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Pedro Calleja, Carles Rafels and Stef Tijs

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Pedro Calleja^{*,†} and Carles Rafels[†].
Department of Actuarial, Financial and
Economic Mathematics and CREB, University of Barcelona,
Av.Diagonal 690, 08034 Barcelona, Spain.

Stef Tijs.
Center and Department of Econometrics and
Operations Research, Tilburg University,
P.O. Box 90153, 5000 LE Tilburg, The Netherlands.

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Abstract

The main objective of the paper is to study the locus of all core selection and aggregate monotonic point solutions of a TU-game: the aggregate-monotonic core. Furthermore, we characterize the class of games for which the core and the aggregate-monotonic core coincide. Finally, we introduce a new family of rules for TU-games which satisfy core selection and aggregate monotonicity.

Keywords: cooperative game theory, core, aggregate monotonicity.

1 Introduction.

The core of a cooperative TU-game, Gillies (1953, 1959), is one of the most important and intuitive solution concepts. Roughly speaking, the core is the set of feasible outcomes that can not be improved upon by any coalition of players. From Bondareva (1963) and Shapley (1967), we know algebraic conditions of the characteristic function that guarantee its non-emptiness. Since the core of a cooperative game may be empty, its generalizations and modifications have been taken into account from the very beginning (for details see Kannai, 1992).

If one is interested in solutions, the core selection property seems natural to be requested. It says that if a game has a non-empty core, then the proposed solution has to belong to it. Among the main two generic single point solutions,

*Corresponding author. E-mail adress: calleja@ub.edu. Telephone: ++34 93 402 90 28. Fax: ++34 93 403 48 92.

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the (pre)nucleolus (Schmeidler, 1969) satisfies this property, while the Shapley value (Shapley, 1953) does not.

Another important and intuitive property for a solution concept is aggregate monotonicity, introduced by Megiddo (1974). Roughly speaking it says that everybody will be weakly better off if only the worth of the grand coalition (efficiency level) grows.

The Shapley value satisfies this property. In fact it satisfies a stronger version: coalitional monotonicity, which is aggregate monotonicity extended to all coalitional values. The nucleolus does not satisfy aggregate monotonicity (Megiddo, 1974), not even on the class of convex games (Hokari, 2000) or on the class of veto balanced games (Arin and Feltkamp, 2005).

From the above results there seems to be some kind of incompatibility between monotonicity properties and core selection for solution concepts. In fact, such an incompatibility was shown by Young (1985) and Housman and Clark (1998). It is proved that there is no point solution concept in the whole class of cooperative TU-games satisfying core selection and coalitional monotonicity properties for $n \geq 4$.

Nevertheless, core selection and aggregate monotonicity are compatible in the class of cooperative TU-games independently of the number of agents involved. This is important, and can be verified by looking at the per-capita prenucleolus, a variant of the classical prenucleolus, defined by means of the per-capita excesses instead of the classical excesses. This point solution concept satisfies core selection and aggregate monotonicity (see for example Moulin, 1988 or Young et al., 1982).

This paper is devoted to analyze the behavior of point solution concepts when we combine core selection and aggregate monotonicity properties.

Clearly, these two properties are independent. However, they are also mutually conditioned. In particular, there may be core elements never selected by point solutions satisfying both properties. Given an arbitrary game, there always exists a minimum worth of the grand coalition from which any level of efficiency gives rise to a balanced game. Any point solution having the core selection property must pick out a core element at this minimum balancedness level game. For balanced games, this might imply, by aggregate monotonicity, that not every core element of the original game could be attainable by a solution satisfying both properties. Consider for instance the three player game¹: $v(i) = 0$ for all $i \in N$, $v(12) = v(13) = 1$, $v(23) = 0$ and $v(123) = 3$, and the core element $(0, 1, 2)$. This core allocation is never attainable by a core selection and aggregate monotonic point solution due to the fact that the core of the minimum balancedness level game reduces to $(1, 0, 0)$.

The point solutions that are candidates to satisfy core selection and aggregate monotonicity are those attainable from the core of the minimum balancedness level game imposing aggregate monotonicity. The set formed for all these points is always well defined for any cooperative game, and consists of a subset of the core of the original game whenever it is balanced. Such a set has

¹As usual, we write $v(i), v(12), \dots$ instead of $v(\{i\}), v(\{1, 2\}), \dots$

been already used (only for balanced games) to study time consistency of solutions for dynamic cooperative games in Dementieva (2004). We call this set the aggregate-monotonic core and we study it in section 3. We prove that this is the locus of all core selection point solutions which are aggregate monotonic. We also determine those games for which the core and the aggregate-monotonic core coincide. In section 4, we introduce sequential-maximization rules, a class of point solutions satisfying core selection and aggregate monotonicity. Finally, in section 5 we conclude with some final remarks.

