A Geometric Description of a Macroeconomic Model with a Center Manifold

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A geometric description of a macroeconomic model with a center manifold*

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Abstract

This paper presents a unified framework of different algorithms to numerically compute high order expansions of invariant manifolds associated to a steady state of a dynamical system. The framework is inspired in the parameterization method of Cabré, Fontich and de la Llave [7], and the semianalytical algorithms proposed by Simó [13], and those of Gomis-Porqueras and Haro [9]. Within this methodology, one can compute high order approximations of stable, unstable and center manifolds. In this last case the use of high order approximations (not just linear) are crucial in understanding the dynamic properties of the model near the steady state. To illustrate the algorithms we consider a model economy introduced by Azariadis, Bullard and Smith [6]. Besides its intrinsic importance, this four dimensional macroeconomic model is an ideal testing ground because it delivers steady states with stable and unstable manifolds (of dimensions 1 or 2), and each of them has also a one dimensional center manifold. Moreover, the numerical computations lead to a further theoretical study of the dynamical system completing some of the results in the original paper.

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1 Introduction

The study of macroeconomic phenomena cannot be completely understood without the analysis of the underlying dynamic process. Many macroeconomic concepts rely on intertemporal trade offs, thus inherently embracing the concepts and methodology of dynamical systems. In modeling economics we must confront the fact that agents, whose behavior are generating the dynamics, are themselves observing and trying to forecast the actual dynamics. In rational expectation models, one tries to derive aggregate behavior from assumptions on tastes, technologies and market structure that are summarized by a dynamical system. In a fully specified environment, the optimal allocation of resources is determined by invariant properties of the resulting dynamical system. These invariant structures organize the motion of the economy as time evolves, giving us an idea of the possible evolution of the macroeconomic observables describing the economy.

A revolutionary contribution to the theory of dynamical systems was made by Poincaré at the end of the 19^{th} century. Before his time, the main objective was finding explicit functions that solved the laws of motion. Poincaré's new proposal was to look at the geometry of solutions instead of explicit solutions. This geometric or topological approach, largely identified with the qualitative theory of differential/difference equations, aims at understanding the asymptotic properties of the system. Given that macroeconomic models are described by difference or differential equations, Poincaré's approach is ideal. The area in economics that has applied this approach the most has been economic growth and business cycle theory. For example, bifurcation theory has been useful in proving the existence both of deterministic cycles[1] as in Benhabib and Rustichini [5], and sunspot equilibria as in Grandmont [11], and Azariadis and Guesnerie [2] among others.

Typically these macroeconomic models seldom have explicit time paths describing the temporal evolution of macroeconomic observables even though the underlying economies are governed by explicit rules relating the various observables. Traditionally, only the evolution near the steady state(s) has been analyzed. Unfortunately, ignoring non linear properties of these manifolds might not be appropriate when studying some economic phenomena. For example, Gomis-Porqueras and Haro [10] show how non linear properties of the invariant manifolds of an economy with credit market frictions can be consistent with recurrent hyperinflations. A linear or local description of the associated manifolds are not able to capture this economic phenomena and other alternative hypothesis, like bounded rationality, are needed to explain recurrent hyperinflations.[2] Another instance where is uninformative to ignore non linear aspects of invariant manifolds is when economies have center manifolds. Under these circumstances, the study of non linear dynamics is essential because linear approximations cannot give any sort of information regarding the underlying dynamic properties. Thus higher order terms are required to study the dynamic predictions of the underlying observables.

The goal of this paper is to provide a unified framework for dealing with power series that can be used to characterize non linear invariant manifolds and normal forms, giving rise to efficient algorithms. The methods are inspired in the parameterization method of Cabré, Fontich and de la Llave [7], the semianalytical algorithms proposed by Simó [13], and the algorithms of Gomis-Porqueras and Haro [9]. Since the power series are just the Taylor series of the functions that parameterize the invariant manifolds or give the normal form

[1] Bifurcation theory deals with a parameterized family of dynamical systems and the qualitative variations in the family if we vary the parameters.

[2] Thus, non linearities in the underlying manifolds of an economy are then consistent with some complex economic phenomena.

transformations, we also want to emphasize that the power series are convergent in many important cases, like for the stable and unstable manifolds, or the case of the so called Poincaré-Dulac normal form of a linearly attracting fixed point.[3] Moreover, even in cases where the power series are divergent, like usually happens for the expansions of center manifolds, they give significant information about the dynamics and the geometry of the corresponding invariant objects. It is then very useful to compute high order approximations of the invariant manifolds, to obtain accurate approximations in large regions of phase space of the system. After all, one can only compute a finite number of terms of the power series expansions, a tedious work that can be done very efficiently by computers, as long as efficient algorithms have been provided.

To illustrate our methodology, we consider a model economy introduced by Azariadis, Bullard and Smith [6], ABS model henceforth. This model studies an old topic in monetary economics; i.e, whether the provision of currency should be an activity left strictly to the government or to private agents. Apart from its intrinsic importance, this 4-dimensional model is an ideal testing ground for our algorithm, since it has two steady states (monetary and non monetary) with a 1-dimensional center manifold as well as stable and unstable manifolds of dimensions 1 or 2. The numerical results lead us to obtain more theoretical results on the ABS model. In particular, we can obtain close form solutions for the underlying dynamics on the center manifold, proving that the dynamics is purely oscillatory, contradicting the results in [6]. As a result, we can easily check the accuracy and performance of our algorithm. Moreover, we also see that the dynamics takes place in a 3-dimensional manifold, reducing the dimension of the problem. We also show the importance of the slow manifolds in shaping the predicted time series for the interest rates.[4] In particular, we discover a connexion between the slow stable and the slow unstable manifolds of the two steady states of the model. Thus, there are economic equilibrium paths going from the monetary steady state and to the non monetary steady state. This equilibrium then predicts hyperinflationary equilibrium paths in which money is used at all dates but the price level tends to infinity, generating equilibrium indeterminacy. This new global result is consistent with Friedman [8] who argued that allowing private provision of close currency substitutes is a recipe for generating indeterminacy of equilibrium.

The paper is organized as follows. In Section 2 we present our framework that allows us to compute the invariant manifolds of an economy. The explanations are completed in Appendices A and B. Section 3 introduces the ABS model, presents the theoretical and numerical results. The proofs of the theoretical results are left to Appendix C. We emphasize that some of the theoretical results have been inspired by the numerical computations. Finally, we present some conclusions in Section 4.

2 On the computation of invariant manifolds

Coming back to the Poincaré's foundations of the theory of dynamical systems, this paper emphasizes two very important goals: (i) the study of the invariant manifolds and (ii) the computation of normal forms. These

[3] In this case, we assume that the power series describing the model are also convergent. That is, we assume that the model is analytic around the steady state. One can also consider cases in which the model is infinitely differentiable, or even just finitely differentiable.

[4] The slow stable (resp. slow unstable) manifold is the invariant manifold inside the stable (resp. unstable) manifold associated to the biggest (resp. smallest) eigenvalue of modulus lower (resp. greater) than 1.

invariant manifolds can be thought as "highways" of the dynamical system that organize the global dynamics of the economy.[5] The theory of normal forms, on the other hand, consists of transforming a dynamical system to a simpler description which is easier to compute and study. These transformations are given by power series around the steady state.[6] The idea of using power series to approximate invariant manifolds is also very fruitful, both from the theoretical point of view and for its numerical applications.[7]

In this section we present the main ideas behind the algorithm that characterizes the associated invariant manifolds of a dynamical system. We relate the graph and the parameterization methods discussed in Gomis-Porqueras and Haro [9] as special cases of this general framework.

The underlying idea behind the methods presented in this section is to compute the invariant manifolds non locally by exploiting an invariance condition that analytically describes them. The first step in achieving our goal is to obtain a local approximation of the manifold where the fixed point belongs to this local description. The second step is to define a fundamental domain around the fixed point where the invariance equation is satisfied with high accuracy. Finally, we iterate this domain expanding the manifold away from the steady state by successive linear approximations which give the non linear properties of the invariant manifold.

Given that macroeconomic models are described by difference or differential equations, the following methodology can be applied to all macroeconomic models that are described by a dynamical system. For illustrative purposes, let us consider a discrete n-dimensional dynamical system given by:

$$z_{t+1} = F(z_t) \ , \tag{1}$$

where $z = (z_1, \ldots, z_n)$ denote coordinates in a open set $\mathcal{B} \subset \mathbb{R}^n$, the *phase space* which is the set of states of the dynamical system, and $F : \mathcal{B} \subset \mathbb{R}^n \to \mathbb{R}^n$ representing a smooth map that describes the evolution of the dynamical system.

In general, a set $S \subset \mathcal{B}$ is said to be invariant under F if $z \in S$ implies $F(z) \in S$. The simplest invariant objects of a dynamical system are the fixed points or steady states and the periodic orbits of F. In the following sections, we consider different invariant sets given by specific parameterizations.

2.1 Parameterization of invariant manifolds

In this section we consider invariant sets described by a particular parameterization. The main observation is that if we have smooth maps $\Phi : \mathcal{U} \subset \mathbb{R}^d \to \mathbb{R}^n$, and $f : \mathcal{U} \subset \mathbb{R}^d \to \mathbb{R}^d$, where $d \leq n$ and $\mathcal{U} \subset \mathbb{R}^d$ is open, such that

$$F(\Phi(u)) - \Phi(f(u)) = 0 \ , \tag{2}$$

then the manifold parameterized by Φ is defined by $\mathcal{W} = \{\Phi(u) \in \mathbb{R}^n \mid u \in \mathcal{U} \subset \mathbb{R}^d\}$. This manifold is invariant under the dynamical system described by equation (1).[8] The coordinates on this manifold are $u = (u_1, \ldots, u_d)$. That is to say, a point on the manifold $\mathcal{W}$ is parameterized by u_t, $z_t = \Phi(u_t)$, which is mapped onto another point $z_{t+1} = \Phi(f(u_t))$ on the manifold by $u_{t+1} = f(u_t)$. In other words, the dynamical process on the invariant manifold $\mathcal{W}$, which is parameterized by Φ, is described by the map f that describes the dynamics of one point

[5] These theoretical "highways" have been used to design space missions in order to minimize fuel consumption.
[6] See [1] for more details.
[7] Both methodologies have been implemented in Celestial Mechanics, see e.g. [13].
[8] We also assume that Φ is an immersion, that is the rank of its differential is d for all the points in its domain.

in the manifold as time evolves so that: $u_{t+1}{=}f(u_t)$. Note that the dynamics of u_t is restricted to belong to $\mathcal{W}$. Thus, we can think of f as a subsystem of F. In the special case when $d = n$, equation (2) suggests that f is conjugate to F, i.e. $f = \Phi^{-1} \circ F \circ \Phi$.

