. Thus, Theorem 2 holds without changes. Theorem 1 also holds with only substituting the competitive quantity for Q_c. It is also straightforward to show that whenever the elasticity is above 1 at the competitive quantity, λ_0 is uniquely defined in this alternative setting, and then λ_0 is the auction price. The only caveat here is that under these assumptions, firms would be capacity constrained at the auction price if and only if their budget constrains also at zero license prices. In any case, the conclusion is unchanged: if demand is not very inelastic, auctions maximize the sum of consumer surplus and revenues even when they reduce output.

We have modeled the firms as setting quantities; in some circumstances investments set quality rather than quantity. For the United States, where some areas lack cell coverage, marginal investments may extend coverage to new areas and therefore increase quality for existing customers who are roaming, while increasing quantity by offering service to new customers. When investment is used to put cell towers closer together, a greater capacity of calls can be carried simultaneously and this corresponds closely to the model's assumptions. The case of 3G represents a mix – an increase in the quality of data transmission but increases in quantity of voice and data. Given that quality is often modeled as a kind of quantity, the present model has a natural, if somewhat strained, quality interpretation.

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[17] This would be the elasticity that would be measured by most empirical tests.

5. *The European 3G Experience*

Licenses for spectrum intended for 3G (third generation) cellular telephony usage were assigned beginning in March 2000 with Spain.[18] The first auction of 3G licenses took place in the United Kingdom and ended in April 2000. The four incumbent GSM operators (Vodafone, BT's O2, France Telecom's Orange, and T-Mobile) and a new entrant, Hutchison, each won a license. Prices were considered astronomically high, because the prices exceeded the prices for the US PCS spectrum, in spite of the US spectrum having fewer constraints on usage. About the same time as the UK auctions, the stock price index of telecoms started declining (see EC 2002, exhibit 26). By the time the next auction took place in the Netherlands, only three months later, telecom firms had lost about 25% of their equity value. This time, each of the five incumbents (KPN, O2, T-Mobile, and Dutchtone, Orange) won a license. A month later, when the German auction closed, telecom share prices had fallen even further, to about two-thirds of their March 2000 value. In Germany 6 licenses were sold.[19] Again, each of the four incumbents (T-Mobile, O2, Vodafone-, and Mobilcom, which was partly owned by France Telecom) obtained a license, and two new firms, Quam (a joint venture of Telefonica and Sonera) and Orange, entered the market. The next auction took place in Italy, in October 2000. By then, the stock market index of European Telco's had already lost more than 40% of its value, as compared to a loss by the American counterparts of about 25% during the same period.

Licenses included obligations to deploy 3G networks with minimum coverage requirements and deadlines. For instance, license holders in the UK were required to

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[18] For a description and analysis of the recent regulation of telephony in Europe, see Grzybowsky (2005).

have a network in place that covered 80% of the UK population by the end of 2007. In Sweden, the conditions of the beauty contest pushed this to 99.98% of the population by the end of 2003. In the Netherlands, the requirements included coverage of at least 60% of the population by 2007, and in Germany 25% of the population by the end of 2003 and 50% by the end of 2005.

Immediately after the first wave of license allocations as the year 2000 ended, the mood in the industry changed. As some say, the internet and telecom bubble burst. The prospect of profitable 3G services receded. If only a few months earlier the market was in the peak of the optimism about the telecom industry, by the end of 2000 and beginning of 2001 the articles in the financial press were filled with comments about the struggling of telecom firms with debt crises. The debt taken to finance the acquisition of licenses was often identified as an important contributory factor of the telecom debt crisis. With the equity markets hostile to telecoms, most European telecoms borrowed a substantial amount of money.[20]

In this landscape of diminished expectations, the launching of 3G services was delayed. In fact, with the unsuccessful exception of Hutchison's 3, the launching of 3G services did not begin until mid-2004. Mobilcom and Quam in Germany and Orange in Sweden had failed to meet their roll-out obligations and consequently had to return their licenses.

