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On the Weights of Nations: Assigning Voting Weights in a Heterogeneous Union

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Abstract

Consider a voting procedure where countries, states, or districts comprising a union each elect representatives who then participate in later votes at the union level on their behalf. The countries, provinces, and states may vary in their populations and composition. If we wish to maximize the total expected utility of all agents in the union, how to weight the votes of the representatives of the different countries, states or districts at the union level? We provide a simple characterization of the efficient voting rule in terms of the weights assigned to different districts and the voting threshold (how large a qualified majority is needed to induce change versus the status quo). Next, in the context of a model of the correlation structure of agents preferences, we analyze how voting weights relate to the population size of a country. We then analyze the voting weights in Council of the European Union under the Nice Treaty and the recent constitution, and contrast them under different versions of our model.

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1 Introduction

Citizens vote occasionally, while their elected representatives vote frequently. This is sensible due to the cost of becoming informed on a myriad of issues and of involving full populations in the innumerable decisions that fully direct democracy would require. While indirect democracy is sensible and prevalent due to the costs of involving full populations in decision making, it introduces distortions in the decision process due to the fact that a single vote by a representative does not completely represent the heterogeneity of votes that would be cast by that representative's constituency.

If districts are small, of similar size, and of similar degrees of heterogeneity, then weighting each representative's vote equally provides a system of indirect democracy that maximizes overall societal welfare. However, for a variety of reasons, there are many systems of indirect democracy that are not structured in this way. A particularly important and timely example is the Council of Ministers of the European Union, a critical decision making body of the EU. That council consists of a single representative from each country in the European Union. The countries differ widely in their population sizes and compositions. Similar examples include the United Nations, the US Senate, and a variety of state and local governments. In any democratic union where the member countries, states, or districts comprising the union may be of different sizes and have different compositions in terms of distributions of citizens' preferences, it makes sense to weight the votes of the representatives.[1] For instance, if countries differ in population and their voting power is not weighted, then small countries might impose decisions that a majority of the affected people are against.

In this paper we take as given that there is a heterogeneous set of countries, states or districts, each having one representative for the purpose of making a collective decision. We characterize the voting rule that maximizes total societal welfare, as measured by the sum of the utilities of all citizens of all the involved countries, states, or districts, subject to the constraint that the district structure is fixed exogenously and possibly heterogeneous. We examine situations where votes are conducted over two alternatives: one referred to as the status quo and the other referred to as a change. We show that an optimal voting rule consists of two things: a weight for each country indicating the relative weight that each country's vote is given; and a threshold or quota, indicating how large the total weight of votes cast in favor of change must be in order for change to be enacted.

One important conclusion of our analysis is that the structuring of the optimal voting weights and thresholds can be treated separately. The optimal weights depend on the distribution of preferences within a country and how that compares to other countries. The threshold depends on the extent of any bias of preferences in terms of total intensity in favor of the status quo compared to change.

The efficient weights can be described intuitively as follows. Consider the vote by a given representative of a country. Suppose that he or she has voted "yes" on a given issue. We can then ask the following question. Given the vote of "yes", what is the surplus of people in the country who favor "yes" over "no"? For instance if 62 percent of the people favor "yes" and 38 percent favor "no", then 24 percent more of the population favor "yes" versus "no". Multiplying this percentage times the population gives us a measure of how much this country would benefit if we choose "yes"

[1] Alternatively, one can think of adjusting the number of representatives that each country, state, or district has.

versus “no”, and how much this country would suffer if we chose the reverse. The efficient voting weight is exactly this expected surplus.[2]

As the general characterization of efficient voting rules depends on the distribution of preferences within each country, we also explore a model of population behavior, which we refer to as the “block model,” which allows us to derive optimal weights as a function of population size. This works by assuming that a country consists of a set of voting blocks of preferences, where citizens within a block are similar and have correlated preferences, while citizens across blocks are uncorrelated. This structure allows us to pinpoint the efficient voting weights and thresholds under two focal scenarios.

After the development of our theoretical model, we examine the model’s implications for the voting system of the Council of Ministers of the European Union, as suggested under both the Nice treaty of 2000 and the Constitutional Convention of 2003, which proposed very different sets of weights and voting thresholds. The Nice Treaty proposed weights that are less than proportional to population size and a relatively high threshold for passage (73.9 percent), while the Constitutional Convention proposed weights that are directly proportional to population size and a lower threshold (65 percent).[3] This leads to a question of what the “right” weights for each of the countries are and how should the threshold be determined? We show that these two conflicting proposals coincide with the two polar cases of our “block model” of population behavior. Which set of weights is more efficient then boils down to an empirical question of preference patterns.

Contribution and Relation to the Literature

It is surprising that the previous literature has not considered the criterion of efficiency (total expected utility) as a guide to determine optimal voting rules for indirect democracy.[4] There is a literature that relates to indirect democracy; however, it approaches the problem from other perspectives. For instance, there is a rich literature in cooperative game theory that examines weighted majority games. The main thread there has been to produce power indices, measuring things such as the relative probabilities that different voters are pivotal, such as the Banzhaf (1965) and Shapley-Shubik (1954) indices, among others. While some researchers have built power measures based on satisfaction (that is, total utility) and contrasted them with power measures built on decisiveness (see for instance Dubey and Shapley (1979), Barry (1980) and Laruelle and Valenciano (2003)), our perspective is still quite different. Our aim is not to measure power or satisfaction or to compare rules under such measures, but instead to study the optimal design of voting rules. To the extent that the previous literature has thought about designing rules, it

[2]Our model allows for heterogeneities in intensities of preferences among voters as well, and the weights adjust for that. Here we are simply describing a special case where intensity among voters is similar, and the full characterization is provided below.

[3]The Convention’s proposal also includes a requirement that at 55 percent of the countries support a measure (Constitution, Title IV, Article I-25), which could also be binding, but less frequently; and that a blocking minority include at least four countries. (There are also special provisions for votes on issues that were not proposed through either the commission or the Union Minister for Foreign Affairs, where the requirement on countries is raised to 72 percent.) As a first (rough) approximation, we ignore these extra constraints in the discussion of the constitution’s proposed weights. As we shall see in the discussion following Theorem 1, more complex voting systems can be optimal (see also Harstad (2005) for a rationalization of dual majority systems).

[4]Rae (1969) analyzed voting rules under this utilitarian perspective of maximizing expected utility or satisfaction rather than decisiveness (see also Badger (1972) and Curtis (1972)), but in the context of direct democracy.

has focussed on equating the power of agents, rather than maximizing the total expected utilities of agents.[5] This dates to the seminal work of Penrose (1946). Depending on the distribution of preferences, these two objectives can lead to quite different voting rules. And, as we show, maximizing total expected utility can result in large inequalities in the treatment of individuals across countries.

