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# A Canonical Representation for the Assignment Game: the Kernel and the Nucleolus


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**Abstract**

The core of an assignment market is the translation, by the vector of minimum core payoffs, of the core of another better positioned market, the matrix of which has the properties of being dominant diagonal and doubly dominant diagonal. This new matrix is defined as the canonical form of the original assignment situation, and it is uniquely characterized by these three properties. The behavior of some well-known cooperative solutions in relation with the canonical form is analyzed. The kernel and the nucleolus of the assignment game, are proved to be the translation of the kernel and the nucleolus of the canonical representative by the vector of minimal core payoffs.

**Keywords** assignment game, core, kernel, nucleolus

## 1 Introduction

In a bilateral assignment market, a product that comes in indivisible units is exchanged for money, and each participant either supplies or demands exactly one unit. The units need not be alike and the same unit may have different values for different participants. From these valuations, a matrix

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can be defined which reflects the profit that can be obtained by each buyer-seller pair if they trade. Assuming that side payments are allowed, Shapley and Shubik (1972) define the assignment game as a cooperative model for this bilateral market and prove the nonemptiness of the core.

In the present paper we introduce a canonical representative for the assignment matrix. The entry related to a mixed pair in the canonical form is given by the second marginal contribution of their partners by a fixed optimal matching. The canonical representative can also be derived from the buyer-seller exact representative introduced in Núñez and Rafels (2002b) and this allows to prove that its definition does not depend on the chosen optimal matching. The main result states that the core of the original market is the translation, by the vector of minimal core payoffs, of the core of its canonical form. Moreover, the canonical representative of the assignment game has dominant diagonal and doubly dominant diagonal, and it is unique with these properties among all the markets with a core that is a translation of the core of the original assignment game.

It follows straightforwardly that the $\tau$-value (Tijs, 1981) of an arbitrary assignment game is the translation, by the vector of minimal core payoffs, of the tau-value of its canonical form.

Also, an easy characterization of the extreme core allocations of those assignment games with dominant diagonal and doubly dominant diagonal matrices is given in Izquierdo et al. (2006), that now, by means of the above translation, provides a characterization of the extreme core allocations of an arbitrary assignment market.

After that, we ask whether other solutions to an assignment market can be obtained by translation of the same solution applied to its canonical form. The question is affirmatively answered in the case of two other well-known solutions, the kernel and the nucleolus of the assignment game.

In Section 2, the basic concepts regarding the assignment model are recalled. In Section 3 we introduce the canonical representative and prove it is the unique dominant diagonal and doubly dominant diagonal assignment matrix among those with a core that is a translation of the core of the original market. The fact that the kernel of an arbitrary assignment market is the translation, by the vector of minimal core payoffs, of the kernel of its canonical form is proved in Section 4. The same property is proved for the nucleolus in Section 5. Section 6 concludes.

## 2 The assignment game

Let us consider a two-sided market with a finite set of buyers $M$ of cardinality $|M| = m$ and a finite set of sellers $M'$ of cardinality $|M'| = m'$, and let $A = (a_{ij})_{(i,j)\in M\times M'}$ be a nonnegative matrix where $a_{ij}$ represents the profit obtained by the mixed–pair $(i,j) \in M \times M'$ if they trade. Let $n = m + m'$ denote the cardinality $|N|$ of $N = M \cup M'$.

The *assignment problem* $(M, M', A)$ consists in looking for an optimal matching between the two sides of the market. A *matching* $\mu \subseteq M \times M'$ between $M$ and $M'$ is a bijection from some $M_0 \subseteq M$ to some $M'_0 \subseteq M'$ such that $|M_0| = |M'_0| = \min\{|M|, |M'|\}$. We write $(i,j) \in \mu$ as well as $j = \mu(i)$ and $i = \mu^{-1}(j)$. We denote the set of matchings between $M$ and $M'$ by $\mathcal{M}(M, M')$. Moreover, we say a buyer $i \in M$ is not assigned by $\mu$ if $(i,j) \notin \mu$ for all $j \in M'$ (and similarly for sellers).

We say a matching $\mu \in \mathcal{M}(M, M')$ is *optimal* for the market $(M, M', A)$ if for all $\mu' \in \mathcal{M}(M, M')$, we have $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$, and will denote the set of optimal matchings by $\mathcal{M}^*_A(M, M')$. Given $S \subseteq M$ and $T \subseteq M'$, we denote by $\mathcal{M}(S,T)$ and $\mathcal{M}^*_A(S,T)$ the set of matchings and optimal matchings of the submarket $(S, T, A_{|S\times T})$ defined by the subset $S$ of buyers, the subset $T$ of sellers and the restriction of $A$ to $S\times T$. If $S = \emptyset$ or $T = \emptyset$, then the only possible matchings are $\mu = \emptyset$ and by convention $\sum_{(i,j)\in\emptyset} a_{ij} = 0$.

By adding null rows or columns if necessary, we will assume that $A$ is square, which means that the assignment problem has as many buyers as sellers.

*Assignment games* were introduced by Shapley and Shubik (1972) as a cooperative model for a two–sided market with transferable utility. Given an assignment problem $(M, M', A)$, the player set is $N = M \cup M'$, and the matrix $A$ determines the characteristic function $w_A$. Given $S \subseteq M$ and $T \subseteq M'$, $w_A(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T)\}$. Notice that a coalition formed only by sellers or only by buyers has worth zero.

Shapley and Shubik proved that the *core*, $C(w_A)$, of the assignment game $(M \cup M', w_A)$ is nonempty and can be represented in terms of any optimal matching $\mu$ of $(M, M', A)$. Once fixed any such optimal matching, $(u,v) \in \mathbf{R}^M \times \mathbf{R}^{M'}$ is in the core if and only if $u_i \geq 0$ for all $i \in M$, $v_j \geq 0$ for all $j \in M'$, $u_i + v_j \geq a_{ij}$ for all $(i,j) \in M \times M'$, $u_i + v_j = a_{ij}$ if $(i,j) \in \mu$ and $u_i = 0$ if $i \in M$ is not matched by $\mu$, while $v_j = 0$ if $j \in M'$ is not matched by $\mu$.

Moreover, the core has a lattice structure with two special extreme points: the *buyers–optimal core allocation*, $(\overline{u}, \underline{v})$, where each buyer attains his maximum core payoff, and the *sellers–optimal core allocation*, $(\underline{u}, \overline{v})$, where each seller does.

From Demange (1982) and Leonard (1983) we know that the maximum core payoff of any player coincides with his or her marginal contribution:

$$\begin{aligned} \overline{u}_i &= w_A(M \cup M') - w_A((M \cup M') \setminus \{i\}) \text{ for all } i \in M\,, \text{ and} \\ \overline{v}_j &= w_A(M \cup M') - w_A((M \cup M') \setminus \{j\}) \text{ for all } j \in M'. \end{aligned} \tag{1}$$

From (1), once fixed $\mu \in \mathcal{M}^*_A(M, M')$, and taking into account that $\underline{u}_i + \overline{v}_{\mu(i)} = a_{i\mu(i)}$, since $(\underline{u}, \overline{v}) \in C(w_A)$, we get that the minimum core payoff of a buyer $i$ who is matched by $\mu$ is

$$\underline{u}_i = a_{i\mu(i)} + w_A((M \cup M') \setminus \{\mu(i)\}) - w_A(M \cup M')\,, \tag{2}$$

while $\underline{u}_i = 0$ if $i$ is not assigned by $\mu$. Similarly the minimum core payoff of a seller $j$ who is matched by $\mu$ is

$$\underline{v}_j = a_{\mu^{-1}(j)j} + w_A((M \cup M') \setminus \{\mu^{-1}(j)\}) - w_A(M \cup M')\,. \tag{3}$$

The two aforementioned extreme core allocations of the assignment game are not, in general, the only ones. We will denote by $Ext(C(w_A))$ the set of extreme points of the core of $(M \cup M', w_A)$.

# 3 The canonical matrix

We associate to each assignment game, a new assignment market the core of which preserves the same structure but has the additional property of being large.[1] This related assignment game will be named the *canonical representative* of the original market. Given an assignment game $(M \cup M', w_A)$, where $N = M \cup M'$, and in order to introduce its canonical form, some notations are needed.