In all countries, firms lobbied for delays in their 3G coverage obligations, and in most places they succeeded. Sweden allowed a year extension on the requirement of (virtually) full population coverage (from the end of 2003 to the end of 2004). Even this

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[19] Twelve blocks were on sale, and each firm could buy two or three of these blocks. Thus, the number of licenses was endogenously determined.

extended deadline was not met. In addition, operators received permission to sharing their networks, so that the originally envisioned structure of one independent, competing network per license was lost. Thus, network sharing agreements among carriers were approved by national governments, including the UK, Sweden, Germany, and the Netherlands. As of February 2005, population coverage of 3G networks had reached only 85% in Sweden, 75% in the UK, and less than 60% in the rest of Europe.

The demand elasticity is a critical input to the theory. Earlier studies in wireless telephony obtain elasticity estimates in the range 0.50-1.0.[21] However, early adopters of cellular telephony probably had relatively inelastic demand, so that demand at lower prices is likely substantially more elastic than these estimates suggest. For instance, using data from 1999 to 2001, Ingraham and Sidak (2004) estimate wireless services price elasticity in the range 1.12 and 1.29. In addition, the demand for 3G services such as video and gaming is likely more elastic than wireless telephony, because the luxury component is larger. Wallenius and Hämäläinen (2002) estimate the elasticity of demand for 3G services to be in the range 1.4-1.7, although their source is not identified. One could argue that the elasticity of the demand for 3G services can be better estimated by measures of the elasticity of internet access than by measures of telephony. The former are consistently higher, typically above 2 (see, for instance, Goolsbee 2006). In a recent study with Finish data obtained from a real-world experiment, Gao, Hyytinen, and Toivanen (2005) estimate the per-minute price elasticity of wireless internet access to range from 1.78 to 2.28 (length of connection) and around 1.59 (number of connections).

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[20] The Economist, January 25, 2001.
[21] See, for instance Rodini et al. (2002) or Hausman (1999), (2000).

If the demand elasticity were 1.5, then auction prices would maximize consumer surplus (including license revenues) for all if license prices accounted for less than a third of the firms' budgets, even if the firms were financially constrained and consumer surplus concave. If, as in the usually assumed case, consumer surplus is a convex function of quantity, then at an elasticity of 1.5, two licenses is sufficient to insure that auctioning is optimal.

A proxy for the firms' budgets is the sum of estimated cost of deploying a 3G network plus the license fee.[22] Taking Western Europe as a whole[23], the total cost of building networks for all licensees (in the 2000-01 sales) has been estimated at 140B €, whereas total cost of licenses was 120B €[24] a ratio of license to total cost of almost ½. However, most of the cost of licenses is accounted for by the British and German auctions, which raised total of 86B €. In the UK, license prices total 36B €, compared to an estimated 21B € needed in network investment. License fees appear close to two-thirds of the total cost (license plus network) of deploying 3G services. Similarly, in Germany the license cost was 50B € and estimated cost of the network only 34B €, so that license fee accounted for 60% of deployment cost.[25] In the rest of the countries that used auctions to assign licenses, the ratio of license fees to total estimated costs ranged from 12% in Greece to 34% in the Netherlands.

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[22] This figure ignores marketing and other costs of operating a network, but also ignores network sharing. As it treats the cost of building a network as fixed, it tends to over-estimate the budget.
[23] All figures used in this paragraph are taken from EC (2002)
[24] Our source is EC (2002). The figures refer to 3G network deployment, and do not include upgrades and replacement investment for 2G networks. The estimated total investment in this category *needed* for the period 2000-2015 is 90B €.
[25] These numbers are corroborated by the experience of O2, the originally BT mobile company. It is estimated that it spent a total 4B £ (approx. 6B €) in building its 3G networks, mainly in the UK and Germany. It spent around 15B € acquiring its British and German licenses. License fees in both markets represented more than 70% of its estimated budget (network cost plus license fees) for both markets.

The formula in Theorem 2 permits calculating the elasticity necessary to justify auction prices, assuming the financing constraints bind.[26] If CS is concave, auction prices maximized consumer surplus in the UK only if the elasticity of demand exceeded 2.7. In Germany, the critical elasticity is about 2.5. These elasticities exceed most estimates for the elasticity of demand for 3G services, suggesting that the prices perhaps were too high to maximize consumer surplus, unless financing constraints for the firms did not bind or consumer surplus is convex. In other countries, however, the critical elasticity is 1.5 or less, suggesting that an auction maximized consumer surplus no matter what assumption is placed on consumer surplus.

