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On Modeling Transport Costs

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Abstract

The interpretation of the loss of utility as transport costs in address models of differentiation poses a methodological difficulty. Transport costs implicitely amounts to assume that there is a good neither included in the differentiated sector nor in the composite (numeraire) good of the economy. We propose to use iceberg-type transport costs to solve this difficulty.

Keywords: Spatial Competition, Iceberg transport costs, Monopoly.

JEL Classification: L12, D42

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1 Introduction.

In this note we tackle the following question: all models of spatial competition have three essential elements. A set of consumers endowed with (separable) preferences over a differentiated good and a composite homogeneous good (numeraire) produced under competitive conditions; a set of firms producing different varieties of the differentiated good, and a function representing the loss of utility borne by a consumer when unable to buy his best preferred variety. A (popular) reinterpretation of the model translates the utility loss into transport costs, so that consumers choose the variety that minimizes the delivered price defined as the sum of the mill price and the transport cost. This interpretation implicitly assumes that there is an industry that does not belong to the differentiated sector and is not included in the composite good. This industry is awkward because its structure is not modeled. It is not specified how transportation is provided, who the agents are in this industry or what their objective functions are. Although it represents an appealing way to study product differentiation, it also contains a methodological issue that has not been addressed so far.

This "difficulty" also appears in other domains of research, such as urban economics (see e.g. Fujita (1995), Abdel-Raman (1994a,b)) or in general equilibrium models of international trade (see e.g. Krugman (1991, 1992), Helpman and Krugman (1988)). Curiously enough, in those areas the way to cope with this issue has been different. Transport costs are formulated in terms of the transported commodity. This modeling was formalized by Samuelson (1954) as "iceberg transport costs" taking up an idea originated in von Thünen (1930).

Surprisingly enough, up to the authors' knowledge, there is no contribution in the literature on spatial competition where transport costs are modeled using the iceberg transport cost technology. Such formulation means that transport costs are dependent on market prices. Several interpretations can be put formard. We can think of the melting phenomenon as a loss of quality, or, in a temporal interpretation, as a lag between the buying of the commodity and its consumption.

To illustrate our point, in section 3 we propose a monopoly and a general melting function. By assuming away competition, we can concentrate in the consequences of modeling transports costs in the iceberg fashion. Next, we apply our analysis to particular specifications of the melting function. The driving force behind the results is that in contrast with traditional models of spatial competition where resources devoted to transportation are lost, under the iceberg modeling these resources are transferred to the firms which, in turn, induces a more elastic demand. Also, given that the resources devoted to transportation are taken into account in the agents' optimization behavior, welfare analysis would be meaningful. Section 4 extends the analysis with an illustration of the effects of the melting function approach in a competitive framework defined by a symmetric duopoly. A section with conclusion closes the paper.

2 Melting vs. Transport Costs.

Let δ be the distance between consumer x and the firm. We denote by $M(\delta)$ the rate at which the commodity melts away per unit of distance at a distance δ of the firm. Given $M(\delta)$ we have to distinguish between the demand an individual addresses to a firm from his consumption. Denote by $q_x(\delta)$ the quantity of the commodity a consumer located at $x \in [0, L]$ needs to buy to consume exactly one unit of the good. With this notation, we can formally define the melting rate M as,

$$M(\delta) = \frac{\partial q_x}{\partial \delta} \frac{1}{q_x}.$$

Finally, denote by $\mu \in (0, 1)$ a constant positively linked to the melting rate per unit of distance.

Definition 1 (Generalized melting function). *A generalized melting function, $h(\delta; \mu)$, specifies the additional demand addressed by an individual located at a distance δ from the firm to be able to consume one unit of the comodity. It is given by,*

$$h(\delta; \mu) = q_x(\delta) - 1, \tag{1}$$

where

$$h(\delta; \mu) > 0, \ \frac{\partial h}{\partial \delta} > 0, \ \frac{\partial h}{\partial \mu} > 0, \ \frac{\partial^2 h}{\partial \delta^2} \geq 0, \ \frac{\partial^2 h}{\partial \mu^2} \geq 0. \tag{2}$$

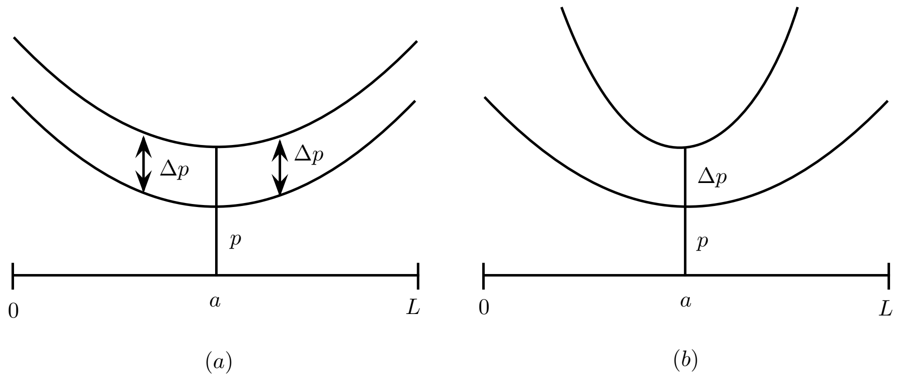

Figure 1: Delivered prices in a spatial model.

Thus, the melting rate is given by,

$$M(\delta) = \frac{\partial h}{\partial \delta} \frac{1}{1 + h(\delta; \mu)}.$$

We could also think of specifying the melting as a function of the distance and the amount of commodity transported. In this case we would have $q_x(1 - f(\delta; \mu)) = 1$. However, we can rewrite this latter demand as the former with $h(\delta; \mu) = \frac{f(\delta; \mu)}{1 - f(\delta; \mu)}$. Therefore, both cases are formally equivalent.

To ease a proper understanding of the role the melting in the modeling, figure 1 illustrates the standard transportation cost and the melting function approaches.

Consider the standard spatial model with convex transport costs. The unit price at a location x is given by $p_x = p + t\delta^\alpha$, where p denotes the unit f.o.b. price and $\alpha > 1$. Note that the slope of p_x with respect to p is one, so that an increase in p translates in exactly the same way to all consumers, i.e. the impact is positive but independent of δ. This situation is depicted in part (a) of figure 1.

