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StrategicRequirements with Indi¤erence: Single-Peaked versus Single-Plateaued Preferences

Dolors Berga and Bernardo Moreno

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Strategic Requirements with Indifference: Single-Peaked versus Single-Plateaued Preferences[1]

Dolors Berga[†]

and

Bernardo Moreno[‡]

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Abstract: We concentrate on the problem of the provision of one pure public good whenever agents that form the society have either single-plateaued preferences or single-peaked preferences over the set of alternatives. We are interested in comparing the relationships between different nonmanipulability notions under these two domains. On the single-peaked domain, under strategy-proofness, non-bossyness is equivalent to convex range. Thus, minmax rules are the only strategy-proof non-bossy rules. On the single-plateaued domain, only constant rules are non-bossy or Maskin monotonic; but strategy-proofness and weak non-bossy are equivalent to strict Maskin monotonicity. Moreover, strategy-proofness and plateau-invariant guarantee convexity of the range.

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Keywords: Strategy-proof, Single-plateaued preferences, Single-peaked preferences, Maskin monotonicity, Non-bossyness, Plateau-invariance.

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[†] Departament d'Economia, Campus de Montilivi, Universitat de Girona, 17071 Girona, Spain. E-mail: dolors.berga@udg.edu.

[‡] Departamento de Teoría e Historia Económica, Facultad de Ciencias Económicas y Empresariales, Campus de El Ejido, 29071 Málaga, Spain. E-mail: bernardo@uma.es

1 Introduction

We consider the problem of the provision of a single pure public good where there are n agents in the society and the set of feasible alternatives is $A = [0, 1]$. Each agent has either single-peaked or else single-plateaued preferences over alternatives. Given that the set of available preferences for each agent is the same, the provided level of public good is chosen by means of a social choice function. We impose some strategic requirements over the decision procedure: strategy-proofness, Maskin monotonicity, non-bossyness (or some variations of them). Strategy-proofness assures us that no individual agent will gain by misrepresenting his true preferences. Maskin monotonicity (a necessary condition for Nash implementation, see Maskin 1977, 1985, and 1999) tells us that no single agent will be able to change the social outcome when changing his preferences in such a way that the lower contour set of the initial outcome under initial preferences is a subset of the lower contour set of the initial outcome under new preferences. In this paper we use the notion of non-bossyness for economies with only pure public goods. Mainly, it says that no agent, by misrepresenting his true preferences, can change the social outcome without changing the (ordinal) utility it assigns to him under his initial preferences.

Concerning preferences domains, Barberà (2007) discusses the extent to which allowing for agents to be indifferent among alternatives may alter the qualitative results that are obtained in social choice theory. Two of the most well-known conditions that guarantee positive results in social choice are Duncan Black's notion of single-peakedness, and the straightforward extension of single-peakedness to allow for indifferences, that is, single-plateaued preference profiles, which allow individuals to be indifferent among several consecutive best alternatives.

Following the idea in Barberà (2007), in our framework of social choice functions with these two domains, we examine logical relations between strategy-proofness, Maskin monotonicity, non-bossyness, or some variations of them. We discuss the relationships between these concepts when preferences profiles satisfy the single-peaked condition and we investigate if such relationships keep holding when we move to single-plateaued preference profiles. In our framework we obtain that non-bossyness, or a weaker version of it that we call weak non-bossyness, turn out to be crucial. In particular, such à la non-bossy conditions allow us to state the relationship between strategy-proofness and conditions à la Maskin monotonicity in our main results (concretely, in Theorems 1 and 2 for single-peakedness and Theorem 3 for single-plateauedness).

Non-bossyness, formally introduced by Satterthwaite and Sonnenschein (1981), has been largely studied in the literature for economies incorporating private goods. However, as far as we know there are very few papers studying such kind of conditions with only public goods. Ritz (1985) defines the condition of "noncorruptability", a strategic requirement on social choice correspondences, and in particular on social choice functions. We name this condition weak non-bossyness and it plays an important role in two of our main results where we relate strategy-proofness and a weak version of Maskin monotonicity (see

Theorems 2 and 3). A social choice function is weakly non-bossy (or equivalently, "noncorruptible") if no agent can, by misrepresenting his true preferences, change the social outcome without changing the value of it for himself. As we argue and claim in the Concluding Remarks section, weak non-bossyness could be very helpful to obtain closed characterization results for strategy-proof social choice function on the domain of single-peaked preferences without convex range.

Non-bossyness and a different weaker version of it have been recently used in Saijo, Sjöström, and Yamato (2007) to analyze double implementation (that is, implementation in both dominant strategies and Nash equilibria) in a general social choice framework incorporating ours. We call their weaker condition quasi non-bossyness since in our framework, and under strategy-proofness, it is in fact equivalent to our non-bossyness.

With the aim of establishing a relationship between strategy-proofness and a type of à la Maskin condition for social choice functions on single-plateaued preferences in subsection 3.2, we consider two ways of relaxing Maskin monotonicity that we call strict Maskin monotonicity and plateau-invariance. The first one weakens the idea of an "improvement" implicit in the very same definition of Maskin monotonicity. Plateau-invariance is part of Moulin's (1984) strong-uncompromisingness (in particular, his parts (iv) and (v)) and not requiring à la Maskin "improvements" to all the alternatives in the range but only to those that are the most preferred alternatives for some agent.

As previously mentioned, in this paper we show that non-bossyness plays an important role combined with strategy-proofness in the characterization of social choice functions in our framework with single-peaked preferences. Theorem 1 shows that under strategy-proofness, a social choice function on single-peaked preferences is non-bossy if and only if its range is convex. Moreover, it also reaffirms the family of minmax rules characterized by Moulin (1980)[1] on a single-peaked domain as an important class of rules. In particular, these rules are also the unique ones that are non-bossy and strategy-proof.

The result in Theorem 1 does not hold on a single-plateaued domain. As we show in Proposition 1, on a single-plateaued domain only constant rules are non-bossy. We may, however, insist on a non-bossy type concept and move to the weaker concept of weak non-bossyness. Then, we are able to relate strategy-proofness and Maskin monotonicity. Theorem 2 shows that on a single-peaked domain a social choice function is strategy-proof and weakly non-bossy if and only if it is Maskin monotonic.

This result does not hold either on the single-plateaued domain. Note that Proposition 1 excludes dictatorial social choice functions, in particular, only constant social choice functions are admissible under Maskin monotonic-

[1] Moulin (1980) characterized on the single-peaked domain the class of minmax rules by using strategy-proofness and peak-onlyness (that is, the best alternative is the unique relevant information of agents' preferences). Ching (1992) used a continuity axiom instead of peak-onlyness (and called the same class of rules as augmented median voter rules) while Sprumont (1995) (in his Theorem 2.4) used convexity of the rule's range to characterize the same class of rules.

ity. Dasgupta, Hammond, and Maskin (1979) also obtain strategy-proofness as a necessary condition for Maskin monotonicity for their rich domains.[2] Given Proposition 1 we show that, for single-plateaued preferences, the relation Maskin monotonicity implies strategy-proofness vacuously holds. To obtain a counterpart result to Theorem 2, we use the two proposed ways of relaxing Maskin monotonicity. Theorem 3 shows that on the single-plateaued domain a social choice function is strategy-proof and weakly non-bossy if and only if it is strictly Maskin monotonic. Finally, we explore the second way to relax Maskin monotonicity, called plateau-invariance. In Proposition 2, we show that on a single-plateaued domain any strategy-proof and plateau-invariant social choice function is weakly non-bossy, thus strictly Maskin monotonic by Theorem 3. With the same flavour as Theorem 1, part (iii) in Proposition 2 guarantees the convexity of the range of any strategy-proof and plateau-invariant rule on single-plateaued preferences.

There are interesting papers in the literature analyzing the relationship among strategy-proofness, Maskin monotonicity, and/or non-bossyness. Barberà and Jackson (1995) state for exchange economies that strategy-proofness and non-bossyness implies coalition strategy-proofness (that is, no coalition of agents can strictly gain by misrepresenting their preferences). Klaus (2001) in a problem of an assignment of indivisible objects with single-dipped preferences obtains an equivalent result adding some model specific conditions to strategy-proofness. In cost sharing models non-bossyness has also appeared as a relevant instrumental condition (see for instance, Moulin, 1994, Serizawa, 1999 and Mutuswami, 2005). Pápai (2000) and Takamiya (2001) showed that non-bossyness joint with strategy-proofness is equivalent to coalition strategy-proofness in the house allocation problem and in the Shapley-Scarf housing market with strict preferences, respectively. Mizukami and Wakayama (2007b) study the relationship between Maskin monotonicity and non-bossyness in a quite general economic framework. They obtain the equivalence between Maskin monotonicity and non-bossyness joint with individual monotonicity whenever preferences domains are weakly monotonically closed. Although Mizukami and Wakayama's model encompasses exchange economies, housing markets, public and private good economies, etc., it does not encompass ours.

As far as we know, only three works deserve our attention concerning the relationship between strategy-proofness and Maskin monotonicity that embed or directly analyze the problem of the provision of only public goods. Shenker (1993) studies the relationship between our three strategic axioms when agents' preferences are monotonically closed in a model dealing simultaneously with both the public and the private goods case. However, his non-bossyness condition is Satterthwaite and Sonnenschein (1981)'s one and thus it is trivially satisfied when analyzing economies with only public goods. As said above we consider an alternative definition that has some bite for public goods.

Another paper closely related to ours is due to Takamiya (2007). He consid-

[2] See Maskin and Sjöström (2002) for a precise statement of this result. See also Bochet and Klaus (2007a) for a discussion and a correction of the relationship and definition of rich domain first proposed in Dasgupta, Hammond, and Maskin (1979).

ers the same framework as ours and defines two sufficient conditions that a domain of a social choice function should satisfy to have the property that coalition strategy-proofness implies Maskin monotonicity and its converse. Our results are independent from his since the domain of single-peaked preferences does not satisfy any of his domain conditions. Moreover, the domain of single-plateaued preferences satisfies only one of his conditions, the one required by Takamiya to establish that Maskin monotonicity implies strategy-proofness. However, we will see that this relationship is vacuous since Maskin monotonicity is very demanding applied to our framework with single-plateaued preferences.

Recently, Bochet and Klaus (2007b) have analyzed the relationship between Maskin monotonicity and strategy-proofness in a general model that covers both the private goods and the public goods setup. Like Takamiya (2007) they introduce two sufficient conditions that a domain of a social choice function should satisfy to guarantee that Maskin monotonicity implies strategy-proofness and the converse. Observe that neither single-plateaued nor single-peaked preferences satisfy their condition to guarantee that strategy-proofness implies Maskin monotonicity. However, both domains satisfy their condition to guarantee that Maskin monotonicity implies strategy-proofness. For single-peaked preferences, we independently state this result (see part of one implication of Theorem 2). As mentioned above, for single-plateaued preferences, we show that this relation is vacuous and trivially holds.

The paper is organized as follows. Section 2 contains the model and definitions while Section 3 encompasses the main results. Some comments and proposals for further research form the Concluding Remarks section (Section 4). We gather the proofs of all results in Section 5.