Given the problems that telecom firms faced with borrowing in the 2001-2005 period, it seems plausible that the firms were financially constrained. But, were they? Financing constraints ought to create a negative correlation between license prices and build-out. In the countries with the two highest prices per capita, the UK and Germany, services were deployed relatively quickly, but services were rapidly rolled out in Luxemborg and Sweden as well, which had low prices. Because the nations with the highest per capita demand will attract higher auction prices, higher budgets and faster deployment, assessing the existence of financing constraints empirically is challenging. Moreover, the effect of financing constraints is generally to slow deployment, rather than reduce it permanently, so that the time of allocation is also important in the attempt to empirically assess financing constraints. Given the small number of countries, and the possibility of endogeneity in the choice of allocation method, an attempt to empirically assess the existence of financing constraints is a daunting task.

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[26] Note that Theorem 2 references the elasticity at the zero profits or auction price, so that the relevant comparison is to the prevailing elasticity estimates, at least in the countries that auctioned the licenses.

6. Conclusion

Both experience with telecommunications companies and corporate finance research indicates that financing constraints are a fact of life in many bidding contexts. In principle, frequent company complaints that high auction prices prevented the rapid rollout of services could have merit. The effect of financing constraints on the deployment of services was examined in the context of a model of hard budgets. Evaluation of the effect of auction prices hinges on relatively inelastic demand, and auctions are optimal even when the firms are financially constrained, provided the auction price isn't too large a fraction of the firms' resources.

Demand for 3G services is probably elastic, as the most relevant evidence suggests. In that case, and even given hard financing constraints, in most countries the auction prices appear to maximize consumer surplus, understood as including license revenue. Thus auctions appear to be the best way to allocate the licenses. Only in the U.K. and Germany some doubts could remain as whether auction prices constrain the rollout of services beyond what is optimal, according to the theory and proposed demand elasticities, and this only if consumer surplus is concave.

In the main body of the paper we have assume homogeneous quantity competition. In our view, the Cournot model captures salient features of the cellular communication market. Capacity is determined by deployment of cell towers in most regions and additional capacity requires either more towers (smaller cells) or more spectrum. As with Kreps and Scheinkman (1983), pricing decisions are made long after capacity choices. Thus, the Cournot model seems like a reasonable starting point for

an analysis of auctions[27]. This is also the main argument that Grzybowski (2005),

among others, advance for analyzing this industry in a Cournot model. With that said,

however, the likely effect of increased prices is not a permanent decrease in capacity

but a temporary decrease in capacity. Our model can be interpreted as one in which

delay is very costly (discount factor close to zero). Consequently, our assumptions

exaggerate the effects of a service reduction, which makes our finding that auctions are

optimal apparently stronger. Nevertheless, it would certainly be worthwhile preparing a

formal model of the dynamic problem.

Auctions have an additional advantage obscured by the symmetry of the model:

auctions select the efficient service providers. Even if demand is relatively inelastic, it

may be desirable to auction in order to select efficiently. However, in such a setting,

auctions could have a perverse effect if the most efficient firms face relatively tighter

financing constraints, because auctions favor both the efficiency and large budgets.

Nevertheless, we expect that the advantages of auctions over random selection are

greater when firms are differentiated than in our simple model.

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[27] While there is some differentiation in services, for the vast majority of users, cellular telephony is little
differentiated.

References

Amir, Rabah, and Natalia Lazzati (2007) "Cournot Oligopoly with Network Effects", mimeo, University of Arizona.

Benoit, J.P., and Vijay Krishna, "Multiple-Object Auctions with Budget Constrained Bidders," The Review of Economic Studies, January 2001, vol. 68, no. 1, pp. 155-179.

Che, Yeon-Koo and Ian Gale, "Standard Auctions with Financially Constrained Bidders," Review of Economic Studies, 65, 1998, 1-22.

Clementi, Gian Luca and Hugo Hopenhayn, "A Theory of Financing Constraints and Firm Dynamics," Quarterly Journal of Economics, 2006, 229-265.

Commission of the European Communities, "Communication from the Commission to the Council, the European Parliament, The Economic and Social Committee, and the Committee of the Regions: Towards the Full Roll-Out of Third Generation Mobile Services", Brussels 11.6.2002, COM(2002) 301 Final.

