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Who's Who in Networks. Wanted: The Key Player

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Abstract

Finite population non-cooperative games with linear-quadratic utilities, where each player decides how much action she exerts, can be interpreted as a network game with local payoff complementarities, together with a globally uniform payoff substitutability component and an own-concavity effect. For these games, the Nash equilibrium action of each player is proportional to her *Bonacich centrality* in the network of local complementarities, thus establishing a bridge with the sociology literature on social networks. This Bonacich-Nash linkage implies that aggregate equilibrium increases with network size and density. We then analyze a policy that consists of targeting the *key player*, that is, the player who, once removed, leads to the optimal change in aggregate activity. We provide a geometric characterization of the key player identified with an *inter-centrality measure*, which takes into account both a player's centrality and her contribution to the centrality of the others.

Keywords: Social networks, peer effects, centrality measures, policies.

JEL Classification: A14, C72, L14.

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1 Introduction

The dependence of individual outcomes on group behavior is often referred to as *peer effects* in the literature.[1] In standard peer effects models, this dependence is homogeneous across members, and corresponds to an *average* group influence. Technically, the marginal utility to one person of undertaking an action is a function of the average amount of the action taken by her *peers*. Generative models of peer effects, though, suggest that this intragroup externality is, in fact, heterogeneous across group members,[2] and varies across individuals with their level of group exposure.

In this paper, we allow for a general pattern of bilateral influences, and analyze the resulting dependence of individual outcome on group behavior.

More precisely, consider a finite population of players with linear-quadratic interdependent utility functions. Take the matrix of cross derivatives in these players' utilities. Our first task is to decompose additively this matrix of cross effects into an idiosyncratic component, a global interaction component, and a local interaction network. The idiosyncratic effect reflects (part of) the concavity of the payoff function in own efforts. The global interaction effect is uniform across all players, and reflects a strategic substitutability in efforts across all pairs of players. Finally, the local interaction component reflects a (relative) strategic complementarity in efforts that varies across pairs of players. The population wide pattern of these local complementarities is well-captured by a network. This description allows for a clear view of global and local externalities and their sign for a given general pattern of interdependencies.

Based on this reformulation, the paper provides three main results. First, we relate individual equilibrium outcomes to the players' positions in the network of local interactions. Second, we show that the aggregate equilibrium outcome increases with the density and size of the local interaction network. Finally, we characterize an optimal network disruption policy that exploits the geometric intricacies of this network structure.

In network games, the payoff interdependence is, at least in part, rooted in the network structure of players' links.[3] In these games, equilibrium strategies, that subsume the payoff interdependence in a consistent manner, should naturally reflect the players' network embeddedness. When the relative magnitude of global and local externalities for our decomposition of cross effects scale adequately, our network game has a unique and interior Nash equilibrium, proportional to the Bonacich network centrality. This measure has been proposed for nearly two decades in sociology by Bonacich (1987), and counts the number of all paths[4] emanating from a given node, weighted by a decay factor that decreases with the length of these paths.[5] This is intuitively related to the

[1] Durlauf (2004) offers an exhaustive survey of the theoretical and empirical literature on peer effects.

[2] For instance, when job information flows through friendship links, employment outcomes vary across otherwise identical agents with their location in the network of such links (Calvó-Armengol and Jackson, 2004).

[3] See, in particular, the recent literature survey by Jackson (2004).

[4] Not just shortest paths.

[5] It was originally interpreted as an index of influence or power of the actors of a social network. Katz (1953) is a

equilibrium behavior, as the paths capture all possible feedbacks. In our case, the decay factor depends on how others' actions enter into own action's payoff.

The sociology literature on social networks is well-established and extremely active (see, in particular, Wasserman and Faust, 1994). One of the focus of this literature is, precisely, to propose different measures of network centralities and to assert the descriptive and/or prescriptive suitability of each of these measures to different situations.[6] This paper provides a behavioral foundation to the Bonacich's index, thus singling it out from the vast catalogue of network measures.

The relationship between equilibrium strategic behavior and network topology given by the Bonacich measure allows for a general comparative statics exercise. We show that a denser and larger network of local interactions increases the aggregate equilibrium outcome. This is simply because the aggregate number of network paths increase with the number of available connections.

