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Proportional Share Analysis

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Abstract

The purpose of this paper is to generalize the theory of "equal share analysis", developed by Selten in 1972, to the one in which every player has a positive weight. We show that for any positive vector of weights, $\alpha \in \mathbf{R}^N_{++}$, it is always possible to find a coalition structure and a payoff vector forming a proportional regular configuration.

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1 Introduction

The present work stands in cooperative game theory and relies in the notion of "equal share analysis", another insight on cooperative games introduced by Selten (1972). In contrast with most of cooperative game theory analysis, he assumes that any outcome of cooperation cannot be separated from the coalitions that they are going to form, and so, his analysis

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gives a coalition structure and an allocation to the players. His assumptions are expressed in the form of three hypothesis, observed empirically by Selten, about the outcome of a characteristic function game with restricted cooperation (a game).

The first hypothesis is that an *exhaustive coalition structure* will be formed, which means that there are no coalitions with incentives to merge. The second hypothesis asserts that the payoff vector will has a strong tendency to be in the *equal division core*, a set-solution concept which is an extension of the core. A payoff vector is in the equal division core if no coalition can divide its value equally among its members and, in this way, give more to each of the members than what they receive in the payoff vector. After giving a complete and transitive *order of strength* within the players based on the characteristic function, the third hypothesis says that within a coalition a stronger player will not receive less than a weaker player.

In consequence, a result of the game is a *configuration*, that consists of a coalition structure $\mathcal{B} = (B_1, \ldots, B_m)$ which is a partition of the set of players $N = \{1, \ldots, n\}$ into permissible coalitions, and a payoff vector $x = (x_1, \ldots, x_n) \in \mathbf{R}^N$, that can be interpreted as the reward to each player. A configuration that satisfies all three hypothesis above is said to be *regular*. Selten (1972) shows that a regular configuration exists for any game.

The equal division core was proposed by Selten (1972) to explain some outcomes observed in experimental cooperative games. He showed with experimental games from different sources in the literature and from his own experimentation that 76% of the cases conform to all three hypotheses. Moreover, Selten (1987) says that "experimental evidence clearly suggests that equity considerations have a strong influence on observed payoff division".

We can find several other descriptive theories in the literature based on equity considerations, namely equal excess theory (Komorita, 1979) and the equal division kernel (Crott and Albers, 1981). For instance, Klijn et al.(2000) provided five characterizations of the egalitarian solution (Dutta and Ray,1989) for the case of convex games and all of them involve a stability property coming from the concept of the equal division core. More recently, Branzei et al.(2004) introduced and analyzed the egalitarian solution and the equal division core for convex cooperative fuzzy games and Bhattacharya (2004) provides an axiomatic characterization of the equal division core.

In the equal share analysis, Selten (1972) considers that all the players have the same weight. This assumption is in many situations much too restrictive as we will see in Section 5. Following this idea we introduce *the proportional division core* as a generalization of the equal division core. This extension was already mentioned by Selten (1978) with the name of *equity core*, but as far as we know there have not been further studies in this direction. The proportional division core depends on a positive vector α of weights of the players. Roughly speaking, a payoff vector is in the proportional division core if no coalition can divide its value proportionally to α and, in this way, give more to all its members than the amount they receive in the payoff vector.

The main goal of this paper is, following Selten's work, to extend his result of existence of regular configurations to the case where not all the players have the same weight. Besides the concept of the proportional division core, we introduce a complete and transitive *order of strength with respect to* α within the players, which, in case that all players have the same weight, coincides with the order of strength given by Selten (1972). Thus, we consider that a coalition structure and a payoff vector is a *proportional regular configuration* if it satisfies the generalizations of the three hypothesis given by Selten. That is to say, the coalition structure is exhaustive, the payoff vector is in the proportional division core associated to the coalition structure, and it preserves the order of strength with respect to α among the players in the same coalition. Our main result in this paper is that for any game (with or without restricted cooperation) and for any vector $\alpha \in \mathbf{R}^N_{++}$ we can always guarantee the existence of a proportional regular configuration.

The outline of the paper is as follows. In Section 2, we provide the preliminary definitions and notation. In Section 3 we introduce the order of strength with respect to α. In Section 4, we give the existence theorem of proportional regular configurations. In Section 5, we discuss an application to the class of financial cooperative games (Izquierdo and Rafels, 2001). We prove that for this class of games the grand coalition and the proportional solution is always a proportional regular configuration with respect to the capitals invested by the players, but it is not generally a regular configuration *à la* Selten. Since the proportional solution plays an important role into financial cooperative games, this enforces our intuition about the necessity of taking into account the weights of the players.

2 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be the player set. A *coalition* is a subset of N, and $\mathcal{P}$ is the set of permissible coalitions that contains at least all the individual coalitions: $\{i\} \in \mathcal{P}$ for all $i \in N$ and $\emptyset \in \mathcal{P}$. To every coalition $S \in \mathcal{P}$ a real number $v(S)$ is attached, which is called the *worth* of that coalition. We say $(\mathcal{P}, v)$ is a *transferable utility game in characteristic function form with restricted cooperation (a game).* The characteristic function satisfies $v(\emptyset) = 0$ and $v(S) \geq \sum_{i \in S} v(i)$ for all $S \in \mathcal{P}$.