The price paid by a consumer located at x according to the proposed general melting function is given by $p_x = p(1 + h(\delta; \mu))$. We observe that the impact on p_x of an increase in p now is a positive function of δ. That is, $\frac{\partial^2 p_x}{\partial p \partial \delta} > 0$. Also, the impact on p_x is increasing with δ. Generically, this situation is presented in part (b) of figure 1.

In the standard spatial model, a decrease in price increases demand because the demand is downward sloping with respect to price. When transport costs are modeled in the iceberg fashion, a decrease in price has an additional effect on demand. Demand increases not only because it is negatively related to the price but because the extra quantity demanded ("transport cost") is also cheaper. In other words, demand is more elastic than in the standard spatial model.

3 Analysis.

Consider a spatial market described by a line segment of length L. Consumers are evenly distributed on the market with unit density. They are identical in all respects but for their location. A consumer is denoted by $x \in [0, L]$. All consumers have a common reservation price $\overline{p}$. Consumers adjust their demands so that, if positive, they are able to consume exactly one unit of the commodity.

There is a monopolist in the market located at a distance a from the left end of the market. It produces a homogeneous product using a constant marginal cost (zero) technology. Assume, without loss of generality $a \leq L/2$.

The assumptions on $h(\delta; \mu)$ given by (2), imply that the demand addressed by consumer x given by (1), is a symmetric and increasing function around the firm's location and convex in both δ and μ.

The consumers indifferent between buying one unit from the monopolist or stay out of the market (denote them by $z \in [0, a]$ and $y \in [a, L]$) are given by the solution of the following equations:

$$p\big(1 + h(a - z; \mu)\big) = \overline{p} = p\big(1 + h(y - a; \mu)\big). \tag{3}$$

A direct inspection of (3), tells us that since $\dfrac{\partial h}{\partial \delta} > 0$

$$a - z = y - a. \tag{4}$$

In equilibrium, the consumers located at either extreme of the interval describing the market covered by the monopolist must obtain no surplus.

If the monopolist charges a price $p = \overline{p}$ it obtains zero demand regardless of its location. If the monopolist decides to cover the whole market, it will do it efficiently locating at the market center ($a = L/2$) and charging a certain price

$p = \tilde{p}(L/2)$ (see expression (6) below) so that the indifferent consumers are located at zero and L respectively.

For prices $p \in (\tilde{p}(L/2), \overline{p})$, the monopolist leaves some consumers unattended. In this case, for every price p there is a continuum of locations yielding the same demand. Among these, there is always one such that the indifferent consumer x is located at zero. We can thus, characterize demand captured by the monopolist for every price $p \in (\tilde{p}(L/2), \overline{p})$ by the set of consumers with its left bound at zero. This in turn, implies that the feasible locations for the monopolist are $a \in \big[0, \frac{L}{2}\big)$.

Demand addressed to the monopolist is given by,

$$\begin{aligned} D(p) &= \int_z^a \big(1 + h(a-w;\mu)\big)dw + \int_a^y \big(1 + h(w-a;\mu)\big)dw \\ &= 2(a-z) + H(a-z;\mu) + H(y-a;\mu) - 2H(0;\mu), \end{aligned}$$

where we have made use of (4) and $H(\delta;\mu) = \int h(\delta;\mu)d\delta$. From (4) it follows $H(a-x;\mu) = H(y-a;\mu)$ so that

$$D(p) = 2(a - z + H(a-z;\mu) - H(0;\mu)). \tag{5}$$

The price that makes the consumer located at zero indifferent between buying the commodity or staying out of the market is,

$$\tilde{p}(a) = \frac{\overline{p}}{1 + h(a;\mu)}. \tag{6}$$

Evaluating firm's profits at $\tilde{p}(a)$, we obtain,

$$\Pi(a) = 2\tilde{p}(a)\big(a + H(a;\mu) - H(0;\mu)\big). \tag{7}$$

Proposition 1. *Under monopoly, the market will be covered if, for all $a \leq L/2$,*

$$\big(1 + h(a;\mu)\big)^2 \geq \frac{\partial h}{\partial \delta}(a;\mu)\big(a + H(a;\mu) - H(0;\mu)\big). \tag{8}$$

Proof. From (7) we derive

$$\frac{\partial \Pi(a)}{\partial a} = 2\overline{p}\left(1 - \frac{\frac{\partial h}{\partial \delta}(a;\mu)(a + H(a;\mu) - H(0;\mu))}{(1 + h(a;\mu))^2}\right),$$

that is non negative if the condition (8) holds. □

As particular cases we propose the following:

Definition 2 (Melting with Quantity and Distance: MQD)**.** *We say that melting is MQD when it is proportional to the product of the quantity bought and the distance traveled. Formally:*

$$q_x - \mu\delta q_x = 1 \tag{9}$$

and $h(\delta;\mu) = \frac{\mu\delta}{1-\mu\delta}$.

Definition 3 (Melting with Distance: MD)**.** *We refer to MD melting as the situation where the melting is proportional to a power (α) of the distance. Formally:*

$$q_x - \mu\delta^{\alpha} = 1 \tag{10}$$

and $h(\delta;\mu) = \mu\delta^{\alpha}$.

Definition 4 (Samuelson Melting)**.** *Samuelson's melting process considers a constant melting rate with* $h(\delta;\mu) = e^{\mu\delta} - 1$ *and*

$$q_x = e^{\mu\delta} \tag{11}$$

Corollary 1. *(i) Under MQD, the monopolist covers the whole market and locates at its center iff* $\dfrac{L}{2} \leq \dfrac{e-1}{e\mu}$

(ii) Under MD, the monopolist locates at the center and covers all the market if $\alpha \leq 2$.

(iii) Under Samuelson Melting, the monopolist locates at the center and covers all the market.

Proof. See appendix $\square$

4 Melting in oligopolistic markets

We now extend our analysis to the case of oligopolistic markets. We intend to illustrate the effect of the iceberg transport approach in the study of oligopoly. We retain the same model as in the monopoly case, but now we introduce competition between two identical firms except for their location on the market. To ease the illustration we restrict firms to locate symmetrically around the market center, and we will characterize transport costs by a Samuelson-type melting function.

To be precise, we introduce first the necessary notation.