European Commission, "Comparative Assessment of the Licensing Regimes for 3G Mobile Communications in the European Union and their Impact on the Mobile Communications Sector", final report, Brussels, June 25, 2002.

Fazzari, Steven and Michael J. Athey, "Asymmetric Information, Financing Constraints, and Investment," *Review of Economics and Statistics,* 69, No. 3, August 1987, 481-7.

Fazzari, Steven, R. Glenn Hubbard, and Bruce C. Petersen, "Financing Constraints and Corporate Investment," *Brookings Papers on Economic Activity,* Vol 1988, No. 1, 1988, 141-206.

Fazzari, Steven and Bruce Petersen, "Working Capital and Fixed Investrment: New Evidence on Financing Constraints," *Rand Journal of Economics*, 24, No. 3, Autumn 1993, 328-42.

Gao, Maija, Ari Hyytinen, and Otto Toivanen, "Demand for Mobile Internet: Evidence from a Real-World Pricing Experiment", HECER Discussion Paper No. 43, University of Helsinki, 2005

Goolsbee, Austan, "The Value of Broadband and the Deadweight Loss of Taxing New Technology", mimeo, University of Chicago, 2006.

Greenwald, Bruce, Joseph E. Stiglitz and Andrew Weiss, "Informational Imperfections in the Capital Market and Macroeconomic Fluctuations," *American Economic Review Papers and Proceedings,* 74, May 1984, 194-9.Hart, Oliver and John Moore, "Debt and Seniority: An Analysis of the Role of Hard Claims in Constraining Management," *American Economic Review,* 85, No. 3 (June 1995), 567-85.

Greenwood, Jeremy and R. Preston McAfee, "Externalities and Asymmetric Information," *Quarterly Journal of Economics* CVI, no. 1, February 1991: 103-121.

Grzybowsky, Lukasz, "Regulation of Mobile Telephony across the European Union: An Empirical Analysis", Journal of Regulatory Economics, 28, No. 1 (2005) 47-67.

Haan, M. A., and L. A. Toolsema (2003) "License Auctions when Winning Bids are Financed through Debt", mimeo, U. of Groningen.

Hausman, Jerry, "Cellular Telephone, New Products, and the CPI", Journal of Business and Economic Statistics, 17, No. 2 (1999) 188-94.

Hausman, Jerry, " Efficiency Effects on the U.S. Economy from Wireless Taxation", National Tax Journal, 53, No 2 (2000) 733-42.

Ingraham, Allan, and Gregory Sidak (2004) "Do States Tax Wireless Services Inefficiently? Evidence on the Price Elasticity of Demand *", Virginia Tax Review,* 24, No. 2, 249-261.

Jensen, Michael C., "Agency Costs of Free Cash Flow, Corporate Finance and Takeovers," *American Economic Review Papers and Proceedings*, 76, No. 2, (May 1986), 323-9.

Klemperer, Paul, "How (not) to run auctions: The European 3G telecom auctions," *European Economic Review*, 46, 2002, 829-845.

Kreps, David and Jose Scheinkman: "Quantity precommitment and Bertrand competition yield Cournot outcomes," *Bell Journal of Economics* 14,1983, 326-337

Lewis, Tracy and David E. M. Sappington, "Inflexible Rules in Incentive Problems," *American Economic Review,* 79, No.1 March 1989a, 69-84.

Lewis, Tracy and David E. M. Sappington, "Countervailing Incentives in Agency Problems," *Journal of Economic Theory*, 49, 1989b, 294-313.

Love, Inessa, "Financial Development and Financing Constraints: New Evidence from the Structural Investment Model," *Review of Financial Studies*, Fall 2003, 765-791.

McCleland, Sephen, "Telecom Meltdown: Don't Blame Us, Say Forecasters," BusinessWeek online, October 13, 2003.

Myers, Stewart and Nicholas S. Majluf, "Corporate Financing and Investment Decisions when Firms Have Information that Investors Do Not Have," *Journal of Financial Economics,* 13, June 1984, 187-221.

Pitchik, Carolyn, and Andrew Schotter, "Budget Constrained Sequential Auctions, 1986, unpublished manuscript.