We write $\alpha \in \mathbf{R}^N_{++}$ to denote $\alpha \in \mathbf{R}^N$ and $\alpha_i > 0$ for $i = 1, \ldots, n$. We consider $\alpha \in \mathbf{R}^n_{++}$ the player's vector of weights and $\alpha(S) = \sum_{i \in S} \alpha_i$ for any $S \subseteq N$, where $\alpha(\emptyset) = 0$.

A vector $x \in \mathbf{R}^N$ is called a payoff vector to the players. The ith coordinate x_i of x represents the payoff to player $i \in N$.

A *coalition structure* $\mathcal{B} = (B_1, \ldots, B_m)$ is a partition of N into permissible coalitions $B_j \in \mathcal{P},\ j = 1, \ldots, m$, and the *imputation set for the coalition structure* $\mathcal{B}$, $I_{\mathcal{B}}(\mathcal{P}, v)$, is the set of payoff vectors satisfying the following conditions

(i) $\sum_{i \in B_j} x_i = v(B_j),$ for $j = 1, \ldots, m,$

(ii) $x_i \geq v(i),$ for $i = 1, \ldots, n.$

A result of the game is a *configuration* $(B_1, \ldots, B_m;\ x_1, \ldots, x_n)$ that consists of a permissible coalition structure $\mathcal{B} = (B_1, \ldots, B_m)$, and a payoff vector $x = (x_1, \ldots, x_n) \in I_{\mathcal{B}}(\mathcal{P}, v)$.

A coalition structure $(B_1, \ldots, B_m)$ is called *exhaustive* if and only if for any coalition $S \in \mathcal{P}$ which is the union of some B_j, we have

$$(1) \qquad v(S) \leq \sum_{B_j \subseteq S} v(B_j).$$

This means that nothing can be gained by forming a larger permissible coalition from several coalitions of an exhaustive structure.

For every $S \in \mathcal{P}$, we define the *proportional share of* S by $\frac{v(S)}{\alpha(S)}$. Thus, given a game $(\mathcal{P}, v)$ with a permissible coalition structure $\mathcal{B}$ and a vector of weights $\alpha \in \mathbf{R}^n_{++}$, the *proportional division core of* v *associated to* $\mathcal{B}$ *with respect to* α (Vilella, 2004) is the following set

$$(2) \quad PDC^{\alpha}_{\mathcal{B}}(\mathcal{P}, v) = \Big\{x \in I_{\mathcal{B}}(\mathcal{P}, v) \mid \text{for all } S \in \mathcal{P},\ \text{there exists } i \in S \text{ with } x_i \geq \frac{v(S)}{\alpha(S)}\alpha_i\Big\}.$$

For the case where all the weights are equal, we obtain the equal division core concept due to Selten (1972) that we denote by $EDC_{\mathcal{B}}(\mathcal{P}, v) = \{x \in I_{\mathcal{B}}(\mathcal{P}, v) \mid \text{for all } S \in \mathcal{P},\ \text{there exists } i \in S \text{ with } x_i \geq \frac{v(S)}{|S|}\}$.

3 The order

Our main purpose in this section is to establish a transitive and complete order of strength between the players. Following Selten we name it the *order of strength with respect to* α. Firstly we define a transitive but non-complete order of precedence, next we give a process to decrease the game that will allow us to define a complete and transitive order of strength with respect to α.

3.1 Order of precedence with respect to α

A coalition structure $\mathcal{B} = (B_1, \ldots, B_m)$ is a *proportional maximal share structure of a game* $(\mathcal{P}, v)$ if

$$\text{(3)} \qquad \frac{v(B_j)}{\alpha(B_j)} = \max_{C \in \mathcal{Q}_j} \left\{ \frac{v(C)}{\alpha(C)} \right\} \text{ for } j = 1, \ldots, m,$$

$$\text{where } \mathcal{Q}_1 = \mathcal{P} \text{ and } \mathcal{Q}_j = \{S \in \mathcal{P} \mid S \subseteq N \setminus (B_1 \cup \cdots \cup B_{j-1})\}, \text{ for } j = 2, \ldots, m.$$

The *proportional share payoff vector* $x = (x_1, \ldots, x_n)$ associated to the coalition structure $\mathcal{B}$, is defined for all $i \in N$ as

$$x_i = \frac{v(B_j)}{\alpha(B_j)} \alpha_i, \text{ where } i \in B_j.$$

Those payoff vectors associated to proportional maximal share structures will play an important role in the definition of the order of precedence. We denote by $\bar{m}^\alpha(\mathcal{P}, v)$ the set of all proportional share vectors associated to proportional maximal share structures. Formally,

$$\bar{m}^\alpha(\mathcal{P}, v) = \Big\{ x \in \mathbf{R}^N \mid \text{there exists a proportional maximal share structure}$$
$$\mathcal{B} = (B_1, \ldots, B_m), \text{ such that for all } i \in N, \; x_i = \frac{v(B_j)}{\alpha(B_j)} \alpha_i \text{ where } i \in B_j \Big\}.$$

Now we introduce some definitions necessary to define the order of precedence.