The *marginal value* of player $i$ to the coalition $S$, $i \notin S$, is given by $\triangle_i(S, w_A) = w_A(S \cup \{i\}) - w_A(S)$, and the special case where $S = N \setminus \{i\}$ is denoted by $\triangle_i(w_A) = w_A(N) - w_A(N \setminus \{i\})$ and named the *marginal contribution* of player $i$.

[1]For an arbitrary coalitional game $(N, v)$, the core is said to be large (Sharkey, 1982) if for all aspiration $y \in \mathbf{R}^n$ such that $y(S) \geq v(S)$ for all $S \subseteq N$, there exists $x \in C(v)$ that satisfies $x_k \leq y_k$ for all $k \in N$.

The *second difference* of two different players, $i$ and $j$, to coalition $S$ such that $i,j \notin S$ is $\triangle_{ij}(S,w_A) = w_A(S\cup\{i,j\}) + w_A(S) - w_A(S\cup\{i\}) - w_A(S\cup\{j\})$. Notice that $\triangle_{ij}(S,w_A) = \triangle_{ji}(S,w_A)$. The special case where $S = N\setminus\{i,j\}$ is named the *second marginal contribution* of players $i$ and $j$, and it is denoted by $\triangle_{ij}(w_A) = w_A(N) + w_A(N\setminus\{i,j\}) - w_A(N\setminus\{i\}) - w_A(N\setminus\{j\})$.

The marginal value of player $i$ to the coalition $S$ measures the contribution of this player to the coalition $S$. The second difference of players $i$, $j$ to the coalition $S$ measures how the marginal value of one of the players changes with respect to the other player. Notice that $\triangle_{ij}(w_A) = \triangle_i(S\cup\{j\},w_A) - \triangle_i(S,w_A)$ and also $\triangle_{ij}(w_A) = \triangle_j(S\cup\{i\},w_A) - \triangle_j(S,w_A)$.

Once fixed an optimal matching $\mu \in \mathcal{M}^*_A(M,M')$, the *canonical matrix* $A^c$ associated to the assignment problem $(M,M',A)$ is defined, for any mixed pair $(i,j)\in M\times M'$, by the second marginal contribution of their optimal partners.

**Definition 1** *Let $(M\cup M', w_A)$ be an assignment game with as many buyers as sellers and let it be $\mu \in \mathcal{M}^*_A(M,M')$. The* canonical representative *of the assignment game $(M\cup M',w_A)$ is another assignment game with the same set of agents and defined by the matrix $A^c$ where, for all $(i,j)\in M\times M'$,*

$$a^c_{ij} = \triangle_{\mu(i),\mu^{-1}(j)}(w_A)$$

*or equivalently*

$$\begin{aligned} a^c_{ij} = w_A(M\cup M') &+ w_A(M\cup M'\setminus\{\mu^{-1}(j),\mu(i)\}) \\ &- w_A(M\cup M'\setminus\{\mu(i)\}) - w_A(M\cup M'\setminus\{\mu^{-1}(j)\}). \end{aligned}$$

Since $\mu(i)$ and $\mu^{-1}(j)$ belong to opposite sides of the market, they are complements, and by Shapley (1962) their second marginal contribution is non-negative. Thus, $a^c_{ij}\geq 0$ for all $(i,j)\in M\times M'$ and $A^c$ properly defines an assignment market.

The canonical representative may seem to depend on the fixed optimal matching $\mu$, but in fact it does not. To see this, for all $(i,j)\in M\times M'$, let us denote by $a^r_{ij}$ the solution to the following linear programme,

$$\min_{(u,v)\in C(w_A)} u_i + v_j\,. \tag{4}$$

Once fixed any optimal matching, $\mu \in \mathcal{M}^*_A(M,M')$, the solution of the above programme is given by (see p. 430 in Núñez and Rafels, 2002b)

$$a^r_{ij} = a_{i\mu(i)} + a_{\mu^{-1}(j),j} + w_A(M\cup M'\setminus\{\mu^{-1}(j),\mu(i)\}) - w_A(M\cup M')\,. \tag{5}$$

Notice however that, by (4), $a^r_{ij}$ does not depend on the fixed optimal matching, but only on $C(w_A)$.

An assignment game is said to be buyer-seller exact if no matrix entry can be raised without modifying the core of the game. That is, $(M \cup M', w_A)$ is *buyer-seller exact* if and only if for all $i \in M$ and all $j \in M'$ there exists $(u, v) \in C(w_A)$ such that $u_i + v_j = a_{ij}$. For an arbitrary assignment game $(M \cup M', w_A)$, the matrix $A^r = (a^r_{ij})_{(i,j)\in M\times M'}$ just obtained defines the *buyer-seller exact representative* $(M \cup M', w_{A^r})$ of $(M \cup M', w_A)$. This means that it is the unique assignment market with the same core as the initial assignment market, $C(w_{A^r}) = C(w_A)$, and with the property of being buyer-seller exact.

Taking into account expressions (5), (2) and (3), we can easily verify that[2], for all $(i, j) \in M \times M'$,

$$a^c_{ij} = a^r_{ij} - \underline{u}_i - \underline{v}_j. \tag{6}$$

Notice that the right hand side of (6) does not depend on any fixed optimal matching and, as a consequence, the canonical matrix $A^c$ is well-defined since it does not depend on the chosen optimal matching.

If we define another matrix with $m$ rows and $m'$ columns, $\underline{A}$, by $\underline{a}_{ij} = \underline{u}_i + \underline{v}_j$ for all $(i, j) \in M \times M'$, and taking into account the above consideration, the following proposition follows straightforwardly.

**Proposition 2** *Let $(M \cup M', w_A)$ be an assignment game with as many buyers as sellers. The canonical matrix $A^c$ satisfies $A^c = A^r - \underline{A}$.*

The following example illustrates how to obtain the canonical form of an assignment game and also gives some intuition about its properties.

**Example 3** *Consider the $2 \times 2$-assignment market defined by the following matrix $A$:*

| | *1'* | *2'* |
|---|---|---|
| *1* | *3* | *4* |
| *2* | *0* | *2* |

*There is only one optimal matching which is $\mu = \{(1, 1'), (2, 2')\}$ and thus the worth of the grand coalition is $w_A(M \cup M') = 5$. By (1), (2) and (3), the*

[2]The substraction of $\underline{u}_i$ and $\underline{v}_j$ from $a_{ij}$ instead of $a^r_{ij}$ does not guarantee non-negativeness

*buyers-optimal and the sellers-optimal core allocations are* $(\overline{u}, \underline{v}) = (3, 1; 0, 1)$ *and* $(\underline{u}, \overline{v}) = (2, 0; 1, 2)$ *. As a consequence, the vector of minimal core payoffs is* $(\underline{u}, \underline{v}) = (2, 0; 0, 1)$ *.*

*It is easy to check that* $C(w_A)$ *is the convex hull of the three extreme points (2, 0; 1, 2), (3, 0; 0, 2) and (3, 1; 0, 1), and the projection of the core to the space of payoffs to the buyers is depicted in the first picture of Figure 1.*

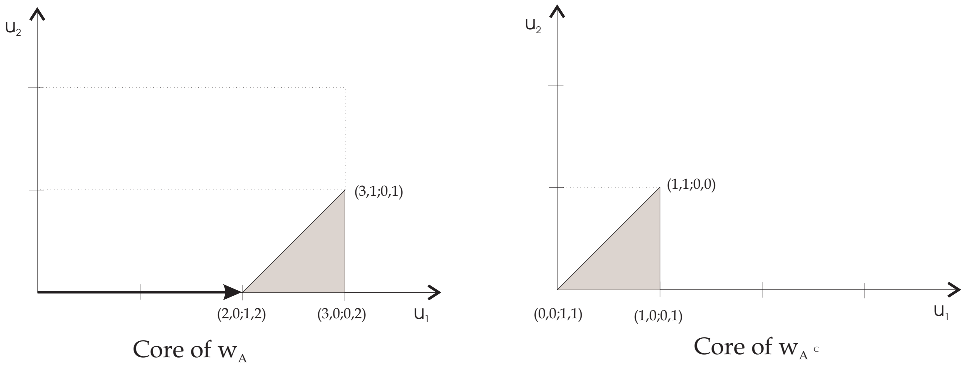


Figure 1: The core of the game in Example 3 and of its canonical form

*We use Definition 1 to obtain the canonical representative of the above assignment market. For instance,*

$$
\begin{aligned}
a^c_{11'} &= w_A(M \cup M') + w_A(M \cup M' \setminus \{1, 1'\}) \\
&\quad - w_A(M \cup M' \setminus \{1\}) - w_A(M \cup M' \setminus \{1'\}) = (5 + 2) - (2 + 4) = 1\,, \\
a^c_{12'} &= w_A(M \cup M') + w_A(M \cup M' \setminus \{2, 1'\}) \\
&\quad - w_A(M \cup M' \setminus \{2\}) - w_A(M \cup M' \setminus \{1'\}) = (5 + 4) - (4 + 4) = 1\,,
\end{aligned}
$$