2 The basic model and definitions

Let $A = [0, 1]$ be the set of *alternatives*[3] that stand for the feasible levels of a public good, and $N = \{1, 2, ..., n\}$ be the set of *agents* (with $n \geqslant 2$) in the society. Let capital letters $S, T \subset N$ denote subsets of agents while small letters s, t their cardinality.

Let $\mathcal{D}$ denote the set of *admissible preferences for each agent*, that will be a subset of continuous and convex preferences on A. Although we restrict to ordinal preferences, we use utility functions to denote them. A preference profile is denoted by $u = (u_1, ..., u_n) \in \mathcal{D}^n$ or also by $u = (u_S, u_{-S}) \in \mathcal{D}^n$ when we want to stress the role of a coalition $S \subset N$, and then $u_S \in \mathcal{D}^s$ and $u_{-S} \in \mathcal{D}^{n-s}$ denote the preferences of agents in S and in $N \backslash S$, respectively.

In each definition, $\mathcal{D}$ will refer to either one of the following domains: the set of single-plateaued preferences (denoted by $\mathcal{F}$) or else the set of single-peaked preferences (denoted by $\mathcal{S}$). That is, each property will be defined for two different domains: $\mathcal{F}$ and $\mathcal{S}$.

We now define a *single-plateaued preference* for any agent $i \in N$.

[3] All of our proofs can be easily adapted for a finite set of alternatives and for $\mathbb{R}$.

Definition 1 *A preference u_i is single-plateaued if there exist $p^-(u_i)$, $p^+(u_i) \in A$, $p^-(u_i) \leqslant p^+(u_i)$ such that $[p^-(u_i), p^+(u_i)] = \{x \in A : u_i(x) \geqslant u_i(y)$, for all $y \in A\}$, and for any $y, z \in A$ such that $y < z \leqslant p^-(u_i)$, or $p^+(u_i) \leq z < y$, then $u_i(z) > u_i(y)$.*

The set of best alternatives for agent i according to u_i, that is $[p^-(u_i), p^+(u_i)]$, is denoted by $\tau(u_i)$ and called the *plateau* of u_i. The domain of *single-peaked preferences*, denoted by $\mathcal{S}$, is a subdomain of $\mathcal{F}$ for which the plateau of a preference u_i is a singleton called the *peak*, say $p(u_i)$. That is, $p^-(u_i) = p^+(u_i) = p(u_i)$ for $u_i \in \mathcal{S}$.

For some $u_i \in \mathcal{D}$ and for some $x < p^-(u_i)$ (respectively, $x > p^+(u_i)$), let $r^{u_i}(x)$ be an alternative in A such that $u_i(x) = u_i(r^{u_i}(x))$ if it exists and $r^{u_i}(x) = 1$ (respectively, 0) otherwise. Note that when $r^{u_i}(x)$ exists it is unique since single-plateaued preferences do not allow for indifferences between alternatives in the same side of the plateau.

A *social choice function* (or also a *rule*) on $\mathcal{D}^n$ is a function $f : \mathcal{D}^n \to A$. Let A_f denote the range of f.

The most well-known nonmanipulability property is *strategy-proofness*. Strategy-proofness requires that the truth be a dominant strategy and it is a necessary condition for implementation in dominant strategies (Gibbard, 1973). Strategy-proofness assures that the rule will be immune to unilateral strategic behavior.

Definition 2 *A social choice function f on $\mathcal{D}^n$ is strategy-proof if for any $u \in \mathcal{D}^n$, any $i \in N$ and any $v_i \in \mathcal{D}$, $u_i(f(u)) \geqslant u_i(f(v_i, u_{-i}))$. Otherwise, f is said to be manipulable on $\mathcal{D}^n$, concretely, manipulable by i at u via v_i.*

It is worth mentioning that if f is strategy-proof on $\mathcal{D}^n$ then A_f is closed (see Step 1 of Theorem 2's proof in Zhou, 1991 and Lemma 1 in Barberà and Jackson, 1994).

Another well-known condition in implementation theory is *Maskin monotonicity*, a necessary condition for Nash implementation.[4] A social choice function is said to be Maskin monotonic if the outcome to be chosen by the function does not vary whenever each individual switches his preference keeping or improving the relative ranking of that outcome.

Definition 3 *A social choice function f on $\mathcal{D}^n$ is Maskin monotonic if for any $u \in \mathcal{D}^n$ and $v_i \in \mathcal{D}$, if $L(f(u), u_i) \subseteq L(f(u), v_i)$ then $f(v_i, u_{-i}) = f(u)$, where $L(x, u_i) = \{y \in A : u_i(x) \geq u_i(y)\}$.*[5]

Another property on rules related to the strategic behavior of agents is non-bossyness. The usual notion of non-bossyness introduced by Satterthwaite and Sonnenschein (1981) is trivially satisfied for any social choice function in economic environments with only public goods. We use the notion of *non-bossyness*,

[4] See Maskin (1977) and Muller and Satterthwaite (1977).

[5] An alternative definition more appropriate to work with weaker versions of Maskin monotonicity is as follows. We say that f satisfies Maskin monotonicity if for any $u, v \in \mathcal{D}^n$ and for any $i \in N$, for any $x \in A$, if $u_i(f(u)) \geqslant u_i(x)$ implies $v_i(f(u)) \geqslant v_i(x)$ then $f(v_i, u_{-i}) = f(u)$.

recently used by Saijo, Sjöström, and Yamato (2007), and stronger than the one introduced by Ritz (1985) and called "noncorruptability", that has some bite for public goods. Mainly, a non-bossy social choice function is a rule for which no individual can, by misrepresenting his preferences, change the social outcome without changing the value of it for himself. Formally:

Definition 4 *A social choice function f on $\mathcal{D}^n$ is non-bossy if for any $i \in N$, for any $u_i, v_i \in \mathcal{D}$, and $u_{-i} \in \mathcal{D}^{n-1}$, $u_i(f(u)) = u_i(f(v_i, u_{-i}))$ implies $f(u) = f(v_i, u_{-i})$.*

We say that an agent can boss another agent around if, by changing her announced utility, she can change the social outcome without changing her own utility.[6]

Two different natural weaker versions of non-bossyness can be considered. None of them have been analyzed a lot in the literature of only public goods. Saijo, Sjöström, and Yamato (2007) considered *quasi non-bossyness*[7] that, joint with strategy-proofness, assures in their general framework the possibility of dominant strategy implementation via the associated direct revelation mechanism. *Weak non-bossyness* was originally defined by Ritz (1985) as "noncorruptability" in a work where he studies the relationship between Arrow social welfare functions and social choice correspondences. Below, we formally define both properties.

Definition 5 *A social choice function f on $\mathcal{D}^n$ is quasi non-bossy if for any $i \in N$, for any $u_i, v_i \in \mathcal{D}$, and $u_{-i} \in \mathcal{D}^{n-1}$, if $f(u) \neq f(v_i, u_{-i})$ then there is some $w_{-i} \in \mathcal{D}^{n-1}$ such that $u_i(f(u_i, w_{-i})) \neq u_i(f(v_i, w_{-i}))$.*

That is, we say that an agent is quasi a boss if, by changing her announced utility, she can change the social outcome without changing her own utility under both his original preferences independently of others' preferences.

Definition 6 *A social choice function f on $\mathcal{D}^n$ is weakly non-bossy if for any $i \in N$, for any $u_i, v_i \in \mathcal{D}$, and $u_{-i} \in \mathcal{D}^{n-1}$, $u_i(f(u)) = u_i(f(v_i, u_{-i}))$ and $v_i(f(u)) = v_i(f(v_i, u_{-i}))$ implies $f(u) = f(v_i, u_{-i})$.*

Mainly, an agent can weakly boss another agent around if, by changing her announced utility, she can change the social outcome without changing her own utility under both his original and final preferences.

Non-bossyness implies both weak non-bossyness and quasi non-bossyness by definition. However, the converse does not hold in our framework: neither weak

[6] We could have adapted other existing formulations of non-bossyness for economies with private goods. For example, instead of requiring that $f(u)$ be equal to $f(v_i, u_{-i})$, we could require that $f(u)$ be indifferent to $f(v_i, u_{-i})$ for all agents. Obviously this makes the condition weaker. However, it is easy to check that on $\mathcal{S}^n$ and $\mathcal{F}^n$ under strategy-proofness both non-bossy conditions are equivalent and we can use them indistinctly.

[7] The authors call it weak non-bossyness. See also Mizukami and Wakayama (2007a).

non-bossyness nor quasi non-bossyness imply non-bossyness. For both single-peaked and single-plateaued preferences, consider the rule defined in Example 2 below (see the details in the example).

A natural question refers to establishing in our framework a relationship between weak non-bossyness and quasi non-bossyness. As we will see below as a consequence of Lemma 2 in the Appendix and part (iv) of Theorem 1 in Section 3, under strategy-proofness, quasi non-bossyness and non-bossyness are equivalent while weak non-bossyness is strictly weaker than non-bossyness.

With the aim of establishing a relationship between strategy-proofness and a type of à la Maskin condition for social choice functions on $\mathcal{F}^n$ in subsection 3.2, we consider two ways of relaxing Maskin monotonicity that we call *strict Maskin monotonicity* and *plateau-invariance*. One weakens the idea of an "improvement" implicit in the very same definition of Maskin monotonicity. The other one, not requiring à la Maskin "improvements" to all the alternatives in the range but only to those that are the most preferred alternatives for some agent.

We say that x improves à la Maskin with respect to y for agent i when moving from u_i to v_i if $u_i(x) \geq u_i(y)$ then $v_i(x) \geq v_i(y)$. Note that a Maskin monotonic social choice function requires that for any alternative in the range, say $f(u)$, improving with respect to any other alternative for all agent when moving from u_i to v_i, then the outcome does not change.

We say that x strictly improves with respect to y for agent i when moving from u_i to v_i if $u_i(x) = u_i(y)$ then $v_i(x) \geq v_i(y)$ and if $u_i(x) > u_i(y)$ then $v_i(x) > v_i(y)$. With these ideas in mind we present our two properties related to Maskin monotonicity. We say that a social choice function is strict Maskin monotonic if the outcome to be chosen by the function does not vary whenever each individual switches his preference keeping or strictly improving the relative ranking of that outcome.

Definition 7 *A social choice function f on $\mathcal{D}^n$ is strictly Maskin monotonic if for any $i \in N$, for any $u_i, v_i \in \mathcal{D}$, $u_{-i} \in \mathcal{D}^{n-1}$, and for all $x \in A$, $u_i(f(u)) = u_i(x)$ implies $v_i(f(u)) \geqslant v_i(x)$ and $u_i(f(u)) > u_i(x)$ implies $v_i(f(u)) > v_i(x)$ then $f(v_i, u_{-i}) = f(u)$.*

Observe that Maskin monotonicity implies strict Maskin monotonicity by definition. Moreover, strict Maskin monotonicity is equivalent to Maskin monotonicity on single-peaked preferences (see Lemma 1 below) but not on single-plateaued ones (see the Concluding Remarks section).