Perhaps the closest predecessor to the theoretical part of our work is that of Felsenthal and Machover (1999), who also study the design of two-stage voting rules from an optimization perspective. Their objective is to minimize the expected difference between the size of the majority and the number of supporters of the chosen alternative.[6] That objective differs from maximizing total expected utility as it does not account for the surplus of voters in favor of an alternative when the majoritarian alternative is selected, but only accounts for the deficit when the majoritarian alternative is not selected.[7]

Finally, there is also a literature has examined the European Union's decision-making and brought ideas from weighted games to assess the relative power of different countries under the Nice Treaty (e.g., see Laruelle (1998), Laruelle and Widgrén (1998), Sutter (2000), Baldwin, Berglöf, Giavazzi, and Widgrén (2001), Bräuninger and König (2001), Galloway (2001), Leech (2002), and some of the references cited there). As the foundations of our analysis of voting rules differ from the previous literature and power indices, so does our analysis of the Nice Treaty and the new Constitution. Among other things, we identify conditions on the correlation structure of citizens' preferences that would justify the various rules that have been proposed, something which does not appear previously.

Since writing this, we have become aware of independent work by Feix, Lepelley, Merlin, and Rouet (2004), Bovens and Hartmann (2004), and Beisbart, Bovens, and Hartmann (2004) that examine efficiency as an issue in the definition of a voting rule. However, the works are (completely) complementary.[8]

2 A Simple Example

We begin by presenting a simple example that gives a preview of some of the issues that arise in designing an efficient voting rule. The example shows why in some cases it will be efficient to use

[5]There are exceptions in the recent literature (e.g., Aghion and Bolton (2003), Barbera and Jackson (2004), Harstad (2004), Casella (2005)), but these approach the problem from very different perspectives.

[6]See Felsenthal and Machover for an illuminating discussion of their objective, and some of the imprecisions in the previous literature.

[7]While these two perspectives differ, they lead to the same weights in the particular case of large countries of i.i.d. voters, where the weights are proportional to the square root of a country's population size, as originally suggested by Penrose (1946) from an even different perspective. The setting with a large number of i.i.d. voters is special and not so realistic - especially for applications such as to the European Union. Our analysis applies to a more general model, and we find that the weights that maximize total expected utility usually differ from the square root of population size.

[8]Our characterization results, block models, and fitting the models to the EU data differ from their analyses. Bovens and Hartmann examine combinations of maximin and utilitarian (efficient) measures and examine when degressive proportionality is justified, and Beisbart, Bovens, and Hartmann (2004) provide a Pareto ranking of several variations on potential rules for the EU council of ministers under assumptions on mean expected utility. Feix, Lepelley, Merlin, and Rouet (2004) contrast two models of voter preferences: Impartial Culture and Impartial Anonymous Culture, and show through simulations that these can lead to different optimal voting rules.

weights that are not proportional to population.

Example **1** *Non-Proportional versus Proportional Weights*

Consider a world with three countries. Countries 1 and 2 have populations of one agent each. Country 3 has a population of three agents.

Each agent has an equal probability of supporting alternative a as alternative b. An agent gets a payoff of 1 if their preferred alternative is chosen, and -1 if the other alternative is chosen. Thus, total utility can be deduced simply by keeping track of the number of agents who support each alternative.

First, let us consider a situation where we weight countries in proportion to their populations and then use a threshold of 50% of the total weight. That would result in weights of $w = (1, 1, 3)$ and a threshold of 2.5. This reduces to letting country 3 choose the alternative.

Here it is possible for a minority of agents to prefer an alternative and still have that be the outcome. For instance, if two agents in country 3 prefer a, and all other agents prefer b, then a is still chosen.

Let us compare this to the efficient weights - that is, those that maximize the total expected utility. Here those weights turn out to be (1,1,1.5), and the threshold is 1.75. Thus, this voting rule is equivalent to one vote per country. The proof that this is the efficient rule comes from our characterization theorem below, but we can see the improvement in utility directly.

First, note that it is still possible for a minority of agents to prefer a and a majority to prefer b, but to still have a selected. For instance, this happens if agents in countries 1 and 2 prefer a, but agents in country 3 all prefer b. Despite the fact that the rule is not always making the correct choice in terms of maximizing the total utility, there is an important distinction between the efficient rule and the proportional rule here. Fewer configurations of preferences under the efficient weights lead to incorrect (minority-preferred) decisions.

Let us list configurations that are problematic in terms of agents preferences, where the last three agents are the agents in country 3.

The only way that a can be the outcome and only be preferred by a minority under the efficient weights is when preferences are (a;a;b,b,b).

However, under the weights that are proportional to population there are three preference configurations that can lead to a being chosen when preferred by a minority. These are (b;b;a,a,b), (b;b;a,b,a) and (b;b;b,a,a).

When we compute the total expected utility (summed across all agents) it is 1.75 under the efficient weights compared to 1.5 under the population weights, which reflects this difference in potential incorrect decisions.

This example is clearly a very stark one. It illustrates some of the ideas that we will run across in what follows. More generally, the characterization of the efficient rule will depend on many considerations including the distribution of agents' preferences, the way in which representatives of a country act, and the configuration of countries. In some cases weights that are proportional to population are efficient, while in other cases non-proportional weights are efficient. We now turn to that more general analysis.

3 The Model

Decisions and Agents

A population of agents is divided into m countries.

Country i consists of n_i agents and we denote this set by C_i. The total number of agents is $n = \sum_i n_i$.

Although we use the language of a union of "countries," the model equivalently applies to any voting procedure where different groups elect representatives who then vote on their behalf.

These agents must make a decision between two alternatives that we label a and b. We think of b as the status quo, and a as change.

A state of the world is a full description of all agents' preferences over the two alternatives. Given that there are only two alternatives, we need only keep track of the difference in utility that a agent has for alternatives a and b. Thus, without loss of generality we normalize things so that agent j gets a utility of u_j if a is chosen and a utility of 0 if b is chosen. Thus, for instance, if u_j is positive, then j prefers a.

So, a state of the world is a vector $u \in I\!R^n$, with element u_j being the difference between agent j's valuations for a and b. The uncertainty regarding the state is described by some distribution function F, and all expectations in what follows are respect to this distribution.

A Two Stage Voting Procedure

The decision making process is described as follows.

The First Stage

In the first stage u is realized and each country's representative decides on whether to vote for a or vote for b. Generally, the representative's vote will be related to the preferences of the agents in the representative's country.