Two players $i, j \in N, i \neq j$, are *undistinguished with respect to* $\alpha \in \mathbf{R}^N_{++}$, and we denote this by $i \backsimeq^v_u j$, if and only if there are two vectors $x, y \in \bar{m}^\alpha(\mathcal{P}, v)$ such that $x_i > x_j$ and $y_i < y_j$.

Two players $i, j \in N$ are *equal in precedence with respect to* $\alpha \in \mathbf{R}^N_{++}$, and we denote this by $i \backsimeq^v_p j$, if and only if it is possible to find a chain of players $i_1, i_2, \ldots, i_k$, beginning with $i_1 = i$ and ending with $i_k = j$, such that

$$i = i_1 \backsimeq^v_u i_2 \backsimeq^v_u \cdots \backsimeq^v_u i_{k-1} \backsimeq^v_u i_k = j.$$

Observe that, two neighbouring players i_r, i_{r+1} are undistinguished and the chain of players allows repetitions of players. Obviously, the relation of equality in precedence is transitive and symmetric. We write $i \not\simeq_p^v j$ if two players $i, j \in N$ are not equal in precedence with respect to α.

Given two players $i, j \in N$, $i \neq j$, we say that *player i strictly precedes player j with respect to $\alpha \in \mathbf{R}^N_{++}$*, and we denote this by $i \succ_p^v j$, if i and j are not equal in precedence w.r.t. α and there is at least one $x \in \bar{m}^\alpha(\mathcal{P}, v)$, such that $x_i > x_j$.

The following Lemma [1] summarizes some technical properties that are useful to prove Proposition 3.2. The proof follows directly.

Lemma 3.1. *Let $(\mathcal{P}, v)$ be a game, $\alpha \in \mathbf{R}^N_{++}$ a vector of weights and $i \succ_p j$. Then we have,*

(a) There is no $x \in \bar{m}^\alpha(\mathcal{P}, v)$ where $x_j > x_i$.

(b) Player j does not strictly precede player i (i.e. $j \not\succ_p i$).

(c) For all $x \in \bar{m}^\alpha(\mathcal{P}, v)$, $x_i \geq x_j$ and at least one is strict.

(d) If there is $x \in \bar{m}^\alpha(\mathcal{P}, v)$ such that $x_j > x_i$, then $i \not\succ_p j$.

With the following example we illustrate the previous definitions and we show that the condition (c) of Lemma 3.1 is not enough to establish the strict order of precedence.

Example 1. *Consider the following four-player game v where all coalitions are permissible, i.e. $\mathcal{P} = 2^N$,*

$$\begin{array}{llll} & v(12)=0, & & \\ v(1)=0, & v(13)=1, & v(123)=0, & \\ v(2)=0, & v(14)=0, & v(124)=3, & v(1234)=0, \\ v(3)=0, & v(23)=0, & v(134)=3, & \\ v(4)=0, & v(24)=2, & v(234)=0, & \\ & v(34)=0. & & \end{array}$$

[1] If there is no confusion, and in order to simplify notation, we will omit the game v in $i \backsimeq_u^v j$, $i \backsimeq_p^v j$ and $i \succ_p^v j$.

Giving weights $\alpha = (1,1,1,1)$ to the players, we obtain the following three proportional maximal share structures with the corresponding proportional payoff vectors,

$$(\{1,2,4\},\{3\};\ (1,1,0,1))$$

$$(\{1,3,4\},\{2\};\ (1,0,1,1))$$

$$(\{2,4\},\{1,3\};\ (0.5,1,0.5,1))$$

Denote the payoff vectors by $x = (1,1,0,1),\ y = (1,0,1,1),\ z = (0.5,1,0.5,1)$. Since $y_1 > y_2$ and $z_1 < z_2$, it follows that $1 \simeq_u 2$, and so $1 \simeq_p 2$. Since $x_2 > x_3$ and $y_2 < y_3$ it follows that $2 \simeq_u 3$ and then $2 \simeq_p 3$. Since $1 \simeq_p 2$ and $2 \simeq_p 3$, then $1 \simeq_p 3$. Notice that players 1 and 3 are not undistinguished but they are equals in precedence. Moreover we have, $x_1 > x_3$, $y_1 = y_3$, $z_1 = z_3$. Hence, Lemma 3.1 (c) is not enough to establish the strict order of precedence.

There is no chain of undistinguished players between 1 and 4, so they are not equals in precedence. Nevertheless, $z_4 > z_1$, $x_1 = x_4$ and $y_1 = y_4$, therefore $4 \succ_p 1$. By the same reasoning we see that $4 \succ_p 2$ and $4 \succ_p 3$. Thus, we can say that $4 \succ_p 3 \simeq_p 2 \simeq_p 1$.