Lemma 1 *Any strictly Maskin monotonic social choice function on $\mathcal{S}^n$ is Maskin monotonic.*

The second condition that we call plateau-invariance was part of Moulin's (1984) strong-uncompromisingness (in particular, his parts (iv) and (v)). In his Lemma 5, Moulin (1984) shows that his Generalized Condorcet winner choice functions satisfy plateau-invariance.

Definition 8 *A social choice function f on $\mathcal{D}^n$ is plateau-invariant if for any $i \in N$ and $u_{-i} \in \mathcal{D}^{n-1}$, the following holds:*
(1) for any $u_i \in \mathcal{F} \diagdown \mathcal{S}$, $v_i \in \mathcal{F}$, if either $f(u) \in Interior[\tau(u_i)]$ and $f(u) \in \tau(v_i)$ or if $f(u) = p^e(u_i) = p^e(v_i)$, for e being either $\{-, +\}$, then $f(u) = f(v_i, u_{-i})$, and
(2) for any $u_i, v_i \in \mathcal{S}$ such that $f(u) = p(u_i) = p(v_i)$, then $f(u) = f(v_i, u_{-i})$.

Note that for our two domains, strategy-proofness implies the second part of the definition of plateau-invariance. Moreover, for single-peakedness, part (1) does not apply and part (2) is implied by peak-onlyness. Thus, strategy-proofness implies plateau-invariance on $\mathcal{S}^n$ but the converse does not hold (see Example 6 below). For single-plateaued preferences, neither strategy-proofness implies plateau-invariance nor the converse (see the following examples).

Example 1 *Let $n \geq 2$. Then, for any $u \in \mathcal{F}^n$, define the social choice function f as follows:*

$$f(u) = \begin{cases} \max\limits_{i \in N} \{p^-(u_i)\} & \text{if } \bigcap_{i \in N} \tau(u_i) = \varnothing, \\ \max\limits_{i \in N} \{p^-(u_i)\} & \text{if } \bigcap_{i \in N} \tau(u_i) \neq \varnothing \text{ and } \frac{1}{2} \notin \bigcap_{i \in N} \tau(u_i), \\ \frac{1}{2} & \text{if } \bigcap_{i \in N} \tau(u_i) \neq \varnothing \text{ and } \frac{1}{2} \in \bigcap_{i \in N} \tau(u_i). \end{cases}$$

Observe that f is strategy-proof but it does not satisfy plateau-invariance. To show the failness of the last condition, let $u \in \mathcal{F}^n$ such that $\tau(u_j) = [\frac{1}{3}, \frac{2}{3}]$ for any $j \in N \backslash \{3\}$ and $\tau(u_3) = \left[\frac{1}{3}, \frac{1}{3} + \varepsilon\right]$, $\varepsilon \in \mathbb{R}^+$ being such that $\frac{1}{3} + \varepsilon < \frac{1}{2}$ and let $v_3 \in \mathcal{F}$ such that $\tau(v_3) = [\frac{1}{3}, \frac{2}{3}]$. Then, $f(u) = \frac{1}{3} = p^-(u_3)$ but $f(v_3, u_{-3}) = \frac{1}{2} \neq f(u)$.
It is also interesting to observe that f has a convex range and it satisfies weak non-bossyness.

Example 2 *Let $\alpha \in A$. Then, for any $u \in \mathcal{F}^n$, let $f(u) = \alpha$ if there exist $k \in N$ such that $\alpha \in \tau(u_k)$. Otherwise, (that is, if for any $i \in N$, $\alpha \notin \tau(u_i)$) compute $A(u) = \{l \in N : d[\tau(u_l), \alpha] \leq d[\tau(u_i), \alpha] \text{ for all } i \in N\}$, where $d[\tau(u_l), \alpha] = \min_{x \in \tau(u_l)} |x - \alpha|$. Let a be the smallest $j \in A(u)$ and let $\varepsilon(u_a) > 0$ such that $\alpha + \varepsilon(u_a) < p^-(u_a)$ if $\alpha < p^-(u_a)$ and $\alpha - \varepsilon(u_a) > p^+(u_a)$ if $\alpha > p^+(u_a)$. Then, let $f(u) = \varepsilon(u_a) + \alpha$ when $u_a(1) > u_a(0)$, and $f(u) = \alpha - \varepsilon(u_a)$ when $u_a(1) \leq u_a(0)$.*
This social choice function satisfies plateau-invariance and weak non-bossyness (note that it also satisfies quasi non-bossyness). However, since f is manipulable, by Theorem 3 below, f does not satisfy strict Maskin monotonicity. To show that manipulations by single agents exist, suppose that $\alpha = \frac{3}{8}$ and let $u \in \mathcal{F}^n$ such that $p(u_1) = p(u_2) = 0$, and u_3 such that $\tau(u_3) = [\frac{1}{2}, \frac{3}{4}]$ and $u_3(0) > u_3(1)$. Then, $f(u) = \frac{3}{8} - \varepsilon(u_3)$. Let v_3 such that $\tau(v_3) = [\frac{1}{2}, \frac{3}{4}]$ and $v_3(0) < v_3(1)$. Then, $f(v_3, u_{-3}) = \frac{3}{8} + \varepsilon(u_3)$. Observe that agent 3 manipulates f at u via v_3.[8]

[8] This example can be easily adapted and defined for single-peaked preferences. The same conditions hold and are violated, and similar profiles work to show manipulation by single agents. Note that non-bossyness does not hold either: Let $u \in \mathcal{S}^n$ such that $p(u_1) = p(u_2) = 0$,

By means of these two previous examples, we can also show that neither strict Maskin monotonicity implies plateau-invariance nor the converse on $\mathcal{F}^n$. The idea is that while strict Maskin monotonicity requires a strict improvement for all alternatives in the range, plateau-invariance alternatively requires an improvement but only for some subset of alternatives in the range.

3 Main Results

Along the paper we are interested in establishing the relationship between *strategy-proofness*, *Maskin monotonicity* and *non-bossyness,* and the variations of them defined in Section 1: quasi non-bossyness, weak non-bossyness, strict Maskin monotonicity and plateau-invariance. In particular, we ask whether there are relationships between them and whether these relationships rely on the preference domains under study ($\mathcal{F}$ or $\mathcal{S}$). We first consider the single-peaked domain and then we analyze the single-plateaued one.

3.1 Single-peaked preferences

For the case of single-peaked preferences, there exist lots of rules simultaneously satisfying our three main strategic requirements. See the following very well-known example.

Example 3 *Let $n \geq 3$ be odd. Then, for any $u \in \mathcal{S}^n$, define the social choice function f as follows:*

$$f(u) = med\{p(u_1), ..., p(u_n)\}.$$

The median voter rule is strategy-proof (see Moulin, 1980). It is easy to check that f also satisfies non-bossyness and Maskin monotonicity.

First observe that although there exist rules satisfying all the properties, any couple of them may be logically independent. What we do in this subsection is to obtain for $\mathcal{S}^n$ the exact relationship between our three basic strategic requirements.

By means of Examples 4 and 5 below, observe that for social choice functions on $\mathcal{S}^n$ neither strategy-proofness nor Maskin monotonicity alone imply non-bossyness, nor the converse. However, non-bossyness turns out to be crucial as a complement to strategy-proofness for single-peaked preferences.

Example 4 *Let $a, b \in A$, $a < b$, and $n \geq 2$. Then, for any $u \in \mathcal{S}^n$, define the social choice function f as follows:*

$$f(u) = \begin{cases} a & \textit{if } \#\{i \in N : u_i(a) \geqslant u_i(b)\} \geq \#\{i \in N : u_i(a) < u_i(b)\}, \textit{ and} \\ b & \textit{otherwise.} \end{cases}$$

and u_3 such that $p(u_3) = \frac{1}{2}$ and $u_3(0) > u_3(1)$.Then, $f(u) = \frac{3}{8} - \varepsilon(u_3)$. Let v_3 such that $p(v_3) = \frac{5}{8}$ and $v_3(0) < v_3(1)$. Then, $f(v_3, u_{-3}) = \frac{3}{8} + \varepsilon(u_3)$. Note that we can impose that and $r^{u_3}(\alpha - \varepsilon(u_3)) = \alpha + \varepsilon(v_3)$. Then, non-bossyness is violated.

Observe that f is strategy-proof and it satisfies Maskin monotonicity. However it is not non-bossy. Suppose that n is odd (a similar example could be obtained for n even). Let $u \in \mathcal{S}^n$ be such that for $\frac{n-1}{2}$ agents (say, set S_0), $p(u_j) = 0$, for $\frac{n-1}{2}$ agents $p(u_j) = 1$ (say, set S_1), and for the last agent (say, agent 1) $u_1(a) = u_1(b)$. Let $v_1 \in \mathcal{S}$ be such that $p(v_1) = b$. Then, $f(u) = a$ and $f(v_1, u_{-1}) = b$. Note that $u_1(f(u)) = u_1(f(v_1, u_{-1}))$ but $f(u) \neq f(v_1, u_{-1})$. Using the same preferences we can show that f violates quasi non-bossyness.

Example 5 *Let $a, b \in A$, $a \neq 0$, $a < b$, and $n \geq 2$. Then, for any $u \in \mathcal{S}^n$, define the social choice function f as follows:*

$$f(u) = \begin{cases} a & \text{if } u_1(a) > u_1(b),\ u_1(0) \neq u_1(b), \text{ and } u_1(0) \neq u_1(a), \\ b & \text{if } u_1(b) > u_1(a),\ u_1(0) \neq u_1(b), \text{ and } u_1(0) \neq u_1(a), \text{ and} \\ 0 & \text{if } u_1(a) = u_1(b), \text{ or } u_1(0) = u_1(b), \text{ or } u_1(0) = u_1(a). \end{cases}$$

Note that f is non-bossy however it is manipulable and it does not satisfy Maskin monotonicity. To show that strategy-proofness fails, consider $u_1, v_1 \in \mathcal{S}$ such that $u_1(a) = u_1(b)$ (thus $p(u_1) \in (a, b)$) and $p(v_1) \in (a, b)$, $v_1(a) > v_1(0) > v_1(b)$. Then, $f(u) = 0$ and $f(v_1, u_{-2}) = a$ and agent 1 would manipulate f at u via v_1. Note also that $L(f(u), u_1) = \{0\} \cup [r^{u_1}(0), 1]$ which is a subset of $L(f(u), v_1) = \{0\} \cup [r^{v_1}(0), 1]$ since $r^{v_1}(0) < r^{u_1}(0)$. Thus, Maskin monotonicity does not hold.

Note that the range of the social choice functions in the previous examples is not convex. That the range of a social choice function be a closed interval has happened to be quite important in the characterization of strategy-proof social choice functions on the single-peaked domain (see Moulin, 1980, Ching, 1992, and Sprumont, 1995). From the above examples, one could think that there should be a close relationship between convexity of the range and non-bossyness under strategy-proofness. In fact, as shown in Theorem 1, under strategy-proofness, convexity of the range and non-bossyness are equivalent. The next theorem however tells us much more.

