Centre de Referència en Economia Analítica

Barcelona Economics Working Paper Series

Working Paper nº 275

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April 7, 2006

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Abstract

Given an assignment market, we introduce a set of vectors, one for each possible ordering on the player set, which we name the max-payoff vectors. Each one of these vectors is obtained recursively only making use of the assignment matrix. Those max-payoff vectors that are efficient turn up to give the extreme core allocations of the market. When the assignment market has large core (that is to say, the assignment matrix is dominant diagonal and doubly dominant diagonal) all the max-payoff vectors are extreme core allocations.

Keywords assignment game, core, extreme core points, max-payoff vectors

1 Introduction

The assignment game (Shapley and Shubik, 1972) is a cooperative model for a two-sided market where side payments are allowed. In this market a product that comes in indivisible units is exchanged for money, and each participant either supplies or demands exactly one unit. The units need not be alike and the same unit may have different values for different participants.

*We gratefully acknowledge the financial support from the Barcelona Economics Program of CREA, from Ministerio de Educación y Ciencia and FEDER under grant SEJ2005-02443/ECON and from Generalitat de Catalunya under grant 2005SGR00984.

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From these valuations, a matrix can be written which reflects the profit that can be obtained by each buyer-seller pair if they trade. Shapley and Shubik prove that the core of the assignment game is nonempty, and has a lattice structure.

The first analysis of the extreme core allocations of the assignment game is due to Balinsky and Gale (1987). There, they show how to check, by means of the connectedness of a graph, whether a core allocation is an extreme point. Also, attainable upper and lower bounds for the number of extreme core allocations are provided. After that, Hamers *et al.* (2002) prove that every extreme core allocation of an assignment game is a marginal worth vector, although not all marginal worth vectors are in the core of the assignment game, since these games are not convex in general.

In Núñez and Rafels (2003) we characterize the set of extreme core allocations of the assignment game as the set of reduced marginal worth vectors. The reduced marginal worth vectors are inspired in the classical marginal worth vectors with the difference that, for a fixed permutation on the player set, a reduction of the game is performed before each player is paid her marginal contribution to her set of predecessors. Moreover, for convex games, reduced marginal worth vectors coincide with the marginal worth vectors, and thus this characterization provides a unified approach to the class of convex games and the class of assignment games with regard to the extreme core allocations.

However, to compute a reduced marginal worth vector of an assignment game is quite cumbersome, since before determining the payoff to each agent we must reduce the game by the procedure due to Davis and Maschler. In our paper we present the set of max-payoff vectors, also one for each possible ordering of the set of agents. For a fixed ordering on the player set, the corresponding max-payoff vector is obtained recursively, only making use of the assignment matrix. It turns up that every extreme core allocation of the assignment game is a max-payoff vector. In fact, the set of extreme core allocations of the assignment game coincides with the subset of max-payoff vectors that are efficient. Only when the assignment matrix has dominant diagonal and doubly dominant diagonal, the whole set of max-payoff vectors coincides with the set of extreme core allocations of the market. The definition of dominant diagonal and doubly dominant diagonal assignment matrix is due to Solymosi and Raghavan (2001), and these two properties together characterize those assignment games with large core (and those assignment game which are exact).

In Section 2, notations and the main known facts related to the assignment game are recalled. In Section 3 we introduce the set of max-payoff vectors of an assignment game and prove it contains the set of extreme core allocations. In fact the extreme core allocations of an assignment game are those max-payoff vectors which are efficient. The particular case of assignment games with a matrix that is dominant diagonal and doubly dominant diagonal is analyzed in Section 4.

2 The assignment game

Let us consider a two-sided market with a finite set of buyers M of cardinality $|M| = m$ and a finite set of sellers M' of cardinality $|M'| = m'$, and let $A = (a_{ij})_{(i,j)\in M\times M'}$ be a nonnegative matrix where a_{ij} represents the profit obtained by the mixed–pair $(i,j) \in M \times M'$ if they trade. Let $n = m + m'$ denote the cardinality $|N|$ of $N = M \cup M'$.