2 Notation and preliminaries

A cooperative TU-game (a game) is a pair (N, v) (v for short) where $N = \{1, \dots, n\}$ is the set of players and $v : 2^N \rightarrow \mathbb{R}$ the characteristic function, with $v(\emptyset) = 0$; $v(S)$ is the worth of coalition S . For any coalition $S \subseteq N$, $|S|$ denotes the number of players in S , and e_S the characteristic vector of \mathbb{R}^N associated to coalition S , i.e. $e_{S,i} = 1$ if $i \in S$ and $e_{S,i} = 0$ if $i \notin S$; we use e_i instead of $e_{\{i\}}$ if no confusion arises. By G^N we denote the space of all TU-games with player set N . One of the main purposes of the theory of cooperative games is to study solutions or allocations of the total amount that players can achieve together. A vector $x \in \mathbb{R}^N$ distributing the worth of the grand coalition, i.e. $\sum_{i \in N} x_i = v(N)$,

is called a preimputation or an efficient vector. The preimputation set of a game (N, v) is denoted by $I^*(v)$. Formally, a point solution concept (a point solution for short) is a function $\alpha : G^N \rightarrow \mathbb{R}^N$, such that $\alpha(v) \in I^*(v)$ for any $v \in G^N$.

The core, $C(v)$, of a game (N, v) (Gillies, 1959) consists of the payoff vectors satisfying coalitional rationality and efficiency, formally,

$$C(v) = \{x \in \mathbb{R}^N : x(S) \geq v(S) \text{ for all } S \subset N \text{ and } x(N) = v(N)\},$$

where $x(S) = \sum_{i \in S} x_i$, by convention $x(\emptyset) = 0$ and by \subset we denote strict set inclusion, whereas \subseteq denotes weak set inclusion.

A collection $C = \{S_1, \dots, S_r\}$ of non empty subsets of N is said to be balanced if there exist positive constants $\gamma_1, \dots, \gamma_r \in \mathbb{R}_{++}$, the balancing coefficients of C , such that $\sum_{j:i \in S_j} \gamma_j = 1$ for all $i \in N$. A minimal balanced collection is

a balanced collection which balancing coefficients are unique or equivalently a balanced collection such that no proper subcollection is balanced (see Owen, 1995). We denote by \mathcal{C}_m^N the set of all minimal balanced collections over N . A game is said to be balanced if the following inequality holds:

$$\sum_{j=1}^r \gamma_j v(S_j) \leq v(N),$$

for all $C = \{S_1, \dots, S_r\} \in \mathcal{C}_m^N$. By B^N we denote the set of all balanced games.

According to a well known theorem (Bondareva, 1963 and Shapley, 1967) the core of a game is non-empty if and only if the game is balanced.

Convex games (Shapley, 1971), are an important class of balanced games. A game $v \in G^N$ is said to be convex if $v(T) + v(S) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$.

We are interested in combining two properties of point solutions. The first one is core selection. A point solution α is said to satisfy the core selection property (**CS**) if whenever the game is balanced, $v \in B^N$, then $\alpha(v) \in C(v)$.

The second property is aggregate monotonicity (**AM**). A point solution is said to satisfy aggregate monotonicity (Megiddo, 1974) if for any two games, $v, v' \in G^N$, with $v(S) = v'(S)$ for all $S \subset N$ and $v(N) < v'(N)$, it holds that $\alpha(v) \leq \alpha(v')$, where \leq in \mathbb{R}^N is the standard partial order, i.e. $x \leq y$ if $x_i \leq y_i$, for all $i \in N$. Aggregate monotonicity states that if only the value of the grand coalition grows, no player can suffer from it.

In the next section, the concept of large core plays an important role. A game (N, v) is said to have a large core (Sharkey, 1982) if for every vector $y \in \mathbb{R}^N$ with $y(S) \geq v(S)$ for all $S \subseteq N$ there exists a core element $x \in C(v)$ with $x \leq y$. Convexity of a game is a sufficient condition for largeness of the core (Sharkey, 1982).

A vector y satisfying the conditions $y(S) \geq v(S)$ for all $S \subset N$ (note that we take all $S \neq N$) will be called an *upper vector* of the game (N, v) . The set of all upper vectors of the game (N, v) is denoted by $U(v)$; formally, $U(v) = \{y \in \mathbb{R}^N : y(S) \geq v(S) \text{ for all } S \subset N\}$. The following theorem due to van Gellekom et al. (1999) connects largeness of the core with the extreme points of $U(v)$. Given a convex set $A \subseteq \mathbb{R}^N$, we say that $x \in A$ is an extreme point of A if $y, z \in A$ and $x = \frac{1}{2}y + \frac{1}{2}z$ imply $y = z$.

Theorem 1 (*van Gellekom et al., 1999*) *Let (N, v) be a balanced game. Then (N, v) has a large core if and only if $z(N) \leq v(N)$ for all extreme points z of $U(v)$.*

The above theorem together with the next interesting result from Ichiishi (1990) are the tools to prove one of the main results of the paper. Ichiishi (1990) introduces the *extended exact envelope* of a balanced game v as the function $\bar{v} : \mathbb{R}_+^N \rightarrow \mathbb{R}$ defined by $\bar{v}(p) = \min_{x \in C(v)} p \cdot x$, where $p \cdot x$ denotes the Euclidean scalar product of p and x , $\sum_{i \in N} p_i x_i$. And the result states the following:

Theorem 2 (*Ichiishi, 1990*) *Let v, w be balanced games, and let \bar{v} and \bar{w} be their extended exact envelopes respectively. Then, the following two conditions are equivalent:*

- (1) *For every $y \in C(w)$ there exists $x \in C(v)$ such that $x \leq y$.*
- (2) *$\bar{v}(p) \leq \bar{w}(p)$ for every $p \in \mathbb{R}_+^N$.*

Moulin (1990) introduces totally large cores. A game (N, v) has a totally large core if each one of the subgames (S, v^S) has a large core (for all $\emptyset \neq S \subseteq N$). Here a subgame (S, v^S) of a game (N, v) is a game with player set S and characteristic function $v^S(T) = v(T)$ for all $T \subseteq S$. Moulin (1990) connects convexity and totally largeness of the core.