Two firms a, and b are located at points a and $b = 1 - a$ where $a \in [0, 1/2)$. Let p_a and p_b denote their respective prices, and $\bar{p}$ the common reservation price for consumers. We assume $\bar{p}$ to be high enough (but finite) so that all consumers can afford purchasing from one of the firms (in other words, the market is fully covered). Let $\mu = 1$ in the Samuelson melting function. Then, the price paid by a consumer located at a distance δ from firm i is given by $p_i e^{\delta}$. As $\delta < 1/2$, the assumption of a high enough consumers' willingness to pay translates into

$$\bar{p} > p_i e^{\frac{1}{2}}. \tag{12}$$

4.1 Consumers' decision problem

We start by characterizing the decision problem of a consumer located at a point $x \in [0, 1]$. Such consumer would be indifferent between patronizing either firm if, for a given price pair (p_a, p_b), (s)he satisfies,

$$p_a e^{x-a} = p_b e^{1-a-x}.$$

That is,

$$x(p_a, p_b) = \frac{1}{2}\Big(1 - \ln \frac{p_a}{p_b}\Big). \tag{13}$$

Accordingly, demand captured by both firms is given by,

$$D_a(p_a, p_b) = \int_0^a e^{a-s} ds + \int_a^{x(p_a,p_b)} e^{s-a} ds = -2 + e^a + e^{\frac{1}{2}-a}\sqrt{\frac{p_b}{p_a}}, \tag{14}$$

$$D_b(p_a, p_b) = \int_{x(p_a,p_b)}^{1-a} e^{1-a-s} ds + \int_{1-a}^{1} e^{s-1+a} ds = -2 + e^a + e^{\frac{1}{2}-a}\sqrt{\frac{p_a}{p_b}}. \tag{15}$$

Note that aggregate demand is given by,

$$D_a(p_a, p_b) + D_b(p_a, p_b) = -4 + 2e^a + e^{\frac{1}{2}-a}\Big(\sqrt{\frac{p_a}{p_b}} + \sqrt{\frac{p_b}{p_a}}\Big).$$

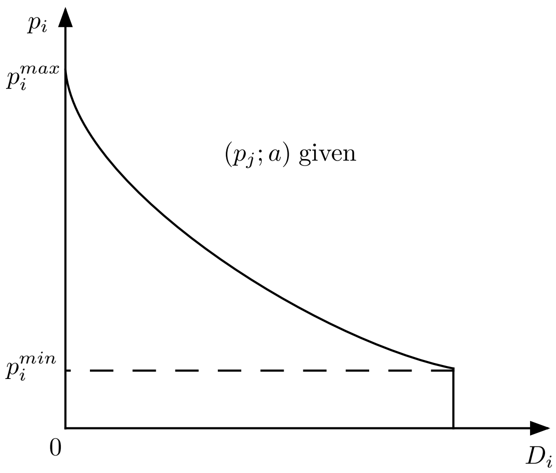

Figure 2: Firm i's contingent demand.

Given the symmetry of the set up, we can focus on a generic firm i with demand

$$D_i(p_i, p_j) = -2 + e^a + e^{\frac{1}{2}-a}\sqrt{\frac{p_j}{p_i}}$$

Note that firm i's market share will be positive if, given p_j its price is such that the indifferent consumer is located at $x(p_i, p_j) \geq 0$. That is, firm i will capture some consumers whenever it quotes a price $p_i \leq p_i^{max}$ where p_i^{max} is defined as the solution of $x(p_i, p_j) = 0$. Similarly, firm i will capture all the market when quoting a price $p_i \leq p_i^{min}$, where p_i^{min} is the solution of $x(p_i, p_j) = 1$. Formally,

$$p_i^{max} = ep_j, \quad \text{and} \quad p_i^{min} = \frac{p_j}{e}.$$

Figure 2 represents firm i's contingent demand. It is continuous, decreasing in p_i and convex in $[p_i^{min}, p_i^{max}]$.

Also, given $\bar{p}$, and recalling that the price paid by a consumer located at a distance δ from the firm is given by $p_i e^\delta$, it follows that $p_i e^\delta < \bar{p}$. As we assume symmetric locations, it is necessary for firms to cover the market that,

$$p_i < \frac{\bar{p}}{e^a}. \tag{16}$$

From (12) and (16), and since e^a is increasing in a, it follows,

$$\frac{\bar{p}}{p_i} > \max\{e^{\frac{1}{2}}, e^a\} = e^{\frac{1}{2}}. \tag{17}$$

4.2 Firms' decision problem

Firms produce their differentiated variety with a constant returns to scale technology, common to both firms and represented by a constant marginal cost $c > 0$. They aim at maximizing profits by choosing non-cooperatively their prices.

Firm i's profits are given by,

$$\Pi_i(p_i, p_j) = (p_i - c)\Big(-2 + e^a + e^{\frac{1}{2}-a}\sqrt{\frac{p_j}{p_i}}\Big) \tag{18}$$

It is straightforward to compute,

$$\frac{\partial \Pi_i}{\partial p_i} = -2 + e^a + e^{\frac{1}{2}-a}\Big(\frac{p_j}{p_i}\Big)^{\frac{1}{2}}\Big(\frac{p_i + c}{2p_i}\Big), \tag{19}$$

$$\frac{\partial^2 \Pi_i}{\partial p_i^2} = -\frac{1}{4}e^{\frac{1}{2}-a}p_j^{\frac{1}{2}}p_i^{-\frac{5}{2}}(p_i + 3c) < 0. \tag{20}$$

Accordingly, firm i's profit function is continuous and concave in $[p^{min}, p^{max}]$.

From the inspection of the first order condition (19), it should be apparent that this price game has a symmetric solution at $p_a = p_b = p$. Note also, that we (implicitly) assume $p_i > c$ so that profits are well defined. In turn. this implies from (17) that,

$$\frac{\bar{p}}{c} > e^{\frac{1}{2}}, \tag{21}$$

an expression that will be useful below.