The *assignment problem* (M, M', A) consists in looking for an optimal matching between the two sides of the market. A *matching* $\mu \subseteq M \times M'$ between M and M' is a bijection from some $M_0 \subseteq M$ to some $M'_0 \subseteq M'$ such that $|M_0| = |M'_0| = \min\{|M|, |M'|\}$. We write $(i,j) \in \mu$ as well as $j = \mu(i)$ and $i = \mu^{-1}(j)$. We denote the set of matchings between M and M' by $\mathcal{M}(M, M')$. Moreover, we say a buyer $i \in M$ is not assigned by μ if $(i,j) \notin \mu$ for all $j \in M'$ (and similarly for sellers).

We say a matching $\mu \in \mathcal{M}(M, M')$ is *optimal* for the market (M, M', A) if for all $\mu' \in \mathcal{M}(M, M')$, we have $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$, and will denote the set of optimal matchings by $\mathcal{M}^*_A(M, M')$. Given $S \subseteq M$ and $T \subseteq M'$, we denote by $\mathcal{M}(S,T)$ and $\mathcal{M}^*_A(S,T)$ the set of matchings and optimal matchings of the submarket $(S, T, A_{|S\times T})$ defined by the subset S of buyers, the subset T of sellers and the restriction of A to $S \times T$. If $S = \emptyset$ or $T = \emptyset$, then the only possible matching is $\mu = \emptyset$ and by convention $\sum_{(i,j)\in\emptyset} a_{ij} = 0$.

Assignment games were introduced by Shapley and Shubik (1972) as a cooperative model for a two–sided market with transferable utility. Given an assignment problem (M, M', A), the player set is $N = M \cup M'$, and the matrix A determines the characteristic function w_A. Given $S \subseteq M$ and $T \subseteq M'$, $w_A(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T)\}$. Notice that a coalition formed only by sellers or only by buyers has worth zero.

Shapley and Shubik proved that the *core*, $C(w_A)$, of the assignment game $(M \cup M', w_A)$ is nonempty and can be represented in terms of any optimal matching μ of (M, M', A). Once fixed any such optimal matching,

$(u,v) \in \mathbf{R}^M \times \mathbf{R}^{M'}$ is in the core if and only if $u_i \geq 0$ for all $i \in M$, $v_j \geq 0$ for all $j \in M'$, $u_i + v_j \geq a_{ij}$ for all $(i,j) \in M \times M'$, $u_i + v_j = a_{ij}$ if $(i,j) \in \mu$ and $u_i = 0$ if $i \in M$ is not matched by μ, while $v_j = 0$ if $j \in M'$ is not matched by μ.

Moreover, the core has a lattice structure with two special extreme points: the *buyers–optimal core allocation*, $(\overline{u}, \underline{v})$, where each buyer attains his maximum core payoff, and the *sellers–optimal core allocation*, $(\underline{u}, \overline{v})$, where each seller does.

From Demange (1982) and Leonard (1983) we know that the maximum core payoff of any player coincides with his or her marginal contribution:

$$\begin{aligned} \overline{u}_i &= w_A(M \cup M') - w_A((M \cup M') \setminus \{i\}) \text{ for all } i \in M\,, \text{ and} \\ \overline{v}_j &= w_A(M \cup M') - w_A((M \cup M') \setminus \{j\}) \text{ for all } j \in M'. \end{aligned} \tag{1}$$

From (1), once fixed $\mu \in \mathcal{M}^*_A(M, M')$, and taking into account that $\underline{u}_i + \overline{v}_{\mu(i)} = a_{i\mu(i)}$, since $(\underline{u}, \overline{v}) \in C(w_A)$, we get that the minimum core payoff of a buyer i which is matched by μ is

$$\underline{u}_i = a_{i\mu(i)} + w_A((M \cup M') \setminus \{\mu(i)\}) - w_A(M \cup M')\,, \tag{2}$$

while $\underline{u}_i = 0$ if i is not assigned by μ. Similarly the minimum core payoff of a seller j which is matched by μ is

$$\underline{v}_j = a_{\mu^{-1}(j)j} + w_A((M \cup M') \setminus \{\mu^{-1}(j)\}) - w_A(M \cup M')\,. \tag{3}$$

The two aforementioned extreme core allocations of the assignment game are not, in general, the only ones. We will denote by $Ext(C(w_A))$ the set of extreme points of the core of $(M \cup M', w_A)$.