Notice that in this example we have found a transitive and complete binary relation of precedence with respect to α in one step, but this is not a general fact. Sometimes, we will have players which cannot be compared in this first step. Then, as we will see in the next subsection, we should decrease the game and follow with other steps.

Considering that, for any $i, j \in N$, $i \succsim_p j$ if and only if either $i \succ_p j$ or $i \simeq_p j$, in the following proposition we state, and the proof is left to the reader, that $\succsim_p$ is transitive.

Proposition 3.2. *Let $(\mathcal{P}, v)$ be a game and $\alpha \in \mathbf{R}^N_{++}$ the players' vector of weights. If $i \succsim_p j$ and $j \succsim_p k$, then $i \succsim_p k$ for any $i, j, k \in N$.*

Unfortunately, we still may have non-comparable players, thus the precedence relation is generally incomplete.

Following Selten (1972), we call *peers* the players who are incomparable by the precedence relation $\succsim_p$. Formally, two players $i, j \in N, i \neq j$ are called *peers with respect to* α if and only if they are not equal in precedence ($i \not\simeq_p j$) and $x_i = x_j$ for all $x \in \bar{m}^\alpha(\mathcal{P}, v)$.

To make the precedence relation complete, we will define a decreased game.

3.2 Proportional decreased game with respect to α

In order to compare the players who are incomparable in the original game, we decrease the game by steps. In every step of this process we decrease the previous game by giving to the coalitions with the higher proportional share the proportional share just below. To do this, consider the set of proportional shares $\frac{v(S)}{\alpha(S)}$ for all non-empty coalitions $S \in \mathcal{P}$. The different values obtained can be ordered as $b_1 < b_2 < \cdots < b_r,\ r \geq 2$. Notice that in the special case where $\frac{v(S)}{\alpha(S)} = b_1$, for all $S \in \mathcal{P},\ S \neq \emptyset$, we do not have any decreased game.

The *proportional decreased game* v_1 *of* v is the game $(\mathcal{P}, v_1)$, defined by

$$v_1(S) = \begin{cases} v(S) & \text{if } \frac{v(S)}{\alpha(S)} < b_r, \\ \\ \alpha(S)\ b_{r-1} & \text{if } \frac{v(S)}{\alpha(S)} = b_r, \end{cases}$$

for all $S \in \mathcal{P}$.

We denote by v_k the decreased game of v_{k-1} for $k = 2, \ldots, r-1$, and we call it the $k-th$ *proportional decreased game* of v.

By repeating this process $r-1$ times, the remaining proportional shares are $b_1 < b_2$, and we obtain the last decreased game $(\mathcal{P}, v_{r-1})$, where $\frac{v_{r-1}(S)}{\alpha(S)} = b_1$ for all $S \in \mathcal{P}$. Therefore, $v_{r-1}(S) = b_1 \alpha(S),$ for all $S \in \mathcal{P}$.

Due to the structure of the decreased game it is not difficult to see that in the last proportional decreased game $(\mathcal{P}, v_{r-1})$, the proportional share payoff vector for any coalition structure $(B_1, \ldots, B_m)$ is $x_i = b_1 \alpha_i$, for all $i \in N$.

3.3 Order of strength with respect to α

The aim is to define a complete preference relation which analogously to Selten (1972) we call *order of strength with respect to α*. To this end, we use the proportional decreased game to compare players who are incomparable in the original game. By repeating this process in a finite number of steps we obtain a complete and transitive order of strength of the players.

Like Selten (1972) says, we could complete this precedence relation by considering two players as equally strong whenever they are peers, but then there are some cases in which this relationship does not reflect obvious differences between some players. Therefore, we define a more refined *order of strength* by considering that one of two peers is stronger than the other if he or she precedes the other in the proportional decreased game.

Since every proportional maximal share structure of the original game is also a proportional maximal share structure of the proportional decreased game, it seems natural to consider the proportional decreased game in order to compare the players who are peers in the original game.

Let $(\mathcal{P}, v)$ be a game, $\alpha \in \mathbf{R}^N_{++}$ the weights of the players, and $(\mathcal{P}, v_k)$, for $k = 1, \ldots, r-1$, the family of its decreased games. A player $i \in N$ is *stronger or indifferent than a player $j \in N$ with respect to α*, $i \succsim_s j$, if and only if one of the following three conditions holds:

1. Player i precedes player j with respect to α in the original game, (i.e. $i \succsim_p^v j$).
2. There exists $k \in \{1, \ldots, r-1\}$ such that i, j are peers in the original game v and also in v_t, for $t = 1, \ldots, k-1$, and player i precedes player j with respect to α in the k-th proportional decreased game v_k, (i.e. $i \succsim_p^{v_k} j$).
3. Players i and j are peers in the original game v and in all the decreased games v_t for $t = 1, \ldots, r-1$.

In the next theorem we prove that $\succsim_s$ is a transitive and complete binary relation on

$N = \{1, \ldots, n\}$.