4.3 Symmetric price equilibrium

The candidate symmetric price equilibrium (under full market coverage) of this game is obtained from (19) after substituting p_i by p. It is given by,

$$p^*(a) = \frac{ce^{\frac{1}{2}-a}}{4 - 2e^a - e^{\frac{1}{2}-a}}. \tag{22}$$

This candidate equilibrium price is well-defined only for some values of the location parameter a. In particular, note that $p^*(a) > 0$ for $a \in [0, 0.35)$, and $p^*(a) < 0$ for $a \in (0.35, 0.5)$, and

$$\lim_{a \to (0.35)^-} p^*(a) = \infty, \quad \text{and} \quad \lim_{a \to (0.35)^+} p^*(a) = -\infty.$$

Also, note that $p^*(a)$ is increasing and convex in a and $p^*(0) > c$. Thus, $p^*(a)$ is a candidate equilibrium price only if $a < 0.35$. However, as we are characterizing the equilibrium under full market coverage, we have to ensure that the most distant consumer to the firm faces a price leaving him(her) indifferent between patronizing the firm or staying out of the market. To identify such a price, recall that all consumers have a reservation price $\bar{p}$. Also, the most distant consumer to, say, firm a is located at zero when $a \geq 1/4$, and is located at 1/2 when $a \leq 1/4$. Accordingly, using (16), we obtain that the price at which the most distant consumer is indifferent between buying and not buying is given by

$$\begin{cases} \frac{\bar{p}}{e^a} \text{ if } a \geq \frac{1}{4} \\ \frac{\bar{p}}{e^{\frac{1}{2}-a}} \text{ if } a \leq \frac{1}{4} \end{cases} \tag{23}$$

From (22) and (23) we obtain that the symmetric price equilibrium under market coverage is given by,

$$p^* = \begin{cases} \min\left\{ \frac{ce^{\frac{1}{2}-a}}{4-2e^a-e^{\frac{1}{2}-a}},\ \frac{\bar{p}}{e^a},\ \frac{\bar{p}}{e^{\frac{1}{2}-a}} \right\} & \text{if } a < 0.35 \\ \frac{\bar{p}}{e^a} & \text{if } a \geq 0.35 \end{cases} \tag{24}$$

That is, as long as $p^*(a)$ is low enough to capture the most distant consumer, the firm follows, when $a < 0.35$, the profit maximizing price. However, it may happen that for a given location of the firm, the most distant consumer finds $p^*(a)$ too high, so that the full price of the purchase, $p^*(a)e^{\delta}$ (with $\delta = \max\left\{a, (\frac{1}{2} - a)\right\}$), lies above his(her) reservation price. Then, for those locations the firm will keep the market covered charging the price given by (23). Hence, p^* is defined as the lower envelope of the prices given by (22) and (23).

Note that

Figure 3 summarizes the discussion. For any $a \in [0, \frac{1}{2})$ the lower envelope is characterized by the ratio $\bar{p}/c$.

Four different configurations may arise. We can characterize them by comparing the values of the price functions at $a = 0$ and at $a = 1/4$. To ease the

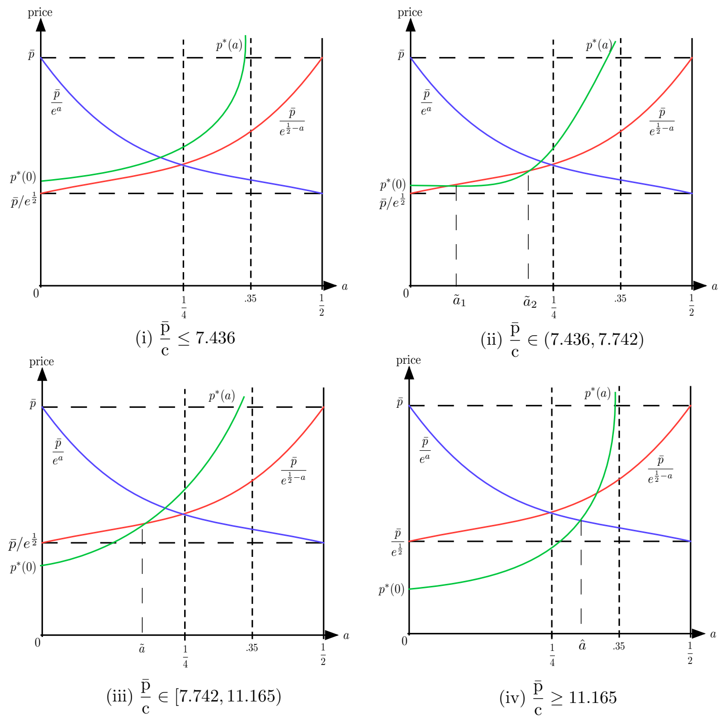

Figure 3: Symmetric equilibrium price profile

characterization, note that

$$p^*(0) = \frac{c}{0.213}, \quad p^*(\frac{1}{4}) = \frac{c}{0.115};$$
$$\frac{\bar{p}}{e^{\frac{1}{2}}} = \frac{\bar{p}}{1.649}, \quad \frac{\bar{p}}{e^{\frac{1}{4}}} = \frac{\bar{p}}{1.284}.$$

Accordingly,

$$p^*(0) > \frac{\bar{p}}{e^{\frac{1}{2}}} \iff \frac{\bar{p}}{c} < 7.742,$$
$$p^*(\frac{1}{4}) > \frac{\bar{p}}{e^{\frac{1}{4}}} \iff \frac{\bar{p}}{c} < 11.165.$$

The four cases depicted in figure 3 are characterized by,

Case (i) is described by $p^*(0) > \frac{\bar{p}}{e^{\frac{1}{2}}}$ and $p^*(\frac{1}{4}) > \frac{\bar{p}}{e^{\frac{1}{4}}}$. This implies $\frac{\bar{p}}{c} < 7.742$. This bound is to be qualified by the argument of case (ii). There we argue that the actual bound is $\frac{\bar{p}}{c} \leq 7.436$. Also, in section 4.5 below, we will identify a lower bound for $\frac{\bar{p}}{c}$ given by $e^{\frac{1}{2}}$. Recalling (21), Case (i) can be described by $\frac{\bar{p}}{c} \in (e^{\frac{1}{2}}, 7.436]$.

Case (ii) is fairly subtle. The curve $p^*(a)$ crosses the curve $\frac{\bar{p}}{e^{\frac{1}{2}-a}}$ twice. Note that $\frac{\partial p^*(a)}{\partial a}\Big|_{a=0} = 0 < \frac{\bar{p}}{e^{\frac{1}{2}}} = \frac{\partial \frac{\bar{p}}{e^{\frac{1}{2}-a}}}{\partial a}\Big|_{a=0}$. Accordingly, it must be the case that $p^*(0) > \frac{\bar{p}}{e^{\frac{1}{2}}}$. The subtlety of the argument appears when verifying whether both crossings occur to the left of $a = 1/4$, or the second crossing occurs to the right of $a = 1/4$. Let us consider this latter situation first. It means that $p^*(\frac{1}{4}) < \frac{\bar{p}}{e^{\frac{1}{4}}}$. This implies $\frac{\bar{p}}{c} < 7.742$ and $\frac{\bar{p}}{c} > 11.165$. Thus, this case cannot arise, and it must be the case that both crossings happen to the left of $a = 1/4$. To see the intuition of this case, assume, for the sake of the argument, that $\bar{p}$ is fixed. Start with a high enough value of c. This places us in case (i) above. Now, lowering gradually the value of c, the curve $p^*(a)$ shifts downwards. For some value of c both curves, $p^*(a)$ and $\frac{\bar{p}}{e^{\frac{1}{2}-a}}$ will be tangent. Such tangency occurs where both slopes coincide, yielding $\frac{\bar{p}}{c} = 7.436$. Therefore, case (ii) appears for $\frac{\bar{p}}{c} \in (7.436, 7.742)$, and accordingly, case (i) as stated above is characterized by $\frac{\bar{p}}{c} \leq 7.436$.