When the Nash-Bonacich linkage holds, the variance of equilibrium actions reflects the variance of network centralities. In this case, a planner may want to remove a few suitably selected targets from the local interaction network, so as to alter the whole distribution of outcomes. To characterize the network optimal targets, we propose a new measure of network centrality, the *inter-centrality measure*, that does not exist in the social network literature. Players with the highest inter-centrality are the *key players* whose removal results in the maximal decrease in overall activity.

Contrary to the Bonacich centrality measure, this new centrality measure does not derive from strategic (individual) considerations, but from the planner's optimality (collective) concerns. Bonacich centrality fails to internalize all the network payoff externalities agents exert on each other, while the inter-centrality measure internalizes them all. Indeed, removing a player from a network has two effects. First, less players contribute to the aggregate activity level (direct effect), and second, the network topology is modified, the remaining players thus adopting different actions (indirect effect). As such, the inter-centrality measure accounts not only for individual Bonacich centralities but also for cross-contributions across them. In particular, the key player is not necessarily the player with the highest equilibrium outcome.

Section 2 presents the model. Sections 3 contains the equilibrium analysis, and Section 4 the network-based policy. Section 5 contains a number of applications, including crime networks, R&D collaboration links in oligopoly markets, and conformist behavior. Section 6 discusses a number of extensions.

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seminal reference.

[6] See Borgatti (2003) for a discussion on the lack of a systematic criterium to pick up the "right" network centrality measure for each particular situation.

2 The model

2.1 The game

Each player $i = 1, ..., n$ selects an effort $x_i \geq 0$, and gets a payoff $u_i(x_1, ..., x_n)$. We focus on bilinear payoff functions of the form:

$$u_i(x_1, ..., x_n) = \alpha_i x_i + \frac{1}{2}\sigma_{ii}x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j, \tag{1}$$

strictly concave in one's effort, that is, $\partial^2 u_i / \partial x_i^2 = \sigma_{ii} < 0$. We set $\alpha_i = \alpha > 0$ and $\sigma_{ii} = \sigma$, identical for all players. Net of bilateral influences, players have thus the same payoffs.

Bilateral influences are captured by the cross derivatives $\partial^2 u_i / \partial x_i \partial x_j = \sigma_{ij}$, $i \neq j$. They depend on the pair of players considered, and can be of either sign. When $\sigma_{ij} > 0$, an increase in effort from j triggers a downwards shift in i's response. We say that efforts are strategic complements from i's perspective within the pair (i, j). Reciprocally, when $\sigma_{ij} < 0$, efforts are strategic substitutes from i's perspective within the pair (i, j).

Let $\underline{\sigma} = \min\{\sigma_{ij} \mid i \neq j\}$ and $\overline{\sigma} = \max\ \{\sigma_{ij} \mid i \neq j\}$.

We assume that $\sigma < \min\{\underline{\sigma}, 0\}$. When $\underline{\sigma} \geq 0$, this is simply the concavity of payoffs in own efforts. When $\underline{\sigma} < 0$, this requires that own marginal returns decrease with the level of x_i at least as much as cross marginal returns do.

Let $\Sigma = [\sigma_{ij}]$ be the square matrix of cross effects:

$$\Sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}.$$

We use Σ as a short-hand for the simultaneous move $n-$player game with payoffs (1) and strategy spaces $\mathbb{R}_+$.

2.2 The decomposition of cross effects

We decompose the matrix Σ additively into an idiosyncratic concavity component, a global (uniform) substitutability component, and a local complementarity component, in the following way.

Let $\gamma = -\min\{\underline{\sigma}, 0\} \geq 0$. If efforts are strategic substitutes for some pair of players, then $\underline{\sigma} < 0$ and $\gamma > 0$. Otherwise, $\underline{\sigma} \geq 0$ and $\gamma = 0$. As we shall see below, the parameter γ accounts for the global substitutability of efforts across all pairs of players.