Pitchik, Carolyn, and Andrew Schotter, "Perfect Equilibrium in Budget-Constrained Sequential Auctions: An Experimental Study," *Rand Journal of Economics*, 19, 1988, 363-88.

Rodini, Mark, Ward, Michael Robert and Woroch, Glenn A., "Going Mobile: Substitutability Between Fixed and Mobile Access" (December 2002). Haas School of Business, Center for Research on Telecommunications Policy Working Paper CRTP-58.

Stiglitz, Joseph E, and Andrew Weiss, "Credit Rationing in Markets with Imperfect Information," *American Economic Review*, 71, June 1981, 393-410.

Wallenius, Eero and Timo Hämäläinen, "Pricing Model for 3G/4G Networks," The 13th IEEE International Symposium on Personal, Indoor and Mobile Radio Communications, Volume: 1, 2002, pp. 187 -191.

Whited, Toni, "Debt, Liquidity Constraints, and Corporate Investment: Evidence from Panel Data," *Journal of Finance,* 47, 1992, 1425-60.

Appendix

Proof of Lemma 1: For the moment, ignore the financing constraint. Fix the output of other firms at Q_{-i}, so that profits are

$$\pi = p(q + Q_{-i})q - cq - \lambda.$$

If the constraint does not bind, the second derivative of profits is

$$\frac{\partial^2 \pi}{(\partial q)^2} = qp''(q + Q_{-i}) + 2p'(q + Q_{-i}).$$

If $p''(q + Q_{-i}) < 0$, $\dfrac{\partial^2 \pi}{(\partial q)^2} < 0$ since p is a demand curve. If $p''(q + Q_{-i}) > 0$,

$$\frac{\partial^2 \pi}{(\partial q)^2} = qp''(q + Q_{-i}) + 2p'(q + Q_{-i}) < (q + Q_{-i})p''(q + Q_{-i}) + 2p'(q + Q_{-i}) < 0 \text{ by (2)}.$$

Either way, π is globally concave, so the Kuhn-Tucker condition characterizes a maximum.

The Kuhn-Tucker condition is

$$\frac{\partial \pi}{\partial q} = qp'(q + Q_{-i}) + p(q + Q_{-i}) - c \begin{cases} = 0 & \text{if } cq + \lambda < B \\[2mm] \geq 0 & \text{if } cq + \lambda = B \end{cases}$$

or

$$qp'(Q) + p(Q) - c \begin{cases} = 0 & \text{if } cq + \lambda < B \\[2mm] \geq 0 & \text{if } cq + \lambda = B \end{cases}$$

or

$$p(Q)\left(1 - \frac{q}{Q}\frac{1}{\varepsilon(Q)}\right) - c \begin{cases} = 0 & \text{if } cq + \lambda < B \\[2mm] \geq 0 & \text{if } cq + \lambda = B \end{cases}$$

Note that, if firm i is constrained, then any firm producing the quantity q less than q_i satisfies

$$p(Q)\left(1 - \frac{q}{Q}\frac{1}{\varepsilon(Q)}\right) - c > p(Q)\left(1 - \frac{q_i}{Q}\frac{1}{\varepsilon(Q)}\right) - c \geq 0.$$

Thus, if one firm is constrained, they are all constrained. That is, either no firm, or all firms, are constrained. Consequently, (2) entails that any equilibrium is symmetric, and satisfies

$$p(Q)\left(1 - \frac{1}{n\varepsilon(Q)}\right) - c \begin{cases} = 0 & \text{if } cQ/n + \lambda < B \\ \\ \geq 0 & \text{if } cQ/n + \lambda = B \end{cases}.$$

Note that

$$\frac{d}{dQ}p(Q)\left(1 - \frac{1}{n\varepsilon(Q)}\right) - c = \frac{1}{n}\frac{d}{dQ}\left(np(Q) + Qp'(Q)\right) = \frac{1}{n}\left((n+1)p'(Q) + Qp''(Q)\right)$$

$$= \frac{1}{n}\left((n-1)p'(Q) + 2p'(Q) + Qp''(Q)\right) \leq \frac{1}{n}\left(2p'(Q) + Qp''(Q)\right) \leq 0 \text{, by (2).}$$

Thus, there is a unique Cournot equilibrium.