Case (iii) is described by $p^*(0) < \frac{\bar{p}}{e^{\frac{1}{2}}}$ and $p^*(\frac{1}{4}) > \frac{\bar{p}}{e^{\frac{1}{4}}}$. Thus, $\frac{\bar{p}}{c} \in (7.742, 11.165)$. It is also easy to verify that $p^*(0) > c$.

Case (iv) is described by $p^*(0) < \frac{\bar{p}}{e^{\frac{1}{2}}}$ and $p^*(\frac{1}{4}) \leq \frac{\bar{p}}{e^{\frac{1}{4}}}$. This implies $\frac{\bar{p}}{c} \geq 11.165$.

Thus, we have a complete characterization of the symmetric price equilibrium of the game, summarized in the following

Proposition 2. *In a duopoly model with Samuelson melting, there is a symmetric* *price equilibrium. It is characterized by*

$$p^* = \begin{cases} \begin{cases} \frac{\bar{p}}{e^{\frac{1}{2}-a}}, & \textit{for } a \in [0, \frac{1}{4}] \\ \frac{\bar{p}}{e^a}, & \textit{for } a \in [\frac{1}{4}, \frac{1}{2}] \end{cases} & \textit{if } \frac{\bar{p}}{c} \in (e^{\frac{1}{2}}, 7.436] \\ \begin{cases} p^*(a), & \textit{for } a \in [\tilde{a}_1, \tilde{a}_2] \\ \frac{\bar{p}}{e^{\frac{1}{2}-a}}, & \textit{for } a \in [0, \tilde{a}_1] \cup [\tilde{a}_2, \frac{1}{4}] \\ \frac{\bar{p}}{e^a}, & \textit{for } a \in [\frac{1}{4}, \frac{1}{2}] \end{cases} & \textit{if } \frac{\bar{p}}{c} \in (7.436, 7.742) \\ \begin{cases} p^*(a), & \textit{for } a \in [0, \tilde{a}] \\ \frac{\bar{p}}{e^{\frac{1}{2}-a}}, & \textit{for } a \in [\tilde{a}, \frac{1}{4}] \\ \frac{\bar{p}}{e^a}, & \textit{for } a \in [\frac{1}{4}, \frac{1}{2}] \end{cases} & \textit{if } \frac{\bar{p}}{c} \in [7.742, 11.165) \\ \begin{cases} p^*(a), & \textit{for } a \in [0, \hat{a}] \\ \frac{\bar{p}}{e^a}, & \textit{for } a \in [\hat{a}, \frac{1}{2}] \end{cases} & \textit{if } \frac{\bar{p}}{c} \geq 11.165 \end{cases}$$

where $\tilde{a}_1, \tilde{a}_2$ and $\tilde{a}$ are the solutions of $p^*(a) = \frac{\bar{p}}{e^{\frac{1}{2}-a}}$ in cases (ii) and (iii) respectively, and $\hat{a}$ is the solution of $p^*(a) = \frac{\bar{p}}{e^a}$ in case (iv).

Note that given $\frac{\bar{p}}{c}$, the equilibrium is unique. Also, p^* is defined for all $a \in [0, \frac{1}{2}]$, and is continuous in a, but not necessarily differentiable in all a. Accordingly, demand and profit functions are also continuous in a.

4.4 Location

To study the optimal (symmetric) location of the firm, recall that price equilibrium is characterized, given $\frac{\bar{p}}{c}$, by the lower envelope of the price functions, $p^*(a)$, $\frac{\bar{p}}{e^a}$, and $\frac{\bar{p}}{e^{\frac{1}{2}-a}}$.

We will proceed in two steps. First, we will study the behavior of the profit function under each of the price functions. then we will characterize the equilibrium location according to the scenario induced by $\frac{\bar{p}}{c}$.

4.5 Study of the profit function $\Pi(a)$

Profits of the firm are defined, as usual, as the mark-up of the equilibrium price over the (constant) marginal cost on the demand, that is,

$$\Pi(a) = (p^* - c)D(a) = (p^* - c)(-2 + e^a + e^{\frac{1}{2}-a}),$$

where we have made use of (14). We want to study the behavior of this profit function with respect to a,

$$\frac{d\Pi(a)}{da} = \frac{dp^*}{da}D(a) + (p^* - c)\frac{dD(a)}{da},$$
$$\frac{d^2\Pi(a)}{da^2} = \frac{d^2p^*}{da^2}D(a) + 2\frac{dp^*}{da}\frac{dD(a)}{da} + (p^* - c)\frac{d^2D(a)}{da^2},$$

where

$$\frac{dD(a)}{da} = e^a - e^{\frac{1}{2}-a}, \quad \frac{d^2D(a)}{da^2} = e^a + e^{\frac{1}{2}-a}.$$

Lemma 1. *Assume $p^* = \frac{\bar{p}}{e^a}$. Then, the profit function $\Pi(a)$ is decreasing in a.*

Proof. Profits are defined for $a \in (\frac{1}{4}, \frac{1}{2})$ as,

$$\Pi(a) = \Big(\frac{\bar{p}}{e^a} - c\Big)(-2 + e^a + e^{\frac{1}{2}-a}).$$

Note that (21) ensure that profits are well-defined. As

$$\frac{dp^*}{da} = -\frac{\bar{p}}{e^a}, \text{ and } \frac{d^2p^*}{da^2} = \bar{p}e^a,$$

it follows,

$$\frac{d\Pi(a)}{da} = \frac{1}{e^{2a}}\Big[2\bar{p}\Big(e^a - e^{\frac{1}{2}}\Big) - c\Big(e^{3a} - e^{\frac{1}{2}+a}\Big)\Big] < 0$$

No general result can be obtained on the concavity or convexity of the profit function. □