Theorem 3 (Moulin, 1990) *The game (N, v) is convex if and only if it has a totally large core.*

3 The aggregate-monotonic core.

Given an arbitrary game v we focus on the set of efficient allocations which a point solution should pick out to hold core selection and aggregate monotonicity. With this aim we first define the *root* game associated to v .

Definition 4 *The root game v_r of a given game $v \in G^N$ is defined by $v_r(S) = v(S)$ for all $S \subset N$ and $v_r(N) = \min_{x \in \mathbb{R}^N} \{x(N) : x(S) \geq v(S) \text{ for all } S \subset N\}$. Moreover, a game v is said to be rooted if it coincides with its root game v_r , $v = v_r$.*

Note that the *root* game coincides with the original one in all coalitional values except the grand coalition. Instead, we take the minimum level of efficiency in order to get balancedness. Indeed,

$$v_r(N) = \max_{\substack{C=\{S_1, \dots, S_r\} \in \mathcal{C}_m^N \\ C \neq \{N\}}} \left\{ \sum_{j=1}^r \gamma_j v(S_j) \right\},$$

and the maximum is always attained in a minimal balanced collection.

The *root* game (N, v_r) is uniquely determined and can be alternatively described as $v_r = v + \varepsilon_r \cdot u_N$, where $\varepsilon_r = \min \{\varepsilon \in \mathbb{R} : v + \varepsilon \cdot u_N \in B^N\}$, and u_S , $\emptyset \neq S \subseteq N$ is the element of the well known unanimity basis of the linear space G^N , where $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise. Note also that a game v can be rewritten in terms of its *root* game, in fact,

$$v = v_r + (v(N) - v_r(N)) \cdot u_N,$$

where the coefficient $(v(N) - v_r(N))$ does not need to be positive. In fact, if $v(N) \geq v_r(N)$ then $C(v) \neq \emptyset$, while if $v(N) < v_r(N)$ then $C(v) = \emptyset$.

Next we define the central concept of the paper which is the aggregate-monotonic core.

Definition 5 *The aggregate-monotonic core of (N, v) , $\mathcal{AC}(v)$, is defined by*

$$\mathcal{AC}(v) = C(v_r) + (v(N) - v_r(N)) \cdot \Delta_n,$$

where Δ_n denotes the unit-simplex, i.e. $\Delta_n = \{x \in \mathbb{R}_+^N : x_1 + \dots + x_n = 1\}$.

Let us point out that the aggregate-monotonic core is well defined since v_r is always a balanced game. The aggregate-monotonic core suggests a new approach when looking for solutions of a cooperative phenomenon. Indeed, now a cooperative situation can be faced as a problem of allocating the worth of the grand coalition at cooperation birth, that is in the root game, and afterwards

bring this allocation up or down to the final efficiency level in a reasonable (monotonic) way.

By following this procedure one can easily check that an allocation in the aggregate-monotonic core of v is an allocation in the core of v , in case the core is non empty, i.e. $\mathcal{AC}(v) \subseteq C(v)$ whether $v \in B^N$. Moreover, from the definition it follows easily that if $v(N) \geq v_r(N)$ then $\mathcal{AC}(v) = \bigcup_{y \in C(v_r)} \{x \in I^*(v) : x \geq y\}$, while if $v(N) \leq v_r(N)$ then $\mathcal{AC}(v) = \bigcup_{y \in C(v_r)} \{x \in I^*(v) : x \leq y\}$.

The relevance of the aggregate-monotonic core is given in next proposition where we denote by \mathbb{S}_{AMCS}^N the set of all point solutions $\alpha : G^N \rightarrow \mathbb{R}^N$ which satisfy core selection and aggregate monotonicity.

Theorem 6 *Given an arbitrary game $v \in G^N$, the aggregate-monotonic core is the locus of all core selection and aggregate monotonic point solutions, that is $\mathcal{AC}(v) = \{\alpha(v) : \alpha \in \mathbb{S}_{AMCS}^N\}$.*

Proof. Given an arbitrary game v , we first show that $\{\alpha(v) : \alpha \in \mathbb{S}_{AMCS}^N\} \subseteq \mathcal{AC}(v)$. It is enough to see that for an arbitrary point solution $\alpha : G^N \rightarrow \mathbb{R}^N$ satisfying the core selection and aggregate monotonicity properties, then $\alpha(v) \in \mathcal{AC}(v)$. Let v_r be the *root* game of v . First, if $v(N) = v_r(N)$ then $v = v_r$ and clearly $\mathcal{AC}(v) = C(v)$, hence by **CS**, $\alpha(v) \in \mathcal{AC}(v)$. Second, if $v(N) > v_r(N)$ by **CS**, $\alpha(v_r) \in C(v_r)$, moreover by **AM**, $\alpha(v) \geq \alpha(v_r)$; since $\alpha(v) \in I^*(v)$ it follows that $\alpha(v) \in \mathcal{AC}(v)$. Finally, if $v(N) < v_r(N)$ by **CS**, $\alpha(v_r) \in C(v_r)$, and by **AM**, $\alpha(v) \leq \alpha(v_r)$; since $\alpha(v) \in I^*(v)$ it follows that $\alpha(v) \in \mathcal{AC}(v)$ and we are finished.