Theorem 1 *Let f be a strategy-proof social choice function on $\mathcal{S}^n$. Then the following statements are equivalent:*
(i) f is non-bossy,
(ii) f is quasi non-bossy,
(iii) f is an minmax rule,[9]
(iv) f has a convex range.

[9] A social choice function f is a minmax rule if there exist a list of parameters $(a_S)_{S \subseteq N} \in A^{2^n}$ satisfying that for any $S, T \subseteq N$, $S \supseteq T$, then $a_S \leq a_T$ and for any $u \in \mathcal{S}^n$,

$$f(u) = \min_{S \subseteq N} \left\{ \max_{i \in N} \{p(u_i), a_S\} \right\}.$$

The class of minmax rules is equivalent to the class of augmented median voter rules due to Ching (1992) and the class of generalized median voter schemes defined for one good as in Barberà, Gul, and Stachetti (1993).

To illustrate the relevance of the result in Theorem 1, observe first (as illustrated in Example 4) that strategy-proofness implies neither non-bossyness, nor quasi non-bossyness, nor convexity of the range. Second, note that at the light of Theorem 2.4 in Sprumont (1995) (which uses Ching's characterization result and convexity of the range), we can establish the characterization result of the well-known class of minmax rules using non-bossyness (or equivalently, quasi non-bossyness) instead of other well-known requirements such as peak-onlyness (used in Moulin, 1980).

Alternatively, on $\mathcal{S}^n$ and under strategy-proofness, quasi non-bossyness, non-bossyness, and also convexity of the range are equivalent. Moreover, the class of minmax rules are the only strategy-proof rules with convex range.

It is worth mentioning that the result in Theorem 1 does not hold relaxing non-bossyness with weak non-bossyness. Observe that the rule defined in Example 4 satisfies weak non-bossyness and strategy-proofness but it does not have a convex range.

A natural question that arises is if strategy-proofness can be replaced by Maskin monotonicity in Theorem 1. To answer this question let us first analyze the relationship between strategy-proofness and Maskin monotonicity for social choice functions on $\mathcal{S}^n$.

We will show below that Maskin monotonicity implies strategy-proofness but the converse does not hold. As it is illustrated by means of Example 4, there exist social choice functions on $\mathcal{S}^n$ that are strategy-proof and weakly non-bossy (the latter is straightforward by definition), that additionally satisfy Maskin monotonicity. As we show in the following statement, the relationship between strategy-proofness joint with weak non-bossyness with respect to Maskin monotonicity stated in the social choice function presented in Example 4 is not a particular feature of that rule.

Theorem 2 *A social choice function f on $\mathcal{S}^n$ is strategy-proof and weakly non-bossy if and only if f is Maskin monotonic.*

Among the previous literature, other papers have also established the relationship between Maskin monotonicity and strategy-proofness. We now discuss some of them in order to clarify the relevance of the previous result.

The classical result by Muller and Satterthwaite (1977) asserts that on the unrestricted strict preference domain, a social choice function is strategy-proof if and only if it is Maskin monotonic (also called in the literature strong positive association). Muller and Satterthwaite (1977) also show that when allowing for weak orderings in the domain, only one implication holds. Concretely, strategy-proofness implies Maskin monotonicity, but not the converse.

Barberà and Peleg (1990) in the framework with a metric space as set of alternatives and preferences being continuous utility functions showed that strategy-proofness does not imply Maskin monotonicity. However, a weaker version of it, which they call modified strong positive association, is necessary though not sufficient for strategy-proofness (see their Lemma 4.8).

Another existing result goes in the other direction. That is, strategy-proofness is a necessary condition for Maskin monotonicity. Dasgupta, Hammond, and

Maskin (1979) (see Footnote 3) considered a general set of alternatives and "rich" domains and they obtain strategy-proofness as a necessary condition for Maskin monotonicity.

In our framework, by Theorem 2 above, on $\mathcal{S}^n$ Maskin monotonicity implies strategy-proofness. However, the converse does not hold as Example 7 below illustrates. With single-peaked preferences, Maskin monotonicity and strategy-proofness are equivalent under weak non-bossyness. Note that to show the relationship between strategy-proofness and Maskin monotonicity in Theorem 2 we can not use the results stated in Takamiya (2007) and in Bochet and Klaus (2007b). The authors consider frameworks encompassing ours and define two sufficient conditions that a domain of a social choice function should satisfy to have the property that strategy-proofness (in fact, coalition strategy-proofness in Takamiya, 2007) implies Maskin monotonicity and its converse. However, one can check that the domain of single-peaked preferences satisfies neither Takamiya's condition A nor condition B. Concerning Bochet and Klaus (2007b), the domain of single-peaked preferences satisfies only their condition R1 that assures that Maskin monotonicity implies strategy-proofness. Independently of their result, we also establish such relationship as part of our Theorem 2.

It is also interesting to observe that the result in Theorem 2 is tied. Or equivalently, as we show in the following examples, strategy-proofness and weak non-bossyness are also independent properties of social choice functions in our framework.

Example 6 *Let $a \in A$ and $n \geq 2$. Then, for any $u \in \mathcal{S}^n$ define f as follows:*

$$f(u) = r^{u_1}(a).$$

Observe that f is weakly non-bossy, but it is not strategy-proof. Suppose, without loss of generality, that $a = \frac{1}{2}$. Let $u_1 \in \mathcal{S}$ be such that $r^{u_1}(\frac{1}{2}) = \frac{1}{4}$ and $v_1 \in \mathcal{S}$ be such that $r^{v_1}(\frac{1}{2}) = \frac{1}{3}$. Let $u_{-1} \in \mathcal{S}^{n-1}$. Therefore, $f(u) = \frac{1}{4}$ and $f(v_1, u_{-1}) = \frac{1}{3}$. Note that $u_1(f(v_1, u_{-1})) > u_1(f(u))$.

Example 7 *Let $a, b \in A$, $a < b$, $n \geq 2$. Then, for any $u \in \mathcal{S}^n$, define f as follows:*

$$f(u) = \begin{cases} a & \text{if } \#\{i \in N : u_i(a) > u_i(b)\} > \#\{i \in N : u_i(a) < u_i(b)\}, \\ b & \text{if } \#\{i \in N : u_i(a) > u_i(b)\} < \#\{i \in N : u_i(a) < u_i(b)\}, \\ a & \text{if } \#\{i \in N : u_i(a) > u_i(b)\} = \#\{i \in N : u_i(a) < u_i(b)\} > 0, \text{ or} \\ & \text{if } \forall i \in N,\ u_i(a) = u_i(b) \text{ and} \\ & \#\left\{i \in N : p(u_i) \leq \frac{a+b}{2}\right\} > \#\left\{i \in N : p(u_i) > \frac{a+b}{2}\right\}, \\ b & \text{otherwise.} \end{cases}$$

Observe that f is strategy-proof. However it is not weakly non-bossy. Suppose that n is odd (a similar example could be obtained for n even). Let $u \in \mathcal{S}^n$ be such that for any agent $i \in N$, $u_i(a) = u_i(b)$, $p(u_1) < \frac{a+b}{2}$, $p(u_j) = \frac{a+b}{2}$ for $j \in S_0 \subset N \backslash \{1\}$, being S_0 a set of $\frac{n-1}{2}$ agents, and $p(u_k) > \frac{a+b}{2}$ for $k \in S_1 = N \backslash (S_0 \cup \{1\})$ being S_1 a set of $\frac{n-1}{2}$ agents. Consider also $v_1 \in \mathcal{S}$ such *that $v_1(a) = v_1(b)$ but $p(v_1) > \frac{a+b}{2}$. Therefore, $f(u) = a$ and $f(v_1, u_{-1}) = b$. Note that $u_1(f(u)) = u_1(f(v_1, u_{-1}))$ and $v_1(f(v_1, u_{-1})) = v_1(f(u))$ but $f(u) \neq f(v_1, u_{-1})$ which means that weakly non-bossy is violated. The same profiles show that f violates Maskin monotonicity. Note that $L(u_1, f(u)) = L(v_1, f(u))$ but $f(u) \neq f(v_1, u_{-1})$.*

Before going to the next subsection where the relationships between our strategic requirements for $\mathcal{F}^n$ are studied, note that Theorem 2 allows us to state that strategy-proofness can be replaced by Maskin monotonicity in Theorem 1.

3.2 Single-plateaued preferences

The statement we did for single-peakedness about "*the existence of lots of rules simultaneously satisfying our three main strategic requirements*" is completely false for the case of single-plateaued preferences. Although we can obtain lots of examples of strategy-proof rules (see Berga, 1998) it is impossible to find non-constant social choice functions satisfying either one of the other two strategic requirements: non-bossyness or Maskin monotonicity. Furthermore, it is also impossible to match strategy-proofness and quasi non-bossyness unless we get constant rules. Such kind of impossibility results are embedded in the following proposition.

Proposition 1 *There is no social choice function f on $\mathcal{F}^n$ with $\#A_f \geqslant 2$ if one of the following statements holds:*
(i) f is non-bossy,
(ii) f is Maskin monotonic,
(iii) f is strategy-proof and quasi non-bossy.

Note that Proposition 1 excludes dictatorial social choice functions, in particular, only constant social choice functions are compatible with either Maskin monotonicity, or non-bossyness, or else strategy-proofness joint with quasi non-bossyness. The result in part (ii) can be also obtained as a corollary to Saijo (1987)'s Theorem. In that paper, Saijo analyzes the relationship between Maskin monotonicity and the constancy of a rule in a more general framework that encompasses ours. In particular, he uses a condition on A_f called "dual dominance" that any social choice function must satisfy to guarantee that the only Maskin monotonic rules are constant ones.[10] The proof of part (ii) of Proposition 1 is essentially identical to Saijo's one. We incorporate it in the Appendix for sake of completeness. A similar conclusion is obtained in Bochet and Klaus (2007b).

Note that, like for $\mathcal{S}^n$, weak non-bossyness is strictly weaker than non-bossyness under strategy-proofness with single-plateaued preferences. By Proposition 1 above, only constant rules are non-bossy while there exist non-constant

[10] In his general framework, A is any set of social alternatives and for any $i \in N$, E_i is the set of agent $i's$ preferences, where E_i is any subset of complete and transitive binary relations on A (that is, weak orderings). We can obtain our framework defining $A = [0, 1]$, and for any $i \in N$, $E_i = \mathcal{F}$. Note also that it is easy to see that any social choice function on $\mathcal{F}^n$ over A satisfies "dual dominance". See his Example 1.

rules satisfying strategy-proofness and weak non-bossyness, for instance the rule presented in Example 1.

Concerning the relationship between strategy-proofness and Maskin monotonicity, again, observe on the one hand that Muller and Satterthwaite (1977)'s equivalence between Maskin monotonicity and strategy-proofness does not hold for $\mathcal{F}^n$ (there exist lots of non-constant strategy-proof rules on $\mathcal{F}^n$ while only constant ones are Maskin monotonic). On the other hand, note that we can not use the results stated in Takamiya (2007) and in Bochet and Klaus (2007b). We can check that the domain of single-plateaued preferences satisfies only condition B (not condition A) in Takamiya's Theorem 2 and only condition R1 (not condition R2) in Bochet and Klaus' Theorem 1 to establish that Maskin monotonicity implies strategy-proofness. Thus, by statement (ii) in Proposition 1 above, the results stated in Theorem 2 in Takamiya (2007) and in Theorem 1 in Bochet and Klaus (2007b) are completely vacuous and trivially satisfied when applied to single-plateaued preferences.