Lemma 2. *Assume $p^* = \frac{\bar{p}}{e^{\frac{1}{2}-a}}$. Then, the profit function $\Pi(a)$ is increasing in a.*

Proof. Profits are defined for $a \in (0, \frac{1}{4})$ as,

$$\Pi(a) = \Big(\frac{\bar{p}}{e^{\frac{1}{2}-a}} - c\Big)(-2 + e^a + e^{\frac{1}{2}-a}),$$

Note that (21) ensure that profits are well-defined. As

$$\frac{dp^*}{da} = \frac{d^2p^*}{da^2} = \frac{\bar{p}}{e^{\frac{1}{2}-a}},$$

it follows,

$$\frac{d\Pi(a)}{da} = e^{a-\frac{1}{2}}\Big[2\bar{p}\Big(e^a - 1\Big) - c\Big(e^{\frac{1}{2}} - e^{1-2a}\Big)\Big] > 0$$

No general result can be obtained on the concavity or convexity of the profit function. □

Lemma 3. *Assume* $p^* = \frac{ce^{\frac{1}{2}-a}}{4-2e^a-e^{\frac{1}{2}-a}}$. *Then,* $\Pi(a)$ *has a minimum at* $\bar{a} \approx 0.05219266638$ *and is convex in* a.

Proof. Profits are defined as,

$$\Pi(a) = \Big(\frac{ce^{\frac{1}{2}-a}}{4-2e^a-e^{\frac{1}{2}-a}} - c\Big)(-2+e^a+e^{\frac{1}{2}-a}),$$

where

$$\frac{dp^*}{da} = \frac{4ce^{\frac{1}{2}-a}(e^a-1)}{\big(4-2e^a-e^{\frac{1}{2}-a}\big)^2}$$

$$\frac{d^2p^*}{da^2} = \frac{-4c\big(4e^{\frac{1}{2}-a} - 6e^{\frac{1}{2}} + e^{1-2a} + 4e^{\frac{1}{2}+a} - 2e^{1-a}\big)}{\big(4-2e^a-e^{\frac{1}{2}-a}\big)^3}$$

Accordingly,

$$\frac{d\Pi(a)}{da} = -\frac{2c(-2+e^a+e^{\frac{1}{2}-a})(6e^{\frac{1}{2}-a} - 3e^{\frac{1}{2}} - 4e^a + 2e^{2a} - e^{1-2a})}{(4-2e^a-e^{\frac{1}{2}-a})^2}$$

The first term in brackets in the numerator is always positive for $a \in [0, 0.35]$. The second term in brackets has a root at $\bar{a} \approx 0.05219266638$, is positive for smaller values of $\bar{a}$ and negative for larger values of $\bar{a}$.

$$\frac{d^2\Pi(a)}{da^2} = \frac{2c\Lambda(a)}{-(4-2e^a-e^{\frac{1}{2}-a})^3} > 0,$$

because $\Lambda(a) < 0 \; \forall a \in (0, 0.35)$, where

$$\begin{aligned}\Lambda(a) = 48e^{\frac{1}{2}-a} + 24e^{3a} + 12e^{\frac{3}{2}-3a} - 4e^{4a} - e^{2-4a} - 48e^{2a} - 52e^{1-2a} + 32e^a \\ - 21e - 96e^{\frac{1}{2}} + 60e^{\frac{1}{2}+a} + 64e^{1-a} - 12e^{\frac{1}{2}+2a} - 6e^{\frac{3}{2}-2a}\end{aligned}$$

□

4.6 Location equilibrium

Once the behavior of the profit function has been fully identified, we can characterize the (symmetric) location equilibrium of the game. For a given ratio $\frac{\bar{p}}{c}$, this equilibrium is unique, as the following proposition states.

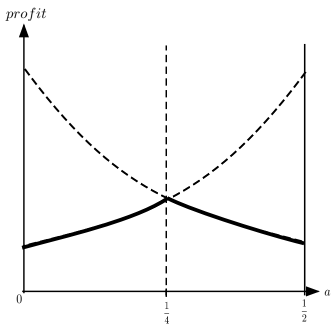

Figure 4: Location equilibrium for $\frac{\bar{p}}{c} \in (e^{\frac{1}{2}}, 7.436)$

Proposition 3. *(i) Assume* $\frac{\bar{p}}{c} \in (e^{\frac{1}{2}}, 11.165)$. *Then, there is a symmetric location equilibrium at* $a = \frac{1}{4}$.

(ii) Assume $\frac{\bar{p}}{c} \geq 11.165$. *Then, there is a symmetric location equilibrium at* $a = \hat{a}$, *where* $\hat{a}$ *is the solution of* $p^*(a) = \frac{\bar{p}}{e^a}$

Proof. See appendix $\square$

The intuition of the proposition goes as follows:

(i) Assume $\frac{\bar{p}}{c} \in (e^{\frac{1}{2}}, 7.436)$. From proposition 2, lemma 1 and lemma 2, the profit function has the shape ahown in figure 4. Accordingly, the maximum value of the profit function is reached at $a = \frac{1}{4}$. Note that at such point, the profit function is not differentiable.

(ii) Assume $\frac{\bar{p}}{c} \in (7.436, 7.742)$. From proposition 2, lemma 1, lemma 2, and lemma 3, two possible scenarios illustrated in figure 5, may arise according to the relative positions of $\tilde{a}_1, \tilde{a}_2$ with respect to 0.052. Recall that $\tilde{a}_1$ and $\tilde{a}_2$ are the real solutions of $p^*(a) = \frac{\bar{p}}{e^{\frac{1}{2}-a}}$. Scenario 1 yields a unique maximum at $a = \frac{1}{4}$. Scenario 2 shows two local maxima at $a = \tilde{a}_1$ and at $a = \frac{1}{4}$. In the appendix it is shown that the latter one is also a global equilibrium.