In this paper, we consider invariant manifolds attached to a fixed point z_* of the system given by equation (1); i.e, $F(z_*){=}z_*$. Local information regarding the underlying dynamics is given by the linearization of F near z_* which is given by:

$$v_{t+1} = Av_t \ , \tag{3}$$

where $A = \mathrm{D}F(z_*)$. These vectors v can be thought as small perturbations near the fixed point and equation (3) describes their evolution.[9] But this linear information is not enough for describing the global dynamics of the system nor even for describing the local dynamics near the fixed point if this has center manifolds (associated to eigenvalues of modulus 1).

Given an eigenspace W of dimension d, the goal is to characterize the linear and non linear components of the associated invariant manifold $\mathcal{W}$ and describe the dynamics on it. Thus the problem of finding the invariant manifolds for F is equivalent to solving the functional equation given by (2) for both Φ and f.[10] This system of invariance equations is infradetermined; i.e, we have $n + d$ unknowns, the components of the maps Φ and f, and just n equations, the components of $F \circ \Phi - \Phi \circ f$. Hence, we cannot expect a unique solution for equation (2). Thus, we can parameterize the invariant manifold $\mathcal{W}$ using two basics techniques: the *graph* and the *parameterization* approach.

In the *graph* method, one considers a splitting of the z variables (possibly after a linear change of coordinates), say $z = (x, y)$ where $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{n-d}$. Then one tries to find a parameterization of the form $\Phi(u) = (u, \psi(u))$, where $\psi : \mathcal{U} \to \mathbb{R}^{n-d}$. Hence, the invariant manifold is described (locally) as a graph given by $y = \psi(x)$. Notice that if we denote F_x and F_y as the x and y components of F, respectively, the invariance equation can be rewritten as follows:

$$\psi(F_x(x, \psi(x))) = F_y(x, \psi(x)) \ .$$

Moreover, the dynamics on the manifold is given by $f(x) = F_x(x, \psi(x))$. Notice also that in this case we have $n - d$ equations and $n - d$ unknowns (the components of ψ).

In the *parameterization* method, one fixes f and then solves equation (2) for Φ. In this case, we have n equations and n unknowns (the components of Φ). But how do we determine f? The main idea in [7] is to choose f *as simple as possible*. For instance, in case that the manifold is attached to a steady state, we could try to find f just as a linear mapping $f(u) = A_1 u$. The method has the flavor of the Theory of Normal Forms, described by Poincaré, since one tries to find a normal form for the dynamics on the invariant manifold.[11] Notice that normal forms correspond to the particular case $d = n$.

As we can see, the *parameterization* method looks for an adapted parameterization of the invariant manifold, while the graph method represents the manifold as a graph. Thus the graph method is more rigid than the parameterization method, because one cannot consider (local) manifolds with folds, that is cannot deal with

[9] The eigenspaces of A are linear manifolds invariant under equation (3).

[10] In [7] several important results on existence of a variety of manifolds attached to fixed points are obtained. Some numerical implementations and applications to macroeconomic models appear in [9] and [10].

[11] See [1] for an exposition of the theory.

manifolds that are correspondences. On the other hand, the parameterization method is more flexible because is able to compute invariant manifolds described by correspondences.

2.1.1 Formal computations

There are several aspects one can consider when studying the invariance equation given by (2). For example, one can consider the known terms of F and the unknowns terms of Φ and f as multivariate power series. Thus, as long as one is able to solve the equations formally or algebraically, one is interested in knowing if these expansions correspond to truly Taylor series of smooth functions or even if the Taylor series are convergent.

Since these power series represent the Taylor series of the corresponding mappings Φ and f, in principle having as many terms as possible can provide us with very accurate approximations of the invariant manifolds. Notice that, in general, just linear approximations (that are given by first order Taylor polynomials) give poor approximations valid in very small neighborhoods of the steady state.

Let us start now with the main ideas and introduce some notation. For the sake of simplicity we assume that the fixed point z_* is at the origin; i.e., $F(0) = 0$. The Taylor series of F around the fixed point is of the form:

$$F(z) = Az + \sum_{k\geq 2} F^{[k]}(z) \ , \tag{4}$$

where each term $F^{[k]}(z)$ is an n-vector whose components are homogeneous polynomials of order k in n variables $z = (z_1, \ldots, z_n)$. These homogeneous polynomials are the *normalized derivatives* of order k given by:

$$F^{[k]}(z) = \frac{1}{k!}\mathrm{D}^k F(0)(z, \overset{k}{\ldots}, z) \ .$$

In particular, $F^{[1]}(z) = \mathrm{D}F(0)z = Az$. We will also use the notation:

$$F^{[k]}(z) = \sum_{|\ell|=k} F_\ell z^\ell,$$

where the coefficients F_ℓ are n vectors, and the subindices are n-tuples $\ell = (\ell_1, \ldots, \ell_n)$ of order $k = |\ell| = \ell_1 + \cdots + \ell_n$. Here we use the standard multi-index notation $z^\ell = z_1^{\ell_1} \ldots z_n^{\ell_n}$ and also denote $F^{[\leq k]}(z_t) = \sum_{i=1}^k F^{[i]}(z_t)$ as the truncated Taylor series up to order k, that is the Taylor polynomial of degree k.[12]

Thus, the unknown map Φ is an n-vector of power series in d variables, $u = (u_1, \ldots, u_d)$, describing the invariant manifold and the unknown map f is a d-vector of power series in d variables that describes the dynamics on the manifold. Both of these power series have zero constant terms which are given by:

$$\Phi(u) = P_1 u + \sum_{k\geq 2} \Phi^{[k]}(u) \ ,$$

and

$$f(u) = A_1 u + \sum_{k\geq 2} f^{[k]}(u) \ ,$$

where $P_1 = \mathrm{D}\Phi(0)$ is a $n \times d$ matrix and $A_1 = \mathrm{D}f(0)$ is a $d \times d$ matrix.

The goal of this algorithm is then to compute recursively the homogeneous polynomials in $\Phi^{[k]}$ and $f^{[k]}$ from the invariance equation given by (2), $F(\Phi(u)) - \Phi(f(u)) = 0$, starting from the first order P_1 and A_1. At

[12] We also use the notation $F^{[<k]}$ as meaning $F^{[\leq k-1]}$.

first order, one obtains the equation $AP_1 = P_1 A_1$ which emphasizes that the linear manifold W generated by P_1 is invariant under the linearization of F, given by A. The matrix A_1 represents the linear map restricted to such linear manifold W. So, the first order equation is a very natural one, since it emphasizes that the manifold $\mathcal{W}$ is tangent to the linear space W.

In the recursive process, one assumes that has already computed all the terms (the derivatives) of the unknowns Φ and f up to order $k-1$, and wants to compute the terms of order k. If one considers the expansions up to order k of $F(\Phi(u)) - \Phi(f(u)) = 0$ one obtains the following:

$$A\Phi^{[k]}(u) - \Phi^{[k]}(A_1 u) - P_1 f^{[k]}(u) - R^{[\leq k]}(u) + \cdots = 0 \ , \tag{5}$$

where $R^{[\leq k]}(u)$ are terms up to degree k coming from the known terms of $\Phi^{[<k]}(u)$ and $f^{[<k]}(u)$ and the dots represent terms of higher order. The resulting equation that we have to solve at the step k is a linear system given by:

$$A\Phi^{[k]}(u) - \Phi^{[k]}(A_1 u) - P_1 f^{[k]}(u) = R^{[k]}(u) \ , \tag{6}$$

where the unknowns are the coefficients of the homogeneous polynomials in $\Phi^{[k]}(u)$ and $f^{[k]}(u)$.

The linear system given by equation (6) is known in the dynamical systems literature as the *homological equation*. We observe that this system is also infradetermined. It is worth mentioning that the solution of (6) depends only on the properties of the linearizations A and A_1, in particular of their eigenvalues. A first result on the computation of the expansions corresponding to the invariant manifold is the following proposition.

Proposition 1 *Assume that W^1 is an invariant space of A, generated by the columns of the matrix P_1, and let A_1 be the matrix such that $AP_1 = P_1 A_1$, $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of A_1, and $\mu_1, \ldots \mu_{n-d}$ be the rest of eigenvalues completing the spectrum of A.*

- *If $\lambda^m - \mu_j \neq 0$ for all $|m| \geq 2$ and $j = 1, \ldots, n-d$, then the homological equation can be solved, and we can compute all the terms of the expansions of $\Phi(u)$ and $f(u)$ recursively. The solutions are not unique. A particular parameterization is of the graph form:*

$$\Phi(u) = P_1 u + P_2 \psi(u) \ , \tag{7}$$

where P_2 is a $n \times (n-d)$ matrix whose columns complete those of P_1 to a basis of $\mathbb{R}^n$, and ψ is a $(n-d)$-vector of power series.

- *Moreover, if $\lambda^m - \lambda_i \neq 0$ for all $|m| \geq 2$ and $i = 1, \ldots, d$, then the homological equation can be solved even when choosing $f^{[k]}(u) = 0$. Then we can compute all terms of the expansions of Φ be choosing $f(u) = A_1 u$, so that:*

$$F(\Phi(u)) = \Phi(A_1 u) \tag{8}$$

Proof: See Appendix A.

As we can see from this Proposition, there are just algebraic obstructions to either solving the homological equation or reducing the dynamics on the manifold to a linear mapping. These kinds of obstructions are known in the literature as *resonances*. In our context, there are two types of resonances:

- The *primary resonances* are the indices m and j, with $|m| \geq 2$ and $j = 1, \ldots, n-d$, such that $\lambda^m - \mu_j = 0$. These are the obstructions to the solution of the homological equation, and to the existence of the invariant manifold $\mathcal{W}$.

- The *secondary resonances* are the indices m and i, with $|m| \geq 2$ and $i = 1, \ldots, d$, such that $\lambda^m - \lambda_i = 0$. These are obstructions to the linearization of the dynamics on the invariant manifold $\mathcal{W}$.

These two types of resonances are very different. On the one hand, the absence of primary resonances let us easily construct the invariant manifold. In particular, a solution is guaranteed if the eigenvalues of A_1 have modulus smaller than one (stable manifold), greater than one (unstable manifold), or equal to one (center manifold). So, the methodology covers the standard cases.[13]

On the other hand, the secondary resonances are the ones that appear in the theory of normal forms.[14] These are obstructions to the linearization of the dynamics on the invariant manifold, but not on its construction. Notice that if the eigenvalues λ_i of A_1 are all of them of modulus smaller (bigger) than one, the number of possible secondary resonances is finite, and we can reduce the dynamics to a polynomial. This is the Poincaré-Dulac normal form of the dynamics on the invariant manifold.