Given the impossibility results in Proposition 1 if we are interested in establishing a relationship between strategy-proofness, a kind of à la non-bossy condition, and/or a kind of à la Maskin monotonicity condition, we should concentrate on weaker versions of them. Concretely, we use weak non-bossyness and strict Maskin monotonicity or plateau-invariance, respectively.

By Examples 2 and 9, we can confirm the independence of strategy-proofness and weak non-bossyness for single-plateaued preferences.

With the aim of establishing a relationship between strategy-proofness and a type of à la Maskin monotonicity condition for social choice functions on $\mathcal{F}^n$, we first consider strict Maskin monotonicity. Below we state for the single-plateaued domain, the counterpart result to Theorem 2, that is, the one relating strategy-proofness and Maskin monotonicity on the single-peaked case using non-bossyness.

Theorem 3 *A social choice function f on $\mathcal{F}^n$ is strategy-proof and weakly non-bossy if and only if f is strictly Maskin monotonic.*

Note that by Proposition 1, we can not replace strict Maskin monotonicity by Maskin monotonicity in Theorem 3: the rule in Example 1 is strategy-proof, weakly non-bossy but it is not constant. Moreover the result stated in Theorem 3 is tied. We can replicate for single-plateaued preferences Examples 6 and 7 above in order to show the independence of strategy-proofness and weak non-bossyness (alternatively, see Examples 2 and 9).

Now we are interested in studying the power that plateau-invariance joint with strategy-proofness give to social choice functions on $\mathcal{F}^n$. As we see by means of the following result, these two conditions imply weak non-bossyness, strict Maskin monotonicity, and also convexity of the range. Plateau-invariance is very powerful. It allows us to state clear relationships between strategy-proofness and both the weaker versions of non-bossyness and strict Maskin monotonicity.

Proposition 2 *Let f be a strategy-proof and plateau-invariant social choice function on $\mathcal{F}^n$. Then, (i) f is weakly non-bossy, (ii) f is strictly Maskin monotonic, and (iii) f has a convex range.*

Note that the converse of these three results do not hold. Concretely, observe first that we can not adapt Theorem 1 for single-plateaued preferences using weak non-bossyness (see the Concluding Remarks section). However, one could think that plateau-invariance would help to obtain that counterpart result; that is, that under strategy-proofness, plateau-invariance and convex range are equivalent. Note that Proposition 2 tells us that this is not true either and Example 1 above presents a strategy-proof rule on $\mathcal{F}^n$ with convex range but violating plateau-invariance.

Second, observe that under strategy-proofness, plateau-invariance is equivalent to neither weak non-bossyness nor to strict Maskin monotonicity. Example 8 presents a strategy-proof and weakly non-bossy, and thus, strictly Maskin monotonic social choice function that violates plateau-invariance.

Example 8 *Let $a, b \in A$, and $a < b$, $n \geq 2$. Then, for any $u \in \mathcal{F}^n$, define f as follows:*

$$f(u) = \begin{cases} a & \textit{if } u_1(a) \geqslant u_1(b), \textit{ and} \\ b & \textit{otherwise.} \end{cases}$$

Observe that f is strategy-proof and weak non-bossy (thus, strictly Maskin monotonic). However the range is not convex. Observe also that f is not plateau-invariant. Let $u_1 \in \mathcal{F}$ be such that $p^-(u_1) > a$ and $p^+(u_1) = b$ and $v_1 \in \mathcal{F}$ be such that $\tau(v_1) = [a, b]$. Let $u_{-1} \in \mathcal{F}^{n-1}$. Therefore, $f(u) = b$ and $f(v_1, u_{-1}) = a$, which contradicts part (1) in the definition of plateau-invariance.

4 Concluding Remarks

To conclude we first summarize our main results and mention some questions for further research. In the framework of the provision of a single pure public good with single-peaked or single-plateaued preferences, we establish the equivalence between strategy-proofness joint with weak non-bossyness and strict Maskin monotonicity (see Theorems 2 and 3). For single-peakedness, we can go further and we obtain the well-known class of minmax rules as the unique strategy-proof rules satisfying non-bossyness (see Theorem 1). We also show that strategy-proofness can be replaced by Maskin monotonicity. For single-plateaued preferences, we obtain constant rules as the unique ones satisfying either Maskin monotonicity, or non-bossyness, or else strategy-proofness and quasi non-bossyness (see Proposition 1). We also identify a condition called plateau-invariance that joint with strategy-proofness guarantees convexity of the range (see Proposition 2). All our proofs can be adequately adapted to show that our results, except the one related to the characterization in part (iii) of Theorem 1, hold whenever we consider weakly single-peaked or else weakly single-plateaued preferences.[11] Roughly speaking, weakly refers to allowing for indifferences in the same side of the peak or the plateau. Cantala (2004) analyzes the particular subclass of weakly single-peaked preferences where one plateau at the lowest feasible level of utility is considered in both sides of the peak. He obtains the extreme minmax rules (a subclass of minmax rules whose outcome is always an agent's peak) as the unique strategy-proof and efficient rules. Note that with weakly single-peaked preferences, the class of strategy-proof rules with convex range (since we can show peak-onlyness) could be a strict subset of the class of minmax rules, not necessarily all.

Given the results in Theorems 1 and 2, a natural further research work is to obtain the characterization class of all strategy-proof and weakly non-bossy rules on $\mathcal{S}^n$. As Theorem 1 indicates the relaxation from non-bossy to weakly non-bossy includes rules with non-convex range as part of the expected characterization class. We already know that the class of strategy-proof and weakly non-bossy rules must be a subclass of Barberà and Jackson (1994)'s rules.

As we have seen non-bossyness turns out to be crucial in the characterization of strategy-proof rules with convex range for single-peakedness. Note also that on single-peaked preferences there does not exist a closed characterization of strategy-proof rules with non convex range. Barberà and Jackson (1994) obtain a characterization via a class of tie-breaking rules. However, neither in the single-peaked nor in the single-plateaued preferences domain, weak non-bossyness guarantees convex range. As we show in part (iii) of Proposition 2, strategy-proofness joint with plateau-invariance assures convexity of the range for single-plateuedness. Thus, one can think that plateau-invariance opens a possibility to obtain a closed characterization result in the line of those obtained for single-peakedness but for the case of plateaux.

In the rest of the section we pose some comments noting that the unique result valid for both domains is the one in Theorem 3. We do it showing that other relationships established in the paper hold only for one of our two domains. We begin showing that the results in Lemma 1, Theorems 1 and 2 do only hold for the single-peaked domain. Moreover, the results in Propositions 1 and 2 do only hold for the single-plateaued domain.

First, the result in Lemma 1 saying that strict Maskin monotonicity implies Maskin monotonicity does not hold for single-plateaued preferences. By Proposition 1 there is no Maskin monotonic rule but Example 8 proposes ones satisfying strict Maskin monotonicity.

Second, the equivalence between non-bossyness and convex range under strategy-proofness does not hold for social choice functions on $\mathcal{F}^n$. Any constant rule (the unique non-bossy ones on $\mathcal{F}^n$) has obviously convex range but there exist other strategy-proof rules with convex range. Only constant minmax

[11] Formally, a preference u_i is *weakly single-plateaued* if there exist $p^-(u_i)$, $p^+(u_i) \in A$, $p^-(u_i) \leqslant p^+(u_i)$ such that $[p^-(u_i), p^+(u_i)] = \{x \in A : u_i(x) \geqslant u_i(y)$, for all $y \in A\}$, and for any $y, z \in A$ such that $y < z \leqslant p^-(u_i)$, or $p^+(u_i) \leq z < y$, then $u_i(z) \geq u_i(y)$. We say that u_i is *weakly single-peaked* if it is a weakly single-plateaued preference such that $[p^-(u_i), p^+(u_i)]$ is degenerated to a single point, the peak.

rules are non-bossy. Moreover, as we show by means of the following example and Example 8, with single-plateaued preferences and under strategy-proofness, neither convex range implies weak non-bossyness nor the converse as one could think trying to state a counterpart result of Theorem 1.

Example 9 *(see Extended median voter schemes defined in Berga, 1998)* *Let $n \geq 2$. For any B closed interval in A, let $g^3(B) = \left(g_1^3(B), g_2^3(B), g_3^3(B)\right)$ where $g_1^3(B) = \min_{x \in B} x$, $g_2^3(B) = \frac{1}{2}(\min_{x \in B} x + \max_{x \in B} x)$, and $g_3^3(B) = \max_{x \in B} x$. Let $\{a_j\}$ be a set of $(3n+1)$ parameters in A such that $2(n+1)$ have value 1 and $(n-1)$ have value 0. For any $u \in \mathcal{F}^n$, define f as follows:*

$$f(u) = med\left\{g^3(\tau(u_1)), ..., g^3(\tau(u_n)), a_1, ..., a_{3n+1}\right\}.$$

By definition, f has closed and convex range. By Berga (1998), f is strategy-proof. However, f does not satisfy weak non-bossyness. To show that f violates weak non-bossyness (thus, strict Maskin monotonicity by Theorem 3) and plateau-invariance, let $n = 3$, $u \in \mathcal{F}^3$ such that $\tau(u_1) = \frac{1}{2}$, $\tau(u_2) = [\frac{1}{3}, \frac{2}{3}]$, and let $u_3, v_3 \in \mathcal{F}$ such that $\tau(u_3) = [\frac{3}{4}, 1]$ and $\tau(v_3) = [\frac{2}{3}, 1]$. Note that $f(u_3, u_{-3}) = \frac{7}{8} \in Interior[\tau(v_3)] \cap \tau(u_3)$, but $f(v_3, u_{-3}) = \frac{5}{6} < f(u_3, u_{-3})$. Moreover note that $f(v_3, u_{-3}) = \frac{5}{6} \in \tau(u_3) \subset \tau(v_3)$ but $f(u_3, u_{-3}) \neq f(v_3, u_{-3})$.

Third, the conclusion in Theorem 2 does not hold either for social choice functions on $\mathcal{F}^n$. In Example 1, we present a strategy-proof and weakly non-bossy social choice function on $\mathcal{F}^n$ that by Proposition 1 is not Maskin monotonic since it is not constant.

Fourth, Theorem 1 is obviously false for $\mathcal{S}^n$. Consider the median voter rule defined for single-peaked preferences with an odd number of agents. This rule is strategy-proof, Maskin monotonic, non-bossy, and quasi non-bossy, however, it is not constant.