The representative's voting behavior is represented by a function $r_i : I\!R^n \to \{a, b\}$, which maps the state into a vote. The notation $r_i(u) = a$ denotes that the representative of country i votes for a, and $r_i(u) = b$ denotes that the representative votes for b. We assume that r_i depends only on the preferences of agents within country i.

It is important to emphasize that this formulation allows for many different ways in which the representative's vote could depend on the state of agents' preferences. It is conceivable that the representative is an existing politician who polls the population, or that the representative is a dictator, bureaucrat, etc., who might decide on how to vote quite differently. Later in the paper we will consider the prominent special case where the representative votes in accordance with a majority of the population.

We also reiterate that we are taking as given that the countries are fixed, as if one can adjust the districting then one can get as close to direct democracy as one wishes, and the issue of optimal weights is no longer a concern. Nonetheless, the model applies in many contexts.

The Second Stage

In the second stage, the votes of the representatives are aggregated according to a weighted voting rule.

An important class of voting rules is that of *qualified majority* rules. In such rules, the vote of the representative of country i is given a weight $w_i \in I\!R_+$. The tally of votes for a is the sum of the w_i's of the representatives who cast votes for a, and similarly for b. Alternative a is selected if its tally of weights exceeds the qualified majority threshold (denoted $\beta \in [0, \sum_i w_i]$), alternative b is selected if the tally of weights for a is less than the qualified majority threshold, and ties are broken by the flip of a fair coin.

Let $v : I\!R^n \to \{0, \frac{1}{2}, 1\}$ denote the outcome of this two stage voting procedure as a function of the state. Here $v(u) = 1$ is interpreted as meaning that alternative a is chosen, $v(u) = 0$ means that alternative b is chosen, and $v(u) = \frac{1}{2}$ denotes that a tie has occurred and a coin is flipped.

We let V denote the set of all such weighted voting rules with qualified majorities.

Given this coding of $v(u)$, the utility of agent j in state u can now be written as $v(u) \times u_j$. Thus the total utility summed across all agents in all countries is

$$v(u) \sum_j u_j,$$

and the total expected utility of the union using a voting rule v is denoted

$$E\left[\sum_j v(u) u_j\right].$$

Equivalent Voting Rules

We must recognize that different weights and thresholds can lead to the same voting rule, and so voting rules will only be defined up to an equivalence class of weights and thresholds.

Beyond defining two different pairs of weights and thresholds to be equivalent if their induced voting rules always make the same choices, we need a coarser requirement for our main results due to the fact that tie-breaking is not completely tied down under efficient voting rules.

Let us say that a profile of voting weights and threshold w, β with induced voting rule v is *equivalent up to ties* to a profile of voting weights and threshold w', β' with induced voting rule v' if $v(u) = v'(u)$ for all u such that $v'(u) \neq \frac{1}{2}$.

This is not quite an equivalence relationship, as it allows v to break ties in a different way from v'.[9]

To see why we define equivalence only up to ties consider a simple example. There are two countries and each consists of a single agent whose utilities take on values in $\{-1, 1\}$. Let w' be (1,1) and the threshold be 1. Note that the induced voting rule v' would be efficient for this example. When things are unanimous, v' picks the unanimous choice, but when u_1 and u_2 are of opposite signs, the rule flips a coin and so $v'(u) = \frac{1}{2}$. Alternative weights $w = (1 + \varepsilon, 1)$ with a threshold of $1 + \frac{\varepsilon}{2}$ would also be efficient, but would favor the first agent in the case of a tie. Thus, its induced voting rule v would be more resolute than v', but would make the same choices in any case where efficiency was at stake.

Equivalent voting weights and thresholds can be rescalings of each other, but also might not be. For instance with three countries, $w = (3, 2, 2)$ with a threshold of 3.5 is equivalent to $w' = (1, 1, 1)$ with a threshold of 1.5 - they both select the alternative that at least two countries to voted for.

[9]This is an asymmetric relationship: v can be equivalent up to ties with v' while the reverse might not hold.

4 Efficient Voting Rules

Let us consider the problem of assigning the weights and setting the threshold of the qualified majority in a manner so that the resulting voting rule maximizes the expected sum of the utilities of all agents in the union.

In this regard, the best one could hope for would be to choose a when $\sum_j u_j > 0$ and b when $\sum_j u_j < 0$. With the two-stage procedure this optimum will generally not be realized, since we lose information in a two stage procedure. In the second stage we see only the votes of the representatives, which includes only indirect information about the preferences of agents.

Efficient Voting Rules

Efficient voting rules are those designed to capture as much information as possible. In particular, we can still ask which $v \in V$ maximizes

$$E\left[\sum_j v(u)u_j\right].$$

We call such a voting rule an *efficient* voting rule.

4.1 A Full Characterization of Efficient Voting Rules

Consider a voting rule that works as follows. For each country assign two weights: one for a votes and one for b votes.

Country i's weight for a votes is

$$w_i^a = E\left[\sum_{k \in C_i} u_k \middle| r_i(u) = a\right],$$

and its weight for b votes is

$$w_i^b = -E\left[\sum_{k \in C_i} u_k \middle| r_i(u) = b\right].$$

The efficient voting rule $v^E(u)$ is then defined by

$$v^E(u) = \begin{cases} 1 & \text{if } \sum_{i:r_i(u)=a} w_i^a > \sum_{i:r_i(u)=b} w_i^b, \\ 0 & \text{if } \sum_{i:r_i(u)=a} w_i^a < \sum_{i:r_i(u)=b} w_i^b, \\ \frac{1}{2} & \text{if } \sum_{i:r_i(u)=a} w_i^a = \sum_{i:r_i(u)=b} w_i^b. \end{cases}$$

THEOREM 1 *If u_j is independent of u_k when j and k are in different countries, then a voting rule is efficient if and only if it is equivalent up to ties to v^E.*

The proof appears in the appendix.

The intuition behind the theorem is straightforward. Conditional on a vote of $r_i(u) = a$, $w_i^a = E\left[\sum_{k \in C_i} u_k \middle| r_i(u) = a\right]$ is the estimate of the total utility support for a in country i. By weighting country's votes in proportion to these expectations, the voting rule chooses the alternative that will result in the highest total utility based on what can be inferred from the votes of the representatives. Despite its simple proof, we feel that this characterization of efficient voting rules is important. We can see this both in terms of some of its implications, as well as its application. Before turning to the application to the European Union, let us discuss a few of the implications of the formula.