Let $\lambda = \overline{\sigma} + \gamma \geq 0$. We assume that $\lambda > 0$. This is a generic property, as $\lambda = 0$ if and only if $\underline{\sigma} = \overline{\sigma}$, and this equality is not robust to small perturbations of the coefficients $\underline{\sigma}$ and $\overline{\sigma}$.[7] Let also $g_{ij} = (\sigma_{ij} + \gamma)/\lambda$, for $i \neq j$, and set $g_{ii} = 0$. By construction, $0 \leq g_{ij} \leq 1$. The

[7] The set of parameters σ_{ij}s for which $\underline{\sigma} = \overline{\sigma}$ has Lebesgue measure zero in $\mathbb{R}^{n(n-1)}$.

parameter g_{ij} measures the relative complementarity in efforts from i's perspective within the pair (i, j) with respect to the benchmark value $-\gamma \leq 0$. This measure is expressed as a fraction of λ, that corresponds to the highest possible relative complementarity for all pairs.

The decomposition is depicted in Figure 1. This is just a centralization (β and λ are defined with respect to γ) followed by a normalization (the g_{ij}s are in $[0, 1]$) of the cross effects. The figure in the upper panel corresponds to σ_{ij} of either sign (the case $\sigma_{ij} \leq 0$, for all $i \neq j$ is similar) while the figure in the lower panel corresponds to $\sigma_{ij} \geq 0$, for all $i \neq j$ (recall that we assume $\sigma < 0$).

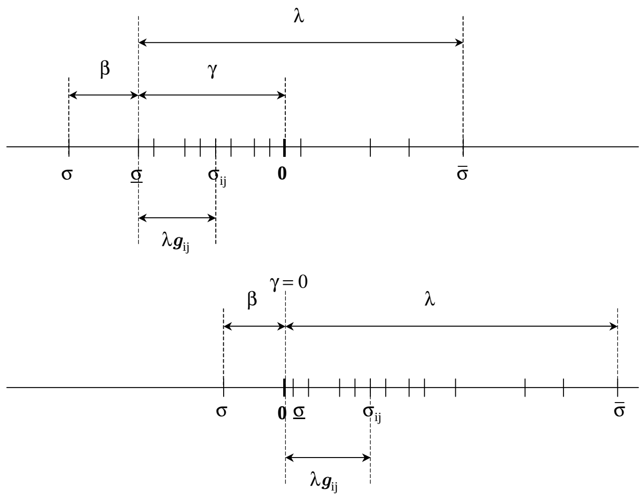


Figure 1

Consider the matrix $\mathbf{G} = [g_{ij}]$. This is a zero diagonal and non-negative square matrix. We interpret it as the adjacency matrix of a network $\mathbf{g}$ that reflects the pattern of existing payoff (relative) complementarities across all pairs of players. When $\sigma_{ij} = \sigma_{ji}$, for all i, j, the matrix $\mathbf{G}$ is symmetric and the network $\mathbf{g}$ is un-directed. When, moreover, cross effects only take two values, that is, $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$, for all $i \neq j$ with $\underline{\sigma} \leq 0$, then $\mathbf{G}$ is a symmetric $(0, 1)-$matrix and the network $\mathbf{g}$ is un-directed and un-weighted. In this case, $\mathbf{g}$ can be represented by a graph without loops nor multiple links,[8] where nodes stand for players and two nodes i and j are directly linked if and only if efforts are relative strategic complements across these two players, that is, $\sigma_{ij} = \sigma_{ji} = \overline{\sigma}$.

Finally, we write the (common) second order derivative in own efforts as $\partial^2 u_i / \partial x_i^2 = \sigma = -\beta - \gamma$, where $\beta > 0$. Given our assumption that $\sigma < \min\{\underline{\sigma}, 0\}$, this is without loss of generality.

[8] A loop is a single direct link starting at i and ending at i, that is, $g_{ii} = 1$. A direct link between i and j is multiple when $g_{ij} \in \{2, 3, ...\}$. The matrix $\mathbf{G}$ is zero diagonal, thus ruling out loops. Besides, $0 \leq g_{ij} \leq 1$ by construction, thus ruling out multiple links. It is important to note, though, that our results also hold for networks with loops and/or multiple links, but the economic intuitions are less appealing in this case.

Proposition 1 *Let* $\mathbf{I}$ *be the* $n-$*square identity matrix, and* $\mathbf{U}$ *the* $n-$*square matrix of ones. Then:*

$$\Sigma = -\beta\mathbf{I} - \gamma\mathbf{U} + \lambda\mathbf{G}, \tag{2}$$

with $\beta > 0$, $\gamma \geq 0$ *and* $\lambda > 0$.