In summary, what we have shown so far is that in order to fully characterize the invariant manifold we need both a representation of the manifold itself as well as the dynamics on it. The algorithm presented in this section leads to a simplification of a finite part of the Taylor expansion of the system at the origin, by means of suitable change of coordinates. On this simpler system, we perform the analysis to gain as much information about the dynamics as possible. In this way we can obtain an asymptotic approximation of the motion of the original system. We are then able to characterize non linear approximations which allows to draw conclusions about the "true" solutions to the invariant manifold and the dynamics on it.[15]

2.1.2 Globalization of the invariant manifold

Once we have computed a local approximation of the invariant manifold, we have to extend or globalize it. In order to do so, we first determine a domain for which the approximation given by the power series is accurate. Since the proposed algorithm employs Taylor series expansions, we have an idea of the associated error of the approximations by looking at higher order terms.

Let $z = \tilde{\Phi}(u)$ be an approximation to the invariant manifold $z = \Phi(u)$, say up to order k, and $u_t = \tilde{f}(u_t)$ an approximation to the dynamics on the manifold $u_{t+1} = f(u_t)$. Since the coefficients of the Taylor polynomials

[13] But, as is emphasized in [7], there are other cases of interest. One can e.g. consider invariant manifolds corresponding to the less contracting directions, called *slow manifolds*, as long as there are no primary resonances.

[14] Notice that one could consider the case $d = n$. So, the theory of normal forms is included in this framework. We emphasize that in the usual way this theory is presented, see e.g. [1], the recursive procedure is done by using a sequence of transformations, instead of computing just one single transformation recursively. The methodology presented here is then more suitable for computer implementation.

[15] There are several results in the dynamical systems literature that associate "true" invariant manifolds to the expansions given above. These results are the classical theorems of the stable/unstable manifolds and the center manifold (see e.g. [12]), and the more recent results on non-resonant manifolds using the parameterization method [7], that include the stable/unstable manifolds theorem. See [4] for proofs of how the coefficients increase for the case of the one dimensional center manifold of a parabolic-hyperbolic steady state.

are affected by computer round off errors, we prefer writing $\tilde{\Phi}(u)$, $\tilde{f}(u)$ instead of $\Phi^{[\leq k]}(u)$, $f^{[\leq k]}$ to explicitly take into account this fact.

Fixing an arbitrary small tolerance $\varepsilon>0$, we define the corresponding domain of validity by:

$$D_\varepsilon = \{u \in \mathbb{R}^d \mid \|F(\tilde{\Phi}(u)) - \tilde{\Phi}(\tilde{f}(u))\| \leq \varepsilon\} ,$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^n$, for instance the supremum norm.

In order to extend the invariant manifold and capture more information regarding the non linear properties we iterate the points of this domain. If the manifold we are calculating is attracting, we iterate using F^{-1}. On the other hand, whenever it is repelling, we use F. If there are eigenvalues of modulus 1, we have a center manifold, we have to make an analysis of the higher order terms of the manifold in order to know the stability properties of the center manifold. In the next section we explicitly address this situation in the simplest case. We emphasize that, even in cases where the dynamics on the center manifold is attracting or repelling (although with a small), the slowness of the motion makes more difficult the process of globalization, so high order local expansions are very useful in enlarging the domain of validity.

2.1.3 Stability analysis on an invariant manifold

The first order terms of a stable or unstable invariant manifold are able to give enough information to determine if trajectories tend to or escape from the steady state. If the eigenvalues of A_1 are all of moduli smaller than 1, the manifold associated to such a subspace is attracting, and the dynamics on the manifold is asymptotically stable to the fixed point. An important particular case is when the fixed point is attracting, with all the eigenvalues of modulus smaller than one, and A_1 is associated to the bigger eigenvalue. This corresponds to the case of a slow manifold that captures the dynamics near the fixed point. Similar considerations can be done if the eigenvalues of A_1 have modulus bigger than one.

In the case that A_1 contain eigenvalues of modulus equal to 1, existence of a center manifold, the situation is more difficult because the first order term says nothing about the stability in the neutral directions. Even if we want to have a very "rough" approximation of the underlying dynamics, one needs to consider non linear terms of the invariant manifold. Thus we need to consider higher order terms.

An illuminating example is the case in which A_1 is a one dimensional matrix so that $A_1 = 1$, and the eigenvalues of A_2 are all of modulus different from 1. In this case, the graph method and the parameterization method for computing the center manifold are equivalent, since all the terms are *secondary resonances*.[16] Hence, the dynamics on the center manifold is of the form:

$$f(u) = u + f_2 u^2 + f_3 u^3 + \dots$$

where f_N is the first non-zero term in the expansion of f. Notice that if we just consider up to first order we have that $f(u) = u$ and all points are fixed (at first order), a non generic situation. Thus, we need to consider higher order terms to determine the stability of the center manifold. Let us assume that f_N is the first non-zero term of the expansion of f, so it is of the form $f(u) = u + f_N u^N + \dots$. In this formulation, the

[16] If one uses a more suitable parameterization method, one can get a polynomial dynamics on the center manifold of the form $f(u) = u + a\ u^N + b\ x^{2N-1}$. This refinement appears in [3] and [4], using normal form techniques that go back to [14].

stability of the dynamics on the center manifold relies on the sign of the coefficient f_N and the parity of N. If N is even the manifold is attracting in one branch and repelling in the other branch. If N is odd and if $f_N > 0$ the center manifold is repelling and if $f_N < 0$ the manifold is attracting.

When the one dimensional center manifold is attached to the eigenvalue -1, then using the graph method one obtains an expression for the dynamics of the form:[17]

$$f(u) = -u + f_2 u^2 + f_3 u^3 + \dots$$

The analysis of the stability of the motion in the center manifold reduces to the previous case by just considering the composition; that is:

$$f \circ f = f^2(u) = u - 2(f_3 + f_2^2)u^3 + \dots$$

For instance, if $f_3 + f_2^2 > 0$ the center manifold is attracting and if $f_3 + f_2^2 < 0$ the center manifold is repelling. If $f_3 + f_2^2 = 0$ we have to compute higher order terms to know the stability on the center manifold.

In summary, in order to determine how the system is going to evolve near the center manifold higher order approximations regarding the dynamics on the manifold are always needed. A linear analysis is not going to be able to determine if orbits near the center manifold are going to tend or move away from it. As a result, a linear approximation is not going to describe the time series of the underlying macroeconomic observables near a center manifold.

2.2 Preserved quantities and invariant manifolds

In the previous section we have considered invariant manifolds that are described using a suitable parameterization. We can also consider manifolds that are defined implicitly by a set of equations. To do so, notice that given a smooth map $H : \mathcal{B} \subset \mathbb{R}^n \to \mathbb{R}^{n-m}$, and given by: $H_0 \in \mathbb{R}^{n-m}$, the level set

$$\Sigma_{H_0} = \{z \in \mathcal{B} \mid H(z) = H_0\} \tag{9}$$

is an m-dimensional manifold if for all each points $z \in \Sigma_{H_0}$ the rank of the differential $\mathrm{D}H(z)$ is maximal (i.e., $n - m$).[18] Then, just notice that the level set Σ_{H_0} is invariant under the dynamical system (1) iff

$$H(z) = H_0 \Rightarrow H(F(z)) = H_0 \ .$$

One way of creating invariant level sets is finding *preserved quantities* for the dynamical system. We say that the map H is a preserved quantity for the dynamical system given by (1) iff for all $z \in \mathcal{B}$, $H(F(z)) = H(z)$. As a result, given an initial state z_0 of the dynamical system, its motion $z_t = F^t(z_0)$ is restricted to the level set of $H_0 = H(z_0)$.

The existence of preserved quantities is useful since it can reduce the dimension of the problem. The computation of preserved quantities for a particular system is far from being trivial, and it is very model dependent. Historically, they have been used in continuous dynamical systems in order to look for explicit

[17] If one uses the parameterization method explained in this paper, one obtains an expansion containing only odd terms.

[18] This is just an straightforward application of the Implicit Function Theorem.

solutions when there are enough preserved quantities.[19] Possibly the most important family of differential equations with preserved quantities are the Hamiltonian systems, defined by a Hamiltonian function that is the preserved quantity.[20]

The existence of preserved quantities restricts also the regions of phase space in which asymptotic invariant manifolds can exist. For instance, the points on the stable and unstable manifolds of a steady state z_* belong to the level set of z_*. This is a consequence (in fact, a particular case) of the following result.

Proposition 2 *Let $H : \mathcal{B} \to \mathbb{R}^{n-m}$ be a smooth map, preserved by the dynamical system* (1)*. Let $z_* \in \mathcal{B}$ be an steady state of* (1)*. Then, for all $z \in \mathcal{B}$ s.t. $\lim_{t\to+\infty} F^t(z) = z_*$ or $\lim_{t\to-\infty} F^t(z) = z_*$, then $z \in \Sigma_{H_*}$ where $H_* = H(z_*)$.*

Proof: See Appendix B.

As we can see, finding preserved quantities in a dynamical system can substantially reduce the computational costs of obtaining the invariant manifolds.

3 Application to an economic model

In order to illustrate the methodology presented in the previous sections, we study ABS economy. The ABS model addresses the following important monetary questions: (1) Do we "need" the government to provide money, or can we rely on "the market" to produce a well-functioning monetary arrangement? (2) Does an efficient monetary system require a mix of private and government money, or should the government be a monopoly provider of currency and close currency substitutes?

Besides its intrinsic importance, the ABS economy is a four dimensional macroeconomic model and is an ideal testing ground because it delivers steady states with stable and unstable manifolds (of dimensions 1 or 2), and each of them has also a one dimensional center manifold. Moreover, the economic model can be transformed so that we can obtain close form solutions for the underlying dynamics on the center manifold. As a result, we can easily check the accuracy and performance of our algorithm.

Let us first briefly describe the ABS economic environment. The economy is inhabited by heterogeneous agents with spatial separation and limited communication where privately-issued liabilities may circulate, either by themselves, or alongside a stock of outside money. These agents in the ABS economy live for three periods and inhabit in two different locations. All young generations are identical in size and composition. In each period agents are endowed with a single non storable consumption good, e_j for lenders and w_j for borrowers where the index j denotes the age of the endowment good. For simplicity, ABS assume that $e_1 = e>0$, $e_2=e_3=0$, $w_2 = w>0$ and $w_1=w_3=0$, and that $w < e$. Furthermore, lenders and borrowers care only about young and middle aged consumptions.

[19]For instance, the existence of enough preserved quantities in the Two Body Problem about the motion of two particles which interact through gravitation, such as the linear momentum, the angular momentum and the energy, is crucial to integrate the equations and solve the problem and, in particular, derive mathematically the Kepler's laws of motion of the planets around the Sun that were found by observation.

[20]Hamiltonian systems appear in Geometrical Optics and Analytical Mechanics.