The assumption of the independence of voters' preferences across countries is restrictive and is important to the conclusions of the theorem. Without this assumption, the estimation of one country's utility for a given alternative would depend on the full profile of votes of all countries. In that case an optimal voting rule would no longer be a weighted rule, but a rule that was a much more complex mapping between vectors of votes and decisions, as each country's vote would convey information about the preferences of all countries' electorates.

Just to get a feeling for how such rules might work, consider an example with three countries, where $1 = w_1^a = w_1^b = w_2^a = w_2^b = w_3^a$, but $w_3^b > 2$. In this case, the efficient rule is to choose b whenever country 3 votes for b, and otherwise to operate under majority rule.[10] Note that this rule cannot be represented as an ordinary weighted voting rule where each country is just given some weight and there is some threshold needed for change. There is an asymmetry between how country 3's vote for b is treated compared to all other votes.

4.2 Bias and Qualified Majority Voting

In many contexts, especially where b is interpreted as the status quo, there might be some asymmetry in the way that we treat alternatives.

Let us say that country i is *biased* with bias $\gamma_i > 0$ if

$$E\left[\sum_{k\in C_i} u_k \,|r_i(u) = b\right] = -\gamma_i E\left[\sum_{k\in C_i} u_k \,|r_i(u) = a\right].$$

A country's bias captures how different our expectations are concerning how much the country's voters care about a over b when their representative votes for a, compared to our expectations about how much the country's voters care about b over a when their representative votes for b.

In cases where there is a common bias factor γ across countries, Theorem 1 has the following corollary. In that case, $w_i^b = \gamma w_i^a$, and then the efficient voting rule can be written as a qualified-majority rule.

Corollary **1** *Suppose that u_j is independent of u_k when j and k are in different countries, and that each country has the same bias factor γ. A weighted voting rule is efficient if and only if it is equivalent up to ties to a weighted voting rule with weights*

$$w_i^* = E\left[\sum_{k\in C_i} u_k \middle| r_i(u) = a\right].$$

and a qualified majority threshold of $\frac{\gamma \sum_i w_i^*}{\gamma+1}$.

[10]This suggests that features such as in the U.N. Security Council voting rule where core countries hold a veto, might have some rationalization where there is some heterogeneity among countries and bias between status quo and change.

It is important to note that the threshold depends on the bias γ, while the weights are determined by the expectations that come from each country. Thus one can judge whether a rule's weights are optimal independently of the threshold, and vice versa.

We emphasize that there are no assumptions other than the common bias behind this theorem, and yet we obtain an essentially unique characterization of efficient voting rules and a strong form of separability of weights and thresholds. As in Theorem 1, the efficient decision is the one that maximizes the expected utility of the population conditional on what can be gleaned from the votes of representatives. The weights correspond to the expected utility differential in a given country based on the observance of the representative's vote. The voting threshold simply adjusts for the bias of the scaling of what is learned from yes versus no.

First, the extent to which a country's representative's vote is tied to the utilities of the agents in the country has important consequences. For example, if the representative's vote was purely random and uncorrelated with the utilities of his constituency, then that country's weight would be 0. More generally, the larger support (in net utility) for an alternative that one infers based on a representative's vote for that alternative, the larger the weight that a country receives.

Second, the weights are affected by the distribution of opinions inside a country. In particular, the correlation structure within a country is an important determinant of the expected size of the surplus of utilities for one alternative or the other. For instance, if a country's agents had perfectly correlated opinions (and the representative voted in accordance with them), then a vote for an alternative would indicate a strong surplus of utility in favor of that alternative. The more independent the population's opinions the lower the expected surplus of utility in any given situation. Thus, higher correlation among agents' utilities will generally lead to higher weights.

Third, the efficient weights take into account the intensity of preferences. So, relatively larger utilities lead to relatively larger weights. Thus, a country that cares more intensely about issues is weighted more heavily than a country that cares less, all else held equal. Due to practical and philosophical difficulties with the appraisals of utilities, one might want to be agnostic on this dimension and just treat all u_j's equally in the sense of only assigning them values of +1 or -1. We do this in the following section. Then accounting for utilities amounts to counting supporters.

Fourth, because of all the things that lie behind the calculations of the weights, the relation between the size of countries and their relative weights is ambiguous. For example, a large country with a representative who is a dictator whose vote is uncorrelated with his population's preferences receives a smaller weight than a smaller country with a representative whose vote is very responsive to his population's preferences.

The following example illustrates the relation between bias and the voting threshold, as well as the separability of weights and thresholds.

Example **2** *Bias and Thresholds*

Consider three countries. Countries A and B have 1 voter each, while country C has N_C voters.

Each voter's preferences over a and b are drawn independently. The u_j's take on values either 1 or $-v$ with equal probability.

When v is not 1, then there is a bias in the way that voters see the alternatives a and b. For instance, when $v > 1$, then it means that a voter who prefers b, is hurt more by a choice of a, than a supporter of a when b is chosen.

In this case, the common bias factor across countries is $\gamma = v$.

Theorem 1 now tells us that the voting threshold should be a fraction of $\frac{v}{v+1}$ of the total weight. As v becomes very large, this means that near unanimity for a is required to overturn the status quo b. If $v = 1$, then the threshold is 50 percent.

The voting weights are independent of v: They are $w_A = w_B = 1$ for countries A and B, and via some straightforward calculations:

$$w_C = 2^{-N_C} \sum_{x > \frac{N_C}{2}} (2x - N_C) \frac{N_C!}{x!(N_C - x)!}.$$

This can produce some interesting voting rules.

For instance, suppose that $N_C = 7$. Then C is much larger than the other countries, and $w_C = 2.186$. However, C's "power" still depends on the voting threshold. If $v = 1$, then the threshold is 50 percent, and so C is the only country that has a nontrivial vote. In that case country C dictates. However, if $v = 2$, then the threshold is 2/3 of voting weights. Then, a passes if and only if country C and at least one of A or B votes for a. Either C, or A and B together, can block a and keep the status quo.

This example shows several things: First the separability of how the weights and thresholds are determined. Here, the weights depend on the relative populations of the countries, while the threshold depends on the underlying preference structure in terms of a bias for change versus the status quo. Second, the structure of the voting rule and how it operates ends up depending in important ways on both the threshold and weights.

A prominent case of interest is one where countries are unbiased. Here there is no a priori disposition favoring change or the status quo, and hence simple majority rule is efficient, as stated in the following easy corollary.

Unbiased Countries

Let us say that a country is *unbiased* if

$$E\left[\sum_{k \in C_i} u_k \,|r_i(u) = b\right] = -E\left[\sum_{k \in C_i} u_k \,|r_i(u) = a\right].$$

An unbiased country is one where what we learn about how much a country cares about a from the fact that the country supports a is the same as what we learn about how much a country cares about b from the fact that the country supports b.