To show that $\mathcal{AC}(v) \subseteq \{\alpha(v) : \alpha \in \mathbb{S}_{AMCS}^N\}$, let x be an arbitrary element of the aggregate-monotonic core, i.e. $x \in \mathcal{AC}(v)$. It is enough to show that there exists a point solution $\alpha : G^N \rightarrow \mathbb{R}^N$ satisfying **CS** and **AM** such that $\alpha(v) = x$.

To define α , consider any game $w \in G^N$ such that $w_r \neq v_r$. Let x^{w_r} be an arbitrary element of $C(w_r)$, i.e. $x^{w_r} \in C(w_r)$; then, we define

$$\alpha(w) = x^{w_r} + \frac{w(N) - w_r(N)}{n} \cdot e_N.$$

Next, consider any game $w \in G^N$ such that $w_r = v_r$. We need to distinguish three cases:

- 1) If $v = v_r$, then we define $\alpha(w) = x + \frac{w(N) - v_r(N)}{n} \cdot e_N$.
- 2) If $v(N) > v_r(N)$, then and since $x \in \mathcal{AC}(v)$ there exists $x^{v_r} \in C(v_r)$ such that $x^{v_r} \leq x$. In this case, we define

$$\alpha(w) = x^{v_r} + \frac{w(N) - x^{v_r}(N)}{x(N) - x^{v_r}(N)} \cdot (x - x^{v_r}).$$

- 3) If $v(N) < v_r(N)$, then and since $x \in \mathcal{AC}(v)$ there exists $x^{v_r} \in C(v_r)$ such that $x^{v_r} \geq x$. In this case, we define

$$\alpha(w) = x^{v_r} + \frac{w(N) - x^{v_r}(N)}{x(N) - x^{v_r}(N)} \cdot (x - x^{v_r}).$$

It is straightforward to see that α is a core selection and aggregate monotonic point solution satisfying $\alpha(v) = x$. ■

As one may expect, the aggregate-monotonic core of a game may be a proper subset of the core, even for convex games. One can easily check that for the three players game: $v(i) = 0$ for all $i \in N$, $v(12) = v(13) = 1$, $v(23) = 0$ and $v(123) = 3$, $v_r(N) = 1$ and $\mathcal{AC}(v) \subset C(v)$. Note that the game v is convex, and in fact it corresponds to the bankruptcy game associated to the bankruptcy problem with estate $E = 3$, and claims $c_1 = 3$ and $c_2 = c_3 = 2$ (see O'Neill, 1982 or Thomson, 2003).

A natural question is for which games the aggregate-monotonic core coincides with the core. In the following example we show that both sets might coincide.

Example 7 Let (N, v) be the three player convex game defined by $v = u_{12} + 2 \cdot u_{123}$. It is easy to see that $v_r(N) = 1$ and consequently $v_r = u_{12}$. Moreover, $C(v) = C(u_{12} + 2 \cdot u_{123}) = C(u_{12}) + 2 \cdot C(u_{123}) = C(u_{12}) + 2 \cdot \Delta_3 = \mathcal{AC}(v)$.

Next we show that the coincidence of the core and the aggregate-monotonic core of an arbitrary non-rooted game depends completely on the largeness of the core of its root game. Note that for any root game v_r it holds that $\mathcal{AC}(v_r) = C(v_r)$.

Theorem 8 Let (N, v) be a non-rooted balanced game and let (N, v_r) be its root game. Then $\mathcal{AC}(v) = C(v)$ if and only if (N, v_r) has a large core.

Proof. Let us prove first the if part; since v is a non-rooted balanced game we have $v(N) > v_r(N)$. Take an arbitrary $y \in C(v)$, clearly $y(S) \geq v(S) \geq v_r(S)$ for all $S \subseteq N$; hence y is an upper vector of (N, v_r) . Since (N, v_r) has a large core there exists $x \in C(v_r)$ such that $x \leq y$. Therefore $y \in \mathcal{AC}(v)$. Moreover, $\mathcal{AC}(v) \subseteq C(v)$ and consequently $\mathcal{AC}(v) = C(v)$.

Proving the only if part will require more arguments. Let us suppose that the root game does not have a large core, we will show the fact that $\mathcal{AC}(v) \subset C(v)$. Since v is a non-rooted balanced game we have $v(N) > v_r(N)$. Since v_r has not a large core, by Theorem 1 there exists an extreme point y^* of $U(v_r)$ with

$$y^*(N) > v_r(N). \quad (1)$$

By the fact that y^* is an extreme point of $U(v_r)$ there exists a set of coalitions $\mathcal{S} = \{S_1, \dots, S_n\}$ such that the vectors e_{S_1}, \dots, e_{S_n} form a basis of \mathbb{R}^N and $y^*(S_j) = v_r(S_j)$ for all $j = 1, \dots, n$. From this, observe that for all $i \in N$ there exists a $S \in \mathcal{S}$ with $i \in S$, and consequently there can not be any $x \in C(v_r)$ with $x \leq y^*$.

Define now the game (N, v_{y^*}) by $v_{y^*} = v_r + (y^*(N) - v_r(N)) \cdot u_N$ where coalitional worths do not vary from v_r , but the worth of the grand coalition increases up to $y^*(N)$, i.e. $v_{y^*}(N) = y^*(N)$. Clearly $y^* \in C(v_{y^*})$ since $y^* \in U(v_r)$ and $v_r(S) = v_{y^*}(S)$ for all $S \subset N$. Note also that $v_{y^*}(S) = v(S)$ for all $S \subset N$.