Theorem 3.3. *Let $(\mathcal{P}, v)$ be a game and $\alpha \in \mathbf{R}^N_{++}$ the weights of the players. The order of strength with respect to α, $\succsim_s$, is transitive and complete.*

Proof. It is trivially complete, since given two players either they are comparable in an intermediate step of the process or they are peers until the end of the process. In this case it is not difficult to see that in the last proportional decreased game $(\mathcal{P}, v_{r-1})$ the proportional share payoff vector for any coalition structure is $x_i = b_1\alpha_i$ for all $i \in N$, then we can compare the players by their weights.

Let us prove transitivity. Let $i, j, k \in N$ such that $i \succsim_s j$ and $j \succsim_s k$ then we must prove that $i \succsim_s k$. If $i \succsim^v_p j$ and $j \succsim^v_p k$ then by Proposition 3.2 $i \succsim^v_p k$. Therefore, by definition $i \succsim_s k$. In case that $i \succsim^v_p j$ and j, k are peers in v, it is not difficult to prove that $i \not\approx_p k$ and $i \not\approx_p j$. Since we assumed $i \succsim^v_p j$ and we have $i \not\approx_p j$, then there exists a vector $x^* \in \bar{m}^\alpha(\mathcal{P}, v)$ such that $x^*_i > x^*_j$. Since j and k are peers, it holds $x^*_j = x^*_k$. Therefore $x^*_i > x^*_k$. This proves that $i \succ^v_p k$ and so $i \succsim_s k$. In all other cases the proof follows equivalently. □

In the next section we introduce the *proportional regular configuration with respect to α* and we prove its existence for any game and for any weights system of the players.

4 Existence

A result of the game is a configuration that consists of a coalition structure and a payoff vector. Among them, we define the proportional regular configurations as follows.

Definition. Given a game $(\mathcal{P}, v)$, a configuration $(B_1, \ldots, B_m; x_1, \ldots, x_n)$ is called a *proportional regular configuration with respect to $\alpha \in \mathbf{R}^N_{++}$* if it satisfies the following three hypothesis:

1. The coalition structure $\mathcal{B} = (B_1, \ldots, B_m)$ is exhaustive (see (1)).

2. The payoff vector $x = (x_1, \ldots, x_n)$ is in the proportional division core associated to $\mathcal{B}$ with respect to α (see (2)).

3. The payoff vector x preserves the order of strength w.r.t. α between the players in the same coalition. Thus, if two players i and j belong to the same coalition B_k and $i \succ_s j$, then we have $x_i \geq x_j$.

Before proving the general existence theorem we need to develop some technical results. In the first one we show some relationship between the order of strength $\succsim_s$ w.r.t. α and the weights of the players.

Lemma 4.1. *Let $(\mathcal{P}, v)$ be a game and let $\mathcal{B} = (B_1, \ldots, B_m)$ be a proportional maximal share structure with respect to $\alpha \in \mathbf{R}^N_{++}$ (see (3)), for any $k \in \{1, \ldots, m\}$ and $i, j \in B_k$ we have*

(a) If $v(B_k) > 0$ (respectively if $v(B_k) < 0$) and $\alpha_i > \alpha_j$, then $i \succsim_s j$ (respectively $i \precsim_s j$).

(b) If $v(B_k) > 0$ (respectively if $v(B_k) < 0$) and $i \succ_s j$, then $\alpha_i \geq \alpha_j$ (respectively $\alpha_i \leq \alpha_j$).

Proof. Let us prove (a). Let x be the proportional payoff vector associated to $\mathcal{B}$. For all $k \in \{1, \ldots, m\}$ and all $i \in B_k$,

$$x_i = \frac{v(B_k)}{\alpha(B_k)}\alpha_i.$$

For $i, j \in B_k$, since $\alpha_i > \alpha_j$ and $v(B_k) > 0$, it holds that

$$x_i = \frac{v(B_k)}{\alpha(B_k)}\alpha_i > \frac{v(B_k)}{\alpha(B_k)}\alpha_j = x_j.$$

If there exists $y \in \bar{m}^\alpha(\mathcal{P}, v)$ such that $y_i < y_j$, we obtain $i \simeq^v_u j$ or equivalently $i \simeq_s j$. In the other case $i \succ^v_p j$ and then $i \succsim_s j$.

To prove (b), take into account (a) and the completeness of $\succsim_s$. $\square$

Next we see that for any proportional maximal share structure, the corresponding proportional division core is non-empty.