In each period, in location 1 there are $N/2$ young lenders and γN young borrowers, where $\gamma \in [1/2, 1)$. In location 2 there are $N/2$ lenders and $(1-\gamma)N$ borrowers.[21] Furthermore, some lenders in this economy may move, lenders who are born in location 1 (2) move to location 2 (1) when middle aged and then remain in the new location until they are old. On the other hand, borrowers spend their entire life in the same location, thus borrowers and lenders that interact when young will never meet again. The ABS framework also considers spatial separation and limited communication across locations. These frictions then imply that trade can only occur between agents who are in direct contact with each other. In particular, young borrowers in any location would like to acquire resources from young lenders.[22]

The economic environment in ABS allows then the possibility of having two kind of liabilities: one period and two period maturities. At each date there will be newly issued liabilities of each type in both locations. In particular, trade must occur as follows: young borrowers acquire resources at time t from young lenders in exchange for claims. Lenders take these claims to their next location at $t+1$ to the next generation of lenders in exchange for goods. At $t+2$ these now newly middle aged lenders bring the claims to the original issuers. But how do borrowers redeem these two period claims when old when they have no old age endowment? Well they must acquire claims on resources at $t+1$ when they are middle aged. Thus transactions mediated through circulation liabilities require middle aged borrowers to save in order to redeem their two period circulating liabilities. These features of the environment force intertemporal trading to be intermediated in part by private liabilities issued by borrowers. These are taken to another location and exchanged for goods before they are brought back to the original issuer for redemption.

Summarizing, trade in the ABS model requires that young borrowers issue some long maturity and some short term liabilities which coexist with a constant stock of fiat money. This initial stock of money, which is constant over time, is held by middle aged borrowers. Long term maturity liabilities are sold to young lenders who take them elsewhere and trade them. Short term liabilities are sold to middle aged borrowers who acquire them as a method of honoring their own liability issues or they can hold fiat currency between periods. Short maturity liabilities do not circulate. Note that lenders can hold circulating liabilities of the borrowers and in addition can hold government issued fiat currency. As we can see, fiat money can be viewed as a substitute for the private liabilities issued by borrowers.[23]

3.1 The dynamical system

In order to obtain the underlying dynamical system of the ABS economy, one needs to be a bit more specific regarding tastes as well as market structure in order to derive the aggregate behavior of the economy. The ABS model assumes logarithmic preferences and perfect competition yielding the following 4-dimensional dynamical

[21] Note that by setting $\gamma \geq 1/2$ location 1 is relatively large and the "excess demand" for credit in location 1 is relatively large at any rate of interest.

[22] In the absence of spatial separation and limited communication, the third period would not involve economic activity.

[23] For further details we refer the reader to the original ABS article.

system:

$$R_{(1,1),t+1} = \frac{1}{1+\alpha-\alpha R_{(2,1),t}} = \frac{p_{1,t}}{p_{1,t+1}} \tag{10}$$

$$R_{(2,2),t+1} = \frac{1}{1+\beta-\beta R_{(1,2),t}} = \frac{p_{2,t}}{p_{2,t+1}} \tag{11}$$

$$R_{(1,2),t+1} = R_{(2,2),t+1} \cdot \frac{R_{(2,2),t}}{R_{(2,1),t}} = \frac{p_{1,t}}{p_{2,t+1}} \tag{12}$$

$$R_{(2,1),t+1} = R_{(1,1),t+1} \cdot \frac{R_{(1,1),t}}{R_{(1,2),t}} = \frac{p_{2,t}}{p_{1,t+1}} \tag{13}$$

where $p_{i,t}$ represents the price level for agents in location i at time t, $R_{(i,j),t}$ denotes the gross one period rate of return for the one period liabilities for agents moving from location i at time t to location j at $t+1$, and the parameters α and β are defined as follows: $\alpha = \frac{e}{2\gamma w}$ and $\beta = \frac{e}{2(1-\gamma)w}$. We recall that $\frac{1}{2} \leq \gamma < 1$ and $e > w$, so we have $\alpha \leq \beta$ and $\frac{w}{e} < 1$.

As we can see, the consumption and saving behavior of agents are fully characterized once the different gross one period rate of returns are known. The evolution of the ABS economy is further restricted by the fact that the two locations have different prices and there are no-arbitrage opportunities. Thus the rates of returns on the liabilities are related as follows:

$$\frac{R_{(1,2),t+1}}{R_{(2,2),t+1}} = \frac{R_{(2,2),t}}{R_{(2,1),t}} \ , \ \frac{R_{(2,1),t+1}}{R_{(1,1),t+1}} = \frac{R_{(1,1),t}}{R_{(1,2),t}} \ . \tag{14}$$

These dynamic relations between rates of return across locations impose further restrictions on the temporal evolution of the economic observables predicted by the ABS model economy. In order to characterize the asymptotic properties of the ABS economy, we have to characterize the steady states of the economy and their associated manifolds.

3.2 Steady State Equilibria

In order to find the fixed points of ABS we have to restrict the gross rates of return so that $R_{(i,j),t}$=$R_{(i,j),t+1}$ for a given i, j and for all t. The resulting steady states $(R_{(1,1)}, R_{(2,2)}, R_{(1,2)}, R_{(2,1)})$ satisfy the following relations:

$$R_{(1,2)}R_{(2,1)} = (R_{(1,1)})^2 = (R_{(2,2)})^2 \ . \tag{15}$$

which explicitly relates the gross one period rate of return for agents moving from location 1 at time t to location 2 at $t+1$ to the returns for agents that do not move across locations. In the original paper, ABS characterized the steady states, and also analyzed their linear stability. We state below their findings in Propositions 3 and 4, and we present the proofs in Appendix C.

Proposition 3 *The ABS economy has two steady states:*
(i) A monetary steady state with a common rate return such that

$$R_{(1,1)} = R_{(2,2)} = R_{(1,2)} = R_{(2,1)} = 1 \ ,$$

and a common price level in both locations;

(ii) a non monetary steady state with rates of return

$$R_{(1,1)} = R_{(2,2)} = R_* \ , R_{(1,2)} = \frac{1}{\beta}\left(1+\beta-\frac{1}{R_*}\right) \ , R_{(2,1)} = \frac{1}{\alpha}\left(1+\alpha-\frac{1}{R_*}\right) \ ,$$

where R_* is the only root of the cubic equation

$$\alpha\beta R^3 + \alpha\beta R^2 - (1+\alpha+\beta)R + 1 = 0 \tag{16}$$

that is bigger than $\frac{1}{1+\alpha}$. Moreover: $\frac{w}{e} \le R_* < 1$.

Proof: See Appendix C.

As we can see, Proposition 3 highlights the fact that the non monetary steady state corresponds to $R_{(1,1)} = R_{(2,2)} = R_*$, with $\max\left(\frac{1}{1+\alpha}, \frac{w}{e}\right) \le R_* < 1$. Furthermore, notice that $\frac{1}{1+\alpha} < \frac{w}{e}$ if and only if $1 < \frac{w}{e} + \frac{1}{2\gamma}$. Moreover, in the symmetric case $\gamma = \frac{1}{2}$, $\alpha = \beta = \frac{e}{w}$, the non monetary steady state is given by $R_{(1,1)} = R_{(2,2)} = R_{(1,2)} = R_{(2,1)} = \frac{w}{e} > \frac{w}{w+e} = \frac{1}{1+\alpha}$.

The results on the linear stability of both monetary and non monetary steady states are summarized by the following proposition.

Proposition 4 *The steady states of the ABS economy satisfy:*

- *The four eigenvalues of the Jacobian in the monetary steady state corresponding to $R = 1$ are real. Moreover, it has one central eigenvalue $\lambda_c = -1$, one stable eigenvalue $\lambda_s \in]-1, 0[$, and two unstable eigenvalues $\lambda_{u,1} < \lambda_{u,2} \in]1, \infty[$. Moreover, $\lambda_{u,1} = 1/R_*$.*
- *The four eigenvalues of the Jacobian in the non monetary steady state corresponding to $R = R_*$ are real. Moreover, it has one central eigenvalue $\lambda_c = -1$, two stable eigenvalues $\lambda_{s,1} \in]-1, 0[$ and $\lambda_{s,2} \in]0, 1[$ and one unstable eigenvalue $\lambda_u \in]1, \infty[$. Moreover, $\lambda_{s,2} = R_*$.*

Proof: See Appendix C.

As we can see, regardless of the specific calibration both monetary and non monetary steady states have a center manifold since -1 is one of the eigenvalues. Moreover, Proposition 4 emphasizes that the monetary steady state has a 1-dimensional center manifold, a 1-dimensional stable manifold and a 2-dimensional unstable manifold. Similarly, the non monetary steady state has a 1-dimensional center manifold, a 2-dimensional stable manifold and a 1-dimensional unstable manifold. The underlying dynamics of the macroeconomic observables of the ABS system are affected by the properties of these invariant manifolds. A local analysis will not able to determine how points near the center manifold evolve over time. In order to do so we need to consider higher order terms of the associated manifolds.

3.3 Invariant manifolds: formal expansions

Now that we have identified the steady states of the ABS system we would like to characterize the associated manifolds. The underlying manifolds can give us an idea of the potential predicted time series as we move away from the steady states. In this section we compute high order approximations of the invariant manifolds associated to the two steady states of the ABS system.

To be more specific, we fix the parameters so that e =1, w =0.3 and γ =0.6 as in the original ABS model.[24] The corresponding coordinates of the monetary steady state are z_{m}=(1.000000, 1.000000, 1.000000, 1.000000) which correspond to the different interest rates. The associated eigenvalues are λ_s=-0.6679793, λ_c=-1.000000, $\lambda_{u,1}$=3.203277, $\lambda_{u,2}$=5.409147. Notice that if $\lambda_{u,1}^m = \lambda_{u,2}$, then $m = 1.450033$, that is not an integer number greater or equal than 2. As a result, there are no secondary resonances on the unstable manifold and we can reduce the dynamics on such a manifold to a linear dynamics given by the diagonal matrix diag$(\lambda_{u,1}, \lambda_{u,2})$. Moreover, apart from the classical fast (or strong) unstable manifold associated to $\lambda_{u,2}$, we can also construct a slow unstable manifold associated to $\lambda_{u,1}$. We emphasize that for parameters close to those given above the dynamics on the unstable manifold is also linearizable, and it also contains fast and strong unstable manifolds.

The coordinates of the non monetary steady state are z_{n}=(0.3121803, 0.3121803, 0.4712136, 0.2068204) which correspond to the different interest rates. The associated eigenvalues are $\lambda_{s,1}$=-0.2085300, $\lambda_{s,2}$=0.3121803, λ_c=-1.000000, λ_u= 1.688629. Since there are no resonances between the stable eigenvalues, that is there is no an integer $m \geq 2$ such that $\lambda_{s,2}^m = \lambda_{s,1}$, the dynamics on the stable manifold is linearizable. Moreover, we can also associate fast stable and slow stable manifolds of dimension 1 to the eigenvalues $\lambda_{s,1}$ and $\lambda_{s,2}$, respectively. Notice also that the dynamics on the fast stable manifold converges oscillatorially to the non monetary steady state.