In the following, we structure the only if part of the proof in three different cases:

Case I) $v_{y^*}(N) = v(N)$. In this case $v_{y^*} = v$ and from the above discussion clearly $y^* \in C(v_{y^*})$ although $y^* \notin \mathcal{AC}(v_{y^*})$ due to the fact that there is no $x \in C(v_r)$ such that $x \leq y^*$. This finishes case I.

Case II) $v(N) < v_{y^*}(N)$. We know that $y^* \in C(v_{y^*})$ and there is no $x \in C(v_r)$ such that $x \leq y^*$. Hence, by Theorem 2 there exists $p \in \mathbb{R}_+^N$ such that $\bar{v}_r(p) > \bar{v}_{y^*}(p)$; that is, $p \cdot \hat{x} = \min_{x \in C(v_r)} p \cdot x > \min_{y \in C(v_{y^*})} p \cdot y = p \cdot \hat{y}$ where $\hat{x} \in C(v_r)$ and $\hat{y} \in C(v_{y^*})$.

Define the vectors $z_\lambda = \lambda \hat{x} + (1 - \lambda) \hat{y}$ for all $\lambda \in (0, 1)$ and let $\hat{\lambda}$ be such that $z_{\hat{\lambda}}(N) = v(N)$, indeed $0 < \hat{\lambda} = \frac{\hat{y}(N) - v(N)}{\hat{y}(N) - \hat{x}(N)} = \frac{v_{y^*}(N) - v(N)}{v_{y^*}(N) - v_r(N)} < 1$ since $v_r(N) < v(N) < v_{y^*}(N)$. Clearly, $p \cdot z_{\hat{\lambda}} < p \cdot \hat{x}$. Furthermore, $z_{\hat{\lambda}}(S) = \hat{\lambda} \hat{x}(S) + (1 - \hat{\lambda}) \hat{y}(S) \geq v(S)$ for all $S \subset N$ since $\hat{x} \in C(v_r)$, $\hat{y} \in C(v_{y^*})$ and $v_r(S) = v_{y^*}(S) = v(S)$, hence $z_{\hat{\lambda}} \in C(v)$.

Finally, to finish with case II note that $\bar{v}_r(p) = p \cdot \hat{x} > p \cdot z_{\hat{\lambda}} \geq \min_{z \in C(v)} p \cdot z = \bar{v}(p)$. Again, by Theorem 2 there exists $z \in C(v)$ such that there is no $x \in C(v_r)$ with $x \leq z$, from which we conclude that $z \notin \mathcal{AC}(v)$.

Case III) $v(N) > v_{y^*}(N)$. We will define from y^* which is an extreme point of $U(v_r)$ a vector \bar{y}^* such that $\bar{y}^* \in C(v)$ and $\bar{y}^* \notin \mathcal{AC}(v)$. To do it, let $i \in N$ be such that

$$|\{S \in \mathcal{S} : i \in S\}| \leq |\{S \in \mathcal{S} : j \in S\}| \text{ for all } j \in N. \quad (2)$$

Notice that this player always exists, since we take one of the players belonging to the least minimum number of coalitions from the set $\mathcal{S} = \{S_1, \dots, S_n\}$.

Now, define $\bar{y}^* = y^* + (v(N) - v_{y^*}(N)) \cdot e_i$; note that $\bar{y}_j^* = y_j^*$ for all $j \neq i$, so we give all the extra surplus of the grand coalition to player i . It follows easily that $\bar{y}^* \in C(v)$ since $\bar{y}^*(N) = v(N)$ and $\bar{y}^*(S) \geq y^*(S) \geq v(S)$ for all $S \subset N$ due to $y^* \in U(v_r)$ and $v_r(S) = v(S)$ for all $S \subset N$.

To finish the proof let us see that $\bar{y}^* \notin \mathcal{AC}(v)$. Suppose on the contrary that there exists $x \in C(v_r)$ with $x \leq \bar{y}^*$.

Now, by R we denote the set of players different from player i , for which there is a coalition in \mathcal{S} that does not include player i , i.e. $R = \bigcup_{S \in \mathcal{S}: i \notin S} S$. Notice that $R \neq \emptyset$. Suppose the contrary, if $R = \emptyset$, then $i \in S$ for all $S \in \mathcal{S}$, and consequently $|\{S \in \mathcal{S} : i \in S\}| = n$. Since by definition $|\{S \in \mathcal{S} : i \in S\}| \leq |\{S \in \mathcal{S} : j \in S\}|$ for all $j \in N$ it follows that $|\{S \in \mathcal{S} : j \in S\}| = n$ for all $j \in N$, which contradicts the fact that the corresponding set of characteristic vectors forms a basis of \mathbb{R}^N .

First, we will show that $x_j = \bar{y}_j^* = y_j^*$ for all $j \in R$. Let j be an arbitrary player in R , there exists a coalition $S \in \mathcal{S}$ with $j \in S$ and $i \notin S$ such that $\bar{y}^*(S) = y^*(S) = v_r(S) = v(S)$. Since $x \in C(v_r)$ it follows that $x(S) \geq v_r(S) = v(S) = \bar{y}^*(S)$. Moreover $x(S) \leq \bar{y}^*(S)$ due to $x \leq \bar{y}^*$. Hence $x(S) = \bar{y}^*(S)$, but since $x \leq \bar{y}^*$ it follows that $x_j = \bar{y}_j^* = y_j^*$ for all $j \in R$.