*and similarly* $a^c_{21'} = 0$ *and* $a^c_{22'} = 1$ *. Thus the canonical representative* $A^c$ *is*

| | *1'* | *2'* |
|---|---|---|
| *1* | *1* | *1* |
| *2* | *0* | *1* |

*Notice that the core of this canonical representative is the triangle with vertices (0, 0; 1, 1), (1, 0; 0, 1) and (1, 1; 0, 0). Again we depict its projection to the buyers' space of payoffs to see that this core is a translation of the core of the original assignment market. Since the origin is in the core of the canonical form, the vector of translation is the vector of minimal core payoffs:* $C(w_A) = \{(\underline{u}, \underline{v})\} + C(w_{A^c})$ *.*

*Moreover, different to the initial market* $(M \cup M', w_A)$*, the core of the canonical representative* $(M \cup M, w_{A^c})$ *is large.*

Assignment games with *large core* have been characterized by Solymosi and Raghavan (2001) by means of two properties of the assignment matrix: dominant diagonal and doubly dominant diagonal. An assignment game $(M \cup M', w_A)$ with as many buyers as sellers has *dominant diagonal* if and only if, once placed an optimal matching in the diagonal, $a_{ii} \geq a_{ij}$ for all $j \in M'$ and $a_{ii} \geq a_{ji}$ for all $j \in M$. As these authors point out, this is equivalent to saying that each agent has a null minimal core payoff. Since the property of having null core payoff does not depend on the optimal matching placed on the diagonal, it follows that, for all $\mu \in \mathcal{M}^*_A(M, M')$, the assignment game has dominant diagonal if and only if, for all $i_* \in M$, $a_{i_*\mu(i_*)} \geq a_{i_*j}$ for all $j \in M'$ and $a_{i_*\mu(i_*)} \geq a_{i\mu(i_*)}$ for all $i \in M$.

An assignment game with as many buyers and sellers and an optimal matching placed on the diagonal has *doubly dominant diagonal* if and only if $a_{ij} + a_{kk} \geq a_{ik} + a_{kj}$ for all $i, j, k \in M$. It is proved in Núñez and Rafels (2002b) that this property characterizes those matrices with the property of buyer-seller exactness, that is, those where each matrix entry is attained by some core element. Since buyer-seller exactness is a property of the game which does not depend on the optimal matching placed on the diagonal, for each $\mu \in \mathcal{M}^*_A(M, M')$, the assignment game $(M \cup M', w_A)$ has doubly dominant diagonal if and only if $a_{ij} + a_{k\mu(k)} \geq a_{i\mu(k)} + a_{kj}$ for all $i, j, k \in M$.

Recall that the translation of a convex game $C \subseteq \mathbf{R}^n$ by the vector $t \in \mathbf{R}^n$ is the set of vectors $x \in \mathbf{R}^n$ such that $x = t + y$ with $y \in C$. Formally, $\{t\} + C = \{t + y \mid y \in C\}$.

The next theorem gives a characterization of the canonical representative.

**Theorem 4** *Let* $(M \cup M', w_A)$ *be an assignment game with as many buyers as sellers. The matrix* $A^c$ *is unique among those matrices* $A'$ *such that*

1. $A'$ *has dominant diagonal,*
2. $A'$ *has doubly dominant diagonal and*
3. $C(w_A)$ *is a translation of* $C(w_{A'})$*.*

PROOF: We first prove that $A^c$ satisfies the three properties in the statement of the theorem. To prove the property of translation of the cores, recall first that $C(w_A) = C(w_{A^r})$. Let us see first that both matrices $A^r$ and $A^c$ have the same set of optimal matchings (and as a consequence, they have at least one optimal matching in common). It happens that $\mu \in \mathcal{M}^*_{A^r}(M,M')$ if and only if, for all $\mu' \in \mathcal{M}(M,M')$ it holds $\sum_{(i,j)\in\mu} a^r_{ij} \geq \sum_{(i,j)\in\mu'} a^r_{ij}$ , and this is equivalent to

$$\sum_{(i,j)\in\mu} a^r_{ij} - \sum_{i\in M} \underline{u}_i - \sum_{j\in M'} \underline{v}_j \geq \sum_{(i,j)\in\mu'} a^r_{ij} - \sum_{i\in M} \underline{u}_i - \sum_{j\in M'} \underline{v}_j$$

which is equivalent to $\sum_{(i,j)\in\mu} a^c_{ij} \geq \sum_{(i,j)\in\mu'} a^c_{ij}$ for all $\mu' \in \mathcal{M}(M,M')$, and this is equivalent to the fact that $\mu$ is also an optimal matching for $(M,M',A^c)$.

Now, let us see that $C(w_A) = \{(\underline{u},\underline{v})\}+C(w_{A^c})$, or equivalently $C(w_{A^r}) = \{(\underline{u},\underline{v})\}+C(w_{A^c})$. Take $(u,v) \in \mathbf{R}^M \times \mathbf{R}^{M'}$ and define $(u',v') \in \mathbf{R}^M \times \mathbf{R}^{M'}$ by

$$u'_i = u_i - \underline{u}_i\,, \text{ for all } i \in M \text{ and } v'_j = v_j - \underline{v}_j\,, \text{ for all } j \in M'.$$

We claim that $(u,v) \in C(w_{A^r})$ if and only if $(u',v') \in C(w_{A^c})$. To see that, notice first that, since $(\underline{u},\underline{v})$ is the vector of minimal core payoffs of $(M \cup M', w_A)$, if $(u,v) \in C(w_{A^r})$ then $u_i \geq \underline{u}_i$ and $v_j \geq \underline{v}_j$, and thus we get $u'_i \geq 0$ and $v'_j \geq 0$. Conversely, if $u'_i$ and $v'_j$ are non-negative, since this implies $u_i \geq \underline{u}_i$ and $v_j \geq \underline{v}_j$, non-negativity also holds for $u_i$ and $v_j$. Moreover, since for all $i \in M$ and $j \in M'$ $u'_i + v'_j = u_i + v_j - \underline{u}_i - \underline{v}_j$ and $a^c_{ij} = a^r_{ij} - \underline{u}_i - \underline{v}_j$, we have that $u_i + v_j \geq a^r_{ij}$ if and only if $u'_i + v'_j \geq a^c_{ij}$. Take now $\mu \in \mathcal{M}^*_{A^c}(M,M')$ and recall that $\mu$ is also an optimal matching for $A^r$. Then, if $(i,j) \in \mu$, we have that $u_i + v_j = a^r_{ij}$ if and only if $u'_i + v'_j = a^c_{ij}$.

From the argument above follows straightforwardly that, since $w_{A^r}$ is buyer-seller exact, $w_{A^c}$ is also buyer-seller exact and thus $A^c$ has doubly dominant diagonal. Notice only that if $(u,v) \in C(w_{A^r})$ satisfies $u_i + v_j = a^r_{ij}$, then $(u',v') = (u,v) - (\underline{u},\underline{v})$ belongs to $C(w_{A^c})$ and satisfies $u'_i + v'_j = a^c_{ij}$.

It remains to prove that $A^c$ has dominant diagonal. Once proved that $C(w_{A^r})$ is the translation of $C(w_{A^c})$ by the vector $(\underline{u},\underline{v})$, it follows that, if we denote by $(\underline{u}^c, \overline{v}^c)$ and $(\overline{u}^c, \underline{v}^c)$ the sellers-optimal and the buyers-optimal core allocation of $(M \cup M', w_{A^c})$, then $(\underline{u}, \overline{v}) = (\underline{u},\underline{v}) + (\underline{u}^c, \overline{v}^c)$ and $(\overline{u}, \underline{v}) = (\underline{u},\underline{v}) + (\overline{u}^c, \underline{v}^c)$. Then, $\underline{u}_i = \underline{u}_i + \underline{u}^c_i$ for all $i \in M$, and $\underline{v}_j = \underline{v}_j + \underline{v}^c_j$ for all $j \in M'$. Thus, every agent has a null minimal core payoff in the game $w_{A^c}$ which means that $A^c$ has dominant diagonal.