Both steady states have one dimensional center manifolds, associated to the eigenvalue $\lambda_c = -1$. Thus orbits in the center manifold oscillate around the corresponding steady state, but we do not know their asymptotic behavior; i.e, if they converge or not to the steady state. In order to determine the behavior of the motion on the center manifold, a study of higher order terms is mandatory.

Table 1 displays the coefficient of the Taylor series up to 10^{th} order both for the dynamics and parameterization of the stable manifold of the monetary steady state.[25] Table 2 shows the corresponding results for the center manifold.[26]

Since we use the parameterization method, the numerical computation produces linear dynamics on the stable manifold; i.e., $f(u) = \lambda_s u$, which is reflected in the second column of Table 1. In the last two columns of Table 1 we give an idea of the accuracy of the expansions. The column labeled "domain" corresponds to the validity domain of the expansion up to order k with a tolerance $\varepsilon = 10^{-6}$. "Length" represents the length of the corresponding curve; i.e., the image under $\Phi^{[\leq k]}$ of the validity domain. Notice that both the interval and the length grow as we increase k. All computations are obtained in double precision arithmetics. Moreover, the Taylor series coefficients corresponding to the different manifolds reported in the different Tables are obtained using the parameterization method.

Table 2 presents the coefficients of the expansions corresponding to the center manifold for the monetary steady state. Since we use the parameterization method, in principle the expansion of the dynamics of the center manifold is of the form: $f(u) = -u + f^{[3]}u^3 + f^{[5]}u^5 + \dots$, that is only has odd order terms. Our numerical results yield coefficients $f^{[3]}$,$f^{[5]}$,..., smaller than 10^{-18}, below the round-off error of the computer.

[24]The dynamical properties associated with this example are not exclusive to these particular parameters, and the methodology can be applied to other calibrations.

[25]Even though we have computed the one and two dimensional invariant manifolds of the monetary and non monetary steady states up to order 30.

[26]We emphasize that all the computation of the expansions have been carried out in less than one second using nowadays computers, thanks to the high performance of our implementations.

k	$f^{[k]}$	$\Phi_1^{[k]}$	$\Phi_2^{[k]}$	$\Phi_3^{[k]}$	$\Phi_4^{[k]}$	domain	length
0	0.000000e+00	0.000000e+00	0.000000e+00	0.000000e+00	0.000000e+00		
1	-6.679793e-01	1.000000e+00	0.000000e+00	0.000000e+00	0.000000e+00	[-1.2500e-03,1.2500e-03]	2.5000e-03
2	0.000000e+00	-3.462864e-02	2.681350e-01	2.602246e-02	9.633629e-03	[-3.2500e-02,3.2500e-02]	6.5004e-02
3	0.000000e+00	7.624036e-02	4.967390e-03	4.820844e-04	1.784698e-04	[-7.9375e-02,7.9375e-02]	1.5888e-01
4	0.000000e+00	-1.249909e-03	2.070675e-02	2.009587e-03	7.439578e-04	[-1.9875e-01,1.9938e-01]	4.0016e-01
5	0.000000e+00	5.838347e-03	7.655094e-04	7.429256e-05	2.750343e-05	[-2.8500e-01,2.8875e-01]	5.7989e-01
6	0.000000e+00	1.206414e-05	1.606155e-03	1.558771e-04	5.770639e-05	[-4.4188e-01,4.4750e-01]	9.1231e-01
7	0.000000e+00	4.490866e-04	8.860896e-05	8.599485e-06	3.183567e-06	[-5.3375e-01,5.4563e-01]	1.1207e+00
8	0.000000e+00	9.247159e-06	1.251258e-04	1.214344e-05	4.495554e-06	[-6.9063e-01,7.0563e-01]	1.4865e+00
9	0.000000e+00	3.469793e-05	9.130467e-06	8.861103e-07	3.280419e-07	[-7.7188e-01,7.9625e-01]	1.6972e+00
10	0.000000e+00	1.353743e-06	9.789055e-06	9.500261e-07	3.517038e-07	[-9.1500e-01,9.4563e-01]	2.0792e+00

Table 1: Coefficients corresponding to the stable manifold of the monetary steady state.

Similar results are obtained when computing the expansions of the center manifold for the non monetary steady state.

k	$f^{[k]}$	$\Phi_1^{[k]}$	$\Phi_2^{[k]}$	$\Phi_3^{[k]}$	$\Phi_4^{[k]}$	domain	length
0	0.000000e+00	0.000000e+00	0.000000e+00	0.000000e+00	0.000000e+00		
1	-1.000000e-00	0.000000e+00	1.000000e+00	0.000000e+00	0.000000e+00	[0.0000e+00,0.0000e+00]	0.0000e+00
2	0.000000e+00	8.000741e-02	2.479119e-01	-1.022280e-01	-2.534513e-02	[-6.8750e-03,6.8750e-03]	1.3750e-02
3	-1.084202e-19	-1.215280e-02	0.000000e+00	-6.413486e-02	-2.313605e-02	[-3.0000e-02,3.0000e-02]	5.9999e-02
4	0.000000e+00	-2.243779e-02	-4.342071e-02	-5.396960e-02	-1.596927e-02	[-6.2500e-02,6.2500e-02]	1.2499e-01
5	-5.590417e-20	1.490997e-03	0.000000e+00	-2.407681e-02	-8.927910e-03	[-1.0875e-01,1.0938e-01]	2.1809e-01
6	0.000000e+00	9.379552e-03	-6.533255e-04	-2.654547e-02	-7.096582e-03	[-1.4875e-01,1.4875e-01]	2.9733e-01
7	-2.879912e-20	-1.431981e-03	0.000000e+00	-1.460054e-02	-5.535222e-03	[-1.9938e-01,2.0000e-01]	3.9901e-01
8	0.000000e+00	-2.383643e-03	-7.533388e-03	-1.774142e-02	-4.924297e-03	[-2.3875e-01,2.3813e-01]	4.7611e-01
9	-2.202286e-20	-1.753388e-05	0.000000e+00	-9.535741e-03	-3.649439e-03	[-2.8438e-01,2.8563e-01]	5.6893e-01
10	0.000000e+00	1.720851e-03	-2.636964e-03	-1.256799e-02	-3.372125e-03	[-3.1813e-01,3.1688e-01]	6.3316e-01

Table 2: Coefficients corresponding to the center manifold of the monetary steady state.

The numerical analysis then suggests that the dynamics on both center manifolds is of the form $f(u) = -u$; i.e, the dynamics is purely oscillatory. Thus, orbits on the center manifold are two periodic.[27] This fact is rigorously proved in Proposition 6.

3.4 The 2-periodic system and dynamics on the center manifolds

The previous numerical results suggest that the dynamics on both center manifolds of the two steady states are 2-periodic. In other words, there are curves (the center manifolds) of 2 periodic points. In this section we prove that this insight is correct. Before proving this fact, we also derive some interesting properties of the ABS model.

[27] In their original paper, Azariadis, Bullard and Smith (2001) studied numerically the stability of the dynamics along the center manifolds for 1000 economies chosen randomly in $e = 1, w \in]0,1[, \gamma \in [\frac{1}{2}, 1[$ for both steady states. The authors claim that for all the cases they studied, the dynamics on the center manifold of the monetary is asymptotically stable. In particular, they claim that trajectories on the center manifold approach the steady state with a oscillatory motion, and the oscillations may only dampen very slowly over the time. This result is, as we will see, inaccurate. The dynamics on the center manifold is stable, but not asymptotically stable. In fact, it is purely oscillatory both for the monetary and non monetary steady states.

Let us start by considering the 2-period ABS system, the ABS^2 system henceforth, which is given by:

$$R_{(1,1),t+2} = \frac{1}{1+\alpha - \frac{\alpha}{1+\alpha-\alpha R_{(2,1),t}} \cdot \frac{R_{(1,1),t}}{R_{(1,2),t}}} \ , \tag{17}$$

$$R_{(2,2),t+2} = \frac{1}{1+\beta - \frac{\beta}{1+\beta-\beta R_{(1,2),t}} \cdot \frac{R_{(2,2),t}}{R_{(2,1),t}}} \ , \tag{18}$$

$$R_{(1,2),t+2} = R_{(2,2),t+2} \cdot \frac{1+\alpha-\alpha R_{(2,1),t}}{1+\beta-\beta R_{(1,2),t}} \cdot \frac{R_{(1,2),t}}{R_{(1,1),t}} \ , \tag{19}$$

$$R_{(2,1),t+2} = R_{(1,1),t+2} \cdot \frac{1+\beta-\beta R_{(1,2),t}}{1+\alpha-\alpha R_{(2,1)),t}} \cdot \frac{R_{(2,1),t}}{R_{(2,2),t}} \ . \tag{20}$$

The fixed points of the ABS^2 system correspond to 2-periodic orbits of the ABS system. We now prove that the ABS^2 has a preserved quantity. In other words, the ABS^2 system has several 3-dimensional invariant manifolds which are the levels sets of the preserved quantity. Among them, there is one 3-dimensional manifold that is in fact invariant under the action of the ABS system.

Proposition 5 *The 2-period value of* $Q = \frac{R_{(1,1)}R_{(2,2)}}{R_{(1,2)}R_{(2,1)}}$ *is constant along the evolution of the economy. That is to say, for each positive value of a constant* s*, the 3-dimensional manifold* Σ_s *defined implicitly by:*

$$\frac{R_{(1,1)}R_{(2,2)}}{R_{(1,2)}R_{(2,1)}} = s \ ,$$

is invariant under the ABS^2 *system. Moreover, if for a given time* t *we have that*

$$\frac{R_{(1,1),t}R_{(2,2),t}}{R_{(1,2),t}R_{(2,1),t}} = s$$

then

$$\frac{R_{(1,1),t+1}R_{(2,2),t+1}}{R_{(1,2),t+1}R_{(2,1),t+1}} = \frac{1}{s} \ .$$

The manifold Σ_s *is mapped onto* $\Sigma_{s^{-1}}$ *under the action of ABS system. In particular,* Σ_1 *is invariant under the ABS system.*

Proof: See Appendix C.

Note that $Q = \frac{R_{(1,1)}R_{(2,2)}}{R_{(1,2)}R_{(2,1)}}$ is a measure indicating if the arbitrage condition is satisfied in the ABS model or not. Given the underlying assumptions of the ABS model, economies that satisfy $Q \neq 1$ are not an equilibrium since the no arbitrage condition does not hold. The only relevant preserved quantity consistent with the ABS environment is then when $Q = 1$ which yields a 3-dimensional invariant manifold Σ_1.