Corollary **2** *Suppose that u_j is independent of u_k when j and k are in different countries, and that each country is unbiased. A profile of voting weights and a threshold is efficient if and only if it is equivalent up to ties to the weights*

$$w_i^* = E\left[\sum_{k \in C_i} u_k \middle| r_i(u) = a\right]$$

and the 50% threshold of $\frac{\sum_i w_i^*}{2}$.

In order to apply the theory and calculate weights as a function of a country's population, we now introduce a model that is more specific about the distribution of agents' preferences and how representatives vote.

5 A Block Model

We now specialize the model to a highly stylized setting, that allows us to explore the characterization of optimal voting rules in more detail. While the assumptions in this stylized setting are clearly strong enough to rule out some important considerations and to miss some aspects of reality, we feel this is still a useful exercise as it gives us some important intuitions as to how the optimal weights depend on the distribution of voters' preferences within and across countries. We call this stylized model the "block model," and it works as follows.

First, we treat agents' utilities equally, in the sense that we only account for them as +1 or -1, and will disregard personal intensities. This may be defended on grounds of practicality, but also more philosophically as an equal treatment condition. We also examine a case where each agent has an equal probability of supporting either alternative.

Second, we assume that representatives vote for the alternative that has a majority of support in their country.

Third, we make the following specific assumptions about the distribution of the utilities of agents. We consider a world where each country is made up of some number of blocks of constituents, where agents within each constituency think alike - that is have perfectly correlated preferences, and where agents across constituencies think independently. We take the blocks within a country to be of the same size.

These assumptions are a stylized version of what we generally see. They reflect the fact that countries are often made up of some variety of constituencies, within which agents tend to have very highly correlated preferences. For instance, the farmers in a country might have similar opinions on a wide variety of issues, as will union members, intellectuals, etc. The block model is a very useful way to introduce correlation among voters preferences, and simple variations of it provide interesting and pointed specifications of optimal voting rules.

By adjusting the size and number of blocks in a country we obtain varying expressions for the efficient weights of that country.

Efficient Weights in the Block Model

In the block model, we let N_i be the number of blocks in country i. In most applications the numbers N_i are likely to be relatively small. Then letting p_i be the size of each block, then we obtain the following expression for the efficient weight of country i.

$$w_i^B = p_i 2^{-N_i} \sum_{x > \frac{N_i}{2}} (2x - N_i) \frac{N_i!}{x!(N_i - x)!}. \tag{1}$$

There are two prominent variations on the block model that we consider in what follows.

We call the first variation the *fixed-size-block model.* In this variation, blocks are of a fixed size across all countries. In this case, a country's population can be measured in blocks, and a larger country has more blocks than a smaller one. Here the p_i's are the same across all countries.

We call the second variation the *fixed-number-of-blocks model* . In this variation, all countries have the same number of blocks, and the size of the blocks in a given country adjust according the country's population size. Here the N_i's are the same across all countries.

Thus, we get the following expressions for the efficient weights in the two specializations of the block model.

Efficient Weights in the Fixed-size-block Model

Given that the population size of a block (p_i) is the same across all countries, these can be cancelled out, and the weights in the fixed-size block model, w_i^a, reduce to:

$$w_i^{FS} = 2^{-N_i} \sum_{x > \frac{N_i}{2}} (2x - N_i) \frac{N_i!}{x!(N_i - x)!}. \tag{2}$$

These weights are graphed as follows, as a function of the number of blocks in the country.[11]

Figure 1 here

Efficient Weights in the Fixed-number-of-blocks model

In the fixed-number-of-blocks model, as the number of blocks (N_i) are the same in all countries, the difference in the weights then comes only in how many agents are represented in a block. When calculating the weights, the weights turn out to be directly proportional to the population size of the countries. Thus,

$$w_i^{FN} = p_i. \tag{3}$$

We note that for large numbers of blocks, the weights in the fixed-size-block model vary with the square root of the number of blocks, which is consistent with weights originally proposed by Penrose (1946),[12] while for small numbers of blocks they diverge from this.

Asymmetries and Non-Monotonicities in Expected Utilities

Our perspective has been to maximize the sum of expected utilities, and in the block model as we have only looked at the sign of utilities, this amounts to maximizing the expected number of agents who are in agreement with the alternative chosen. What we emphasize here is that this is quite different from trying to equalize expected utilities across agents. In particular, efficient rules necessarily treat agents asymmetrically, depending on the size of the country they live. Let us examine this in more detail for the two variations on the block model.

Let us compare the expected utilities of agents living in two countries of different population size, under the efficient voting rule in the two variations of the block model.

[11] Note that the weights are the same for 1 and 2 blocks, 3 and 4 blocks, etc. This is reflective of the expectation in (2) as well as Corollary 2.

[12] See also Felsenthal and Machover (1999), as discussed in the introduction. Here we end up with similar expressions, but only in one specific version of the block model, and only for large populations with relatively small blocks, and for quite different reasons. More generally, the weights we obtain will differ from the square root, especially when the number of blocks is small or when we leave the fixed-size-block model.

PROPOSITION 1 *In the fixed-number-of-blocks model, agents living in the larger country have expected utilities which are at least as large as agents living in the smaller country; and whenever the two countries weights are not equivalent*[13] *then the agents in the larger country have a strictly higher expected utility. In the fixed-size-block model, the comparison of expected utilities of agents across countries can go either way depending on the specifics of the context.*

The proof of the proposition is straightforward. We offer a simple argument for the fixed-number-of-blocks model, and an example showing ambiguity for the fixed-size-block model.

In the fixed-number-of-blocks model, any agent's block in any country has exactly the same probability of agreeing with the agent's representative's vote. Thus, the expected utilities of agents in different countries differ only to the extent that their representatives receive different weights. As larger countries have larger weights, the claim in the proposition follows directly.[14]

To see the ambiguity in the fixed-size-block model let us examine an example. Consider a union of three countries. Let us examine the expected utilities of the agents as we vary the number of blocks in the various countries.[15]

Table 1

Populations of Countries in Blocks	Efficient Voting Weights	Expected Utility of a Agent in Country 1 or 2	Expected Utility of a Agent in Country 3
(1,1,1)	(1,1,1)	.5	.5
(1,1,3)	(1,1,1.5)∼(1,1,1)	.5	.25
(1,1,5)	(1,1,1.875)∼(1,1,1)	.5	.1875
(1,1,7)	(1,1,2.186)∼(0,0,1)	0	.3125
(2,2,7)	(1,1,2.186)∼(0,0,1)	0	.3125
(3,3,7)	(1.5,1.5,2.186)∼(1,1,1)	.25	.15625

There are some interesting things to note here. The changes in voting weights result in non-monotonicities in expected utilities in several ways. In the cases of (1,1,3) and (1,1,5), a agent in country 1 or 2 has a higher utility than a agent in country 3. However, once country 3 hits a

[13]Two countries weights are equivalent if there exists a set of weights that lead to the same voting rule where these two countries weights are identical.