By adding null rows or columns if necessary, we will assume from now on that A is square, which means that the assignment problem has as many buyers as sellers.

Following Solymosi and Raghavan (2001), an assignment game with as many buyers as sellers $(M \cup M', w_A)$ has *dominant diagonal* if and only if, once placed an optimal matching in the diagonal, $a_{ii'} \geq a_{ij'}$ for all $j' \in M'$ and $a_{ii'} \geq a_{ji'}$ for all $j \in M$. As these authors point out, this is equivalent to saying that each agent has a null minimal core payoff. Since the property of having null core payoff does not depend on the optimal matching placed on the diagonal, it follows that, for all $\mu \in \mathcal{M}^*_A(M, M')$, the assignment game has dominant diagonal if and only if, for all $i_* \in M$, $a_{i_*\mu(i_*)} \geq a_{i_*j}$ for all $j \in M'$ and $a_{i_*\mu(i_*)} \geq a_{i\mu(i_*)}$ for all $i \in M$.

As mentioned in the introduction, Solymosi and Raghavan (2001) introduce another important property of the assignmen matrices, namely dominant diagonal dominance. An assignment game with as many buyers as sellers $(M \cup M', w_A)$ has *doubly dominant diagonal* if and only if, once chosen an optimal matching $\mu \in \mathcal{M}^*_A(M, M')$, $a_{ij} + a_{k\mu(k)} \geq a_{i\mu(k)} + a_{\mu(k)j}$ for all $i, j, k \in M$. This property is proved in Núñez and Rafels (2002) to characterize those assignment games with the property of being *buyer-seller exact* An assignment game is buyer-seller exact if, for each mixed pair $(i, j) \in M \times M'$ there exists a core allocation $x \in C(w_A)$ such that $x_i + x_j = a_{ij}$. Then, the fact that an assignment matrix is doubly dominant diagonal does not depend on the fixed optimal matching.

The two aforementioned properties together, that is to say, having dominant diagonal and doubly dominant diagonal, characterize those assignment games with large core and also those assignment games which are exact, as it is proved in Solymosi and Raghavan (2001).

3 The max-payoff vectors

With the aim of determining the extreme core allocations of the assignment game we introduce a set of vectors, the *max-payoff vectors*, one of them for each possible ordering in the set of agents. An *ordering* θ on $N = M \cup M'$ is a bijection from $\{1, 2, \ldots, n\}$ to $N = M \cup M'$. We then denote an ordering θ by $(k_1, k_2, \ldots, k_n)$ where, for all $i \in \{1, 2, \ldots, n\}$, $k_i = \theta(i)$ is the agent that occupies place i. The set of predecessors of agent $k_* \in M \cup M'$ in the ordering θ is $P^\theta_{k_*} = \{k \in M \cup M' \mid \theta^{-1}(k) < \theta^{-1}(k_*)\}$, and the set of all orderings on N is denoted by $\mathcal{S}_N$.

Definition 1 *Let $(M \cup M', w_A)$ be an assignment game. For all $\theta \in \mathcal{S}_N$, the max-payoff vector $x^\theta(A) \in \mathbf{R}^n$ related to θ is defined by $x^\theta_{k_1}(A) = w_A(k_1) = 0$ and, for all $r \in \{2, \ldots, n\}$,*

$$x^\theta_{k_r}(A) = \begin{cases} \max_{i \in P^\theta_{k_r} \cap M} \{0, a_{ik_r} - x^\theta_i(A)\} & \text{if} \quad k_r \in M', \\ \max_{j \in P^\theta_{k_r} \cap M'} \{0, a_{k_r j} - x^\theta_j(A)\} & \text{if} \quad k_r \in M. \end{cases}$$

Notice that to obtain the vectors $x^\theta(A)$ the characteristic function is not needed, but only the assignment matrix. When no confusion regarding the assignment matrix can arise, we simple write x^θ. Once fixed and ordering, the first agent receives a null payoff and, after that, the payoff to the other agents is defined by recurrence: each agent receives the maximum profit he can obtain by trading with one of his predecessors on the opposite side of the market, after paying this partner what the vector x^θ has already allocated to him.