The steady states of the ABS^2 system correspond to the 2-periodic states of the ABS system that satisfy the following equations:

$$R_{(1,1)} = \frac{1}{1+\alpha - \frac{\alpha}{1+\alpha-\alpha R_{(2,1)}} \cdot \frac{R_{(1,1)}}{R_{(1,2)}}} \tag{21}$$

$$R_{(2,2)} = \frac{1}{1+\beta - \frac{\beta}{1+\beta-\beta R_{(1,2)}} \cdot \frac{R_{(2,2)}}{R_{(2,1)}}} \tag{22}$$

$$R_{(1,2)} = R_{(2,2)} \cdot \frac{1+\alpha-\alpha R_{(2,1)}}{1+\beta-\beta R_{(1,2)}} \cdot \frac{R_{(1,2)}}{R_{(1,1)}} \tag{23}$$

$$R_{(2,1)} = R_{(1,1)} \cdot \frac{1+\beta-\beta R_{(1,2)}}{1+\alpha-\alpha R_{(2,1))}} \cdot \frac{R_{(2,1)}}{R_{(2,2)}} \ . \tag{24}$$

Note that the equations (23) and (24) are both equivalent to the following equation:

$$R_{(1,1)}\left(1+\beta-\beta R_{(1,2)}\right)=R_{(2,2)}\left(1+\alpha-\alpha R_{(2,1)}\right) \tag{25}$$

which reemphasizes the fact there is a relation between the different gross interest rates across locations that are preserved every two periods. Note then that we have four unknowns and three equations (21), (22) and (25), that (possibly) define a 1-dimensional object in the 4-dimensional space, formed by the 2-periodic orbits of the ABS system. In order to determine a solution, we need an extra condition. A natural one is to fix a level set Σ_1, that is invariant under the ABS^2 system. So, on each level set Σ_1 we will look for 2-periodic orbits of the ABS system as suggested by the following proposition.

Proposition 6 *The 2-periodic orbits in the manifold Σ_s are obtained by solving the following equations*

$$(1+\alpha)(1+\beta)R_{(1,2)}R_{(2,1)}s=\left(1+\frac{\alpha R_{(2,1)}s}{1+\beta-\beta R_{(1,2)}}\right)\left(1+\frac{\beta R_{(1,2)}s}{1+\alpha-\alpha R_{(2,1)}}\right)\ , \tag{26}$$

$$(1+\beta)(1+\beta-\beta R_{(1,2)}+\alpha sR_{(2,1)})=(1+\alpha)(1+\alpha-\alpha R_{(2,1)}+\beta sR_{(1,2)})\ , \tag{27}$$

for $R_{(1,2)}$ and $R_{(2,1)}$. The explicit relations are given by:

$$R_{(1,1)}=\frac{1}{1+\alpha}\left(1+\frac{\alpha R_{(2,1)}s}{1+\beta-\beta R_{(1,2)}}\right)\ , \tag{28}$$

$$R_{(2,2)}=\frac{1}{1+\beta}\left(1+\frac{\beta R_{(1,2)}s}{1+\alpha-\alpha R_{(2,1)}}\right)\ . \tag{29}$$

Proof: See Appendix C.

As we can see, by eliminating, for instance, $R_{(2,1)}$ from (26) and (27), we obtain a quartic equation for $R_{(1,2)}$. This allows us to have an explicit solution for the gross interest rate for agents moving from location 1 at time t to location 2 at $t+1$. Moreover, Proposition 6 shows that the ABS system has curves of 2-periodic points, parameterized by s. These include of course the monetary and non monetary states, that correspond to $s=1$. This is a non generic feature of the ABS system.[28] As a result, the center manifolds of both steady states are contained in these curves, and the dynamics on the center manifolds is purely oscillatory. Moreover, the center manifolds are not relevant in the ABS model under no-arbitrage conditions, since they are transversal to the no-arbitrage manifold Σ_1.

3.5 The reduced system

So far we have shown that by considering the ABS^2 system we are able to find a preserved quantity that can help us simplify the study of its evolution. Moreover, we have seen that the level set Σ_1 is invariant under the ABS system, and contains the monetary and non monetary steady states and their corresponding stable and unstable manifolds. [29] More importantly, given the relationship between rates of return and the no-arbitrage conditions, we have that the initial states of the economy satisfy the following:

$$R_{(1,2),0}=R_{(2,2),0}\frac{p_{(1),0}}{p_{(2),0}}\ ,R_{(2,1),0}=R_{(1,1),0}\frac{p_{(2),0}}{p_{(1),0}}\ ,$$

[28]Notice that this result comes from the fact that the four equations that determine the periodic orbits can be reduced to three equations.

[29]See Proposition 2.

for given initial price levels $p_{(1),0}, p_{(2),0}$. Hence,

$$\frac{R_{(1,1),0}R_{(2,2),0}}{R_{(1,2),0}R_{(2,1),0}} = 1 \ ,$$

and the initial states belong to the level set Σ_1, and as a consequence the rates of return evolve on Σ_1, that is

$$\frac{R_{(1,1),t}R_{(2,2),t}}{R_{(1,2),t}R_{(2,1),t}} = 1 \ ,$$

for all t.

In summary, the evolution of the model under the no-arbitrage conditions takes place in the 3-dimensional manifold Σ_1, thus reducing the dimension of the system. We refer to this invariant manifold Σ_1 as the no-arbitrage manifold.

As a result, we just need to consider three coordinates to represent the dynamics of the ABS model in Σ_1. For instance, we can focus on $(R_{(1,1)}, R_{(2,2)}, R_{(1,2)})$. Moreover, since the stable manifold of the non monetary steady state and the unstable manifold of the monetary steady state are 2-dimensional in a 3-dimensional space, these manifolds separate different types of dynamical behavior of the system.[30] In particular, if these manifolds intersect each other, the intersections would be generically 1-dimensional curves.[31] In the following section we pursue this geometric study.

3.6 Invariant manifolds: globalization

As we have demonstrated in the previous section, the dynamics on the center manifolds is purely periodic. This particular feature of the ABS model is ideal because one can easily check the accuracy and performance of our algorithm. In fact, we emphasize that the rigorous results were inspired by the numerical results. In our numerical computation, we find that all the coefficients corresponding to the dynamics of the center manifold up to 100^{th} order are zero within the numerical accuracy.

We have also seen that, under no-arbitrage conditions of the economy, the relevant invariant object is the 3-dimensional manifold Σ_1. The center manifolds are transversal to Σ_1 and do not affect the dynamics in Σ_1. Moreover, the monetary and non monetary steady states and their corresponding asymptotic manifolds (stable and unstable, fast and slow), are contained in Σ_1. Thus, the transition dynamics, which always lie on Σ_1, is influenced by the stable and unstable manifolds. The dynamics on the 1-dimensional and 2-dimensional non resonant manifolds (stable, unstable, fast and slow) can be reduced to a linear form, even if the manifolds themselves are not linear at all.

The analysis of the non linear invariant manifolds of the ABS economy is summarized in Figure 1 and 2. In Figure 1 we depict all the relevant manifolds of the ABS model inside the 3-dimensional manifold Σ_1, using the coordinates $R_{(1,1)}$, $R_{(2,2)}$ and $R_{(1,2)}$.

The monetary steady state, labeled with m, has a 2 dimensional unstable manifold, colored with red, and a 1 dimensional stable manifold, colored with blue. The slow and fast unstable 1 dimensional manifolds inside

[30] This is thanks to the existence of preserved quantities. In general, these manifolds would be 2-dimensional in a 4-dimensional space.

[31] This is, again, due to the existence of preserved quantities. In general, 2-dimensional manifolds in a 4-dimensional space intersect in O-dimensional points.

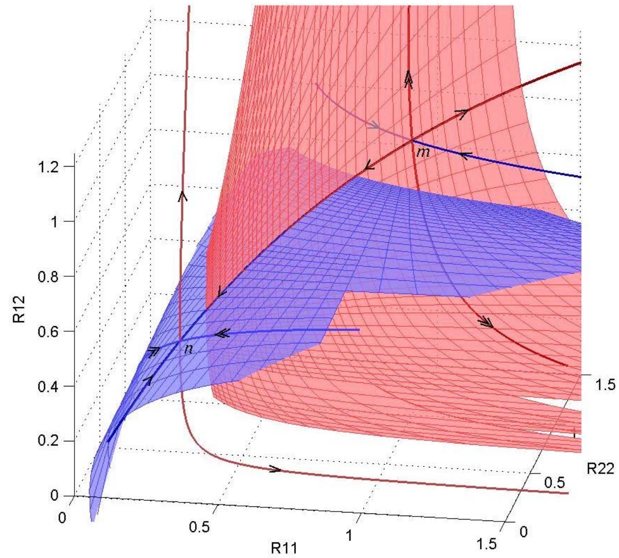

Figure 1: Invariant manifolds of the ABS model.

the unstable manifold are designated with single and double arrows, respectively. The non monetary steady state, labeled with n, has a 2 dimensional stable manifold, colored with blue, and a 1 dimensional unstable manifolds, colored with red. The slow and fast stable 1 dimensional manifolds inside the stable manifold are designated with single and double arrows, respectively.

As in ABS, the coexistence of publicly and privately issued liabilities is not a source of indeterminacy only when we are near the monetary steady state. Indeterminacies do arise in a neighborhood of the non monetary steady state. The global analysis of the invariant manifolds of the ABS economy predicts a new source for equilibrium indeterminacy previously not found by the original paper. In particular, as we can see from Figure 1, the unstable manifold of the monetary steady and the stable manifold of the non monetary steady state intersect in a curve. The points on the curve are known as *heteroclinic* points. These points move under the evolution of the economy from the monetary steady state to the non monetary steady state. An interesting fact of the ABS model is that this *heteroclinic* connection between the steady states is constituted by the slow stable manifold of the non monetary steady state and the slow unstable manifold of the monetary steady state. In other words, the 1-dimensional slow manifolds coincide. This is what is observed in Figure 1. As a result, the predicted time series are such that the interest rates are going to decrease over time, generating a new source for equilibrium indeterminacy. To gain more insight, Figure 2 displays the actual time series of the four rates of return of an heteroclinic point.

The rates of return associated with this heteroclinic point are such that these returns move from those in the monetary steady state to those in the non monetary steady state. Thus, *heteroclinic* points predict hyperinflationary equilibrium paths in which money is used at all dates but the price level tends to infinity. The predicted time series of an *heteroclinic* point cannot be generated when performing a local analysis.

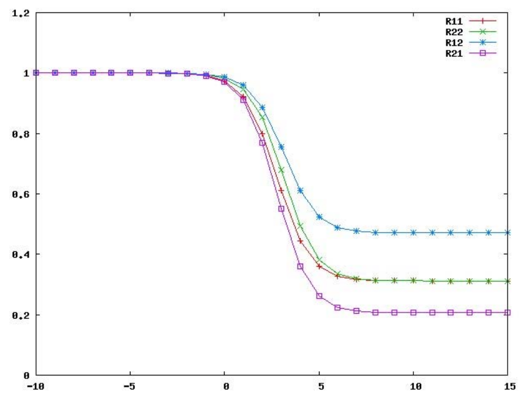

Figure 2: Time series of the rates of return of an heteroclinic point.