Fifth, the results in Proposition 2 do not hold for a single-peaked domain. To show it consider Example 7 above and remember that, as we already mentioned in Section 1, plateau-invariance is implied by strategy-proofness on $\mathcal{S}^n$. Then, on the one hand, f in Example 7 satisfies strategy-proofness (thus, plateau-invariance) but it is not weakly non-bossy. On the other hand, if f in Example 7 was strictly Maskin monotonic, by Lemma 1 it would be Maskin monotonic. Then, by Theorem 2, it should satisfy weak non-bossyness which is false. That is, plateau-invariance on $\mathcal{S}^n$ (which is implied by strategy-proofness), implies neither weak non-bossy, nor strict Maskin monotonicity, nor convexity of the range.

Finally, observe that since strict Maskin monotonicity implies strategy-proofness on our two domains, then any strict Maskin monotonic rule on $\mathcal{D}^n$ has closed range.

5 Proofs

We devote this section to the proofs of all results. We start showing the equivalence between strict Maskin and Maskin monotonicity for single-peaked preferences.

Proof of Lemma 1. Let f be a strict Maskin monotonic social choice function on $\mathcal{S}^n$. Let $u \in \mathcal{S}^n$ and $v_i \in \mathcal{S}$ such that $L(f(u), u_i) \subseteq L(f(u), v_i)$ (say, condition B). To show that Maskin monotonicity holds, we must obtain that $f(v_i, u_{-i}) = f(u)$. Without loss of generality, suppose that $p(u_i) \geq f(u)$. Note that on $\mathcal{S}^n$, this fact joint with condition B imply that $p(v_i) \in [f(u), r^{u_i}(f(u))]$ and $r^{v_i}(f(u)) \leq r^{u_i}(f(u))$. Thus, it is trivial to see that the following two conditions hold: (i) for any $x \in A$ such that $u_i(f(u)) = u_i(x)$ then it also holds that $v_i(f(u)) \geq v_i(x)$ and (ii) for any $x \in A$ such that $u_i(f(u)) > u_i(x)$ then it also holds that $v_i(f(u)) > v_i(x)$. Thus, conditions of strict Maskin monotonicity holds and thus $f(v_i, u_{-i}) = f(u)$ which ends the proof. ■

Let us now prove the main results, all presented in Section 3. First, we show Theorem 1, that is, the equivalence of non-bossyness, convex range, and quasi non-bossyness under strategy-proofness for single-peaked preferences. We need the definition of *uncompromisingness* first introduced by Border and Jordan (1983).

Definition 9 *We say that f satisfies uncompromisingness if the following holds: Pick any $u \in \mathcal{D}^n$ and set $f(u) = z$. For all $j \in N$ and all $v_j \in \mathcal{D}$, we have $f(v_j, u_{-j}) = f(u)$ if either $z < p^-(u_j)$ and $z \leq p^-(v_j)$, or else $z > p^+(u_j)$ and $z \geq p^+(v_j)$.*

To show the equivalence between quasi non-bossyness and convex range for single-peaked preferences we need the following intermediate result that also holds for single-plateaued preferences. It says that strategy-proofness and quasi non-bossyness impose some additional property on f (say, Property L) which is, in fact, stronger than Maskin monotonicity by definition.

Lemma 2 *If a social choice function f on $\mathcal{D}^n$ is strategy-proof and quasi non-bossy then the following property (say, Property L) holds:*

"For all $u_i, v_i \in \mathcal{D}$ such that $L(x, u_i) \subseteq L(x, v_i)$ for all $x \in A_f$ we have that $f(u) = f(v_i, u_{-i})$ for all $u_{-i} \in \mathcal{D}^{n-1}$".

Proof of Lemma 2. Let f be a strategy-proof and quasi non-bossy social choice function. Let $u_i, v_i \in \mathcal{D}$ be such that $L(x, u_i) \subseteq L(x, v_i)$ for all $x \in A_f$. Note that by strategy-proofness, $u_i(f(u)) \geqslant u_i(f(v_i, u_{-i}))$ and $v_i(f(v_i, u_{-i})) \geqslant v_i(f(u))$ for all $u_{-i} \in \mathcal{D}^{n-1}$. Since $L(x, u_i) \subseteq L(x, v_i)$ for all $x \in A_f$, we have that $L(f(u), u_i) \subseteq L(f(u), v_i)$, and therefore $u_i(f(u)) \geqslant u_i(f(v_i, u_{-i}))$ implies that $v_i(f(u)) \geqslant v_i(f(v_i, u_{-i}))$. Then $v_i(f(u)) = v_i(f(v_i, u_{-i}))$ for all $u_{-i} \in \mathcal{D}^{n-1}$. By quasi non-bossyness, we have that $f(u) = f(v_i, u_{-i})$ for all $u_{-i} \in \mathcal{D}^{n-1}$. ■

Proof of Theorem 1.

We proceed showing the two following if and only if relationships: (i) f is non-bossy$\Longleftrightarrow$ (iv) f has convex range and (ii) f is quasi non-bossy $\Longleftrightarrow$ (iv) f has convex range. Note that the relationship (iii) f is an minmax rule $\Longleftrightarrow$ (iv)

f has convex range, has been already stated in the literature (see Theorem 2.4 in Sprumont, 1995).

$(i) \Longleftrightarrow (iv)$ Let f be a strategy-proof social choice function on $\mathcal{S}^n$. We first show that if f is non-bossy then it has convex range. By contradiction suppose that there exist $x, y \in A_f$, $x < y$, such that $(x, y) \subseteq A \backslash A_f$. Let $u, v \in \mathcal{S}^n$ such that $f(u) = x$ and $f(v) = y$. Let $w_i \in \mathcal{S}$ be such that $p(w_i) \in (f(u), f(v))$ and $w_i(f(u)) = w_i(f(v))$. Under u we distinguish three type of agents, $N^1 = \{i \in N$ such that $u_i(f(u)) = u_i(f(v))\}$, $N^2 = \{j \in N$ such that $u_j(f(u)) > u_j(f(v))\}$ and $N^3 = \{k \in N$ such that $u_k(f(v)) > u_k(f(u))\}$. We distinguish three cases:
Case 1. If any $i \in N^1$ announces w_i, by non-bossyness $f(w_i, u_{-i}) = f(u)$. Therefore, $f(w_{N^1}, u_{N^2}, u_{N^3}) = f(u)$.
Case 2. If any $j \in N^2$ announces w_j, $f(w_{N^1}, w_j, u_{N^2 \backslash \{j\}}, u_{N^3}) \in \{f(u), f(v)\}$ by strategy-proofness. Suppose to get a contradiction that $f(w_{N^1}, w_j, u_{N^2 \backslash \{j\}}, u_{N^3})=$ $f(v)$. By non-bossyness, $f(w_{N^1}, u_{N^2}, u_{N^3})= f(w_{N^1}, w_j, u_{N^2 \backslash \{j\}}, u_{N^3})= f(v)$, contradicting that $f(u) \neq f(v)$. Therefore $f(w_{N^1}, w_j, u_{N^2 \backslash \{j\}}, u_{N^3}) = f(u)$, and $f(w_{N^1}, w_{N^2}, u_{N^3}) = f(u)$.
Case 3. If any $k \in N^3$ announces w_k, by strategy-proofness at $(w_{N^1}, w_{N^2}, w_k, u_{N^3 \backslash \{k\}})$, $f(w_{N^1}, w_{N^2}, w_k, u_{N^3 \backslash \{k\}}) \in \{f(u), f(v)\}$. Suppose to get a contradiction that $f(w_{N^1}, w_{N^2}, w_k, u_{N^3 \backslash \{k\}}) = f(v)$. By non-bossyness, we obtain that $f(w_{N^1}, w_{N^2}, u_{N^3})$ $= f(w_{N^1}, w_{N^2}, w_k, u_{N^3 \backslash \{k\}}) = f(v)$, contradicting that $f(u) \neq f(v)$. Therefore $f(w_{N^1}, w_{N^2}, w_k, u_{N^3 \backslash \{k\}}) = f(u)$ and $f(w_{N^1}, w_{N^2}, w_{N^3}) = f(w) = f(u)$. Beginning from v, and from a similar argument than before we get that $f(w) = f(v)$ and we get the desired contradiction.
We now show that if f has convex range then it is non-bossy. By contradiction suppose that there exist i, u_i, $v_i \in \mathcal{S}$ and $u_{-i} \in \mathcal{S}^{n-1}$ such that $f(u) \neq f(v_i, u_{-i})$, and $u_i(f(u)) = u_i(f(v_i, u_{-i}))$. Suppose without loss of generality that $f(u) < f(v_i, u_{-i})$. Observe first that $f(u)$ and $f(v_i, u_{-i})$ are different from $p(u_i)$. By single-peakedness, $p(u_i) \in (f(u), f(v_i, u_{-i}))$. By strategy-proofness, $p(v_i) > f(u)$. By Lemma 2.1 in Sprumont (1995), we have that f is peak-only. Then, his Fact 2 and peak-onlyness imply uncompromisingness of f. Thus, it is straightforward to show that $f(v_i, u_{-i}) = f(u)$ and we get a contradiction.

$(ii) \Longleftrightarrow (iv)$ Let f be a strategy proof and quasi non-bossy social choice function. By contradiction, suppose that A_f is not convex. Then, there exist $a, b \in A_f$, $a < b$, such that $(a, b) \subseteq A \backslash A_f$. Let $v_i \in \mathcal{S}$ be such that $p(v_i) \in (a, b)$ and $v_i(a) = v_i(b)$, $u_i \in \mathcal{S}$ be such that $p(u_i) = a$ and $L(x, u_i) \subseteq L(x, v_i)$ for all $x \in A_f$, and $w_i \in \mathcal{S}$ be such that $p(w_i) = b$ and $L(x, w_i) \subseteq L(x, v_i)$ for all $x \in A_f$. Since f is strategy proof, we have that it is unanimous on the range. Therefore, $f(u_1, u_2, .., u_n) = a$, and $f(w_1, w_2, .., w_n) = b$. Beginning from u and changing one individual each time we have from Lemma 2 that $f(v_1, v_2, .., v_n) = a$. Beginning from w and changing one individual each time we have from Lemma 2 that $f(v_1, v_2, .., v_n) = b$. We get a contradiction with f being a social choice function.
The converse, that is, that any strategy-proof rule with convex range is quasi non-bossy, is straightforward by the implication $(iv) \Rightarrow (i)$ shown above. If f is

strategy-proof and has convex range is non-bossy and thus quasi non-bossy (by definition). ■

Second, we show that Maskin monotonicity is strictly stronger than strategy-proofness on $\mathcal{S}^n$. Moreover, they are equivalent adding weak non-bossyness.