With no restrictions on the assignment matrix, the set of max-payoff vectors, $\{x^\theta\}_{\theta\in\mathcal{S}_N}$, contains all the extreme core allocations of the game. That is, for every extreme core allocation it is possible to find an order θ on the player set such that the given extreme coincides with x^θ.

Theorem 2 *Let $(M\cup M', w_A)$ be an assignment game with as many buyers as sellers. Then*

$$Ext(C(w_A)) \subseteq \{x^\theta\}_{\theta\in\mathcal{S}_N}.$$

Proof. Take $x \in Ext(C(w_A))$ and consider the tight graph $G^{w_A}(x)=(V,E)$ with set of vertices $V=M\cup M'$ and set of edges $E=\{\{i,j\}\mid i\in M\,,\, j\in M'\,,\, x_i+x_j=a_{ij}\}$. Such a graph can be associated to any core allocation and by Hamers *et al.* (2002) we know that $x\in Ext(C(w_A))$ if and only if each connected component of $G^{w_A}(x)$ contains at least one player k such that $x_k=0$.

Fix $\mu\in\mathcal{M}_A^*(M,M')$ and select $k_1\in M\cup M'$ such that $x_{k_1}=0$. If $k_1\in M$, define $k_2=\mu(k_1)$, and if $k_1\in M'$, define $k_2=\mu^{-1}(k_1)$. We define $\theta(1)=k_1$ and $\theta(2)=k_2$. We see now that, for any ordering θ on these conditions, $x^\theta_{k_1}=x_1$ and $x^\theta_{k_2}=x_2$. To prove this, let us assume, without loss of generality, that $k_1\in M$. It is straightforward that $x^\theta_{k_1}=0=x_{k_1}$. Moreover,

$$x^\theta_{k_2}=\max\{0,a_{k_1k_2}-x^\theta_{k_1}\}=\max\{0,a_{k_1k_2}-x_{k_1}\}=a_{k_1k_2}-x_{k_1}=x_{k_2}$$

where the third equality is due to the fact that $(k_1,k_2)\in\mu$ and $x\in C(w_A)$, and thus $a_{k_1k_2}-x_{k_1}=x_{k_2}\geq 0$.

Assume, by induction hypothesis, that the first $2r$ players in the order θ are defined, $\theta(l)=k_l$ for $1\leq l\leq 2r$, in such a way that, for all $l\in\{1,2,\ldots,2r\}$ and odd, if $k_l\in M$ then $k_{l+1}=\mu(k_l)$ and if $k_l\in M'$ then $k_{l+1}=\mu^{-1}(k_l)$. Assume also that, for all $l\in\{1,2,\ldots,2r\}$, $x^\theta_{k_l}=x_{k_l}$. Let us prove that players in the positions $2r+1$ and $2r+2$ in the order θ can be chosen that meet the same conditions.

Case 1: If there exists some agent $k\in(M\cup M')\setminus\{k_1,k_2,\ldots,k_{2r}\}$ connected to some other agent in $\{k_1,k_2,\ldots,k_{2r}\}$, let us say there exist $k\in(M\cup M')\setminus\{k_1,k_2,\ldots,k_{2r}\}$ and $k^*\in\{k_1,k_2,\ldots,k_{2r}\}$ such that $\{k,k^*\}\in E$, then define $k_{2r+1}=k$ and $k_{2r+2}=\mu(k)$ if $k\in M$ and $k_{2r+2}=\mu^{-1}(k)$ if $k\in M'$.

Let us consider now any order θ such that $\theta(l) = k_l$ for all $1 \leq l \leq 2r{+}2$ and prove that $x^\theta_{k_{2r+1}} = x_{k_{2r+1}}$ and $x^\theta_{k_{2r+2}} = x_{k_{2r+2}}$. Let us assume, without loss of generality, that $k_{2r+1} \in M$.