When performing a local analysis, hyperinflationary equilibrium paths can only be generated for a very small set of initial conditions; i.e., only those near the non monetary steady state. On the other hand, when we consider a global analysis we observe that there is a much broader set of initial conditions that are consistent with hyperinflationary equilibrium paths. The existence of these *heteroclinic* points allows for an economy to transition from points near a monetary steady state to those in the non monetary steady state, generating indeterminacy of equilibrium. Moreover, these paths are not oscillatory, thus not generating “excess” volatility.

As opposed to the linear results of ABS economy, a global analysis supports the idea that the use of private liabilities is conducive to indeterminacy of equilibrium when valued outside liabilities are present. This new global result is consistent with Friedman [8] who argued that allowing private provision of close currency substitutes is a recipe for generating indeterminacy of equilibrium.

It is apparent from these numerical explorations that once we contemplate the non linear properties of the dynamical system, the corresponding phase space can become quite complicated because of the possible intersections of the stable and unstable manifolds. It is thanks to the non linearities of these manifolds that we can capture new dynamical phenomena not observed when performing a linear analysis.

4 Conclusions

Macroeconomic models are in many cases described by difference or differential equations explicitly defining a dynamical system. In this paper we have presented a unified framework that helps us characterize the associated invariant manifolds of a given dynamical system. In particular, we are able to relate the graph and the parameterization approach as special cases of this general framework. While in the graph method one tries to simplify the local representation of the manifold, in the parameterization method one tries to simplify the local representation of its dynamics.

To illustrate our methodology, we consider a model economy introduced by Azariadis, Bullard and Smith [6] which studies whether the provision of currency should be an activity left strictly to the government or to private agents. We analyze the stability properties of this economy that has a center, stable and unstable manifolds. This model economy is an ideal testing ground for our algorithm because it delivers a center manifold which requires a global analysis. Moreover, the economic model can be transformed so that we can obtain close form solutions for the underlying dynamics on the center manifold. As a result, we can easily check the accuracy and performance of our algorithm.

Finally, we implement our algorithm and compute the corresponding high order approximations of the associated invariant manifolds of the Azariadis, Bullard and Smith [6] model economy. These non linear invariant manifolds are able to generate new predicted time series that cannot be detected when performing a local analysis. In particular, we show the existence of heteroclinic points that predict hyperinflationary equilibrium paths of economies transitioning from the monetary steady state to the non monetary one, generating indeterminacy of equilibrium. This new global result is consistent with Friedman [8] who argued that allowing private provision of close currency substitutes is a recipe for generating indeterminacy of equilibrium.

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A Invariant manifolds and power series

In this section we will extend the explanations given in Section 2.1.1, including the proof of Proposition 1. For ease of exposition and without loss of generality let us consider a change of coordinates so that:

- the origin is a fixed point so that z_*=0 and $F(0)$= 0;
- the eigenspace W is horizontal, that is the Jacobian at the origin $\mathrm{D}F(0)$ is block triangular and is given by:
$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} ,$$
where A_1, A_2 and B are matrices of dimensions $d \times d$, $(n-d) \times (n-d)$ and $d \times (n-d)$, respectively.

To make the splitting of the phase space more evident, we introduce the notation z =(x, y), where x =$(x_1, \ldots, x_d)$, y =$(y_1, \ldots, y_{n-d})$, and rewrite the dynamical system as follows:

$$\begin{aligned} x_{t+1} &= F_1(x_t, y_t) = A_1 x_t + B y_t + \sum_{k\geq 2} F_1^{[k]}(x_t, y_t) \ , \\ y_{t+1} &= F_2(x_t, y_t) = \qquad\qquad A_2 y_t + \sum_{k\geq 2} F_2^{[k]}(x_t, y_t) \ . \end{aligned} \tag{30}$$

By rewriting the dynamical system in this particular form, it is clear that the x-plane $W = \{y = 0\}$ is a d-dimensional subspace that is invariant under A.

We are now interested in constructing an invariant manifold $\mathcal{W}$ tangent to the x-plane. We will split also the parameterization $z_t = \Phi(u_t)$ in components: $x_t = \varphi(u_t)$, $y_t = \psi(u_t)$. Since the manifold is tangent to the x-plane, we can choose a parameterization such that $\mathrm{D}\varphi(0) = \mathrm{I}_d$ and $\mathrm{D}\psi(0) = 0$. In these coordinates, the linearization of the dynamics $u_{t+1} = f(u_t)$ on the manifold is given by A_1. As a result, the Taylor expansions of our unknown functions φ, ψ and f around $u = 0$ are of the form

$$\begin{aligned} x = \varphi(u) &= u + \sum_{k\geq 2} \varphi^{[k]}(u) \\ y = \psi(u) &= \qquad \sum_{k\geq 2} \psi^{[k]}(u) \\ f(u) &= A_1 u + \sum_{k\geq 2} f^{[k]}(u) \ . \end{aligned}$$

where $\varphi(u) = (\varphi_1(u), \ldots, \varphi_d(u))$, $\psi(u) = (\psi_1(u), \ldots, \psi_{n-d}(u))$, $f(u) = (f_1(u), \ldots, f_d(u))$. Again, the terms $\varphi^{[k]}(u)$, $\psi^{[k]}(u)$, $f^{[k]}(u)$ denote normalized derivatives of order k, and for instance, $\varphi_1^{[k]}(u)$ is a homogeneous polynomial of order k.

In order to find the unknowns φ, ψ and f, one can compute recursively the terms of their Taylor series. Notice that the starting point of the recursion is the terms of order 1, which are already known. Let us assume then that we have already computed all the terms up to order $k-1$, $\varphi^{[<k]}(u), \psi^{[<k]}(u), f^{[<k]}(u)$, and we want to compute the terms of order k, $\varphi^{[k]}(u), \psi^{[k]}(u), f^{[k]}(u)$.

By formal substitution of the Taylor series expansions into equation (2), and truncation up to order k, we obtain the equation for $\varphi^{[k]}(u), \psi^{[k]}(u), f^{[k]}(u)$: [32]

$$\varphi^{[k]}(A_1 u) - A_1 \varphi^{[k]}(u) + f^{[k]}(u) - B\psi^{[k]}(u) = r^{[k]}(u) \ , \tag{31}$$

$$\psi^{[k]}(A_1 u) - A_2 \psi^{[k]}(u) = s^{[k]}(u) \ , \tag{32}$$

where $r^{[k]}(u)$ corresponds to the known coefficients of the x coordinate and $s^{[k]}(u)$ corresponds to the known coefficients of the y coordinate, that is:

$$r^{[k]}(u) = \left[F_1(\varphi^{[<k]}(u), \psi^{[<k]}(u))\right]^{[k]} - \left[\varphi^{[<k]}(f^{[<k]}(u))\right]^{[k]} \ , \tag{33}$$

$$s^{[k]}(u) = \left[F_2(\varphi^{[<k]}(u), \psi^{[<k]}(u))\right]^{[k]} - \left[\psi^{[<k]}(f^{[<k]}(u))\right]^{[k]} \ . \tag{34}$$

Hence, at each step k, we have to solve a linear system of equations (31), (32) where the unknowns are the coefficients of the homogeneous terms, $\varphi^{[k]}(u), \psi^{[k]}(u), f^{[k]}(u)$. These kinds of systems are known as the *homological equations* in the dynamical system literature. As we will see, there can be algebraic obstructions to solve these equations that in some cases can be overcome. It is worth mentioning that this methodology has the flavor of the theory of normal forms, initiated by Poincaré, and that the algebraic obstructions are known as *resonances*.

In our construction, notice that equations (31), (32) are a sort of block triangular system. Thus, we can first solve equation (32) to compute $\psi^{[k]}(u)$ and then we solve equation (31) to obtain $\varphi^{[k]}(u)$ and $f^{[k]}(u)$. For the sake of simplicity, we will assume that the matrices A_1 and A_2 are diagonal (with possibly complex entries), but most of what is explained below works in the general case that A_1 and A_2 have complex Jordan normal form. Thus, we assume that $A_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$, $A_2 = \mathrm{diag}(\mu_1, \ldots, \mu_{n-d})$.

First, we split (32) into components, so for $j = 1, \ldots, n-d$ we have the following relations:

$$\sum_{|m|=k} \psi_m^j (\lambda^m - \mu_j) u^m = \sum_{|m|=k} s_m^j u^m \ , \tag{35}$$

where we use the multi-index notation for $m = (m_1, \ldots, m_d)$. As long as the so called *non-resonance condition*

$$\text{for all } |m| = k, j = 1, \ldots, n-d, \ \lambda^m - \mu_j \neq 0 \tag{36}$$

holds, we can compute the coefficients of the homogeneous terms of ψ^j by taking

$$\psi_m^j = \frac{s_m^j}{\lambda^m - \mu_j} \ .$$

[32] By "formal substitution" we mean that we operate algebraically the composition of Taylor polynomials, and we kill all the terms of order higher than k. Finally, we match the terms of order k.

We note that if for *some* $|m| \geq 2$ and $j = 1, \ldots,\ n-d$, it happens that $\lambda^m = \mu_j$, we have an algebraic obstruction (a *resonance*) for computing ψ_m^j, so in such a case we cannot compute the terms of the expansion of the manifold (an in general there is not an invariant manifold attached to such a linear subspace!). We will refer to such an obstruction as a *primary resonance*. We moreover emphasize that the obstruction only involves an algebraic relation between the eigenvalues of both A_1 and A_2.

Once we have computed the k order coefficients of $\psi^{[k]}$, we can substitute them in equation (31), and solve for $\varphi^{[k]}$ and $f^{[k]}$. For $i = 1, \ldots, d$ we have the following relations:

$$\sum_{|m|=k} (\varphi_m^i(\lambda^m - \lambda_i) + f_m^i)u^m = \sum_{|m|=k} \bar{r}_m^i u^m \ , \tag{37}$$

where $\bar{r}^{[k]}(u) = r^{[k]}(u) + B\psi^{[k]}(u)$. Notice that, in this case, there are non-unique solutions of the equations (37). This fact has to do with the non-uniqueness of the parameterization of the invariant manifold, even if the manifold itself is unique. There are basically two different ways of solving the previous system of linear equations. The first approach, the graph method, tries to simplify the local representation of the manifold. The second one, the parameterization method, tries to simplify the local representation of the dynamics on the manifold.

Graph method. This method simplifies the local representation of the invariant manifold. In order to do so, we set $\varphi_m = 0$ and $f_m^i = \bar{r}_m^i$ so that in the parameterization of $\mathcal{W}$ has $\varphi(u) = u$ and as a result the manifold $\mathcal{W}$ is (locally) a graph $y = \psi(x)$.