[14]As pointed out by a referee, in the extreme where countries are actually formed as fairly homogeneous entities (as suggested, for instance by Alesina and Spalaore (1997)), we would be in a situation where each country was a single block. This would convey the maximal information possible from a representative's vote, and the overall voting rule would end up being completely efficient.

[15]The calculations are as follows. A agent gets a 1 when his or her preferred outcome is chosen and a -1 if it is not. For a agent in country 1 in the (1,1,1), (1,1,3), and (1,1,5) cases, there is a 3/4 chance at least one of the other countries will prefer the agent's preferred alternative and a 1/4 chance that the other two countries will both favor the other alternative. This leads to 3/4 chance of utility of 1 and 1/4 chance of utility of -1. For a agent in country 3 in the (1,1,3) case, there is a 3/4 chance his or her preferred alternative will match the country's vote and a 1/4 chance it will not. In the first case, there is then a 3/4 chance this will receive a vote from at least one of the other two countries and a 1/4 chance it does not. In the second case, there is a 1/4 chance that the agent's preferred alternative will still be passed by the other two countries and a 3/4 chance it will not. More generally, it is easy to check that the agent's ex ante expected utility conditional on his or her country's vote being in the winning majority is simply $\frac{w_i^*}{n_i}$, and conditional on his or her country's vote being on the losing side is $-\frac{w_i^*}{n_i}$. Then we can just calculate the probability that a given country's vote will be in the winning majority, given the weights.

population of 7, then its weight is such that the votes from countries 1 and 2 are irrelevant. Thus, a agent would rather be in the larger country when the configuration is (1,1,7), while a agent would prefer to be in a smaller country when the configuration is (1,1,3) or (1,1,5). Also, we se that as we increase country 3's population for 3 to 5, its agents' utilities fall, but then increasing the population from 5 to 7 leads to an increase in its agents' utilities. This contrasts with decreases in utilities of agents in the other countries.

This example shows us that there are no regularities that we can state concerning agents' utilities in the fixed-size-block model. The difficulty is that changes in population might dilute a given agents' impact within a country, but might also lead to a relative increase of that country's voting weight. As these two factors move against each other, changes can lead to varying effects.

Another issue that we might consider in addition to comparing agents utilities across countries, is to examine how the overall expected utility varies under efficient voting rules as we change the division of a given population into different districts or countries. This issue is also generally ambiguous, regardless of which version of the block model one considers. For instance, one might conjecture that if we start with one division of a population into districts, and then further subdivide the population into finer districts, we would enhance efficiency since agents would become closer to their representatives. However, this is not always the case. To see this note that if we start with a union of just one district or country, then we essentially have direct democracy. This is the most efficient possible. But then dividing this into several districts or countries would lead to a lower total expected utility under the efficient rule, than having just one district. Now, if we continue to further subdivide the districts, we eventually reach a point where each agent resides in a district of one, which brings us back to direct democracy and full efficiency! Generally, subdivisions lead to conflicting changes: on the one hand having a smaller number of agents within a district gives them a better say in the determination of their representative's vote, but on the other hand their representative is now just one among many. This leads to non-monotonicities and ambiguities of the types discussed above.[16]

6 The European Union

Let us now examine the voting rule to be used in the Council of Ministers of the European Union under the Nice Treaty (December 2000) and compare it to the efficient voting rules under the variations of the block model. Given the stylized nature of the block model, and the fact that the overall decision making process of the European Union goes far beyond votes by the Council of Ministers, this is more of an exploratory exercise than a hard commentary on the EU voting process.

Let us emphasize from the outset that this is not a positive exercise, but rather a normative one. Our analysis provides a normative description of how voting systems should be designed. In examining the various proposals for EU weights, we are not presuming that they are optimal systems. Instead, we discuss which variation of the block model would justify a given proposal.

[16]This leads to a basic tradeoff in structuring an indirect democracy. There are tradeoffs between the cost of involving more voters, and having more precise representation. There has been little exploration of such tradeoffs, either theoretically or empirically, and they might help explain why one sees attempts at more centralization in some cases (e.g., the EU) and yet more decentralization within some states.

What the optimal system is depends on an empirical analysis of the correlation structure of voters' preferences (see Barbera and Jackson (2004a) for a preliminary analysis along those lines).

The voting weights for the European Council of Ministers under the Nice Treaty for the expansion of the EU from 15 to 27 members appear in Table 2.[17] The vote is by qualified majority. At least 255 of the 345 votes (73.9%) must be cast in approval of a proposal for it to pass.[18,19]

Let us examine the efficient voting weights and compare those to the actual weights. The following table provides the actual weights and the efficient weights based on two different sizes of voting blocks.

The efficient weights in the fixed-size-block model are calculated for two different block sizes: 1 million and 2 million. So for instance, in the case of 1 million sized blocks, Germany is seen as having 83 blocks, France as 59, and Italy as 58, etc. This leads to efficient voting weights of 7.3, 6.2 and 6.1 for these countries, respectively.[20] Recall that voting weights are not affected by rescaling. So, we need to rescale the efficient weights to the scale of the actual weights. We find the scaling factor by regressing the actual weights on the efficient weights (with no intercept). This leads to a scaling factor of 4.58 for the case of 1 million sized blocks and 9.01 for the case of 2 million sized blocks. The efficient weights reported below are those directly from (2) multiplied by the scaling factor.

The efficient weights in the fixed-number-of-blocks model are calculated directly by rescaling the population sizes to best fit the actual weights (recall that weights are completely equivalent under rescalings). The scaling factor here is .58.

Table 2 here

The Nice Treaty weights compared to the efficient weights are pictured as follows. A regression of the Nice Treaty weights on the efficient weights under the fixed-size-block model provides an R^2 of 96% for the case of 1 million sized blocks and 95% for the case of 2 million sized blocks (with F-statistics in each case over 600).[21] A regression of the Nice Treaty weights on the efficient weights under the fixed-number-of-blocks model provides an R^2 of .80 and an (F-statistic of 102).

The relationship between the different weights is pictured as follows.