Notice that, from $x \in C(w_A)$, we have $x_{k_{2r+1}} \geq 0$ and $x_{k_{2r+1}} + x_j \geq a_{k_{2r+1}j}$ for all $j \in M'$. Moreover, we know that $x_{k_{2r+1}} + x_{k^*} = a_{k_{2r+1}k^*}$, since $\{k_{2r+1}, k^*\} \in E$. This implies $x_{k_{2r+1}} = a_{k_{2r+1}k^*} - x_{k^*} \geq a_{k_{2r+1}j} - x_j$ for all $j \in M'$. Thus, by induction hypothesis,

$$\begin{aligned} x^\theta_{k_{2r+1}} &= \max_{j \in P^\theta_{k_{2r+1}} \cap M'} \{0, a_{k_{2r+1}j} - x^\theta_j\} \\ &= \max_{j \in P^\theta_{k_{2r+1}} \cap M'} \{0, a_{k_{2r+1}j} - x_j\} = a_{k_{2r+1}k^*} - x_{k^*} = x_{k_{2r+1}} . \end{aligned}$$

Similarly, from $x \in C(w_A)$ we have that $x_{k_{2r+2}} \geq 0$, $x_{k_{2r+1}} + x_{k_{2r+2}} = a_{k_{2r+1}k_{2r+2}}$ and $x_i + x_{k_{2r+2}} \geq a_{ik_{2r+2}}$ for all $i \in M$. This implies $x_{k_{2r+2}} = a_{k_{2r+1}k_{2r+2}} - x_{k_{2r+1}} \geq a_{ik_{2r+1}} - x_i$ for all $i \in M$. Thus, again by induction hypothesis,

$$\begin{aligned} x^\theta_{k_{2r+2}} &= \max_{i \in P^\theta_{k_{2r+2}} \cap M} \{0, a_{ik_{2r+2}} - x^\theta_i\} \\ &= \max_{i \in P^\theta_{k_{2r+2}} \cap M} \{0, a_{ik_{2r+2}} - x_i\} = a_{k_{2r+1}k_{2r+2}} - x_{k_{2r+1}} = x_{k_{2r+2}} . \end{aligned}$$

Case 2: If no agent $k \in (M \cup M') \setminus \{k_1, k_2, \ldots, k_{2r}\}$ is connected to any agent in $\{k_1, k_2, \ldots, k_{2r}\}$, then this means that all agents in $(M \cup M') \setminus \{k_1, k_2, \ldots, k_{2r}\}$ are in different components of the graph $G^{w_A}(x)$ than agents in $\{k_1, k_2, \ldots, k_{2r}\}$. Since in each connected component there exists an agent whose payoff by x is zero, let us take one such agent.

Choose $k \in (M \cup M') \setminus \{k_1, k_2, \ldots, k_{2r}\}$ such that $x_k = 0$. If $k \in M$, let us define $k_{2r+1} = k$ and $k_{2r+2} = \mu(k)$, and if $k \in M'$, let us define $k_{2r+1} = k$ and $k_{2r+2} = \mu^{-1}(k)$. We will see that, for any ordering θ such that $\theta(l) = k_l$ for all $1 \leq l \leq 2r + 2$, it holds $x^\theta_{k_{2r+1}} = x_{k_{2r+1}}$ and $x^\theta_{k_{2r+2}} = x_{k_{2r+2}}$. To do that, assume, without loss of generality, that $k_{2r+1} \in M$.

Since $x \in C(w_A)$ and $x_{k_{2r+1}} = 0$, from $x_{k_{2r+1}} + x_j \geq a_{k_{2r+1}j}$, for all $j \in M'$ it follows that $0 \geq a_{k_{2r+1}j} - x_j$ for all $j \in M'$. Then, by induction hypothesis,

$$\begin{aligned} x^\theta_{k_{2r+1}} &= \max_{j \in P^\theta_{k_{2r+1}} \cap M'} \{0, a_{k_{2r+1}j} - x^\theta_j\} = \\ &= \max_{j \in P^\theta_{k_{2r+1}} \cap M'} \{0, a_{k_{2r+1}j} - x_j\} = 0 = x_{k_{2r+1}} . \end{aligned}$$