(iii) Assume $\frac{\bar{p}}{c} \in [7.742, 11.165)$. From proposition 2, lemma 1, lemma 2, and lemma 3, it follows that there are two possible shapes for the profit function as shown in figure 6. In any of the two situations, the global maximum of

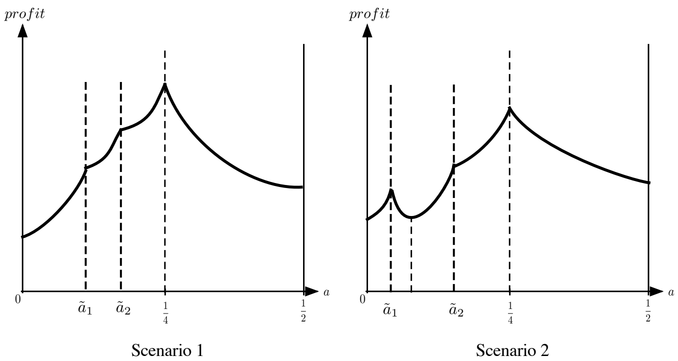

Figure 5: Location equilibrium for $\frac{\bar{p}}{c} \in (7.436, 7.742)$

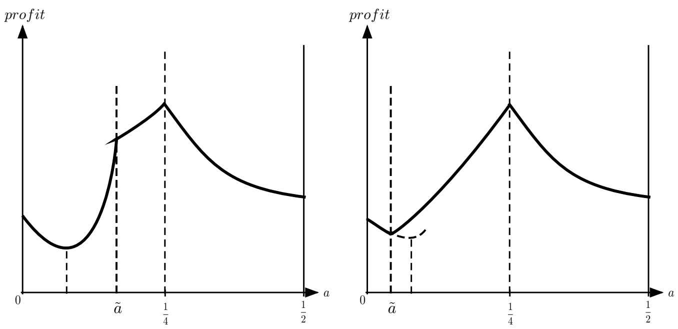

Figure 6: Location equilibrium for $\frac{\bar{p}}{c} \in [7.742, 11.165)$

the profit function is reached at $a = \frac{1}{4}$. Although at $a = 0$ there is a local maximum it is clearly not global because $\Pi(0) < \Pi(\frac{1}{4})$ for $\frac{\bar{p}}{c} > 6.702$. Note that at $a = \frac{1}{4}$, the profit function is not differentiable.

(iv) Assume $\frac{\bar{p}}{c} \geq 11.165$. From proposition 2, lemma 2 and lemma 3, it follows that the profit function has a local maximum at $a = 0$ and a global maximum at $a = \hat{a}$. Figure 7 illustrates.

5 Conclusion.

The motivation of the proposed analysis stems from the apparent inconsistency in the modeling of transport costs in address models of product differentiation. In those models it is assumed that transportation is a commodity different from the differentiated sector object of study and not included in the 'composite good'

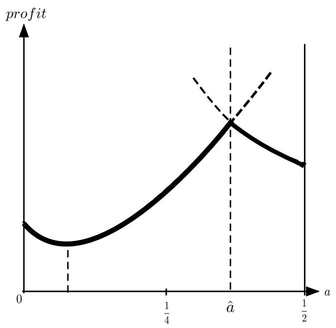

Figure 7: Location equilibrium for $\frac{\bar{p}}{c} \geq 11.165$

summarizing the rest of the economy. Methodologically, this is awkward because transportation becomes an industry whose structure is not modeled. Who are the agents or how transportation is provided is left unspecified. Curiously enough, this way of modeling product differentiation is appealing because the intuitions derived turn out to be quite useful.

We propose to solve this methodological inconsistency of the analysis by measuring transportation in terms of the differentiated commodity, using the so-called *iceberg transport cost* technology introduced by Von Thünen in 1930 and formalized by Samuelson in 1954. To our knowledge, this is the first attempt in the spatial competition literature in proposing such an approach.

The iceberg formulation represents a new way of thinking in spatial competition. To make the reader familiar with it, we first illustrate its implications assuming away all elements of competition. Next, we incorporate strategic interaction with a simple symmetric duopoly model. We can characterize prize and location competition in this new framework in terms of the ratio between the (common) reservation price and the (constant and common) marginal production cost. According to this ratio, we can summarize symmetric equilibrium location decisions in two patterns. Either firms locate at the first and third quartiles respectively, or at points $(\hat{a}, 1 - \hat{a})$, where $\hat{a} \in (\frac{1}{4}, 0.35)$.

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Appendix: Proof of Corollary 1

i) Under MQD melting we have:

$$\frac{\partial h(\delta,\mu)}{\partial \delta} = \frac{\mu}{(1-\mu\delta)^2}$$

and

$$H(\delta,\mu) = \frac{1}{\mu}(-\mu\delta - \ln(1-\mu\delta)).$$

Hence,

$$(1+h(a,\mu))^2 = \frac{1}{(1-\mu a)^2}$$

and

$$\frac{\partial h(a,\mu)}{\partial \delta}(a + H(a,\mu) - H(0,\mu)) = \frac{-\ln(1-\mu a)}{(1-\mu a)^2}.$$

As $\ln(1-\mu a) < 0$ and $\dfrac{d\ln(1-\mu a)}{da} < 0$, if expression (8) in Proposition 1 holds for $a = \frac{L}{2}$, it will also hold for $a < \frac{L}{2}$.

Given that

$$\frac{1}{(1-\mu\frac{L}{2})^2} > \frac{-\ln(1-\mu\frac{L}{2})}{(1-\mu\frac{L}{2})^2} \Leftrightarrow e > \frac{1}{1-\mu\frac{L}{2}} \Leftrightarrow 2\frac{e-1}{e\mu} > L,$$

we obtain, from Proposition 1, the result.

ii) Under MD melting we have:

$$\frac{\partial h(\delta,\mu)}{\partial \delta} = \mu\alpha\delta^{\alpha-1}$$

and

$$H(\delta,\mu) = \frac{\mu\delta^{\alpha+1}}{\alpha+1}.$$

Hence, as

$$(1+h(a,\mu))^2 = 1 + 2\mu a^\alpha + \mu^2 a^{2\alpha}$$

and

$$\frac{\partial h(a,\mu)}{\partial \delta}(a + H(a,\mu) - H(0,\mu)) = \alpha\mu a^\alpha + \frac{\alpha}{\alpha+1}\mu^2 a^{2\alpha},$$

the inequality (8) in Proposition 1 holds for $\alpha \leq 2.$ Accordingly, for $\alpha \leq 2$ the monopolist locates at the center and covers all the market.

iii) Under Samuelson melting we have:

$$\frac{\partial h(\delta,\mu)}{\partial \delta} = \mu e^{\mu\delta}$$

and

$$H(\delta,\mu) = \frac{e^{\mu\delta}}{\mu} - \delta.$$

Hence, given that

$$(1+h(a,\mu))^2 = e^{2\mu a}$$

and

$$\frac{\partial h(a,\mu)}{\partial \delta}.(a + H(a,\mu) - H(0,\mu)) = e^{2\mu a} - e^{\mu a},$$

from Proposition 1 we obtain that the monopolist locates at the center and covers all the market. □

Appendix 2: Proof of Proposition 3

Let us define $(e^{\frac{1}{2}}, 11.165) = (e^{\frac{1}{2}}, 7.436) \cup [7.436, 7.742) \cup [7.742, 11.165)$.