Proof. From the definition of β, γ, λ and $\mathbf{G}$. ∎

The pattern of bilateral influences results from the combination of an idiosyncratic effect, a global interaction effect, and a local interdependence component.

The idiosyncratic effect reflects (part of) the concavity of the payoff function in own efforts. The global interaction effect is uniform across all players (matrix $\mathbf{U}$) and corresponds to a substitutability effect across all pairs of players with value $-\gamma \leq 0$. The local interaction component captures the (relative) strategic complementarity in efforts that varies across pairs of players, with maximal strength λ and population pattern reflected by $\mathbf{G}$. The whole heterogeneity in Σ is thus captured by $\mathbf{G}$.

Let Σ be a matrix of cross effects for some bilinear payoff functions (1). Following the decomposition of Σ in (2), we rewrite these payoffs as:

$$u_i(x_1, ..., x_n) = \alpha x_i - \frac{1}{2}(\beta - \gamma) x_i^2 - \gamma \sum_{j=1}^{n} x_i x_j + \lambda \sum_{j=1}^{n} g_{ij} x_i x_j, \text{ for all } i = 1, ..., n. \tag{3}$$

Let $\lambda^* = \lambda/\beta$ denote the strength of local interactions relative to self-concavity.

2.3 The Bonacich network centrality measure

Before turning to the equilibrium analysis, we define a network centrality measure due to Bonacich (1987) that proves useful for our analysis.

The $n-$square adjacency matrix $\mathbf{G}$ of a network $\mathbf{g}$ keeps track of the direct connections in this network. Indeed, two players i and j are directly connected in $\mathbf{g}$ if and only if $g_{ij} > 0$, in which case $0 \leq g_{ij} \leq 1$ measures the weight associated to this direct connection.

Let $\mathbf{G}^k$ be the kth power of this adjacency matrix with coefficients $g_{ij}^{[k]}$, where k is some nonzero integer. This matrix keeps track of the indirect connections in the network. Indeed, $g_{ij}^{[k]} \geq 0$ measures the number of paths of length $k \geq 1$ in $\mathbf{g}$ between i and j.[9] In particular, $\mathbf{G}^0 = \mathbf{I}$, that is, $g_{ii}^{[0]} = 1$ and $g_{ij}^{[0]} = 0$ for all $i \neq j$.

[9] A *path* of length k from i to j in $\mathbf{g}$ is a sequence $\langle i_0, i_1, ..., i_k \rangle$ of players such that $i_0 = i$, $i_k = j$, $i_p \neq i_{p+1}$, and $g_{i_p i_{p+1}} > 0$, for all $0 \leq p \leq k-1$, that is, players i_p and i_{p+1} are directly linked in $\mathbf{g}$. In fact, $g_{ij}^{[k]}$ accounts for the total weight of all paths of length k, from i to j. When the network is un-weighted, that is, $\mathbf{G}$ is a $(0,1)-$matrix, $g_{ij}^{[k]}$ is simply the number of paths of length k from i to j.

Given a scalar $a \geq 0$ and a network $\mathbf{g}$, we define the following matrix:

$$\mathbf{M}(\mathbf{g}, a) = [\mathbf{I} - a\mathbf{G}]^{-1} = \sum_{k=0}^{+\infty} a^k \mathbf{G}^k.$$

Note that these expressions are all well-defined for low enough values of a.[10] The parameter a is a decay factor that scales down the relative weight of longer paths.

Provided the matrix $\mathbf{M}(\mathbf{g}, a)$ is non-negative, its coefficients $m_{ij}(\mathbf{g}, a) = \sum_{k=0}^{+\infty} a^k g_{ij}^{[k]}$ count the number of paths in $\mathbf{g}$ starting at i and ending at j, where paths of length k are weighted by a^k.

Let $\mathbf{1}$ be the $n-$dimensional vector of ones.