Lemma 4.2. *Let* $(\mathcal{P}, v)$ *be a game,* $\alpha \in \mathbf{R}^N_{++}$ *and* $\mathcal{B} = (B_1 \ldots, B_m)$ *a proportional maximal share structure with respect to* α*. Then, the proportional share payoff vector* $x \in \mathbf{R}^N$ *defined by* $x_i = \frac{v(B_k)}{\alpha(B_k)}\alpha_i$*, for all* $i \in B_k$ *where* $k = 1, \ldots, m$*, belongs to* $PDC^{\alpha}_{\mathcal{B}}(\mathcal{P}, v)$*.*

Proof. Let $\mathcal{B} = (B_1, \ldots, B_m)$ be a proportional maximal share structure with respect to α and $x = (x_1, \ldots, x_n)$ the corresponding proportional share payoff vector. Assume there exists a coalition $S \in \mathcal{P}$ such that $x_i < \frac{v(S)}{\alpha(S)}\alpha_i$ for all $i \in S$, and let $k^* = \min\{k \in \{1, \ldots, m\} \mid S \cap B_k \neq \emptyset\}$. Hence, $S \subseteq N \setminus (B_1 \cup \cdots \cup B_{k^*-1})$ which in case that $k^* = 1$ it is $S \subseteq N$. Let $i \in S \cap B_{k^*}$. Since $\mathcal{B} = (B_1, \ldots, B_m)$ is a proportional maximal share structure, we have

$$x_i = \frac{v(B_{k^*})}{\alpha(B_{k^*})}\alpha_i = \max_{S \in \mathcal{Q}_{k^*}} \left\{\frac{v(S)}{\alpha(S)}\right\}\alpha_i,$$

where $\mathcal{Q}_{k^*} = \{S \in P \mid S \subseteq N \setminus (B_1 \cup \cdots \cup B_{k^*-1})\}$ as defined in (3). Thus, this is a contradiction with $x_i < \frac{v(S)}{\alpha(S)}\alpha_i$ for all $i \in S$, since $S \subseteq N \setminus (B_1 \cup \cdots \cup B_{k^*-1})$ and $S \in \mathcal{P}$. Consequently, we have that for all $S \in \mathcal{P}$ there exists a player $i \in S$ such that

$$x_i \geq \frac{v(S)}{\alpha(S)}\alpha_i.$$

Since x is an imputation for the coalition structure $\mathcal{B}$, we have proved that $x \in PDC^{\alpha}_{\mathcal{B}}(\mathcal{P}, v)$.

□

With the next result we prove that, if a given coalition structure of N is not exhaustive, we can always find a coarser exhaustive coalition structure.

Lemma 4.3. *Given a game* $(\mathcal{P}, v)$ *and a non-exhaustive permissible coalition structure,* $(B_1, \ldots, B_m)$*, there exists* $\{J_1, \ldots, J_k\}$*, a partition of* $M = \{1, \ldots, m\}$ *and an exhaustive permissible coalition structure* $(B'_1, \ldots, B'_k)$ *such that,*

(a) $B'_r = \bigcup_{j \in J_r} B_j$*, for* $r = 1, \ldots, k$*,*

(b) $v(B'_r) \geq \sum_{j \in J_r} v(B_j)$*, for* $r = 1, \ldots, k$*.*

Proof. Consider $\mathcal{Q}$ the set of all permissible coalition structures that we can get from the union of several coalitions from the given coalition structure $(B_1, \ldots, B_m)$. That is to say

$$\mathcal{Q} = \Big\{ (C'_1, \ldots, C'_k) \mid \text{exists a partition } \{J_1, \ldots, J_k\} \text{ of } M, \text{ and } C'_r = \bigcup_{j \in J_r} C_j,\ C'_r \in \mathcal{P}$$

$$\text{and } v(C'_r) \geq \sum_{j \in J_r} v(C_j), \text{ for } r = 1, \ldots, k \Big\}.$$

Since $(B_1, \ldots, B_m) \in \mathcal{Q}$, the set $\mathcal{Q}$ is non-empty. And let $(B'_1, \ldots, B'_k) \in \mathcal{Q}$ be such that $\sum_{i=1}^k v(B'_i)$ is maximal among all the coalition structures in $\mathcal{Q}$.

We prove that $(B'_1, \ldots, B'_k)$ is exhaustive. Assume it is not, then there exists $\emptyset \neq J' \subseteq L = \{1, \ldots, k\}$ such that

$$B = \bigcup_{t \in J'} B'_t \in \mathcal{P} \text{ and } v(B) > \sum_{t \in J'} v(B'_t). \tag{4}$$

It is not difficult to see that $(B, (B'_t)_{t \in L \setminus J'})$ is a coalition structure that belongs to $\mathcal{Q}$.

Moreover, by (4) we have

$$v(B) + \sum_{t \in L \setminus J'} v(B'_t) > \sum_{t \in L} v(B'_t),$$

but this involves a contradiction since we have chosen $(B'_1, \ldots, B'_k) \in \mathcal{Q}$ such that the addition of their worths is maximal. Therefore, $(B'_1, \ldots, B'_k)$ must be exhaustive. □

Finally, the next theorem shows that for any cooperative game, with or without restricted cooperation, and for any vector of weights $\alpha \in \mathbf{R}^N_{++}$ we can always guarantee the existence of at least one proportional regular configuration with respect to α. This is the generalization of the corresponding result in Selten (1972) concerning the equal share concept.