Again, from $x \in C(w_A)$, we have $x_{k_{2r+2}} \geq 0$, $x_{k_{2r+1}} + x_{k_{2r+2}} = a_{k_{2r+1}k_{2r+2}}$ and $x_i + x_{k_{2r+2}} \geq a_{ik_{2r+2}}$ for all $i \in M$. This implies $x_{k_{2r+2}} = a_{k_{2r+1}k_{2r+2}} - x_{k_{2r+1}} \geq a_{ik_{2r+2}} - x_i$ for all $i \in M$. Thus, also by induction hypothesis,

$$\begin{aligned} x^{\theta}_{k_{2r+2}} &= \max_{i \in P^{\theta}_{k_{2r+2}} \cap M} \{0, a_{ik_{2r+2}} - x^{\theta}_i\} \\ &= \max_{i \in P^{\theta}_{k_{2r+2}} \cap M} \{0, a_{ik_{2r+2}} - x_i\} = a_{k_{2r+1}k_{2r+2}} - x_{k_{2r+1}} = x_{k_{2r+2}} . \end{aligned}$$

Thus, for all $x \in Ext(C(w_A))$, we have obtained an ordering $\theta \in \mathcal{S}_N$ such that $x = x^{\theta}$. □

The converse inclusion of that proved in Theorem 2 does not hold in general. To see that, consider for instance the assignment market defined by the matrix

	1'	2'
1	3	4
2	0	2

and notice that the core is the convex hull of the three extreme points (2, 0; 1, 2), (3, 0; 0, 2) and (3, 1; 0, 1). If you take any ordering θ such that buyer 1 is in the first place, $x^{\theta}_1 = 0$ and consequently x^{θ} will never be a core allocation. This assignment market has doubly dominant diagonal but not dominant diagonal.

Consider also the assignment market defined by the matrix

	1'	2'	3'
1	1	0	1
2	1	1	1
3	1	1	1

The core is the segment with extreme points (1, 1, 1; 0, 0, 0) and (0, 0, 0; 1, 1, 1). In this case the matrix has dominant diagonal but not doubly dominant diagonal, since $a_{12'} + a_{33'} < a_{13'} + a_{32'}$. If we take any ordering θ with buyer 1 in the first place and seller 2' in the second place, we have that $x^{\theta}_1 = 0$ and $x^{\theta}_{2'} = 0$, and this can never be achieved in the core of this market.

However, if a max-payoff vector belongs to the core, it must be an extreme point. This is a property that also holds for the marginal worth vectors and the reduced marginal worth vectors in the general domain of arbitrary coalitional games.

Lemma 3 *Let* $(M \cup M', w_A)$ *be an assignment game and* $\theta \in \mathcal{S}_N$ *. If* $x^\theta \in C(w_A)$ *then* $x^\theta \in Ext(C(w_A))$ *.*

Proof. Take $x^\theta \in C(w_A)$ and $\theta = (k_1, k_2, \ldots, k_n)$. By definition of x^θ , notice that for each k_r , $r \in \{1, 2, \ldots, n\}$, there exists a tight core constraint of the form $x^\theta_{k_r} = 0$ or $x^\theta_{k_r} + x^\theta_k = w(\{k_r, k\})$ for some $k \in P^\theta_{k_r}$. This linear system of equalities shows that $x^\theta \in Ext(C(w_A))$. □

In the following theorem we prove that in order to know if a max-payoff vector is an extreme core allocation of the assignment game we only have to check that it is efficient.

Theorem 4 *Let* $(M \cup M', w_A)$ *be an assignment game with as many buyers as sellers. Then*

$$Ext(C(w_A)) = \{x^\theta,\, \theta \in \mathcal{S}_N \mid x^\theta(M \cup M') = w_A(M \cup M')\}\,.$$

Proof. Take x^θ for some $\theta \in \mathcal{S}_N$. For all $(i, j) \in M \times M'$, if $\theta^{-1}(i) < \theta^{-1}(j)$, then, by definition, $x^\theta_j \geq a_{ij} - x^\theta_i$ or equivalently $x^\theta_i + x^\theta_j \geq a_{ij}$. Thus, if x^θ is efficient, $x^\theta(M \cup M') = w_A(M \cup M')$, then it is a core allocation. Now, by Lemma 3, x^θ is an extreme core point. □

Thus, to find the set of extreme core allocations, we have to compute all the max-payoff vectors and then, once chosen an optimal matching μ , select those max-payoff vectors that satisfy $x^\theta_i + x^\theta_j = a_{ij}$ for all $(i, j) \in \mu$.