Lemma 3: *For any vector $(B_1, B_2, \ldots, B_n)$ of budgets, and license price λ there is a unique pure strategy equilibrium output $(q_1, q_2, \ldots, q_n)$. If $B_n - \lambda > cQ_c/n$, then $q_i = Q_c/n$ for all i. If $B_j - \lambda > c Q_c/n > B_{j+1} - \lambda$ for some j<n, then there exists $k \geq j$ such that $cq_i = B_i - \lambda$ for all i>k, $q_i = q$ for all i<k for some value $q > Q_c/n$, and $Q < Q_c$.*

Proof of Lemma 3:

As in the proof of Lemma 1, π is globally concave, so the Kuhn-Tucker condition characterizes a maximum:

$$\frac{\partial \pi}{\partial q} = qp'(q + Q_{-i}) + p(q + Q_{-i}) - c \begin{cases} = 0 & \text{if } cq + \lambda < B \\ \\ \geq 0 & \text{if } cq + \lambda = B \end{cases}$$

or

$$p(Q)\left(1 - \frac{q}{Q}\frac{1}{\varepsilon(Q)}\right) - c \begin{cases} = 0 & \text{if } cq + \lambda < B \\ \\ \geq 0 & \text{if } cq + \lambda = B \end{cases}$$

Assume $B_n - \lambda > cQ_c/n$. Then, if some firm i is constrained, $q_i > Q_c/n$ and any firm producing the quantity q less than q_i satisfies

$$p(Q)\left(1 - \frac{q}{Q}\frac{1}{\varepsilon(Q)}\right) - c > p(Q)\left(1 - \frac{q_i}{Q}\frac{1}{\varepsilon(Q)}\right) - c \geq 0 .$$

Thus, if firm i is constrained, every firm j produces at least the minimum of $B_j - \lambda$ and q_i, which means that $Q > Q_c$. But this contradicts (2). Thus, $B_n - \lambda > cQ_c/n$ no firm is constrained and then the only equilibrium is $q_i = Q_c/n$ for all i.

Now, assume $B_j - \lambda > cQ_c/n > B_{j+1} - \lambda$ for some j<n. First note that $Q < Q_c$ in any equilibrium, in this case. Indeed, otherwise at least one firm i produces more than Q_c/n, and then the Kuhn-Tucker condition is violated for this firm. This immediately implies that $cq_i = B_i - \lambda$ for all i>j, and $Q < Q_c$.

Given a set of firms that are constrained, and therefore their aggregate output, consider the residual demand for other firms. This satisfies the same conditions of the original demand function. Thus, there exists a unique, symmetric output equilibrium for these firms, as in Lemma 1.

Thus, to complete the proof of Lemma 3 we only need to show that there is a unique set of constrained firms. Assume there is some k so that

$$\frac{\partial \pi}{\partial q} = qp'(q_i + Q_{-i}) + p(q_i + Q_{-i}) - c \begin{cases} = 0 & \text{if } i < k \\ \\ \geq 0 & \text{if } i \geq k \end{cases}$$

and

$$q_i = \begin{cases} q & \text{if } i < k \\ \\ \dfrac{B_i - \lambda}{c} & \text{if } i \geq k \end{cases}$$

for some value q. Now assume that for some k'<k, we have some other vector of

outputs for $i \geq k'$, $q'_i = \dfrac{B_i - \pi}{c}$, $q'_i = q'$ for i<k' satisfying the Kuhn-Tucker condition.

Note that $q'_{k'} = \dfrac{B_i - \lambda}{c} > q_{k'}$. Given global concavity of π, this implies both $Q'_{-k'} < Q_{-k'}$ so

that $q > q'$ (since $q_i \leq q_i'$ for all $i \geq k'$) and then $Q'_{-1} > Q_{-1}$. Next we show that this

implies that Q' > Q. Indeed, define $\tilde{q} = Q' - Q_{-1}$. If $Q' \leq Q$, then $\tilde{q} + Q_{-1} \leq Q$, so that

$\tilde{q} \leq q$. Then, from the Kuhn-Tucker conditions, we have that

$$\tilde{q}p'(\tilde{q} + Q_{-1}) + p(\tilde{q} + Q_{-1}) - c = \tilde{q}p'(Q') + p(Q') - c \geq 0$$

and then we conclude that $\tilde{q} \leq q'$ as well. Now, since

$$q' + Q'_{-1} = Q' = \tilde{q} + Q_{-1},$$

this implies that $Q'_{-1} \leq Q_{-1}$. This contradiction proves that Q' > Q.