Theorem 4.4. *For any game* $(\mathcal{P}, v)$ *and all* $\alpha \in \mathbf{R}^N_{++}$*, there always exists a proportional regular configuration with respect to* α*.*

Proof. Let $(B_1, \ldots, B_m)$ be a proportional maximal share structure with respect to α and consider the configuration $(B_1, \ldots, B_m; x_1, \ldots, x_n)$ where

$$x_i = \frac{v(B_k)}{\alpha(B_k)}\alpha_i, \text{ for all } i \in B_k,\ k = 1, \ldots, m.$$

By Lemma 4.2 we have that $(x_1, \ldots, x_n) \in PDC^{\alpha}_{\mathcal{B}}(\mathcal{P}, v)$.

Now we prove that the payoff vector x preserves the order of strength w.r.t. α of the players.

Let $i, j \in B_k$, and $i \succ_s j$. If $v(B_k) \neq 0$, by Lemma 4.1 if $v(B_k) > 0$ then $\alpha_i \geq \alpha_j$ and if $v(B_k) < 0$ then $\alpha_i \leq \alpha_j$. Therefore in both cases the next inequality holds

$$x_i = \frac{v(B_k)}{\alpha(B_k)}\alpha_i \geq \frac{v(B_k)}{\alpha(B_k)}\alpha_j = x_j.$$

If $v(B_k) = 0$, then $x_i = x_j = 0$, thus it holds $x_i \geq x_j$.

Finally, if $(B_1, \ldots, B_m)$ is exhaustive, then $(B_1, \ldots, B_m; x_1, \ldots, x_n)$ is a proportional regular configuration w.r.t. α. If it is not exhaustive, by Lemma 4.3, there exists $J_1, \ldots, J_r$ a partition of $M = \{1, \ldots, m\}$ and an exhaustive permissible coalition structure $(B'_1, \ldots, B'_r)$ such that,

1. $B'_k = \bigcup_{j \in J_k} B_j$, for $k = 1, \ldots, r$,
2. $v(B'_k) \geq \sum_{j \in J_k} v(B_j)$, for $k = 1, \ldots, r$.

Then,

$$\sum_{i \in B'_k} x_i = \sum_{i \in \bigcup_{j \in J_k} B_j} x_i = \sum_{j \in J_k} \sum_{i \in B_j} x_i = \sum_{j \in J_k} v(B_j) \leq v(B'_k), \quad \text{for } k = 1, \ldots, r.$$

Therefore,

$$v(B'_k) - \sum_{i \in B'_k} x_i \geq 0, \text{ for } k = 1, \ldots, r.$$

Now, we define the payoff vector $(y_1, \ldots, y_n)$ where

$$y_i = x_i + \frac{v(B'_k) - \sum_{i \in B'_k} x_i}{|B'_k|}, \quad \text{for all } i \in B'_k \text{ where } k = 1, \ldots, r.$$

We will see that $(B'_1, \ldots, B'_r; y_1, \ldots, y_n)$ is a proportional regular configuration w.r.t. α.

Clearly $y \in I_{\mathcal{B}'}(\mathcal{P}, v)$ and $y \in PDC^{\alpha}_{\mathcal{B}'}(\mathcal{P}, v)$. Moreover, the vector y preserves the order of strength with respect to α. We have to prove that, if $i, j \in B'_k$, and $i \succ_s j$, then $y_i \geq y_j$. Assume that $y_i < y_j$, this implies $x_i < x_j$, where $x \in \bar{m}^{\alpha}(\mathcal{P}, v)$. By Lemma 3.1(d), we have $i \not\succ^v_p j$. Since $x_i < x_j$, for the vector $x \in \bar{m}^{\alpha}(\mathcal{P}, v)$, players i, j are not peers in v. Therefore, either $i \simeq^v_p j$ or $i \prec^v_p j$ which, by definition, implies $i \precsim_s j$ and this is a contradiction with $i \succ_s j$. Thus we have $y_i \geq y_j$ and this proves that $(B'_1, \ldots, B'_r; y_1, \ldots, y_n)$ is a proportional regular configuration w.r.t α. □

Notice that the above theorem gives a constructive way to find some proportional regular configurations.

5 An application

In this section, we apply this proportional share analysis to the class of financial cooperative games (Izquierdo and Rafels, 2001). In this class of games, the set of players is a group of investors $N = \{1, \ldots, n\}$, such that each of them has an amount of money $c_i > 0, i = 1 \ldots, n$ that he or she wishes to invest. We assume that the bank offers a yield that increasingly depends on the amount of money deposited and that investors may combine their resources, $c(S) = \sum_{i \in S} c_i,\ S \subseteq N$, and invest them in the bank. The characteristic function is given by $v(S) = c(S) i(c(S))$ where $i(c(S))$ is the yield that coalition S gets by joining their resources.

Let us denote by $p_c(v) = \left(\frac{v(N)}{c(N)} c_i\right)_{i \in N}$ the proportional allocation of the game. Next we prove that for this class of games $(N; p_c(v))$ is always a proportional regular configuration with respect to the capitals invested by the players.