Figure 2 here

[17]The previous weights for the 15 members were 10 for Germany, France, Italy and the U.K.; 8 for Spain; 5 for Belgium, Greece, the Netherlands, and Portugal; 4 for Austria and Sweden; 3 for Denmark, Ireland and Finland; and 2 for Luxembourg, with 62 of 87 votes (71%) required for approval of a proposal.

[18]There are two other qualifications as well: (i) that the votes represent at least 14 of the 27 countries and (ii) that the votes represent at least 62% of the total population. Calculations by Bräuninger and König (2001) suggest that there are relatively few scenarios in which the weighted vote threshold of 255 votes would be met while one of the other two criteria would fail. It appears that the only impact will be from the population threshold and that this will only involve a few configurations of votes providing a very slight boost in power to Germany and slight decrease in power to Malta. Thus, for practical purposes, these additional considerations are relatively unimportant and the voting weights themselves are the main component of the voting procedure.

[19]There are discrepancies in the Nice Treaty in that some statements imply a threshold of 258 votes and others a threshold of 255 votes. It appears that the correct number is the 255.

[20]Countries with a faction of a block are simply scaled to a corresponding fraction of the efficient weight of 1 for a one block country.

[21]As a comparison, the fit using weights directly proportional to population is only 81%, and so the efficient weights provide a much closer match to the Nice Treaty weights.

Discussion

It is interesting to compare the voting rule under the Nice Treaty to that under the draft of the Constitution produced by the Constitutional Convention in June of 2003, proposed to take affect in November of 2009 (see Article 24). As discussed above, under the proposed voting rule in the Constitution, weights will be proportional to population and the threshold will be 65 percent of the total population. Those weights would not be very efficient if the world is well approximated by the fixed-size-block model, but would be a perfect fit under the fixed-number-of-blocks model.

Thus, we are left with an empirical question. If the world is a good match to the fixed-size-block model then the Nice Treaty weights are almost perfectly efficient, while if the world is a good match to the fixed-number-of-blocks model then the new Constitution's weights are the efficient ones. Of course, these are highly stylized models and it is likely that the world does not conform to either. While it seems clear that countries such as Luxembourg and Malta consist of more than one block, it also seems clear that the smallest countries have fewer voting blocks than the largest ones. This suggests that the weights should be nonlinear, although perhaps not quite to the level suggested by the fixed-size-block model. However, a detailed empirical investigation of voting patterns within the countries of the EU is beyond the scope of this article.[22]

Let us also discuss the voting thresholds. The threshold under the Nice treaty is 73.9% of the weights - which would be efficient if countries have a bias of roughly $\gamma = 3$. This indicates a strong bias for the status quo. In contrast, the threshold of 65% under the Constitution would be efficient if countries have a bias of roughly $\gamma = 1.86$. This is also a bias for the status quo, but a less pronounced one.

In this paper, we have provided a framework for designing and analyzing efficient voting rules in the context of votes by representatives of countries, districts, etc. We have also suggested that the model can be applied to analyzing voting rules such as those of the European Union, and that the relative merits of different rules reduce to readily identifiable hypotheses that are amenable to empirical testing. A careful analysis of the EU voting rules will require richer data and applying our general results beyond the case of the block model. Nevertheless, the theoretical results provided here present a new framework with intuitive characterizations of optimal voting rules, that appears to lend itself well to such an exercise.

There are also other considerations that can be explored in further theoretical analyses. In making decisions, it may be that countries can sometimes include side payments or logroll so that multiple decisions can be made at once. These possibilities can further enhance the efficiency of the decision making and might influence the structure of the voting system that is to be used.[23] One can also consider a voting system's stability. As the rules can be amended, considerations other than efficiency enter the long-run picture, as only certain rules will survive. [24] Another is the issue of fairness or equality. As we have shown, efficient weights do not necessarily lead to

[22]See Barbera and Jackson (2004a) for a preliminary analysis of estimated optimal voting weights based on poll data from the Eurobarometer (2003ab).

[23]For example, see Harstad (2004, 2005) who considers side payments and the ability to invest in projects, Casella (2005) who considers rules specifically designed for votes over sequences of decisions, and Jackson and Sonnenschein (2003) who show that efficiency can be obtained if multiple decisions can be bundled and voted over in a linked manner.

[24]See Barbera and Jackson (2000) and Sosnowska (2002) for an examination of the stability of voting rules.

the same expected utilities for agents in different countries. For instance Proposition 1 showed that larger countries are favored under proportional weights in the fixed-number-of-blocks model . There are also a multitude of other issues that can be considered, such as more alternatives, private information, agenda formation, and risk aversion.

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8 Appendix: Proofs

Proof of Theorem 1: An efficient voting rule maximizes

$$E\left[\sum_k v(u)u_k\right].$$

We can rewrite this as

$$\sum_{r_1,\ldots,r_m} E\left[\sum_k v(u)u_k \,|r_1(u),\ldots,r_m(u)=r_1,\ldots,r_m\right] P(r_1(u),\ldots,r_m(u)=r_1,\ldots,r_m),$$

where $(r_1(u),\ldots,r_m(u) = r_1,\ldots,r_m)$ is the event where the realization of representatives (i.e., votes of the countries) is $(r_1,\ldots,r_m)$. Note that we can write $v(u)$ as a function of $(r_1,\ldots,r_m)$ instead of u. Hence, the total expected utility is

$$\sum_{r_1,\ldots,r_m} E\left[\sum_k v(r_1,\ldots,r_m)u_k \,|r_1(u),\ldots,r_m(u)=r_1,\ldots,r_m\right] P(r_1(u),\ldots,r_m(u)=r_1,\ldots,r_m),$$

Given the independence across countries, we can write this as

$$\sum_{r_1,\ldots,r_m} v(r_1,\ldots,r_m)\left(\sum_i E\left[\sum_{k\in C_i} u_k \,|r_i(u)=r_i\right]\right) P(r_1(u),\ldots,r_m(u)=r_1,\ldots,r_m).$$

It then follows that if we can find a voting rule that maximizes

$$v(r_1,\ldots,r_m)\sum_i E\left[\sum_{k\in C_i} u_k \,|r_i\right] \tag{4}$$

pointwise for each $(r_1, \ldots, r_m)$, then it must be efficient. Moreover, if we find one that leads to a 0 whenever there is indifference between a and b, then and all efficient voting rules must be equivalent to it up to ties.

Note that for any given $(r_1, \ldots, r_m)$, maximizing expression (4) requires setting $v(r_1, \ldots, r_m) = 1$ when

$$\sum_i E\left[\sum_{k\in C_i} u_k \,|r_i\right] > 0 \tag{5}$$

and $v(r_1, \ldots, r_m) = 0$ when

$$\sum_i E\left[\sum_{k\in C_i} u_k \,|r_i\right] < 0, \tag{6}$$

and does not have any requirement in the case that this expression is equal to 0.