Definition 1 *Consider a network* $\mathbf{g}$ *with adjacency* $n-$*square matrix* $\mathbf{G}$ *and a scalar* a *such that* $\mathbf{M}(\mathbf{g}, a) = [\mathbf{I} - a\mathbf{G}]^{-1}$ *is well-defined and non-negative. The vector of Bonacich centralities of parameter* a *in* $\mathbf{g}$ *is:*

$$\mathbf{b}(\mathbf{g}, a) = [\mathbf{I} - a\mathbf{G}]^{-1} \cdot \mathbf{1}.$$

The Bonacich centrality of node i is $b_i(\mathbf{g}, a) = \sum_{j=1}^{n} m_{ij}(\mathbf{g}, a)$, and counts the *total* number of paths in $\mathbf{g}$ starting from i.[11] It is the sum of all loops $m_{ii}(\mathbf{g}, a)$ starting from i and ending at i, and all outer paths $\sum_{j \neq i} m_{ij}(\mathbf{g}, a)$ that connect i to every other player $j \neq i$, that is:

$$b_i(\mathbf{g}, a) = m_{ii}(\mathbf{g}, a) + \sum_{j \neq i} m_{ij}(\mathbf{g}, a).$$

By definition, $m_{ii}(\mathbf{g}, a) \geq 1$, and thus $b_i(\mathbf{g}, a) \geq 1$, with equality when $a = 0$.

3 Nash equilibrium and Bonacich centrality

3.1 Characterization and uniqueness

We now characterize a Nash equilibrium of the game Σ, where equilibrium strategies are proportional to Bonacich centralities in the network of local complementarities $\mathbf{g}$ derived from Σ. We provide conditions such that this equilibrium is unique and interior.

Consider a matrix of cross effects Σ decomposed as in Proposition 1. From now on, we focus on symmetric matrices such that $\sigma_{ij} = \sigma_{ji}$, for all $j \neq i$. Then, the largest eigenvalue $\mu_1(\mathbf{G})$ of $\mathbf{G}$,

[10] Take a smaller than the norm of the inverse of the largest eigenvalue of $\mathbf{G}$.

[11] In fact, $\mathbf{b}(\mathbf{g}, a)$ is obtained from Bonacich (1987)'s measure by an affine transformation. Bonacich defines the following network centrality measure:

$$\mathbf{h}(\mathbf{g}, a, b) = b\,[\mathbf{I} - a\mathbf{G}]^{-1}\,\mathbf{G} \cdot \mathbf{1}.$$

Therefore, $\mathbf{b}(\mathbf{g}, a) = \mathbf{1} + a\mathbf{h}(\mathbf{g}, a, 1) = \mathbf{1} + \mathbf{k}(\mathbf{g}, a)$, where $\mathbf{k}(\mathbf{g}, a)$ is an early measure of network status introduced by Katz (1953). See also Guimerà *et al.* (2001) for a related network centrality measures.

sometimes referred to as the index of the network $\mathbf{g}$, is well-defined. Also, $\mu_1(\mathbf{G}) > 0$ as long as $\sigma_{ij} \neq 0$, for some $j \neq i$.[12]

For all vector $\mathbf{y} \in \mathbb{R}^n$, denote by $y = y_1 + ... + y_n$ the sum of its coordinates.

Theorem 1 *The matrix $[\beta \mathbf{I} - \lambda \mathbf{G}]^{-1}$ is well-defined and non-negative if and only if $\beta > \lambda \mu_1(\mathbf{G})$. Then, the game Σ has a unique Nash equilibrium $\mathbf{x}^*(\Sigma)$, which is interior and given by:*

$$\mathbf{x}^*(\Sigma) = \frac{\alpha}{\beta + \gamma b(\mathbf{g}, \lambda^*)} \mathbf{b}(\mathbf{g}, \lambda^*). \tag{4}$$

Proof. The necessary and sufficient condition for $[\beta \mathbf{I} - \lambda \mathbf{G}]^{-1}$ to be well-defined and non-negative derives from Theorem III*, page 601 in Debreu and Herstein (1953).