Proof of Theorem 2. Let first show that any Maskin monotonic social choice function f on $\mathcal{S}^n$ is strategy-proof and weakly non-bossy. Suppose first that f is not weakly non-bossy. Then there exist $u_i, v_i \in \mathcal{S}$, and $u_{-i} \in \mathcal{S}^{n-1}$, such that $u_i(f(u)) = u_i(f(v_i, u_{-i}))$ and $v_i(f(u)) = v_i(f(v_i, u_{-i}))$ but $f(u) \neq f(v_i, u_{-i})$. Suppose, without loss of generality, that $f(u) < f(v_i, u_{-i})$. By single-peakedness, $p(u_i)$, $p(v_i) \in (f(u), f(v_i, u_{-i}))$. Therefore, $L\left(f(u), u_i\right) \subseteq L\left(f(u), v_i\right)$, and by Maskin monotonicity $f(u) = f(v_i, u_{-i})$, which is a contradiction.
Suppose now that f is not strategy-proof. Then there exist $i \in N$, $u_i, v_i \in \mathcal{S}$, and $u_{-i} \in \mathcal{S}^{n-1}$, such that $u_i(f(v_i, u_{-i})) > u_i(f(u))$. Thus, $f(v_i, u_{-i}) \neq f(u)$. Suppose, without loss of generality, that $p(u_i) < f(u)$. By single-peakedness, $f(v_i, u_{-i}) \in (r^{u_i}(f(u)), f(u))$. Let $w_i \in \mathcal{S}$ be such that $p(w_i) = f(v_i, u_{-i})$ and $L\left(f(u), u_i\right) \subseteq L\left(f(u), w_i\right)$. By Maskin monotonicity, $f(w_i, u_{-i}) = f(u)$. Since $L\left(f(v_i, u_{-i}), v_i\right) \subseteq L\left(f(v_i, u_{-i}), w_i\right) = A$, again by Maskin monotonicity $f(w_i, u_{-i}) = f(v_i, u_{-i})$ and we get a contradiction to the fact that $f(v_i, u_{-i}) \neq f(u)$.
Now, we show that any strategy-proof and weakly non-bossy social choice function is Maskin monotonic. Suppose not. Then there exist $u \in \mathcal{S}^n$, $i \in N$, and $v_i \in \mathcal{S}$ such that $L(f(u), u_i) \subseteq L(f(u), v_i)$ (say condition B) but $f(u) \neq f(v_i, u_{-i})$. Without loss of generality, let $f(u) < f(v_i, u_{-i})$. By strategy-proofness, $u_i(f(u)) \geqslant u_i(f(v_i, u_{-i}))$.
Note that if $f(u) = p(u_i)$ condition B implies that $p(v_i) = f(u)$. By strategy-proofness, $f(v_i, u_{-i}) = f(u)$ which is the desired contradiction.
Now suppose that $f(u) > p(u_i)$. By strategy-proofness, $u_i(f(u)) \geqslant u_i(f(v_i, u_{-i}))$. Then $f(v_i, u_{-i}) \notin (r^{u_i}(f(u)), f(u)]$. In order that condition B holds, $r^{v_i}(f(u)) \geq r^{u_i}(f(u))$ and $p(v_i) \leq f(u)$. This contradicts strategy-proofness since $v_i(f(u)) > v_i(f(v_i, u_{-i}))$.
Finally let $f(u) < p(u_i)$. By strategy-proofness, $f(v_i, u_{-i}) > p(u_i)$ and $r^{u_i}(f(u)) \leq f(v_i, u_{-i})$ (say I1). In order that condition B holds, $r^{v_i}(f(u)) \leq r^{u_i}(f(u))$ (say I2) and thus $p(v_i) < r^{u_i}(f(u))$. If one of the two inequalities (I1) and (I2) hold strictly, that is, if either $r^{u_i}(f(u)) < f(v_i, u_{-i})$ or else $r^{v_i}(f(u)) < r^{u_i}(f(u))$, then we get a contradiction to strategy-proofness since we have that $r^{v_i}(f(u)) < f(v_i, u_{-i})$. Otherwise, if $r^{v_i}(f(u)) = r^{u_i}(f(u)) = f(v_i, u_{-i})$, by weak non-bossyness we obtain that $f(v_i, u_{-i}) = f(u)$ which is the desired contradiction.

■

Previously to the proof of Proposition 1, we need to introduce the well-known notion of an *option set* and an intermediate result stated in Claim 1 below.

Definition 10 *Let f be a social choice function on $\mathcal{D}^n$. The option set of coalition S at $u_{N\setminus S}$ is the set*

$$o(u_{N\setminus S}) = \{x \in A \,\big|\, \text{there exists } u_S \in \mathcal{D}^s \text{ such that } f(u_S, u_{N\setminus S}) = x\}.$$

Claim 1 *Let f be a non-bossy social choice function on $\mathcal{F}^n$ with $\#A_f \geq 2$. Then, (i) $o(u_{-i})$ is a singleton for any $i \in N$ and $u_{-i} \in \mathcal{F}^{n-1}$. Moreover, (ii) $o(u_i)$ is also a singleton for any $i \in N$ and $u_i \in \mathcal{F}$.*

Proof of Claim 1. Let f be a non-bossy social choice function on $\mathcal{F}^n$ with $\#A_f \geq 2$. First, we show part (i) by contradiction. Consider $i \in N$ and $u_{-i} \in \mathcal{F}^{n-1}$ fixed. By contradiction, suppose that $o(u_{-i})$ is not a singleton. Let u_i, $v_i \in \mathcal{F}$ such that $f(u_i, u_{-i}) = x$ and $f(v_i, u_{-i}) = y$ where $x \neq y$. Let $\omega_i \in \mathcal{F}$ such that $\tau(\omega_i) = A$. By non-bossyness, $f(\omega_i, u_{-i}) = x$ and $f(\omega_i, u_{-i}) = y$ which is the desired contradiction.
To show part (ii), let $i \in N$, $u_i \in \mathcal{F}$, and $y \in o(u_i)$. Let $u_{-i} \in \mathcal{F}^{n-1}$ such that $f(u_i, u_{-i}) = y$. Consider any $v_{-i} \in \mathcal{F}^{n-1}$. We must show that $f(u_i, v_{-i}) = y$. We will do it changing one by one agents' preferences from u_{-i} to v_{-i} and applying $n-1$ times part (i) of this claim. Let $j \neq i$, $u_{-\{j\}} \in \mathcal{F}^{n-1}$, by part (i) of this claim, $f(v_j, u_{-j}) = y$. Let $k \neq \{i, j\}$ and let $\overline{u}_{-k} = (v_j, u_{-\{k,j\}})$. Applying part (i) again we obtain that $f(v_j, v_k, u_{-\{j,k\}}) = y$. Repeating this argument we get that $f(u_i, v_{-i}) = y$. ■

Claim 1 is crucial in the proof of part (i) in Proposition 1, that is, to show the inexistence of non-bossy rules on $\mathcal{F}^n$ apart from constant ones. Next we show that any rule on the single-plateaued domain with $\#A_f \geq 2$ is bossy, it violates Maskin monotonicity, and furthermore, constant rules are the only ones satisfying both strategy-proofness and quasi non-bossyness on $\mathcal{F}^n$.

Proof of Proposition 1. (i) Let f be a non-bossy social choice function on $\mathcal{F}^n$ with $\#A_f \geq 2$. Note that by definition of f as a social choice function, given $u \in \mathcal{F}^n$, $f(u) = \bigcap_{i \in N} [o(u_i) \cap o(u_{-i})]$. By Claim 1, both kinds of option sets are singletons. Thus, in order that f be well-defined, given $u \in \mathcal{F}^n$, $f(u) = o(u_i) = o(u_{-i})$ for any $i \in N$. This means that f is a constant function, i.e. for any $u \in \mathcal{F}^n$, $f(u) = x \in A$ (otherwise, if $x = f(u) = o(u_i) = o(u_{-i})$ and $y = f(v) = o(v_i) = o(v_{-i})$, $x \neq y$, then $f(v_i, u_{-i}) = o(v_i) = x$ and $f(v_i, u_{-i}) = o(u_{-i}) = y$ which is the desired contradiction).
(ii) Let f be a Maskin monotonic social choice function on $\mathcal{F}^n$ with $\#A_f \geq 2$. Let $A_f \supseteq \{x, y\}$, where $x \neq y$. Let $u, v \in \mathcal{F}^n$ be such that $f(u) = x$ and $f(v) = y$. Without loss of generality suppose that $x < y$. Let $w \in \mathcal{F}^n$, be such that for all $i \in N$, $\tau(w_i) = [x, y]$. Since f is Maskin monotonic, $f(w_i, u_{-i}) = f(u)$ since $L(x, u_i) \subset L(x, w_i) = A$ for any $i \in N$. Repeating the same argument for any agent $i \in N$, we obtain that $f(w) = f(u)$. Similarly, $f(w) = f(v)$. This implies that $x = y$, a contradiction.
(iii) It is straightforward by part (ii), Lemma 2 above and the fact that Property L implies Maskin monotonicity, as previously noted. ■

We now show that counterpart result of Theorem 2 for $\mathcal{F}^n$. Concretely, on $\mathcal{F}^n$, strategy-proofness and weak non-bossyness are equivalent to strict Maskin monotonicity.

Proof of Theorem 3. We first show that any strategy-proof and weakly non-bossy social choice function on $\mathcal{F}^n$ is strictly Maskin monotonic. Let $i \in N$, $u_i, v_i \in \mathcal{F}$, and $u_{-i} \in \mathcal{F}^{n-1}$ be such that for all $x \in A$, such that