Proposition 5.1. *Let $(\mathcal{P}, v)$ be a financial cooperative game with capitals $c = (c_1, \ldots, c_n) \in \mathbf{R}^N_{++}$ and $N \in \mathcal{P}$. Then, $(N; p_c(v))$ is a proportional regular configuration with respect to c.*

Proof. Since N is a permissible coalition then it is exhaustive. By its own structure, financial cooperative games satisfy $\frac{v(S)}{c(S)} \leq \frac{v(N)}{c(N)}$ for all $\emptyset \neq S \in \mathcal{P}$. Therefore, it is not difficult to see that $p_c(v) \in PDC_N^c(\mathcal{P}, v)$, and the grand coalition N is always a proportional maximal share structure. Therefore, from the proof of Theorem 4.4, the proportional allocation, $p_c(v)$, preserves the order of strength w.r.t. c among the players. □

However, although $(N; p_c(v))$ is a proportional regular configuration and generally the proportional allocation plays an important role into financial cooperative games, the following example shows that this solution is not generally a regular configuration if we analyze the problem *à la* Selten.

Example 2. *Let* $(\mathcal{P}, v)$ *be a three-player game, where the players are a group of investors. The capitals to invest are* $c_1 = 3000$*,* $c_2 = 600$ *and* $c_3 = 3500$*. Suppose that the yield offered by the bank is given by*

$$i(c(S)) = \begin{cases} 0\% & \textit{if} \quad c(S) < 3600, \\ 10\% & \textit{if} \quad 3600 \leq c(S) < 5000, \\ 12\% & \textit{if} \quad c(S) \geq 5000. \end{cases}$$

Let us assume that players 1 *and* 2 *cannot form a coalition unless it includes player one. Thus, the set of permissible coalitions is* $\mathcal{P} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$*. Therefore, the game is,*

$$\begin{aligned} &v(1) = 0, \quad v(12) = 360, \\ &v(2) = 0, \quad v(13) = 780, \quad v(123) = 852, \\ &v(3) = 0. \end{aligned}$$

As the reader may check, the order of strength with respect to $\alpha = (1,1,1)$ is $2 \prec_s 3 \prec_s 1$. Therefore, although N is exhaustive and $p_c(v) = (360, 72, 420)$ is in the equal division core, it does not preserve the order of strength w.r.t $\alpha = (1,1,1)$ since player 1 is stronger than player 3 and $p_{c,1} = 360 < p_{c,3} = 420$. Thus, $(N; p_c(v))$ is not a regular configuration *à la*

Selten, whereas by Proposition 5.1 it is a proportional regular configuration with respect to $c = (3000, 600, 3500)$.

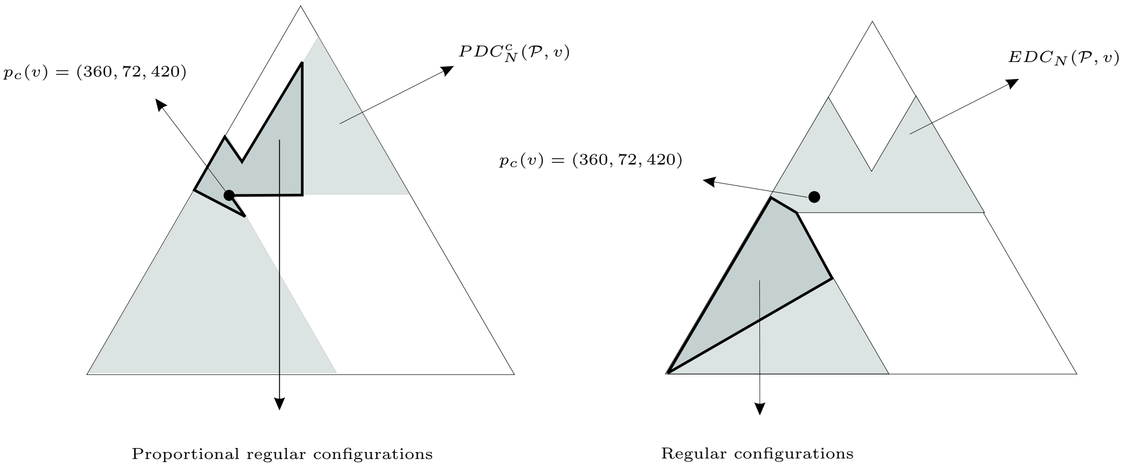

Figure 1: This figure corresponds to the proportional division core and the equal division core of Example 2.

The shadowed zones in the two triangles in Figure 1 represents respectively $PDC_N^c(\mathcal{P}, v)$ and $EDC_N(\mathcal{P}, v)$. Inside, in both cases, with the black lines we have highlighted the subset of payoff vectors that preserve the order of strength in each of the cases. Observe that, in the first triangle this corresponds to the set of payoffs in the $PDC_N^c(\mathcal{P}, v)$ satisfying $x_2 \leq x_1 \leq x_3$, and $p_c(v) = (360, 72, 420)$ belongs to this subset. In the second triangle the shadowed zone corresponds to the payoffs in the $EDC_N(\mathcal{P}, v)$ satisfying $x_2 \leq x_3 \leq x_1$.

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