Consequently, for the allocation x it holds that

$$x(R) = \bar{y}^*(R) = y^*(R). \quad (3)$$

Second, let us see that $x(N \setminus R) \geq y^*(N \setminus R)$.

To this end, let k be an arbitrary player in $N \setminus R$, note that $N \setminus R \neq \emptyset$ since $i \notin R$, and let $S \in \mathcal{S}$ be such that $k \in S$. Since $k \in N \setminus R$, it follows that $i \in S$ (if not, $k \in R$). Therefore, $|\{S \in \mathcal{S} : i \in S\}| \geq |\{S \in \mathcal{S} : k \in S\}|$. However, by hypothesis (2) the equality holds, which means that for any $S \in \mathcal{S}$ such that $i \in S$ then $k \in S$, and consequently all players in $N \setminus R$ go together in any coalition $S \in \mathcal{S}$.

Now, take an arbitrary coalition S in \mathcal{S} such that $i \in S$ then $N \setminus R \subseteq S$. Hence,

$$x(S) = x(S \cap R) + x(S \cap (N \setminus R)) = x(S \cap R) + x(N \setminus R). \quad (4)$$

Moreover, since $x \in C(v_r)$, it follows that

$$x(S) \geq v_r(S), \quad (5)$$

and due to $S \in \mathcal{S}$ we have that

$$v_r(S) = y^*(S) = y^*(S \cap R) + y^*(S \cap (N \setminus R)) = y^*(S \cap R) + y^*(N \setminus R). \quad (6)$$

From (4), (5) and (6), it follows that $x(S \cap R) + x(N \setminus R) \geq y^*(S \cap R) + y^*(N \setminus R)$. Since $x_j = y_j^*$ for all $j \in R$, for allocation x it holds that

$$x(N \setminus R) \geq y^*(N \setminus R). \quad (7)$$

Finally, from (3) and (7) we have that $x(N) = x(R) + x(N \setminus R) \geq y^*(R) + y^*(N \setminus R) = y^*(N) > v_r(N)$, where the last strict inequality follows from (1). This implies a contradiction with $x \in C(v_r)$.

Hence there is no $x \in C(v_r)$ with $x \leq \bar{y}^*$ which implies $\bar{y}^* \notin \mathcal{AC}(v)$, being $\bar{y}^* \in C(v)$. With this we finish case III and end the proof. ■

Note from the above theorem, that for any two arbitrary balanced games $v, w \in B^N$ with $v_r = w_r$ it follows that $\mathcal{AC}(v) = C(v)$ is equivalent to $\mathcal{AC}(w) = C(w)$. The next corollary summarizes these relations.

Corollary 9 *Let v_r be a root game. Then the following three conditions are equivalent:*

- (1) (N, v_r) has a large core.
- (2) For all $\varepsilon \geq 0$, $\mathcal{AC}(v_r + \varepsilon \cdot u_N) = C(v_r + \varepsilon \cdot u_N)$.
- (3) There exists $\varepsilon > 0$ such that $\mathcal{AC}(v_r + \varepsilon \cdot u_N) = C(v_r + \varepsilon \cdot u_N)$.

Proof. Straightforward from Theorem 8. ■

But Theorem 8 does also says something about games with a large core. In fact, it follows easily from the proof of the theorem that whenever an arbitrary

balanced game v has not a large core then there exists $y \in \mathbb{R}^N$, $y(S) \geq v(S)$ for all $S \subseteq N$ such that there is no $x \in C(v)$ with $x \leq y$ at any level of efficiency larger than $v(N)$. Furthermore, the theorem has a nice implication on the additivity of the cores of two games with different efficiency levels:

Corollary 10 *Let (N, v) and (N, w) be balanced games such that $w(S) = v(S)$ for all $S \subseteq N$ and $w(N) < v(N)$. Then $C(v) = C(w) + (v(N) - w(N)) \cdot \Delta_n$ if and only if (N, w) has a large core.*

Proof. The proof follows directly with the same arguments as in the proof of Theorem 8, but now with game w playing the role of the game v_r . ■

Moulin (1990) introduces totally large cores. A game (N, v) has a totally large core if each one of the subgames (S, v^S) has a large core (for all $\emptyset \neq S \subseteq N$). Moulin (1990) connects convexity of the game and total largeness of the core.

We are also interested in the possibility of extending the coincidence of the core and the aggregate-monotonic core to all subgames. With this aim we first define two classes of games. We say that a game (N, v) is a totally root-convex game if (S, v_r^S) is convex for all $S \subseteq N$, where (S, v_r^S) is the root game of the subgame (S, v^S) . Analogously, we say that a game (N, v) is a totally root-large core game if for every subgame the corresponding root game has a large core, i.e. $C(v_r^S)$ is large for all $S \subseteq N$, where (S, v_r^S) is the root game of the subgame (S, v^S) . The next corollary establishes this connection.