$u_i(f(u)) = u_i(x)$ we have that $v_i(f(u)) \geqslant v_i(x)$ and $u_i(f(u)) > u_i(x)$ we have that $v_i(f(u)) > v_i(x)$. By strategy-proofness, $u_i(f(u)) \geqslant u_i(f(v_i, u_{-i}))$ and by the condition in strict Maskin monotonicity letting $x = f(v_i, u_{-i})$, we have that $v_i(f(u)) \geqslant v_i(f(v_i, u_{-i}))$. Again, by strategy-proofness $v_i(f(v_i, u_{-i})) \geqslant v_i(f(u))$. Therefore, $v_i(f(v_i, u_{-i})) = v_i(f(u))$ and $u_i(f(u)) = u_i(f(v_i, u_{-i}))$ (if $u_i(f(u)) > u_i(f(v_i, u_{-i}))$ we would have $v_i(f(u)) > v_i(f(v_i, u_{-i}))$ and i would manipulate f at (v_i, u_{-i}) via u_i). By weak non-bossyness, $f(u) = f(v_i, u_{-i})$.
We now show that if f is strictly Maskin Monotonic then it is weakly non-bossy and strategy-proof. To show weak non-bossyness, consider $i \in N$, u_i, $v_i \in \mathcal{F}$, and $u_{-i} \in \mathcal{F}^{n-1}$ such that $v_i(f(v_i, u_{-i})) = v_i(f(u))$ and $u_i(f(u)) = u_i(f(v_i, u_{-i}))$. If for any $x \in A$, $u_i(f(u)) = u_i(x)$ implies $v_i(f(u)) \geq v_i(x)$ and $u_i(f(u)) > u_i(x)$ implies $v_i(f(u)) > v_i(x)$ or else for any $x \in A$, $v_i(f(u)) = v_i(x)$ implies $u_i(f(u)) \geq u_i(x)$ and $v_i(f(u)) > v_i(x)$ implies $u_i(f(u)) > u_i(x)$, then by strict Maskin monotonicity we get that $f(v_i, u_{-i}) = f(u)$. Otherwise, let $w_i \in \mathcal{F}$ such that $\tau(w_i) = [\min\{f(u), f(v_i, u_{-i})\}, \max\{f(u), f(v_i, u_{-i})\}]$. Then, by strict Maskin monotonicity applied to $[u, (w_i, u_{-i})]$ and also to $[(v_i, u_{-i}), (w_i, u_{-i})]$ we have both that $f(w_i, u_{-i}) = f(u)$ and $f(w_i, u_{-i}) = f(v_i, u_{-i})$. Thus, since f is a social choice function $f(u) = f(v_i, u_{-i})$.
Finally, suppose that f is not strategy-proof. Then, there exists $i \in N$, $u_i, v_i \in \mathcal{F}$, and $u_{-i} \in \mathcal{F}^{n-1}$ such that $u_i(f(v_i, u_{-i})) > u_i(f(u))$. Let, without loss of generality, $f(v_i, u_{-i}) > f(u)$. Note that $f(v_i, u_{-i}) \in (f(u), r^{u_i}(f(u)))$ or else $f(v_i, u_{-i}) \in (f(u), 1]$ if there is no $x > p^+(u_i)$ such that $u_i(f(u)) = u_i(x)$. Let $w_i \in \mathcal{S}$, be such that $p(w_i) = f(v_i, u_{-i})$ and $w_i(f(u)) = w_i(r^{u_i}(f(u)))$. Note that the following holds: if $u_i(f(u)) = u_i(x)$ then $w_i(f(u)) \geqslant w_i(x)$ and if $u_i(f(u)) > u_i(x)$ then $w_i(f(u)) > w_i(x)$. Then, by strict Maskin monotonicity, $f(w_i, u_{-i}) = f(u)$. Since $p(w_i) = f(v_i, u_{-i})$, we have that if $v_i(f(v_i, u_{-i})) = v_i(x)$ then $w_i(f(v_i, u_{-i})) \geqslant w_i(x)$ and if $v_i(f(v_i, u_{-i})) > u_i(x)$ then $w_i(f(v_i, u_{-i})) > w_i(x)$. Again, by strict Maskin monotonicity, $f(v_i, u_{-i}) = f(w_i, u_{-i})$ and we get a contradiction. ■

Finally, in the next proof we show that for social choice functions on $\mathcal{F}^n$, strategy-proofness and plateau-invariance imply both weak non-bossyness and convex range.

Proof of Proposition 2. (i) Let f be a strategy-proof and plateau-invariant social choice function on $\mathcal{F}^n$. Let us show that f satisfies weak non-bossyness. Thus, let $u \in \mathcal{F}^n$, $i \in N$, $v_i \in \mathcal{F}$ such that $u_i(f(u)) = u_i(f(v_i, u_{-i}))$ (A1) and $v_i(f(u)) = v_i(f(v_i, u_{-i}))$ (A2) but $f(u) \neq f(v_i, u_{-i})$. Without loss of generality, suppose that $f(u) < f(v_i, u_{-i})$. Distinguish the following three cases (A, B and C).
<u>Case A</u>: $f(u)$, $f(v_i, u_{-i}) \notin \tau(u_i)$ and $f(u)$, $f(v_i, u_{-i}) \notin \tau(v_i)$. First note that by (A1) and (A2), $\tau(u_i), \tau(v_i) \subset (f(u), f(v_i, u_{-i}))$. Let $\omega_i \in \mathcal{F}$ such that $p^-(\omega_i) = f(u)$ and $p^+(\omega_i) = f(v_i, u_{-i})$. By strategy-proofness, $f(\omega_i, u_{-i}) \in \{f(u), f(v_i, u_{-i})\}$ (otherwise, either i would manipulate f at u via ω_i, or i would manipulate f at (ω_i, u_{-i}) via u_i). Suppose that $f(\omega_i, u_{-i}) = f(u)$ (a similar argument would work for $f(\omega_i, u_{-i}) = f(v_i, u_{-i})$). Let $\overline{v}_i \in \mathcal{F}$ such

that $\tau(\overline{v}_i) = [p^-(v_i), f(v_i, u_{-i})]$. By strategy-proofness, $f(\overline{v}_i, u_{-i}) \in \tau(\overline{v}_i)$ (otherwise, agent i would manipulate f at $(\overline{v}_i, u_{-i})$ via v_i). Moreover, also by strategy-proofness, $f(\overline{v}_i, u_{-i}) = f(v_i, u_{-i})$ (otherwise, agent i would manipulate f at (v_i, u_{-i}) via $\overline{v}_i$). By plateau-invariance, $f(\overline{v}_i, u_{-i}) = f(\omega_i, u_{-i}) = f(u)$ which is the desired contradiction.
Case B: $f(u)$, $f(v_i, u_{-i}) \in \tau(u_i)$ and $f(u)$, $f(v_i, u_{-i}) \in \tau(v_i)$. Without loss of generality, suppose that $f(u) < f(v_i, u_{-i})$. Consider the following subcases: (1) $f(u) \in Interior[\tau(u_i)]$, (2) $f(u) = p^-(u_i) = p^-(v_i)$, (3) $f(u) = p^-(u_i)$, $p^-(v_i) < p^-(u_i)$ and $f(v_i, u_{-i}) \in Interior[\tau(v_i)]$, (4) $f(u) = p^-(u_i)$, $p^-(v_i) < p^-(u_i)$, $f(v_i, u_{-i}) = p^+(v_i) = p^+(u_i)$, (5) $f(u) = p^-(u_i)$, $p^-(v_i) < p^-(u_i)$, $f(v_i, u_{-i}) = p^+(v_i)$, $p^+(u_i) > f(v_i, u_{-i})$. Note that for cases 1 to 4 we obtain that $f(u) = f(v_i, u_{-i})$ straightforwardly by plateau-invariance. For case 5, define $\omega_i \in \mathcal{F}$ such that $\tau(\omega_i) = [p^-(u_i), p^+(v_i)]$. By plateau-invariance, we obtain both that $f(\omega_i, u_{-i}) = f(u)$ and $f(\omega_i, u_{-i}) = f(v_i, u_{-i})$ which means that $f(u) = f(v_i, u_{-i})$.
Case C (identical argument to Case A): $f(u)$, $f(v_i, u_{-i}) \in \tau(u_i)$ and $f(u)$, $f(v_i, u_{-i}) \notin \tau(v_i)$ (a similar argument would follow if $f(u)$, $f(v_i, u_{-i}) \notin \tau(u_i)$ and $f(u)$, $f(v_i, u_{-i}) \in \tau(v_i)$). First note that by (A2), $\tau(v_i) \subset (f(u), f(v_i, u_{-i}))$. Let $\omega_i \in \mathcal{F}$ such that $\tau(\omega_i) = [f(u), f(v_i, u_{-i})]$. By plateau-invariance, $f(\omega_i, u_{-i}) = f(u)$. Let $\omega'_i \in \mathcal{F}$ such that $\tau(\omega'_i) = [p^-(v_i), f(v_i, u_{-i})]$. By plateau-invariance, $f(\omega'_i, u_{-i}) = f(v_i, u_{-i})$. But since $p^+(\omega_i) = p^+(\omega'_i) = f(v_i, u_{-i})$ and $f(\omega'_i, u_{-i}) = f(v_i, u_{-i})$,by plateau-invariance, $f(\omega_i, u_{-i}) = f(\omega'_i, u_{-i})$ which is the desired contradiction.
(*ii*) Note that strict Maskin monotonicity is straightforward by part (i) and Theorem 3.
(*iii*) Let f be a strategy-proof and plateau-invariant social choice function on $\mathcal{F}^n$. We now show that f has convex range. By contradiction suppose that there exist $x, y \in A_f$, $x < y$, such that $(x, y) \subseteq A \backslash A_f$. Let $u, v \in \mathcal{F}^n$ such that $f(u) = x$ and $f(v) = y$. Let $w_i \in \mathcal{F}$ be such that $\tau(w_i) = [f(u), f(v)]$ and $w'_i \in \mathcal{F}$ be such that $\tau(w'_i) = [f(u), f(u) + \varepsilon]$ where $\varepsilon > 0$ and $f(u) + \varepsilon < f(v)$. Consider the following argument.
Given a preference profile u we distinguish three types of agents: $N_u^1 = \{i \in N$ such that $\{f(u), f(v)\} \in \tau(u_i)\}$, $N_u^2 = \{j \in N \backslash N_u^1$ such that $u_j(f(u)) \geqslant u_j(f(v))\}$, and $N_u^3 = \{k \in N$ such that $u_k(f(v)) > u_k(f(u))\}$. We now change agents' preferences from u to w in the following order: first (step 1) change preferences of each agent in N_u^1, second (step 2), change preferences of each agent in N_u^2 and last (step 3) change preferences of each agent in N_u^3.
Step 1. If $i \in N_u^1$ announces w_i, by plateau-invariance $f(w_i, u_{-i}) = f(u)$.
Repeating the same argument $\#N_u^1$ times, changing the preference of a single agent in N_u^1 each time, we obtain that $f(w_{N_u^1}, u_{N_u^2}, u_{N_u^3}) = f(u)$.
Step 2. If $j \in N_u^2$ announces w'_j, by strategy-proofness $f(w_{N_u^1}, w'_j, u_{N_u^2 \backslash \{j\}}, u_{N_u^3}) = f(u)$ (otherwise, j would manipulate f at $(w_{N_u^1}, w'_j, u_{N_u^2 \backslash \{j\}}, u_{N_u^3})$ via u_j). By plateau-invariance, $f(w_{N_u^1}, w_j, u_{N_u^2 \backslash \{j\}}, u_{N_u^3}) = f(w_{N_u^1}, w'_j, u_{N_u^2 \backslash \{j\}}, u_{N_u^3}) = f(u)$. Therefore $f(w_{N_u^1}, w_j, u_{N_u^2 \backslash \{j\}}, u_{N_u^3}) = f(u)$.
Repeating the same argument $\#N_u^2$ times, changing the preference of a single agent in N_u^2 each time, we obtain that $f(w_{N_u^1}, w_{N_u^2}, u_{N_u^3}) = f(u)$.

Step 3. If $k \in N_u^3$ announces w_k, by strategy-proofness $f(w_{N_u^1}, w_{N_u^2}, w_k, u_{N_u^3 \setminus \{k\}}) \in \{f(u), f(v)\}$ (otherwise, k would manipulate f at $(w_{N_u^1}, w_{N_u^2}, w_k, u_{N_u^3 \setminus \{k\}})$ via u_k). Moreover, also by strategy-proofness $f(w_{N_u^1}, w_{N_u^2}, w_k, u_{N_u^3 \setminus \{k\}}) = f(u)$, otherwise, k would manipulate f at $(w_{N_u^1}, w_{N_u^2}, u_{N_u^3})$ via w_k. Repeating the same argument $\#N_u^3$ times, changing the preference of a single agent in N_u^3 each time, we obtain that $f(w_{N_u^1}, w_{N_u^2}, w_{N_u^3}) = f(u)$.

Beginning from v, and using a similar argument than above we get that $f(w) = f(v)$ which is the desired contradiction. ■

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