An interior Nash equilibrium in pure strategies $\mathbf{x}^* \in \mathbb{R}^n_+$ is such that $\partial u_i / \partial x_i(\mathbf{x}^*) = 0$ and $x_i^* > 0$, for all $i = 1, ..., n$. If such an equilibrium exists it then solves:

$$-\Sigma \cdot \mathbf{x} = [\beta \mathbf{I} + \gamma \mathbf{U} - \lambda \mathbf{G}] \cdot \mathbf{x} = \alpha \mathbf{1}. \tag{5}$$

The matrix $\beta \mathbf{I} + \gamma \mathbf{U} - \lambda \mathbf{G}$ is generically non-singular, and (5) has a unique generic solution in $\mathbb{R}^n$, denoted by $\mathbf{x}^*$.[13] Using the fact that $\mathbf{U} \cdot \mathbf{x}^* = x^* \mathbf{1}$, (5) is equivalent to:

$$[\mathbf{I} - \lambda^* \mathbf{G}] \cdot \mathbf{x}^* = \frac{\alpha - \gamma x^*}{\beta} \mathbf{1} \Leftrightarrow \mathbf{x}^* = \frac{\alpha - \gamma x^*}{\beta} \mathbf{b}(\mathbf{g}, \lambda^*),$$

and (4) follows by simple algebra. Given that $\alpha > 0$ and $b_i(\mathbf{g}, \lambda^*) \geq 1$, for all i, it follows that $\mathbf{x}^*$ is interior.

We now establish uniqueness. First, the previous argument shows that $\mathbf{x}^*$ is a unique interior equilibrium. We deal with corner solutions.

Denote by $\beta(\Sigma), \gamma(\Sigma), \lambda(\Sigma)$ and $\mathbf{G}(\Sigma)$ the elements from the decomposition of Σ in Proposition 1. We omit the dependence in Σ when there is no confusion. For all matrix $\mathbf{Y}$, vector $\mathbf{y}$ and set $S \subset \{1, ..., n\}$, $\mathbf{Y}_S$ is the sub-matrix of $\mathbf{Y}$ with rows and columns in S, and $\mathbf{y}_S$ is the sub-vector of $\mathbf{y}$ with rows in S.

Let $S \subset \{1, ..., n\}$. Then $\gamma(\Sigma_S) \leq \gamma(\Sigma), \beta(\Sigma_S) \geq \beta(\Sigma)$ and $\lambda(\Sigma_S) \leq \lambda(\Sigma)$. Also, $\lambda \mathbf{G} = \Sigma + \gamma(\mathbf{U} - \mathbf{I}) - \sigma \mathbf{I}$, and the coefficients in the S rows and columns of $\lambda \mathbf{G}$ are at least as high as the corresponding coefficients in $\lambda(\Sigma_S) \mathbf{G}_S$. Theorem I*, page 600 in Debreu and Herstein (1953) then implies that $\mu_1(\lambda(\Sigma_S) \mathbf{G}_S) \leq \mu_1(\lambda(\Sigma) \mathbf{G})$. Therefore, $\beta(\Sigma) > \lambda(\Sigma) \mu_1(\mathbf{G})$ implies $\beta(\Sigma_S) > \lambda(\Sigma_S) \mu_1(\mathbf{G}_S)$.

Let $\mathbf{y}^*$ be a non-interior Nash equilibrium of the game Σ. Let $S \subset \{1, ..., n\}$ such that $y_i^* = 0$ if and only if $i \in N \backslash S$. Therefore, $y_i^* > 0$, for all $i \in S$. Note that $S \neq \emptyset$, as $\partial u_i / \partial x_i(\mathbf{0}) = \alpha > 0$,

[12] Note that $\mathbf{G}$ is symmetric from the symmetry of Σ. By the Perron-Frobenius theorem, the eigenvalues of a symmetric matrix $\mathbf{G}$ are all real numbers. Also, the matrix $\mathbf{G}$ with all zeros in the diagonal has a trace equal to zero. Therefore, $\mu_1(\mathbf{G}) > 0$ whenever $\mathbf{G} \neq \mathbf{0}$.

[13] The set of parameters β, γ, λ for which $\det(\beta \mathbf{I} + \gamma \mathbf{U} - \lambda \mathbf{G}) = 0$ has Lebesgue measure zero in $\mathbb{R}^3$.

and $\mathbf{0}$ cannot be a Nash equilibrium. Then,

$$-\Sigma_S \cdot \mathbf{y}_S^* = [\beta \mathbf{I}_S + \gamma \mathbf{U}_S - \lambda \mathbf{G}_S] \cdot \mathbf{y}_S^* = \alpha \mathbf{1}_S.$$