(i) Assume $\frac{\bar{p}}{c} \in (e^{\frac{1}{2}}, 7.436)$. This means that, from proposition 2, lemma 1 and lemma 2,

$$\frac{d\Pi(a)}{da} \begin{cases} > 0, & \text{for } a \in [0, \frac{1}{4}] \\ < 0, & \text{for } a \in [\frac{1}{4}, \frac{1}{2}] \end{cases}$$

Accordingly, the maximum value of the profit function is reached at $a = \frac{1}{4}$. Note that at such point, the profit function is not differentiable.

(ii) Assume $\frac{\bar{p}}{c} \in (7.436, 7.742)$. From proposition 2, lemma 1, lemma 2, and lemma 3, two possible scenarios may arise according to the relative positions of $\tilde{a}_1, \tilde{a}_2$ with respect to 0.052. Recall that $\tilde{a}_1$ and $\tilde{a}_2$ are the real solutions of $p^*(a) = \frac{\bar{p}}{e^{\frac{1}{2}-a}}$.

Scenario 1: $0.052 < \tilde{a}_1 < \tilde{a}_2$

$$\frac{d\Pi(a)}{da} \begin{cases} > 0, & \text{for } a \in [0, \frac{1}{4}] \\ < 0, & \text{for } a \in [\frac{1}{4}, \frac{1}{2}] \end{cases}$$

Scenario 2: $\tilde{a}_1 < 0.052 < \tilde{a}_2$

$$\frac{d\Pi(a)}{da} \begin{cases} > 0, & \text{for } a \in [0, \tilde{a}_1] \\ < 0, & \text{for } a \in [\tilde{a}_1, 0.052) \\ > 0, & \text{for } a \in (0.052, \frac{1}{4}] \\ < 0, & \text{for } a \in [\frac{1}{4}, \frac{1}{2}] \end{cases}$$

A potential third scenario characterized by $\tilde{a}_1 < \tilde{a}_2 < 0.052$ can be discarded. We know that the tangency between the curves $p^*(a)$ and $\frac{\bar{p}}{e^{\frac{1}{2}-a}}$ occurs at $a = 0.075628$. Given $\bar{p}$, if we lower the value of c, the curve $p^*(a)$ shifts downwards. Accordingly, $\tilde{a}_2 > 0.075628$.

Scenario 1 yields a unique maximum at $a = \frac{1}{4}$. Scenario 2 shows two local maxima at $a = \tilde{a}_1$ and at $a = \frac{1}{4}$. The latter one is also a global equilibrium. Note that $\tilde{a}_1$ and $\tilde{a}_2$ are the intersection points of the curves, $p^*(a)$ and $\frac{\bar{p}}{e^{\frac{1}{2}-a}}$. Also, lemma 2 tells us that the maximum profit over the curve $\frac{\bar{p}}{e^{\frac{1}{2}-a}}$ is reached at $a = 1/4$. This implies that $\Pi(\tilde{a}_1) < \Pi(\tilde{a}_2)$ because $\tilde{a}_1 < \tilde{a}_2 < \frac{1}{4}$.

(iii) Assume $\frac{\bar{p}}{c} \in [7.742, 11.165)$. From proposition 2, lemma 1, lemma 2, and lemma 3, it follows that

$$\frac{d\Pi(a)}{da} \begin{cases} < 0, & \text{for} \begin{cases} a \in [0, 0.052] & \text{if } \tilde{a} > 0.052 \\ a \in [0, \tilde{a}] & \text{if } \tilde{a} < 0.052 \end{cases} \\ > 0, & \text{for} \begin{cases} a \in [0.052, \frac{1}{4}] & \text{if } \tilde{a} > 0.052 \\ a \in [\tilde{a}, \frac{1}{4}] & \text{if } \tilde{a} < 0.052 \end{cases} \\ < 0, & \text{for } a \in [\frac{1}{4}, \frac{1}{2}] \end{cases}$$

In any of the two situations, the maximum value of the profit function is reached at $a = \frac{1}{4}$. Although at $a = 0$ there is a local maximum it is clearly not global because $\Pi(0) < \Pi(\frac{1}{4})$ for $\frac{\bar{p}}{c} > 6.702$. Note that at $a = \frac{1}{4}$, the profit function is not differentiable.

(iv) Assume $\frac{\bar{p}}{c} \geq 11.165$. From proposition 2, lemma 2 and lemma 3, it follows that

$$\frac{d\Pi(a)}{da} \begin{cases} < 0, & \text{for } a \in [0, 0.0522) \\ > 0, & \text{for } a \in (00522, \hat{a}] \\ < 0, & \text{for } a \in [\hat{a}, \frac{1}{2}] \end{cases}$$

Accordingly, the maximum value of the profit function is reached at $a = \hat{a}$. As in (iii) here we find again a local maximum at $a = 0$. However $\Pi(0) < \Pi(\hat{a})$ because

$$\big(\frac{c}{0.213} - c\big)(-2 + 1 + e^{\frac{1}{2}}) < \big(\frac{\overline{p}}{e^{\hat{a}}} - c\big)(-2 + e^{\hat{a}} + e^{\frac{1}{2} - \hat{a}})$$

where

$$\frac{\overline{p}}{e^{\hat{a}}} = \frac{ce^{\frac{1}{2} - \hat{a}}}{4 - 2e^{\hat{a}} - e^{\frac{1}{2} - \hat{a}}},$$

can be simplified to

$$\frac{(-2 + e^{\hat{a}} + e^{\frac{1}{2} - \hat{a}})^2}{4 - 2e^{\hat{a}} - e^{\frac{1}{2} - \hat{a}}} > 1.198$$

The fraction on the left-hand side is increasing and convex in $a \in (\frac{1}{4}, 0.35)$. Therefore, it has a minimum value of

$$\frac{(-2 + e^{\frac{1}{4}} + e^{\frac{1}{4}})^2}{4 - 2e^{\frac{1}{4}} - e^{\frac{1}{4}}} = 2.181.$$

Hence, it follows that the profit function reaches its global maximum at $a = \hat{a}$. Note that at such point, the profit function is not differentiable. $\square$