Corollary 11 *Let (N, v) be an arbitrary game, the following statements are equivalent:*

- 1) (N, v) is a totally root-large core game
- 2) (N, v) is a totally root-convex game
- 3) $\mathcal{AC}(v^S) = C(v^S)$ for all $\emptyset \neq S \subseteq N$

Proof. 1) \rightarrow 2) Let $\emptyset \neq S \subseteq N$ be an arbitrary coalition; by assumption $C(v_r^S)$ is large. Now, let $\emptyset \neq T \subset S$ be an arbitrary subcoalition of S ; by assumption $C(v_r^T)$ is large and by Theorem 1, $C(v^T)$ is also large. Hence (S, v_r^S) has a totally large core since each one of the subgames (T, v^T) for all $\emptyset \neq T \subset S$ has a large core. Applying Theorem 3 (S, v_r^S) is convex.

2) \rightarrow 3) Since convexity is a sufficient condition for largeness of the core, this implication follows easily applying Theorem 8.

3) \rightarrow 1) It follows easily from Theorem 8. ■

Notice that the convexity of the root game is not enough to characterize those games (N, v) for which $\mathcal{AC}(v^S) = C(v^S)$ for all $\emptyset \neq S \subseteq N$, as the next example shows.

Example 12 *Let (N, v) be the four player game: $v(i) = 0$ for all $i \in N$, $v(12) = v(13) = v(124) = v(134) = 1$, $v(123) = v(1234) = 2$ and $v(S) = 0$ otherwise. The game v is convex, in fact $v = v_r$. However, for the subgame*

associated to coalition $S = \{1, 2, 3\}$, which is also convex, it is easy to check that $C(v^S) = \text{ch}\{(2, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ where ch means convex hull, while $C(v_r^S) = \{(1, 0, 0)\}$, due to $v_r^S(123) = 1$. Hence $\mathcal{AC}(v^S) \subset C(v^S)$.

4 Sequential-maximization rules.

If we look at the better known point solutions, we realize that none of them satisfies core selection and aggregate monotonicity. As has been already noted, the Shapley value and the nucleolus fail to satisfy both properties together. The Tau value (Tijs, 1981) neither possesses the core selection property nor aggregate monotonicity, even in the class of convex games (Hokari and van Gellekom, 2002). The separable cost remaining benefits solution (see James and Lee, 1971) is neither aggregate monotonic nor core selection (Young et al., 1982). As far as we know, only the per-capita prenucleolus satisfies both properties. Then it seems interesting to introduce new point solutions satisfying these two properties.

From Theorem 6 we know that there can be many point solutions compatible with the two properties. In this section we introduce a new family of point solutions satisfying both. To this end, we use the aggregate-monotonic core concept.

The idea behind the solutions we will introduce is to solve a sequential maximization problem over the aggregate-monotonic core according to an ordering on the player set. Moreover, we are also interested in the average of such solutions over the set of orders. In case one uses the core instead of the aggregate-monotonic core and constrained to the class of balanced games the solutions appear to be those studied recently by Tijs (2005); and fail to have the aggregate monotonicity property. Changing the core by the aggregate-monotonic core has to important consequences. First, as the aggregate-monotonic core is non empty for all games, this new solution will be well defined for any $v \in G^N$. Second, it always selects an element of the aggregate-monotonic core, which is important to get core selection and aggregate monotonic solutions.

Let us define formally these sequential maximization rules. For this aim, an ordering $\theta = (i_1, \dots, i_n)$ of N is a bijection from N to N , and we denote by S_N the set of all possible orderings.

Definition 13 *Let v be a game. The sequential-maximization rule on the aggregate-monotonic core associated to the ordering $\theta = (i_1, \dots, i_n) \in S_N$ is the vector $\bar{x}^\theta(v) \in \mathbb{R}^N$ defined by,*

$$\begin{aligned} \bar{x}_{i_1}^\theta(v) &= \max_{x \in \mathcal{AC}(v)} \{x_{i_1}\} \text{ and} \\ \bar{x}_{i_k}^\theta(v) &= \max_{x \in \mathcal{AC}(v)} \{x_{i_k} : x_{i_l} = \bar{x}_{i_l}^\theta(v) \text{ for all } l \in \{1, \dots, k-1\}\} \text{ for } k = 2, \dots, n. \end{aligned}$$

The average sequential-maximization rule (ASM for short), is defined by

$$ASM(v) = \frac{1}{n!} \sum_{\theta \in S_N} \bar{x}^\theta(v)$$

Note that in a sequential-maximization rule the first player in an arbitrary ordering $\theta = (i_1, \dots, i_n)$ maximizes his potential gains over the aggregate-monotonic core, i.e. $\bar{x}_{i_1}^\theta(v) = \max_{x \in AC(v)} \{x_{i_1}\}$. The second player maximizes his payoff over the aggregate-monotonic core restricted to allocations such that $x_{i_1} = \bar{x}_{i_1}^\theta(v)$, i.e. $\bar{x}_{i_2}^\theta(v) = \max_{x \in AC(v)} \{x_{i_2} : x_{i_1} = \bar{x}_{i_1}^\theta(v)\}$. Repeating the process for all players, notice that for the last player the amount he receives is just what is left by the rest of players, i.e. $\bar{x}_{i_n}^\theta(v) = v(N) - \sum_{l=1}^{n-1} \bar{x}_{i_l}^\theta(v)$.

Due to the fact that the aggregate-monotonic core is a non empty compact and convex set, the above sequential-maximization rule is well defined and selects a unique extreme point of the aggregate-monotonic core for any $\theta \in S_N$.