Finally the uniqueness of the matrix $A'$ satisfying the three requirements follows easily. On the one hand, if $C(w_A)$ is a translation of $C(w_{A'})$, since $(M \cup M', w_{A'})$ has dominant diagonal and thus null minimal core payoffs, the vector of translation must be $(\underline{u}, \underline{v})$ . Secondly, once $C(w_{A'})$ is determined by $C(w_{A'}) = \{(-\underline{u}, -\underline{v})\} + C(w_A) = C(w_{A^c})$ , recall from Núñez and Rafels (2002b) that there exists only one assignment game with a given core and with the property of being buyer-seller exact (or equivalently doubly dominant diagonal). $\square$

Recall from Solymosi and Raghavan (2001) that having dominant diagonal and doubly dominant diagonal characterizes those assignment games with large core (and also those assignment games which are exact). Then, the following corollary follows from Theorem 4.

**Corollary 5** *Let $(M \cup M', w_A)$ be an assignment game with as many buyers as sellers. The canonical representative $(M \cup M', w_{A^c})$ is the unique assignment game with a core that is large and is a translation of $C(w_A)$ .*

We now ask which is the behavior of the main cooperative solutions to the assignment game with respect to its canonical representative. Those solutions tightly related to the core behave as expected: the buyers-optimal core allocation of the assignment game is the translation, by the vector of minimal core payoffs, of the buyers-optimal core allocation of the canonical representative. The same happens with the sellers-optimal core allocation. Since the $\tau$-value of the assignment game is the midpoint between the buyers-optimal and the sellers-optimal core allocation (Núñez and Rafels 2002a), the $\tau$-value of the assignment game is also the translation, by the vector of minimal core payoffs, of the buyers-optimal core allocation of the canonical representative.

In Izquierdo et al. (2006) we show that the set of max-payoff vectors coincides with the set of extreme core allocations for those assignment games with large core. To introduce these vectors some notation is needed. An *ordering* $\theta$ on $N = M \cup M'$ is a bijection from $\{1, 2, \ldots, n\}$ to $N = M \cup M'$ . We then denote an ordering $\theta$ by $(k_1, k_2, \ldots, k_n)$ where, for all $i \in \{1, 2, \ldots, n\}$ , $k_i = \theta(i)$ is the agent that occupies place $i$ . The set of predecessors of agent $k_* \in M \cup M'$ in the ordering $\theta$ is $P^{\theta}_{k_*} = \{k \in M \cup M' \mid \theta^{-1}(k) < \theta^{-1}(k_*)\}$ , and the set of all orderings on $N$ is denoted by $\mathcal{S}_N$ .

Let $(M \cup M', w_A)$ be an assignment game. For all $\theta \in \mathcal{S}_N$ , the *max-payoff vector* $x^{\theta}(A) \in \mathbf{R}^n$ related to $\theta$ is defined by $x^{\theta}_{k_1}(A) = w_A(k_1) = 0$ and, for all $r \in \{2, \ldots, n\}$ ,

$$x^{\theta}_{k_r}(A) = \begin{cases} \max_{i \in P^{\theta}_{k_r} \cap M} \{0, a_{ik_r} - x^{\theta}_i(A)\} & \text{if} \quad k_r \in M' , \\ \max_{j \in P^{\theta}_{k_r} \cap M'} \{0, a_{k_r j} - x^{\theta}_j(A)\} & \text{if} \quad k_r \in M . \end{cases}$$

Once fixed and ordering, the first agent receives a null payoff and, after that, each agent receives the maximum profit he can obtain by trading with one of his predecessors on the opposite side of the market, after paying this partner what the vector $x^\theta$ has already allocated to him.

It is proved in Izquierdo et al. (2006) that, for those assignment games with as many buyers as sellers, the set of max-payoff vectors coincides with the set of extreme core allocations of the assignment game if and only if the assignment matrix is dominant diagonal and doubly dominant diagonal.

Now, as a consequence of Theorem 4, the set of extreme core allocations of an arbitrary assignment market turns out to be the translation, by the vector of minimal core payoffs, of the set of max-payoff vectors of its canonical representative.

**Corollary 6** *Let* $(M \cup M')$ *be an assignment game with as many buyers as sellers and let* $A^c$ *be its canonical representative. The set of extreme core allocations is*

$$Ext(C(w_A)) = \{(\underline{u}, \underline{v})\} + \{x^\theta(A^c)\}_{\theta \in \mathcal{S}_N} ,$$

*where* $(\underline{u}, \underline{v})$ *is the vector of minimal core payoffs of the game* $(M \cup M', w_A)$ *.*

# 4 The kernel

In this section we consider another set-solution concept for coalitional games that is known as the kernel of the game. We will see that, as it happens with the core, the kernel of an assignment game is the translation, by the vector of minimal core payoffs, of the kernel of its canonical representative. The kernel $\mathcal{K}(v)$ of a coalitional game with transferable utility $(N, v)$ is a set-solution concept introduced by Davis and Maschler (1965), and it is always nonempty. In the case of a *zero–monotonic game* ( $v(S) \geq v(T) + \sum_{i \in S \setminus T} v(\{i\})$ , for all $T \subseteq S$ ), as it is the case of assignment games, the *kernel* is given by

$$\mathcal{K}(v) = \{z \in \mathbf{R}^N \mid \sum_{k \in N} z_k = v(N) \text{ and } s^v_{ij}(z) = s^v_{ji}(z) \text{ for all } i, j \in N,\, i \neq j\},$$

where the maximum surplus $s^v_{ij}(z)$ of player $i$ over another player $j$ with respect to the allocation $z \in \mathbf{R}^N$ in the game $(N, v)$ is defined by

$$s^v_{ij}(z) = \max\{v(S) - \sum_{k \in S} z_k \mid S \subseteq N \,,\, i \in S \,,\, j \notin S\} \,.$$

Then, the kernel can be understood as the set of all efficient allocations for which all pair of players are in equilibrium.

The first analysis of the kernel of an assignment game, $\mathcal{K}(w_A)$, is carried out by Rochford (1984) who characterizes the intersection between the kernel and the core. She points out that, for those $(u, v) \in C(w_A)$, the maximum excess $s_{ij}(u, v)$ is attained either at a mixed-pair coalition or at a single-player coalition, and the same happens to $s_{ji}(u, v)$. Also Granot and Granot (1992) analyze some properties of the intersection between the kernel and the core of the assignment game. Once Driessen (1998) proves that the kernel of an assignment game is included in the core, the above remarks can be put together to provide a simplified definition of the kernel of an assignment game.

We will denote by $\Phi(A)$ the set of pairs $(i, j) \in M \times M'$ belonging to all optimal matching of the market $(M, M', w_A)$. Notice then that, since we assume $A$ square[3], if $(i, j) \notin \Phi(A)$ then there exists $\mu_1 \in \mathcal{M}^*_A(M, M')$ such that $\mu_1(i) \neq j$ and thus $s_{ij}(u, v) = a_{i\mu_1(i)} - u_i - v_{\mu_1(i)} = 0$ for all $(u, v) \in C(w_A)$. Similarly, there exists $\mu_2 \in \mathcal{M}^*_A(M, M')$ such that $\mu_2^{-1}(j) \neq i$ and thus also $s_{ji}(u, v) = 0$.

As a consequence,

$$\mathcal{K}(w_A) = \{(u, v) \in C(w_A) \mid s_{ij}(u, v) = s_{ji}(u, v)\,,\ \forall (i, j) \in \Phi(A)\,\} \tag{7}$$

where $s_{ij}(u, v) = \max\{-u_i, a_{ij_1} - u_i - v_{j_1}, \forall j_1 \in M' \setminus \{j\}\}$ and $s_{ji}(u, v) = \max\{-v_j, a_{i_1 j} - u_{i_1} - v_j, \forall i_1 \in M \setminus \{i\}\}$.