Notice also that the invariance equation can then be rewritten as follows:

$$\psi(F_1(x, \psi(x))) = F_2(x, \psi(x)) \ ,$$

where the dynamics on the manifold is given by:

$$f(x) = F_1(x, \psi(x)).$$

Parameterization method. This method simplifies the dynamics on the invariant manifold by choosing a suitable parameterization of the manifold. In order to do so, we impose the following conditions:

- If $\lambda^m - \lambda_i \neq 0$, we set
$$\varphi_m^i = \frac{\bar{r}_m^i}{\lambda^m - \lambda_i} \ , \ f_m^i = 0 \ ;$$
- If $\lambda^m - \lambda_i = 0$, we set
$$\varphi_m^i = 0 \ , f_m^i = \bar{r}_m^i \ .$$

We emphasize that if $\lambda^m \neq \lambda_i$ for all $|m| \geq 2$ and $i = 1, \ldots, d$, then we can reduce the dynamics on the manifold to a linear dynamics. The obstructions $\lambda^m = \lambda_i$ are the *secondary resonances*, and their existence avoid the possibility of linearization. Moreover, if the eigenvalues λ_i of A_1 are all of modulus smaller (bigger) than one, then there is only a finite set of possible secondary resonances, and at least one can reduce the dynamics to a polynomial dynamics. See [7]. This resembles the Poincaré-Dulac normal form.

The set of possibilities is not closed with the two above. The only thing we need is to be able to solve the homological equations. The strategy can be pushed much more. For instance, if A_1 is (1), then all the terms in the manifold are resonant, an in principle we cannot eliminate any of the terms of the expansion of f. But if instead of choosing $\varphi_m^i = 0$ at each step we are more sharp, we can be able of eliminating further terms of f an obtain a polynomial. This strategy has been used in [3] and [4].

B Preserved quantities and asymptotic manifolds

In this section we prove Proposition 2. It follows from a simple continuity argument. So, it is enough to assume that the preserved quantity $H : \mathcal{B} \to \mathbb{R}^{n-m}$ is a continuous function.

Assume that $z \in \mathcal{B}$ is such that its orbit $z_t = F^t(z)$ converges to the steady state $z_* \in \mathcal{B}$ in the future. That is, assume that $\lim_{t\to+\infty} F^t(z) = z_*$. Since $H_0 = H(z) = H(z_t)$ for all t, then $H_0 = \lim_{t\to+\infty} H(z_t) = H(z_*)$.

The study of the case $\lim_{t\to-\infty} F^t(z) = z_*$ is analogous.

As a result, all the points that converge asymptotically either in the future or in the past to a steady state, belong to the same level surface of such a steady state. $\square$

C Proofs of the ABS model

Proof of Proposition 3. The steady states $(R_{(1,1)}, R_{(2,2)}, R_{(1,2)}, R_{(2,1)})$ of the ABS model satisfy the equations:

$$R_{(2,1)} = \frac{1}{\alpha}\left(1+\alpha-\frac{1}{R_{(1,1)}}\right) , \tag{38}$$

$$R_{(1,2)} = \frac{1}{\beta}\left(1+\beta-\frac{1}{R_{(2,2)}}\right) , \tag{39}$$

$$R_{(1,2)}R_{(2,1)} = (R_{(1,1)})^2 = (R_{(2,2)})^2 . \tag{40}$$

Since the rates of returns are all positive, we have that $R_{(1,1)} = R_{(2,2)} = R > 0$, and from equations (38) and (39) we obtain that R is a root of the following quartic polynomial

$$f(R) = \alpha\beta R^4 - (1+\alpha)(1+\beta)R^2 + (2+\alpha+\beta)R - 1. \tag{41}$$

Notice that we have to consider the solutions such that $R_{(1,2)}$ and $R_{(2,1)}$ are positive. In particular, $R = 1$ is a solution, thus the corresponding steady state is $R_{(1,1)} = R_{(2,2)} = R_{(1,2)} = R_{(2,1)} = 1$.

Since $f(R) = (R-1)g(R)$, where

$$g(R) = \alpha\beta R^3 + \alpha\beta R^2 - (1+\alpha+\beta)R + 1, \tag{42}$$

the rest of the roots solve the equation $g(R) = 0$. Furthermore, note that $g(-1) = 2+\alpha+\beta > 0$, $g(0) = 1$ and $g\left(\frac{1}{1+\alpha}\right) = -\frac{\beta}{(1+\alpha)^3} < 0$. Hence, the three roots of g are real an they are in the intervals $]-\infty, -1[$, $]0, \frac{1}{1+\alpha}[$ and $]\frac{1}{1+\alpha}, \infty[$.

The root R_* we are interested is the one that is in the third interval, because $R = R_* > \frac{1}{1+\alpha} \geq \frac{1}{1+\beta}$ which implies that both $R_{(1,2)}$ and $R_{(2,1)}$ are positive. Furthermore, under the assumption $e > w$, $\frac{w}{e} \leq R_* < 1$. This follows from the following facts: $g\left(\frac{w}{e}\right) = \left(\frac{1}{4\gamma(1-\gamma)}\right)\left(\frac{w}{e}-1\right) \leq 0$ and $g(1) = 2\alpha\beta - \alpha - \beta = 2\alpha\beta\left(1-\frac{w}{e}\right) > 0$. $\square$

Proof of Proposition 4. In order to study the linear stability at a steady state we first compute the Jacobian matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \alpha R^2 \\ 0 & 0 & \beta R^2 & 0 \\ 0 & \frac{1}{\beta R}\left((1+\beta)-\frac{1}{R}\right) & R\left((1+\beta)-\frac{1}{R}\right) & -\frac{\alpha}{\beta}\cdot\frac{(1+\beta)-\frac{1}{R}}{(1+\alpha)-\frac{1}{R}} \\ \frac{1}{\alpha R}\left((1+\alpha)-\frac{1}{R}\right) & 0 & -\frac{\beta}{\alpha}\cdot\frac{(1+\alpha)-\frac{1}{R}}{(1+\beta)-\frac{1}{R}} & R\left((1+\alpha)-\frac{1}{R}\right) \end{pmatrix} .$$

where either $R = 1$ or $R = R_*$ depending on the steady state we are examining. The associated eigenvalues are the roots of the characteristic polynomial, that is given by:

$$p(\lambda) = (\lambda + 1)q(\lambda) \ ,$$

where

$$q(\lambda) = \lambda^3 + (1 - (2 + \alpha + \beta)R)\lambda^2 + \alpha\beta R^4\lambda + \alpha\beta R^4 \ .$$

Hence, -1 is an eigenvalue of A.

We have to analyze the roots of the cubic polynomial $q(\lambda)$ for the cases $R = 1$ and $R = R_*$.

If $R = 1$,

$$q(\lambda) = \lambda^3 - (1 + \alpha + \beta)\lambda^2 + \alpha\beta\lambda + \alpha\beta = \lambda^3 g\left(\frac{1}{\lambda}\right) \ ,$$

where g is defined in (42). In the proof of Proposition 3, we separated the zeros of g, so $\frac{1}{\lambda}$ is in $]-\infty, -1[$, or in $]0, \frac{1}{1+\alpha}[$ or in $]\frac{1}{1+\alpha}, 1[$. Hence, $\lambda_s \in]-1, 0[$, $\lambda_{u,1} \in]1, 1+\alpha[$ and $\lambda_{u,2} \in]1+\alpha, \infty[$.

If $R = R_*$,

$$q(\lambda) = \lambda^3 + (1 - (2 + \alpha + \beta)R_*)\lambda^2 + \alpha\beta R_*^4\lambda + \alpha\beta R_*^4 \ .$$

Then:

1. $q(-1) = -(2 + \alpha + \beta)R_* < 0$;

2. $q(0) = \alpha\beta R_*^4 > 0$;

3. $q(1) = 2 - (2 + \alpha + \beta)R_* + 2\alpha\beta R_*^4 = \frac{1}{2\alpha\beta}R_*\left(R_*^3 - R_*^2 - R_* + \frac{w}{e}\right) < 0$.

In order to prove the third inequality we proceed as follows. First, notice that $g(R_*) = 0$ (see equation (42)) implies that

$$2 - (2 + \alpha + \beta)R_* = (\alpha + \beta)R_* - 2\alpha\beta R_*^2 - \alpha\beta R_*^3 \ ,$$

which proves

$$q(1) = \frac{1}{2\alpha\beta}R_*\left(R_*^3 - R_*^2 - R_* + \frac{w}{e}\right) \ .$$

Since $\frac{w}{e} \leq R_* < 1$, and the function $R \to R^3 - R^2 - R$ is decreasing in $[0, 1]$, we have

$$R_*^3 - R_*^2 - R_* + \frac{w}{e} \leq \left(\frac{w}{e}\right)^3 - \left(\frac{w}{e}\right)^2 = \left(\frac{w}{e}\right)^2\left(\frac{w}{e} - 1\right) < 0 \ .$$

In fact,

$$q(R_*) = R_*^2 - (1 + \alpha + \beta)R_*^3 + \alpha\beta R_*^4 + \alpha\beta R_*^5 = R_*^2 g(R_*) = 0 \ ,$$

so $\lambda_{s,2} = R_*$ is an eigenvalue of the Jacobian matrix in the non monetary steady state. $\square$

Proof of Proposition 5. The proof is a simple consequence of the fact that

$$\frac{R_{(1,2),t+1}R_{(2,1),t+1}}{R_{(1,1),t+1}R_{(2,2),t+1}} = \frac{R_{(1,1),t}R_{(2,2),t}}{R_{(1,2),t}R_{(2,1),t}} \ ,$$

which can be obtained by multiplying the two formulae in equation (14). □

Proof of Proposition 6. Since we look for the solutions of (21),(22) and (25) that are on the level set Σ_1, we add the extra equation

$$\frac{R_{(1,1)}R_{(2,2)}}{R_{(1,2)}R_{(2,1)}} = 1 \ . \tag{43}$$

From (21) we obtain

$$\begin{aligned}\alpha R^2_{(1,1)} &= R_{(1,2)}(1+\alpha-\alpha R_{(2,1)})((1+\alpha)R_{(1,1)}-1) \\ &= \frac{R_{(2,2)}R_{(1,1)}}{R_{(2,1)}}\frac{R_{(1,1)}}{R_{(2,2)}}((1+\beta-\beta R_{(1,2)})((1+\alpha)R_{(1,1)}-1) \ ,\end{aligned}$$

where in the last equality we use (43). It follows that

$$R_{(1,1)} = \frac{1}{1+\alpha}\left(1+\frac{\alpha R_{(2,1)}}{1+\beta-\beta R_{(1,2)}}\right) \ ,$$

which is (28). A similar argument proves the formula for $R_{(2,2)}$, (29).

Finally, equations (26) and (27) of Proposition 6 are obtained by substituting the formulae (28) and (29) in (25) and (43). □