We show that any sequential-maximization rule is core selection and aggregate monotonic. As a direct consequence we will obtain that the average sequential-maximization rule also satisfies core selection and aggregate monotonicity.

Theorem 14 *Let v be a game. For an arbitrary order $\theta = (i_1, \dots, i_n) \in S_N$, the sequential-maximization rule $\bar{x}^\theta(v)$ and the average sequential-maximization rule $ASM(v)$ are core selection and aggregate monotonic.*

Proof. Any sequential-maximization rule satisfies core selection since by definition $\bar{x}^\theta(v) \in AC(v)$. Moreover, the average sequential-maximization rule also satisfies core selection since also by definition we have that $ASM(v) \in AC(v)$.

The aggregate monotonicity property follows directly once we observe that if $v \in B^N$ then $\bar{x}^\theta(v) = \bar{x}^\theta(v_r) + (v(N) - v_r(N)) \cdot e_{i_1}$ and if $v \notin B^N$ then $\bar{x}^\theta(v) = \bar{x}^\theta(v_r) + (v(N) - v_r(N)) \cdot e_{i_n}$ where $\theta = (i_1, \dots, i_n)$. The interpretation of these relations is natural. Any sequential-maximization rule gives all the surplus (if $v \in B^N$) with respect to the root game to the first player in the order; and all the losses (if $v \notin B^N$) also with respect to the root game to the last player in the order. The proof of these equalities is left to the reader, and basically depends on the fact that $AC(v) = C(v_r) + (v(N) - v_r(N)) \cdot \Delta_n$. With these relations in mind, it is direct to show that any sequential-maximization rule satisfies aggregate monotonicity.

Finally, since all players take the first (last) position according to an ordering the same number of times it follows easily that $ASM(v) = ASM(v_r) + \frac{v(N) - v_r(N)}{|N|} \cdot e_N$, which involves that the average sequential-maximization rule also satisfies aggregate-monotonicity. ■

Let us remark that any convex combination of sequential-maximization rules $\{\bar{x}^\theta(v)\}_{\theta \in S_N}$ will give rise to a core selection and aggregate monotonic point solution.

To finish this section, let us notice that the average sequential-maximization rule does not coincide with the per-capita prenucleolus. The per-capita prenucleolus, $\eta_{PC}(v) \in \mathbb{R}^N$, is the preimputation $x \in I^*(v)$ that lexicographically

minimizes the vector of per-capita excesses $e(S, x) = \frac{v(S) - x(S)}{|S|}$, $\emptyset \neq S \subset N$, when these excesses are arranged in order of descending magnitude.

The per-capita prenucleolus satisfies core selection and also aggregate monotonicity since it divides the profits (deficits) derived from an increasing (decreasing) efficiency level equally among the players (see Moulin, 1988). The average sequential-maximization rule also satisfies these two properties. Nevertheless, they may not coincide, as the following example shows.

Example 15 *Let (N, v) be the three player game defined by $v(i) = 10$ for all $i \in N$, $v(12) = 20$, $v(13) = 10$, $v(23) = 21$ and $v(123) = 31$. Some computation yields to $ASM(v) = (10, 10\frac{1}{2}, 10\frac{1}{2})$, and the per-capita prenucleolus $\eta_{PC}(v) = (10, 10\frac{2}{3}, 10\frac{1}{3})$.*

Hence, for $n \geq 3$ both solutions might not coincide. On the other hand, if the core of the root game reduces to a unique point then both solutions coincide. The axiomatic approach to these solution concepts is left for a subsequent paper.

5 Concluding Remarks.

Throughout this work a new set solution concept has been introduced. Its interest lays not only on its properties, it is the locus in the core of core selection and aggregate monotonic point solution, but also in the fact that it induces a new way of looking at the cooperative phenomenon. In fact, any stable allocation in the root game, extended monotonically to an allocation in the aggregate-monotonic core will have strong arguments to be proposed as a reasonable one.

We have seen that convexity of a game does not necessarily imply the coincidence of the core and the aggregate-monotonic core. In fact, convexity of a game does not imply convexity of its *root* game which is a sufficient condition for largeness of the core (Sharkey, 1982). However, the *root* game associated to a convex game is almost convex, i.e. $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$ for all $i \in N$ and all $S \subset T \subset N \setminus \{i\}$. As a consequence, and since the extreme points of the core of an almost convex game are known, we can derive the extreme points of the aggregate-monotonic core of a convex game. Núñez and Rafels (1998) introduce the reduced marginal worth vectors and show that those are the extreme points of the core of an almost convex game.

From Theorem 8, root games with a large core are of interest. Convexity and subconvexity are sufficient conditions for largeness of the core (Sharkey, 1982), also exactness (Schmeidler, 1972) of a symmetric game is a necessary and sufficient condition for largeness of the core (Biswas et al., 1999). Therefore, a *root* game holding any of these conditions has a large core.

Moreover, some assignment games are root games with a large core. Assignment games were introduced by Shapley and Shubik (1972) as a model for a two-sided market with transferable utility. For this well known class of balanced games, the value of the grand coalition can be seen as the sum of the worths of a set of coalitions forming a partition of the player set. Consequently, any

assignment game is a *rooted* game. But, we can still say something else. From Solymosi and Raghavan (2001) we know that an assignment game has a large core if and only if its corresponding matrix A has a dominant and doubly dominant diagonal. As a consequence, the class of root games with large core seems to be large and rich enough.

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