The above expression substantially reduces the number of equality constraints that are to be taken into account to obtain the kernel of an assignment game. We are now prepared to prove the relationship between the kernel of an assignment game and that of its canonical form.

**Theorem 7** *Let $(M \cup M', w_A)$ be an assignment game with as many buyers as sellers. Then,*

$$\mathcal{K}(w_A) = \{(\underline{u}, \underline{v})\} + \mathcal{K}(w_{A^c})\,,$$

*where $(\underline{u}, \underline{v})$ is the vector of minimal core payoffs of the game $(M \cup M', w_A)$.*

Proof: Since we know from Núñez (2004) that $\mathcal{K}(w_A) = \mathcal{K}(w_{A^r})$, we will in fact prove that $\mathcal{K}(w_{A^r}) = \{(\underline{u}, \underline{v})\} + \mathcal{K}(w_{A^c})$. Moreover, since $C(w_A) =$

[3] This assumption is not restrictive at all since when we add null rows or columns to make the matrix square, the kernel of the new market is obtained from the kernel of the original one by giving a zero payoff to the added dummy agents.

$C(w_{A^r}) = \{(\underline{u}, \underline{v})\} + C(w_{A^c})$ , and we know that the kernel of the assignment game is included in the core, we will prove that, for all $x \in C(w_{A^r})$ , $x \in \mathcal{K}(w_{A^r})$ if and only if $y = x - (\underline{u}, \underline{v}) \in \mathcal{K}(w_{A^c})$ . Recall also from the proof of Theorem 4 that $A^r$ and $A^c$ have the same optimal matchings, which implies $\Phi(A^r) = \Phi(A^c)$ .

Now we prove that, for all $(i_*, j_*) \in \Phi(A^r)$ and all $x \in C(w_{A^r})$ it holds $s^{A^r}_{i_*j_*}(x) = s^{A^c}_{i_*j_*}(y)$ and $s^{A^r}_{j_*i_*}(x) = s^{A^c}_{j_*i_*}(y)$ , where $y = x - (\underline{u}, \underline{v})$ . Two cases will be considered.

Case 1: Assume $\underline{u}_{i_*} = 0$ . Take any $x \in C(w_{A^r})$ and consider $y = x - (\underline{u}, \underline{v})$ .

If $s^{A^r}_{i_*j_*}(x)$ is attained at the single-player coalition, $s^{A^r}_{i_*j_*}(x) = -x_{i_*}$ . Then on one side $-x_{i_*} = -y_{i_*} - \underline{u}_{i_*} = -y_{i_*}$ . Moreover, for all $j \in M' \setminus \{j_*\}$ , $-x_{i_*} \geq a^r_{i_*j} - x_{i_*} - x_j = a^r_{i_*j} - (y_{i_*} + \underline{u}_{i_*}) - (y_j + \underline{v}_j) = a^c_{i_*j} - y_{i_*} - y_j$ . Thus, in this case, $s^{A^r}_{i_*j_*}(x) = s^{A^c}_{i_*j_*}(y)$ , where $y = x - (\underline{u}, \underline{v})$ .

If $s^{A^r}_{i_*j_*}(x)$ is attained at a mixed-pair coalition, $s^{A^r}_{i_*j_*}(x) = a^r_{i_*j_1} - x_{i_*} - x_{j_1} = a^c_{i_*j_1} - y_{i_*} - y_{j_1}$ . Moreover, for all $j \in M' \setminus \{j_*, j_1\}$ , it holds $a^r_{i_*j_1} - x_{i_*} - x_{j_1} \geq a^r_{i_*j} - x_{i_*} - x_j = a^c_{i_*j} - y_{i_*} - y_j$ . Finally, $a^r_{i_*j_1} - x_{i_*} - x_{j_1} \geq -x_{i_*} = -y_{i_*} - \underline{u}_{i_*} = -y_{i_*}$ . Thus, also in this case $s^{A^r}_{i_*j_*}(x) = s^{A^c}_{i_*j_*}(y)$ .

Case 2: Assume now that $\underline{u}_{i_*} > 0$ . We first claim that, under this assumption, for all $x \in C(w_{A^r})$ it holds $s^{A^r}_{i_*j_*}(x) \neq -x_{i_*}$ .

To prove the claim, assume there exists $x \in C(w_{A^r})$ such that $s^{A^r}_{i_*j_*}(x) = -x_{i_*}$ . This means that, for all $j \in M' \setminus \{j_*\}$ , $-x_{i_*} \geq a^r_{i_*j} - x_{i_*} - x_j$ , which means that

$$a^r_{i_*j} \leq x_j \ \text{ for all } \ j \in M' \setminus \{j_*\} . \tag{8}$$

Define then a payoff vector $\tilde{x} \in \mathbf{R}^M \times \mathbf{R}^{M'}$ in the following way: $\tilde{x}_j = x_j$ for all $j \in M' \setminus \{j_*\}$ , $\tilde{x}_{j_*} = x_{i_*} + x_{j_*}$ , $\tilde{x}_i = x_i$ for all $i \in M \setminus \{i_*\}$ and $\tilde{x}_{i_*} = 0$ .

We now prove that $\tilde{x} \in C(w_{A^r})$ . To do that, fix any optimal matching $\mu \in \mathcal{M}^*_{A^r}(M, M')$ and notice that $\tilde{x}_k \geq 0$ for all $k \in M \cup M'$ ; $\tilde{x}_i + \tilde{x}_j = x_i + x_j = a^r_{ij}$ for all $(i,j) \in \mu$ ; $\tilde{x}_i + \tilde{x}_j \geq a^r_{ij}$ for all $i \in M \setminus \{i_*\}$ and $j \in M'$ ; and, by (8), $x_{i_*} + x_j = x_j \geq a^r_{i_*j}$ for all $j \in M \setminus \{j_*\}$ . From $\tilde{x} \in C(w_{A^r})$ follows that $\underline{u}_{i_*} = 0$ in contradiction with the assumption of Case 2.

Our second claim is that, under the assumption that $\underline{u}_{i_*} > 0$ , there exists $j_2 \neq j_*$ such that $\underline{u}_{i_*} + \overline{v}_{j_2} = a^r_{i_*j_2}$ . If that were not the case, $\underline{u}_{i_*} + \overline{v}_j > a^r_{i_*j}$ for all $j \neq j_*$ , we could define $\varepsilon = \min_{j \neq j_*}\{\underline{u}_{i_*} + \overline{v}_j - a^r_{i_*j}, \underline{u}_{i_*}\} > 0$ . Then, the payoff vector $(u, v) \in \mathbf{R}^M \times \mathbf{R}^{M'}$ , where $u_{i_*} = \underline{u}_{i_*} - \varepsilon$ , $v_{j_*} = \overline{v}_{j_*} + \varepsilon$ , $u_i = \underline{u}_i$ for all $i \in M \setminus \{i_*\}$ and $v_j = \overline{v}_j$ for all $j \in M' \setminus \{j_*\}$ , would belong to the core, in contradiction with the definition of $\underline{u}_{i_*}$ as the minimal core payoff of agent $i_*$.

Then, by our first claim, there exists $j_1 \in M' \setminus \{j_*\}$ such that

$$s^{A^r}_{i_*j_*}(x) = \max_{j \in M' \setminus \{j_*\}} \{a^r_{i_*j} - x_{i_*} - x_j\} = a^r_{i_*j_1} - x_{i_*} - x_{j_1} = a^c_{i_*j_1} - y_{i_*} - y_{j_1}.$$

Moreover, for all $j \in M' \setminus \{j_*, j_1\}$, $a^r_{i_*j_1} - x_{i_*} - x_{j_1} \geq a^r_{i_*j} - x_{i_*} - x_j = a^c_{i_*j} - y_{i_*} - y_j$. By our second claim, there exists $j_2 \in M' \setminus \{j_*\}$ such that $\underline{u}_{i_*} + \overline{v}_{j_2} = a^r_{i_*j_2}$. Then, $a^r_{i_*j_1} - x_{i_*} - x_{j_1} \geq a^r_{i_*j_2} - x_{i_*} - x_{j_2} = \underline{u}_{i_*} + \overline{v}_{j_2} - x_{i_*} - x_{j_2} \geq -x_{i_*} + \underline{u}_{i_*} = -y_{i_*}$. This concludes the proof of $s^{A^r}_{i_*j_*}(x) = s^{A^c}_{i_*j_*}(y)$ in Case 2.