By the proof of Theorem 2, notice that, given an extreme core allocation x , to find an order in the player set such that $x = x^\theta$, we can restrict to a particular set of orderings, those such that, once fixed an optimal matching μ , agents occupying an odd place in the ordering are followed by their optimal partners by μ . For $\mu \in \mathcal{M}^*_A(M, M')$, let us denote by $\mathcal{S}^\mu_N$ this subset of orderings on the set of agents,

$$\mathcal{S}^\mu_N = \left\{ \theta = (k_1, \ldots, k_n) \in \mathcal{S}_N \;\middle|\; \begin{array}{l} \text{for all } r \in \{0, 1, 2, \ldots, \frac{n}{2} - 1\} \\ \text{if } k_{2r+1} \in M \text{ then } k_{2r+2} = \mu(k_{2r+1}) \text{ and} \\ \text{if } k_{2r+1} \in M' \text{ then } k_{2r+2} = \mu^{-1}(k_{2r+1}) \end{array} \right\}.$$

Then, from Theorem 2 and Theorem 4 we can state without proof the following result.

Corollary 5 *Let* $(M \cup M', w_A)$ *be an assignment game with as many buyers as sellers and let* μ *be an optimal matching. Then,*

$$Ext(C(w_A)) = \{x^\theta,\, \theta \in \mathcal{S}^\mu_N, \mid x^\theta_i + x^\theta_j = a_{ij} \text{ for all } (i, j) \in \mu\}\,.$$

As an application of the above result, we can obtain the extreme core allocations of the assignment game defined by matrix

$$\begin{array}{c|c|c|} & 1' & 2' \\ \hline 1 & 3 & 4 \\ \hline 2 & 0 & 2 \\ \hline \end{array}$$

For each ordering in $\mathcal{S}_N^\mu$, we compute the max-payoff vector and in boldface we show those max-payoff vectors that are efficient, and thus the extreme core allocations of the assignment market.

$\theta \in \mathcal{S}_N^\mu$	$x^\theta = (u_1, u_2; v_{1'}, v_{2'})$
(1,1',2,2')	(0, 0; 3, 4)
(1,1',2',2)	(0, 0; 3, 4)
(1',1,2,2')	**(3, 0; 0, 2)**
(1',1,2',2)	**(3, 1; 0, 1)**
(2,2',1,1')	**(2, 0; 1, 2)**
(2,2',1',1)	**(3, 0; 0, 2)**
(2',2,1,1')	(4, 2; 0, 0)
(2',2,1',1)	(4, 2; 0, 0)

Notice that it is simpler to obtain the extreme core allocations of the assignment game by this method than by the reduced marginal worth vectors (Núñez and Rafels, 2003). The reduced marginal worth vectors require the computation of successive reduced games, while the max-payoff vectors x^θ are completely obtained from the assignment matrix. In addition to that, to obtain the set of extreme core allocations not all the orderings of the set of agents are needed, but only those orderings where, once fixed an optimal matching, each agent is either preceded or followed by his or her optimal partner.

4 The case of the assignment games with large core

We will now assume that the assignment game $(M \cup M', w_A)$ has dominant diagonal and doubly dominant diagonal. From Solymosi and Raghavan (2001) we know that these two properties together characterize those assignment games which have a large core. The concept of large core was introduced by Sharkey (1982) for arbitrary coalitional games and is based on the notion of aspiration. An *aspiration* is a payoff vector $x \in \mathbf{R}^n$ that satisfies coalitional rationality but may not satisfy efficiency. Thus, an aspiration for the assignment game $(M \cup M', w_A)$ is $x \in \mathbf{R}^M \times \mathbf{R}^{M'}$ such that $x(S) \geq w_A(S)$ for all $S \subseteq M \cup M'$. As it happens with an arbitrary coalitional game, an assignment game has a *large core* if and only if for any aspiration $x \in \mathbf{R}^M \times \mathbf{R}^{M'}$ there exists $y \in C(w_A)$ such that $x_k \geq y_k$ for all $k \in M \cup M'$.