Then,

$$Q'_{-k'} = Q' - \frac{B_i - \lambda}{c} > Q - \frac{B_i - \lambda}{c} > Q - q_{k'} = Q - q,$$

and then concavity of π and the Kuhn-Tucker conditions for the equilibrium resulting in

output Q' contradicts the Kuhn-Tucker condition for the equilibrium resulting in output Q,

i.e.,

$$qp'(q + Q_{k'}) + p(q + Q_{k'}) - c = 0.$$

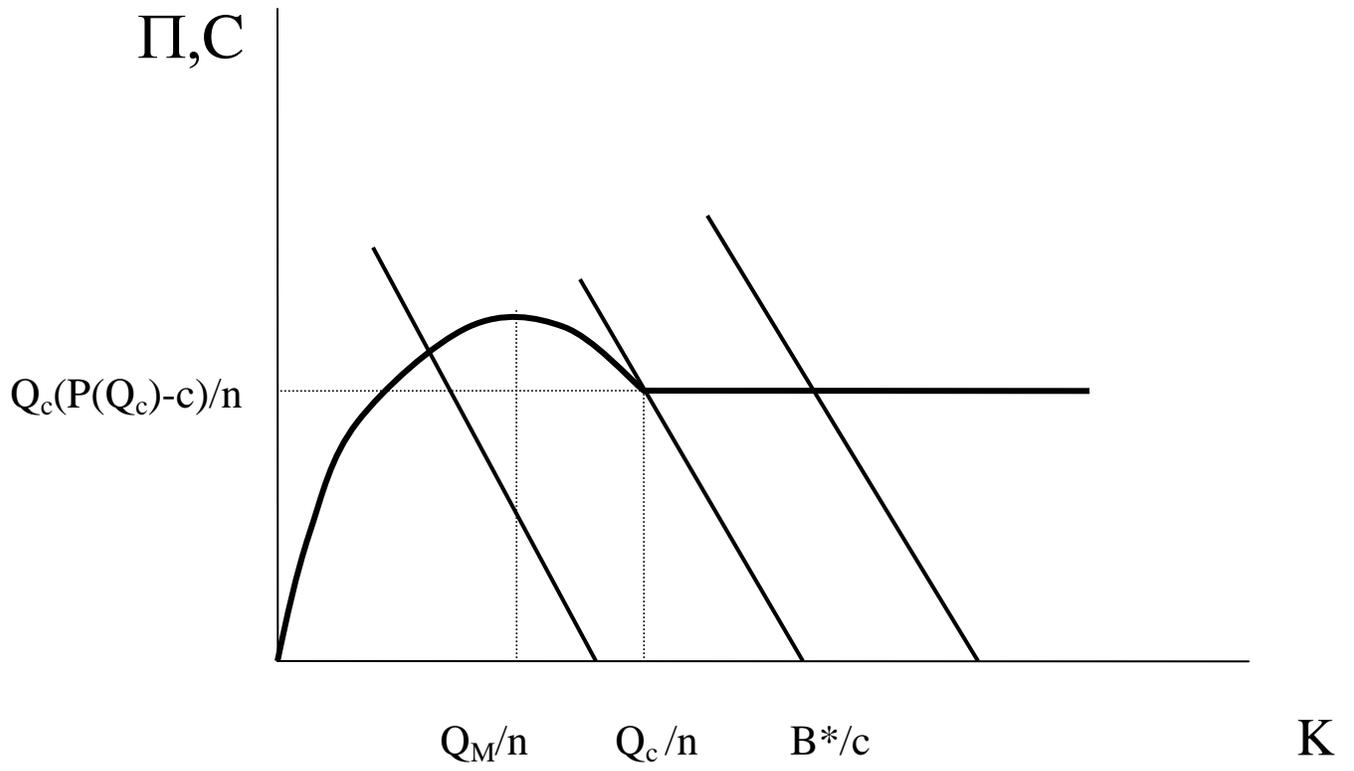This contradiction shows that there is only one equilibrium for each vector of budgets.

Figure 1