The proof of $s^{A^r}_{j_*i_*}(x) = s^{A^c}_{j_*i_*}(y)$ is analogous and thus left to the reader. We have then seen that $x \in \mathcal{K}(w_A)$ if and only if $y = x - (\underline{u}, \underline{v}) \in \mathcal{K}(w_{A^c})$, which finishes the proof of the theorem. □

# 5 The nucleolus

The nucleolus is a single–valued solution concept for TU coalitional games. Let us recall the definition, which is due to Schmeidler (1969). Recall that an imputation of a game $(N, v)$ is a payoff vector $x \in \mathbf{R}^n$ that is efficient, $\sum_{i \in N} x_i = v(N)$, and individually rational, $x_i \geq v(\{i\})$ for all $i \in N$. The set of imputations of $(N, v)$ is denoted by $I(v)$. For all imputation $x$ of $(N, v)$, and all coalition $S \subseteq N$, the *excess of coalition $S$ with respect to $x$* is $e(S, x) = v(S) - x(S)$. Now, for all imputation $x$, let us define the *vector $\theta(x) \in \mathbf{R}^{2^n - 2}$ of excesses* of all coalitions (different from the grand coalition and the empty set) at $x$, arranged in a nonincreasing order. That is to say, for all $k \in \{1, \ldots, 2^n - 2\}$, $\theta_k(x) = e(S_k, x)$, where $\{S_1, \ldots, S_k, \ldots, S_{2^n-2}\}$ is the set of all nonempty coalitions in $N$ different from $N$, and $e(S_k, x) \geq e(S_{k+1}, x)$.

Then the *nucleolus* of the game $(N, v)$ is the imputation $\nu(v)$ which minimizes $\theta(x)$ with respect to the lexicographic order over the set of imputations: $\theta(\nu(v)) \leq_{Lex} \theta(x)$ for all $x \in I(v)$. This means that, for all $x \in I(v)$, either $\theta(\nu(v)) = \theta(x)$ or $\theta(\nu(v)) <_{Lex} \theta(x)$. And $\theta(\nu(v)) <_{Lex} \theta(x)$ if there exists $k \in \{1, 2, \ldots, n\}$ such that $\theta_i(\nu(v)) = \theta_i(x)$ whenever $i < k$ and $\theta_k(\nu(v)) = \theta_k(x)$.

It is easy to see that, whenever the core of a game is nonempty, the nucleolus belongs to it. Moreover, the nucleolus always belongs to the kernel.

An alternative definition of the nucleolus for an arbitrary TU coalitional game is given by Maschler, Peleg and Shapley (1979) as an iterative process that constructs the set of payoffs that lexicographically minimize the vector of ordered excesses. They prove that this set of minimizers is actually a single point, called the *lexicographic center of the game*, which coincides with the nucleolus. Solymosi and Raghavan (1994) present a definition of lexicographic center specialized for assignment games, based on the fact that for assignment games only one–player coalitions and mixed–pair coalitions play a role in the computation of the nucleolus.

As in Núñez (2004), the definition of lexicographic center of an assignment game we present here is slightly different from that of Solymosi and Raghavan (1994); for instance, we define the initial feasible set $X^0$ to be the core of the game. All steps in the proof of Solymosi and Raghavan (1994) can be followed to prove that our definition of lexicographic center of an assignment game also consists of only one point and coincides with the nucleolus.

Let $(M \cup M', w_A)$ be an assignment game and $\mu$ a fixed optimal matching for $(M, M', w_A)$, $\mu \in \mathcal{M}^*_A(M, M')$. Let us consider the set of one–player coalitions and mixed–pair coalitions, that is $\mathcal{C} = \{\{k\} \mid k \in M \cup M'\} \cup \{\{i,j\} \mid i \in M\,,\, j \in M'\}$. We iteratively construct a sequence $(\Delta^0, \Sigma^0), \ldots, (\Delta^{s+1}, \Sigma^{s+1})$ of partitions of $\mathcal{C}$, with $\Sigma^0 \supseteq \Sigma^1 \supseteq \cdots \supseteq \Sigma^{s+1}$, and a sequence $X^0 \supseteq X^1 \supseteq \cdots \supseteq X^{s+1}$ of sets of payoff vectors such that:
Initially $\Delta^0 = \{\{i,j\} \mid (i,j) \in \mu\} \cup \{\{k\} \mid k \in M \cup M' \text{ not matched by } \mu\}$; $\Sigma^0 = \mathcal{C} \setminus \Delta^0$, and $X^0 = C(w_A) = \{x \in \mathbf{R}^M_+ \times \mathbf{R}^{M'}_+ \mid e(S,x) = 0 \text{ for all } S \in \Delta^0\,,\, e(S,x) \leq 0 \text{ for all } S \in \Sigma^0\}$.
For $r \in \{0, 1, \ldots, s\}$ define recursively

1. $\alpha^{r+1} = \min_{x \in X^r} \max_{S \in \Sigma^r} e(S,x)$,
2. $X^{r+1} = \{x \in X^r \mid \max_{S \in \Sigma^r} e(S,x) = \alpha^{r+1}\}$,
3. $\Sigma_{r+1} = \{S \in \Sigma^r \mid e(S,x) \text{ is constant on } X^{r+1}\}$,
4. $\Sigma^{r+1} = \Sigma^r \setminus \Sigma_{r+1}$, $\Delta^{r+1} = \Delta^r \cup \Sigma_{r+1}$,

where $s$ is the last index for which $\Sigma^r \neq \emptyset$. The set $X^{s+1}$ is called the *lexicographic center* of $(M \cup M', w_A)$.

As a consequence, following Solymosi and Raghavan (1994), it can be shown that, for all $r \in \{0, 1, \ldots, s\}$, the payoff set $X^r$ can be written in the following way:

$$X^{r+1} = \left\{ x \in \mathbf{R}^M_+ \times \mathbf{R}^{M'}_+ \;\middle|\; \begin{array}{l} e(S,x) = e(S, \nu(w_A))\,, \text{ for all } S \in \Delta^{r+1} \\ e(S,x) \leq \alpha^{r+1}\,, \text{ for all } S \in \Sigma^{r+1} \end{array} \right\}$$

or equivalently

$$X^{r+1} = \left\{ x \in \mathbf{R}_+^M \times \mathbf{R}_+^{M'} \;\middle|\; \begin{array}{l} x_k = \nu_k(w_A) \text{ for all } \{k\} \in \Delta^{r+1} \\ a_{ij} - x_i - x_j = a_{ij} - \nu_i(w_A) - \nu_j(w_A) \text{ for all } \{i,j\} \in \Delta^{r+1} \\ x_k \geq -\alpha^{r+1} \text{ for all } \{k\} \in \Sigma^{r+1} \\ x_i + x_j \geq a_{ij} - \alpha^{r+1} \text{ for all } \{i,j\} \in \Sigma^{r+1} \end{array} \right\}$$

where $\nu(w_A)$ is the nucleolus of the game $(M \cup M', w_A)$. Thus, for each allocation in $X^{r+1}$, the excess of a coalition in $\Delta^{r+1}$ already equals the excess of this coalition at the nucleolus while, for the coalitions in $\Sigma^{r+1}$, the right hand side of the corresponding core constraint has been increased in $-\alpha^{r+1}$.

In the proof of the following theorem we will consider another collection of payoff sets, one for each step of the iterative procedure. In each step, the excess of the settled coalitions ( $S \in \Delta^{r+1}$ ) will be fixed, and also equal to the corresponding excess at the nucleolus, while for the unsettled coalitions ( $S \in \Sigma^{r+1}$ ) we will keep the corresponding core constraints. We then define a sequence $Z^0 \supseteq Z^1 \supseteq \cdots \supseteq Z^{s+1}$ of sets of payoffs such that: $Z^0 = X^0 = C(w_A)$ and, for each $r \in \{0, 1, \ldots, s\}$,

$$Z^{r+1} = \left\{ x \in \mathbf{R}_+^M \times \mathbf{R}_+^{M'} \;\middle|\; \begin{array}{l} x_k = \nu_k(w_A) \text{ for all } \{k\} \in \Delta^{r+1} \\ a_{ij} - x_i - x_j = a_{ij} - \nu_i(w_A) - \nu_j(w_A) \text{ for all } \{i,j\} \in \Delta^{r+1} \\ x_k \geq 0 \text{ for all } \{k\} \in \Sigma^{r+1} \\ x_i + x_j \geq a_{ij} \text{ for all } \{i,j\} \in \Sigma^{r+1} \end{array} \right\} \tag{9}$$

Notice that, by definition, $Z^0 = X^0 = C(w_A)$ and $X^{r+1} \subseteq Z^{r+1}$ for all $r \in \{0, 1, \ldots, s\}$. Moreover, $Z^{s+1} = X^{s+1}$ and thus $Z^{s+1}$ also coincides with the nucleolus.