Given that $\beta(\Sigma_S) > \lambda(\Sigma_S)\mu_1(\mathbf{G}_S)$, we have:

$$\mathbf{y}_S^* = \frac{\alpha - \gamma y_S^*}{\beta} \mathbf{b}(\mathbf{g}_S, \lambda^*). \tag{6}$$

Now, every player $i \in N \backslash S$ is best-responding with $y_i^* = 0$, so that:

$$\frac{\partial u_i}{\partial x_i}(\mathbf{y}^*) = \alpha + \sum_{j \in S} \sigma_{ij} y_j^* = \alpha - \gamma y_S^* + \lambda \sum_{j \in S} g_{ij} y_j^* \leq 0, \text{ for all } i \in N \backslash S,$$

where the last equality uses the decomposition of Σ. Using (6), we rewrite this inequality as:

$$(\alpha - \gamma y_S^*)(1 + \lambda^* \sum_{j \in S} g_{ij} b_j(\mathbf{g}_S, \lambda^*)) \leq 0, \text{ equivalent to } \alpha - \gamma y_S^* \leq 0.$$

Using (6), we conclude that $y_i^* \leq 0$, for all $i \in S$, which is a contradiction. ∎

When the matrix of cross effects Σ reduces to $\lambda \mathbf{G}$ (that is, $\beta = \gamma = 0$), the game Σ is supermodular and we have a multiplicity of Nash equilibria. If, instead, this matrix reduces to $-\beta \mathbf{I} - \gamma \mathbf{U}$ (that is, $\lambda = 0$), the equilibrium is generically unique. The condition in Theorem 1 requires that the parameter for own-concavity β to be high enough to counter the payoff complementarity captured by $\lambda \mu_1(\mathbf{G})$. Here, λ has to do with the level and $\mu_1(\mathbf{G})$ with the population-wide pattern of positive cross effects. Note that this condition does not impose any bound on the absolute values for these cross effects, but only on their relative magnitude.

The Bonacich-equilibrium expression (4) also implies that:

$$x_i^*(\Sigma) = \frac{b_i(\mathbf{g}, \lambda^*)}{b(\mathbf{g}, \lambda^*)} x^*(\Sigma).$$

Each player contributes to the aggregate equilibrium outcome in proportion to her network centrality. The dependence of individual outcomes on group behavior is referred to as *peer effects*. Here, this intragroup externality is not an average influence. It is *heterogeneous* across members, with a variance related to the Bonacich network centrality.

Remark 1 *Consider the general game $(\Sigma, \boldsymbol{\alpha})$, where $\alpha_i > 0$ differs across players. Then, (4) still holds where $\alpha \mathbf{b}(\mathbf{g}, \lambda^*)$ is replaced by the* weighted *Bonacich centrality measure $\mathbf{b}_{\boldsymbol{\alpha}}(\mathbf{g}, \lambda^*) = [\mathbf{I} - \lambda^* \mathbf{G}]^{-1} \cdot \boldsymbol{\alpha}$.*

Remark 2 *In the proof of Theorem 1, the symmetry of Σ does not play any explicit role but for the existence and positivity of the largest eigenvalue of $\mathbf{G}$. Therefore, the Bonacich-Nash linkage holds for any asymmetric matrix Σ of cross effects with a well-defined positive spectral radius.*[14]

[14] Debreu and Hernstein (1953) provide some general conditions for a matrix to have a well-defined and positive spectral radius.

When $\mathbf{g}$ is an un-directed and un-weighted network, the condition in Theorem 1 can be expressed directly in terms of the number of nodes and links in $\mathbf{g}$, thus dispensing with computing the network index.

Let $g = \sum_{i,j} g_{ij}$ be the sum of all direct links in $\mathbf{g}$. When $\mathbf{G}$ is a $(0,1)-$ matrix, this is twice the number of direct links in the un-directed and un-weighted network $\mathbf{g}$.

Corollary 1 *Suppose that $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$, for all $i \neq j$ with $\underline{\sigma} \leq 0$, and that the network $\mathbf{g}$ induced by the decomposition of Σ in (2) is connected. If $\beta > \lambda\sqrt{g+n-1}$, the only Nash equilibrium of the game Σ is given by (4).*

Proof. From the upper bound on the index of a connected graph in Theorem 1.5, page 5 in Cvetković and Rowlinson (1990). ∎

3.2 Comparative statics

The previous results relate individual equilibrium outcomes to the Bonacich centrality in the network $\mathbf{g}$ of local complementarities. The next result establishes a positive relationship between the aggregate equilibrium outcome and the pattern of local complementarities.