Given the definitions of w_i^a and w_i^b, we can then rewrite (5) and (6) as $v(r_1, \ldots, r_m) = 1$ when

$$\sum_{i:r_i=a} w_i^a - \sum_{i:r_i=b} w_i^b > 0 \tag{7}$$

and $v(r_1, \ldots, r_m) = 0$ when

$$\sum_{i:r_i=a} w_i^a - \sum_{i:r_i=b} w_i^b < 0. \tag{8}$$

This is as defined in v^E, where we flip a coin in the case of a tie. Any efficient voting rule must agree with this one except in the case where this rule results in an expression equal to 0. This concludes the proof of the theorem. ▮

Table 2

Country	Population	Nice Treaty Weights	Fixed-Size Efficient Weights: 1M Sized Blocks	Fixed-Size Efficient Weights: 2M Sized Blocks	Fixed-Number Efficient and Constitution Weights
Germany	82.8	29	33.4	33.4	48.3
U.K.	59.5	29	28.4	27.9	34.7
France	59.3	29	28.4	27.9	34.6
Italy	57.6	29	27.9	27.9	33.6
Spain	40	27	22.9	22.7	23.3
Poland	38.7	27	22.9	22.7	22.6
Romania	22.4	14	16.9	17.5	13.1
Netherlands	15.9	13	14.2	14.3	9.3
Greece	10.6	12	12.4	12.3	6.2
Czech	10.3	12	11.4	12.3	6.0
Belgium	10.2	12	11.4	12.3	5.9
Hungary	10.1	12	11.4	12.3	5.9
Portugal	10	12	11.4	12.3	5.8
Sweden	8.9	10	11.4	9.7	5.2
Bulgaria	7.8	10	10.1	9.7	4.6
Austria	8.1	10	10.1	9.7	4.7
Slovakia	5.4	7	8.7	8.1	3.1
Denmark	5.3	7	8.7	8.1	3.1
Finland	5.2	7	8.7	8.1	3.0
Ireland	3.8	7	6.9	6.5	2.2
Lithuania	3.6	7	6.9	6.5	2.1
Latvia	2.4	4	4.6	6.5	1.4
Slovenia	1.9	4	4.6	6.2	1.1
Estonia	1.4	4	4.6	4.5	.8
Cyprus	0.8	4	3.7	2.6	.5
Luxembourg	0.5	4	2.3	1.6	.3
Malta	0.4	3	1.8	1.3	.2

Figure 1: Weights under the Fixed Number of Blocks Model

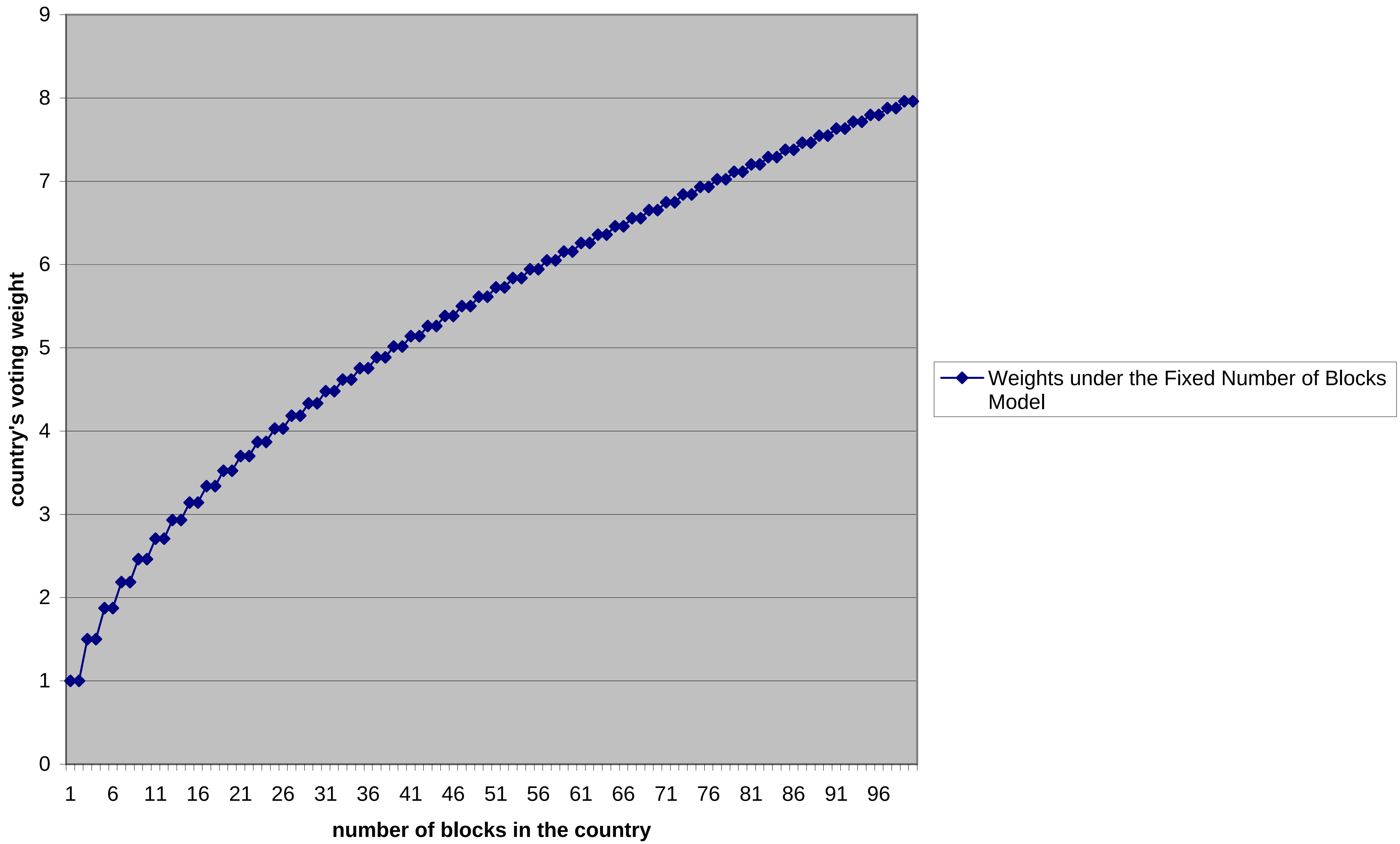

Figure 2: Comparison of weights from the block models and EU Council of Ministers