**Theorem 8** *Let $(M \cup M', w_A)$ be an assignment game with as many buyers as sellers.* [4] *Then,*

$$\nu(w_A) = (\underline{u}, \underline{v}) + \nu(w_{A^c}),$$

*where $(\underline{u}, \underline{v})$ is the vector of minimal core payoffs of the game $(M \cup M', w_A)$.*

Proof: We can assume without loss of generality that $w_A$ is buyer-seller exact ( $A = A^r$ ), since, by Núñez (2004), $\nu(w_A) = \nu(w_{A^r})$. Let us fix $\mu \in \mathcal{M}_A^*(M, M')$. Since $A = A^r$, we have $\mu \in \mathcal{M}_{A^c}^*(M, M')$, as it is justified at the beginning of the proof of Theorem 4. We then denote

[4] It is argued in Núñez (2004) that making the matrix square does not modify the nucleolus of the assignment game, if we drop the null payoff to the added dummy agents.

by $(\Delta^0, \Sigma^0), \ldots, (\Delta^{s+1}, \Sigma^{s+1})$ and $X^0, \ldots, X^{s+1}$ the partitions and payoff sets which define the lexicographic center of $(M \cup M', w_A)$, and by $(\tilde{\Delta}^0, \tilde{\Sigma}^0), \ldots, (\tilde{\Delta}^{s'+1}, \tilde{\Sigma}^{s'+1})$ and $\tilde{X}^0, \ldots, \tilde{X}^{s'+1}$ the partitions and payoff sets which define the lexicographic center of $(M \cup M', w_{A^c})$.

We prove that for all $0 \le r \le s+1$, $\Delta^r = \tilde{\Delta}^r$, $\Sigma^r = \tilde{\Sigma}^r$, and $X^r = \{(\underline{u}, \underline{v})\} + \tilde{X}^r$. We also write $e^A(S, x) = w_A(S) - x(S)$ and $e^{A^c}(S, y) = w_{A^c}(S) - y(S)$, for all $S \subseteq M \cup M'$ and all $x \in C(w_A)$ or $y \in C(w_{A^c})$.

The proof is by induction on $r$. By definition, $\Delta^0 = \tilde{\Delta}^0$, $\Sigma^0 = \tilde{\Sigma}^0$ and $X^0 = \{(\underline{u}, \underline{v})\} + \tilde{X}^0$. Assume now that, for some $r \ge 0$, $\Delta^r = \tilde{\Delta}^r$, $\Sigma^r = \tilde{\Sigma}^r$, and $X^r = \{(\underline{u}, \underline{v})\} + \tilde{X}^r$, and let us show that these equalities hold at step $r+1$. To see this, we prove that, for all $x \in Z^r$ (the set defined in (9)) and $y = x - (\underline{u}, \underline{v})$,

$$\max_{S \in \Sigma^r} e^A(S, x) = \max_{S \in \Sigma^r} e^{A^c}(S, y) = \max_{S \in \tilde{\Sigma}^r} e^{A^c}(S, y), \tag{10}$$

where the last equality follows from the induction assumptions. We will prove (10) in two steps.

**Claim 1:** For all $x \in Z^r$, if $\max_{S \in \Sigma^r} e^A(S, x) = e^A(S^*, x)$, then $e^A(S^*, x) = e^{A^c}(S^*, y)$ for $y = x - (\underline{u}, \underline{v})$.

If the above maximum is attained at a mixed-pair coalition $S^* = \{i, j\} \in \Sigma^r$, where $i \in M$ and $j \in M'$, the assertion in Claim 1 follows straightforwardly since

$$e^A(S^*, x) = a_{ij} - x_i - x_j = a_{ij} - \underline{u}_i - \underline{v}_j - (x_i - \underline{u}_i) - (x_j - \underline{v}_j) = e^{A^c}(S^*, y).$$

Assume then that, for some $x \in Z^r$, $\max_{S \in \Sigma^r} e^A(S, x) = -x_{i^*}$ for some $S^* = \{i^*\} \in \Sigma^r$ (let us assume without loss of generality that $i^* \in M$). If $\underline{u}_{i^*} = 0$, then also $e^A(S^*, x) = -x_{i^*} = -y_{i^*} = e^{A^c}(S^*, y)$. Assume otherwise that $\underline{u}_{i^*} > 0$.

Since $-x_{i^*} \ge a_{ij} - x_i - x_j$ for all $\{i, j\} \in \Sigma^r$, we get

$$(x_i - x_{i^*}) + x_j \ge a_{ij} \text{ for all } \{i, j\} \in \Sigma^r. \tag{11}$$

Define then the graph $G = (V, E)$ with the set of vertices $V = M \cup M'$ and the set of edges $E = \{\{i, j\} \mid \{i, j\} \in \Delta^r\}$. Let $C_{i^*}$ be the connected component of $G$ containing player $i^*$ and denote by $V_{i^*}$ the vertices in this component. Notice that:

a) For all $k \in V_{i^*}$, $\{k\} \notin \Delta^r$. To see that, assume there exists $k \in V_{i^*}$ such that $\{k\} \in \Delta^r$. Since $i^*$ and $k$ are in the same component, there exist $k_1, k_2, \ldots, k_l \in V_{i^*}$ such that the edges $\{k, k_1\}$, $\{k_1, k_2\}$,..., $\{k_l, i^*\}$ all belong to $E$. But then, from $\{k\} \in \Delta^r$ and $\{k, k_1\} \in \Delta^r$ follows that $\{k_1\} \in \Delta^r$ and then recursively that $\{k_2\}, \ldots, \{k_l\}$ and $\{i^*\}$ also belong to $\Delta^r$, in contradiction with $\{i^*\} \in \Sigma^r$.

b) For all $i \in V_{i^*} \cap M$, $x_{i^*} \leq x_i$. This follows from the fact that $\max_{S \in \Sigma^r} e^A(S, x) = -x_{i^*} \geq -x_i$ for all $\{i\} \in \Sigma^r$.

We now define an allocation $\tilde{x} \in \mathbf{R}^M \times \mathbf{R}^{M'}$ by

$$\begin{array}{ll} \tilde{x}_i = x_i - x_{i^*}\,, \text{ for all } i \in V_{i^*} \cap M\,; & \tilde{x}_i = x_i\,, \text{ for all } i \in M\,,\ i \notin V_{i^*}\,; \\ \tilde{x}_j = x_j + x_{i^*}\,, \text{ for all } j \in V_{i^*} \cap M'\,; & \tilde{x}_j = x_j\,, \text{ for all } j \in M'\,,\ j \notin V_{i^*}\,. \end{array}$$

Let us check that $\tilde{x} \in C(w_A)$. It is obvious from b) that $\tilde{x}_k \geq 0$ for all $k \in M \cup M'$. Secondly, if $(i, j) \in \mu$ then $\{i, j\} \in \Delta^r$ and either both players belong to the component $V_{i^*}$ or none of them does. If $i, j \in V_{i^*}$, then $\tilde{x}_i + \tilde{x}_j = x_i - x_{i^*} + x_j + x_{i^*} = x_i + x_j = a_{ij}$ since $x \in Z^r \subseteq C(w_A)$. If $i \notin V_{i^*}$ and $j \notin V_{i^*}$, then also $\tilde{x}_i + \tilde{x}_j = x_i + x_j = a_{ij}$ since $x \in Z^r \subseteq C(w_A)$.