Under the assumption that A has dominant diagonal and doubly dominant diagonal, the next theorem states that all the max-payoff vectors x^θ, for $\theta \in \mathcal{S}_N$, are extreme core allocations. Moreover, the above two properties characterize the assignment games where the coincidence between the set of max-payoff vectors and extreme core points holds.

Theorem 6 *Let $(M \cup M', w_A)$ be an assignment game with as many buyers as sellers. The following statements are equivalent:*

1. *A has dominant diagonal and doubly dominant diagonal.*
2. *$Ext(C(w_A)) = \{x^\theta\}_{\theta \in \mathcal{S}_N}$.*

Proof. Let us assume that A has dominant diagonal and doubly dominant diagonal, and consequently $(M \cup M', w_A)$ has large core. Notice first that, for all $\theta \in \mathcal{S}_N$, x^θ is an aspiration of the assignment game $(M \cup M', w_A)$. To see that, for all $S \subseteq M \cup M'$ take μ_S an optimal matching for the submarket formed by the agents in S, that is $\mu_S \in \mathcal{M}^*_A(S \cap M, S \cap M')$. Then,

$$x^\theta(S) = \sum_{(i,j) \in \mu_S} (x^\theta_i + x^\theta_j) + \sum_{\substack{k \in S \\ k \text{ not matched by} \mu_S}} x^\theta_k \geq \sum_{(i,j) \in \mu_S} a_{ij} = w_A(S),$$

where the inequality follows from Definition 1. Now, since $(M \cup M', w_A)$ has large core, there exists $x \in C(w_A)$ such that $x^\theta_k \geq x_k$ for all $k \in M \cup M'$. But notice that, by Definition 1, for all $i \in M$, either $x^\theta_i = 0$ or there exists $j \in M'$ such that $x^\theta_i + x^\theta_j = a_{ij}$. This means that x^θ_i cannot be decreased and still give a core allocation. Similarly, for all $j \in M'$, either $x^\theta_j = 0$ or there exists $i \in M$ such that $x^\theta_i + x^\theta_j = a_{ij}$, and x^θ_j can neither be lowered and still give a core allocation.

Once we know that $x^\theta = x$ belongs to the core, by Lemma 3 we know it is an extreme point. Then $\{x^\theta\}_{\theta \in \mathcal{S}_N} \subseteq Ext(C(w_A))$ and the other inclusion follows from Theorem 2.

Conversely, assume that all x^θ are extreme core allocations. Take an ordering θ such that all buyers enter in first place and after all sellers enter in an arbitrary order. By Definition 1, $x_i^\theta = 0$ for all $i \in M$. This proves that each buyer has a null minimum core payoff. Similarly, if we take an order where all sellers enter in first place we see that in the corresponding max-payoff vector all sellers receive a null payoff. This means that A is dominant diagonal.

To see that A is also doubly dominant diagonal, for each mixed pair $(i,j) \in M \times M'$, take an ordering θ such that i occupies the first place and j the second one. Again, by Definition 1, $x_i^\theta = 0$ and $x_j^\theta = a_{ij}$. Then, x^θ is a core allocation by assumption and $x_i^\theta + x_j^\theta = a_{ij}$. This means that A is buyer-seller exact and consequently, by Núñez and Rafels (2002), A is doubly dominant diagonal. □

From the theorem above, if we have an assignment game $(M \cup M', w_A)$ with large core (that is A has dominant digonal and doubly dominant diagonal), and once fixed an optimal matching $\mu \in \mathcal{M}_A^*(M, M')$, the set of extreme core allocations coincides with the set of max-payoff vector corresponding to the orderings in $\mathcal{S}_N^\mu$.

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