Finally, if $(i, j) \notin \mu$, there are four possibilities. If $i \in V_{i^*}$ and $j \in V_{i^*}$, then again $\tilde{x}_i + \tilde{x}_j = x_i - x_{i^*} + x_j + x_{i^*} = x_i + x_j \geq a_{ij}$ since $x \in Z^r \subseteq C(w_A)$. If $i \notin V_{i^*}$ and $j \notin V_{i^*}$, then, since $x \in Z^r$ is a core allocation, $\tilde{x}_i + \tilde{x}_j = x_i + x_j \geq a_{ij}$. If $i \in V_{i^*}$ and $j \notin V_{i^*}$, this means that $\{i, j\} \in \Sigma^r$ and thus by (11) we get $\tilde{x}_i + \tilde{x}_j = x_i - x_{i^*} + x_j \geq a_{ij}$. And in the last case, that is $i \notin V_{i^*}$ and $j \in V_{i^*}$, we also get $\tilde{x}_i + \tilde{x}_j = x_i + x_j + x_{i^*} \geq a_{ij} + x_{i^*} \geq a_{ij}$ since $x \in Z^r$ belongs to the core.

Once proved that $\tilde{x} \in C(w_A)$, notice that $\tilde{x}_{i^*} = 0$ in contradiction with $\underline{u}_{i^*} > 0$. This proves Claim 1. It now remains to prove the following claim.

**Claim 2:** For all $x \in Z^r$, if $\max_{S \in \Sigma^r} e^A(S, x) = e^A(S^*, x)$, then $e^{A^c}(S^*, y) = \max_{S \in \Sigma^r} e^{A^c}(S, y)$, where $y = x - (\underline{u}, \underline{v})$.

On the one side, for all $\{i, j\} \in \Sigma^r$, where $i \in M$ and $j \in M'$, $e^{A^c}(S^*, y) = e^A(S^*, x) \geq a_{ij} - x_i - x_j = a^c_{ij} - y_i - y_j$, where the first equality holds from Claim 1. On the other side, for all $\{i\} \in \Sigma^r$ (let us assume without loss of generality that $i \in M$) we have $e^{A^c}(S^*, y) = e^A(S^*, x) \geq -x_i = -y_i - \underline{u}_i$. If $\underline{u}_i = 0$, we have that $e^{A^c}(S^*, y) \geq -y_i$.

Otherwise, that is if $\underline{u}_i > 0$, notice that $Z^r$ preserves the same lattice structure as the core of the assignment game and thus there exists a sellers-optimal allocation in $Z^r$, that we denote by $(\underline{u}^r, \overline{v}^r)$. Since $Z^r \subseteq C(w_A)$, it holds $\underline{u}^r_i \geq \underline{u}_i > 0$. Since $(\underline{u}^r, \overline{v}^r)$ is an extreme point of $Z^r$, $\{i\} \in \Sigma^r$ and $\underline{u}_i > 0$, some other constraint of $Z^r$ involving agent $i$ must be tight at $(\underline{u}^r, \overline{v}^r)$, apart from those related to coalitions in $\Delta^r$. This means that there exists $\{i,j\} \in \Sigma^r$ such that $\underline{u}_i^r + \overline{v}_j^r = a_{ij}$ . Then,

$$\begin{aligned} e^{A^c}(S^*, y) &= e^A(S^*, x) \geq a_{ij} - x_i - x_j = \underline{u}_i^r + \overline{v}_j^r - x_i - x_j \\ &\geq -x_i + \underline{u}_i^r \geq -x_i + \underline{u}_i = -y_i \,. \end{aligned}$$

This proves Claim 2.

Now, once proved that for all $x \in Z^r$ , $\max_{S\in\Sigma^r} e^A(S,x) = \max_{S\in\Sigma^r} e^{A^c}(S,y)$ where $y = x-(\underline{u},\underline{v})$ , notice that since $X^r \subseteq Z^r$ , the above coincidence holds for all $x \in X^r$ . Then, since by induction hypothesis $X^r = \{(\underline{u},\underline{v})\} + \tilde{X}^r$ and $\Sigma^r = \tilde{\Sigma}^r$ ,

$$\alpha^{r+1} = \min_{x\in X^r} \max_{S\in\Sigma^r} e^A(S,x) = \min_{y\in\tilde{X}^r} \max_{S\in\tilde{\Sigma}^r} e^{A^c}(S,y) = \tilde{\alpha}^{r+1} \,.$$

As a consequence,

$$\begin{aligned} X^{r+1} &= \{x \in X^r \mid \max_{S\in\Sigma^r} e^A(S,x) = \alpha^{r+1}\} = \\ &= \{(\underline{u},\underline{v})\} + \{y \in \tilde{X}^r \mid \max_{S\in\tilde{\Sigma}^r} e^{A^c}(S,y) = \tilde{\alpha}^{r+1}\} = \{(\underline{u},\underline{v})\} + \tilde{X}^{r+1} \,. \end{aligned}$$

Moreover, since $-y_i = -x_i + \underline{u}_i$ , $-y_j = -x_j + \overline{v}_j$ and $a_{ij} - x_i - x_j = a^c_{ij} - y_i - y_j$ , where $y = x - (\underline{u},\underline{v})$ , we have that $e^A(S,x)$ is constant on $X^{r+1}$ if and only if $e^{A^c}(S,y)$ is constant on $\tilde{X}^{r+1}$ . Thus, $\Sigma_{r+1} = \tilde{\Sigma}_{r+1}$ , $\Delta^{r+1} = \tilde{\Delta}^{r+1}$ and $\Sigma^{r+1} = \tilde{\Sigma}^{r+1}$ .

Thus, when $X^r$ reduces to one single point for some $r \geq 0$ , the same happens to $\tilde{X}^r$ . As a consequence, the algorithms to compute the lexicographic center of $(M \cup M', w_A)$ and $(M \cup M', w_{A^c})$ finish at the same time, that is $s = s'$ . Then, $X^{s+1} = \{\underline{u},\underline{v})\} + \tilde{X}^{s'+1}$ or equivalently $\nu(w_A) = (\underline{u},\underline{v}) + \nu(w_{A^c})$ . □

# 6 Concluding remarks

In this paper we have related to every assignment market a canonical form, the core of which is the translation of the core of the original market. We first want to remark that in the characterization of this canonical form $A^c$ (Theorem 4) the buyer-seller exact representative $A^r$ plays a fundamental role since it guarantees that the minimal core payoffs of agents $i \in M$ and $j \in M'$ can be subtracted from the worth of the mixed-pair coalition and still have a non-negative profit.

Secondly, the usefulness of the canonical form $A^c$ lies on the fact that the computation of some solutions for the canonical form gives, by means of translation, the same solution for the original market.

This is the case of the extreme core allocations. We know a simple procedure to compute these extreme points which is only valid for assignment games with large core (see Izquierdo et al., 2006). However, once the extreme core points of the canonical representative have been obtained, their translation by the vector of minimal core payoffs are the extreme core points of the original market.

Since the $\tau$-value of the assignment game is the midpoint between the buyers-optimal and the sellers-optimal core allocations, this $\tau$-value is the translation, by the vector of minimal core payoffs, of the $\tau$-value of the canonical representative. Also the kernel of the assignment game is proved in the paper to be the translation by the vector of minimal core payoffs of the kernel of its canonical representative. And the same happens to the nucleolus. Thus, if we had a simple procedure to compute the nucleolus or the kernel of those assignment games with large core, we would obtain by translation the same solution applied to the original assignment market.

Other solution concepts, when applied to assignment markets, do not behave so regularly with respect to the canonical representative: this is the case of the Shapley or the stable sets. As for the Shapley value of an assignment game, it is well-known that it usually does not belong to the core. However, Hoffmann and Sudhölter (2005) have recently proved that exactness (that is to say, being dominant diagonal and doubly dominant diagonal) is a sufficient condition to guarantee that the Shapley value of the assignment game is a core allocation. We can then define a single-valued solution, *the translated Shapley value* $\phi^t(w_A)$, for the assignment game as the translation of the Shapley value of its canonical representative, $\phi(w_{A^c})$, by the vector of minimal core payoffs, that is, $\phi^t(w_A) = (\underline{u}, \underline{v}) + \phi(w_{A^c})$. By Theorem 4, $\phi^t(w_A)$ is a core selection but it will not coincide, in general, with the Shapley value of the assignment game $(M \cup M', w_A)$.

Finally, it seems that the use of the canonical representative will allow us to prove how to determine, just from the assignment matrix, the dimension of the core of an assignment game. Also, this canonical representative seems to be a useful tool to prove the existence of an stable set for an arbitrary assignment market. But these will be the subjects of subsequent papers.

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