For any two matrices Σ and Σ', we write $\Sigma' \gneqq \Sigma$ if $\sigma'_{ij} \geq \sigma_{ij}$, for all i, j, with at least one strict inequality.

Theorem 2 *Let Σ and Σ' symmetric such that $\Sigma' \gneqq \Sigma$. If $\beta > \lambda\mu_1(\mathbf{G})$ and $\beta' > \lambda'\mu_1(\mathbf{G}')$ for the decompositions (2) of Σ and Σ', then $x^*(\Sigma') > x^*(\Sigma)$.*

Proof. When $\Sigma' > \Sigma$, we write $\Sigma' = \Sigma + \lambda\mathbf{D}$, with $d_{ij} \geq 0$ with at least one strict inequality, and λ given by the decomposition (2) of Σ. If $\beta > \lambda\mu_1(\mathbf{G})$ and $\beta' > \lambda'\mu_1(\mathbf{G}')$, Theorem 1 holds, so that $-\Sigma \cdot \mathbf{x}^*(\Sigma) = -\Sigma' \cdot \mathbf{x}^*(\Sigma') = \alpha\mathbf{1}$, and $\mathbf{x}^*(\Sigma), \mathbf{x}^*(\Sigma') > \mathbf{0}$. We compute $-\mathbf{x}^{*t}(\Sigma') \cdot \Sigma \cdot \mathbf{x}^*(\Sigma)$ in two different ways. First, $-\mathbf{x}^{*t}(\Sigma') \cdot \Sigma \cdot \mathbf{x}^*(\Sigma) = \alpha\mathbf{x}^{*t}(\Sigma') \cdot \mathbf{1} = \alpha x^*(\Sigma')$. Second, using the symmetry of Σ', we have**:**

$$\begin{aligned}
-\mathbf{x}^{*t}(\Sigma') \cdot \Sigma \cdot \mathbf{x}^*(\Sigma) &= -\mathbf{x}^{*t}(\Sigma') \cdot \Sigma' \cdot \mathbf{x}^*(\Sigma) + \lambda\mathbf{x}^{*t}(\Sigma') \cdot \mathbf{D} \cdot \mathbf{x}^*(\Sigma) \\
&= \alpha x^*(\Sigma) + \lambda\mathbf{x}^{*t}(\Sigma') \cdot \mathbf{D} \cdot \mathbf{x}^*(\Sigma).
\end{aligned}$$

Using the fact that $\alpha > 0$, we conclude that $x^*(\Sigma') > x^*(\Sigma)$. ∎

In words, the denser the pattern of local complementarities, the higher the aggregate outcome, as players can rip more complementarities in $\mathbf{g}'$ than in $\mathbf{g}$. The geometric intuition for this result is clear. Recall that $b(\mathbf{g}, \lambda^*)$ counts the total number of weighted paths in $\mathbf{g}$. This is obviously an increasing function in $\mathbf{g}$ (for the inclusion ordering), as more links imply more such paths.

Remark 3 *When the decompositions (2) of* Σ *and* Σ' *are such that* $(\alpha, \beta, \gamma, \lambda) = (\alpha', \beta', \gamma', \lambda')$ *and* $\mathbf{G}' \gneq \mathbf{G}$*, then* $\beta' > \lambda' \mu_1(\mathbf{G}')$ *implies that* $\beta > \lambda \mu_1(\mathbf{G})$.[15]

4 A network-based policy

4.1 Finding the key player

In our model, individual equilibrium behavior is tightly rooted in the network structure through (4). The removal of a player from the population, holding the pattern of social interactions among the other players fixed, alters the whole distribution of outcomes.

We provide a simple geometric criterion to identify the optimal target in the population when the planner wishes to reduce (or to increase) optimally the aggregate group outcome.[16]

In what follows, we suppose that Σ is symmetric with $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$ for all $i \neq j$, and $\underline{\sigma} \leq 0$. In this case, the decomposition of Σ in (2) yields a $(0,1)-$adjacency matrix $\mathbf{G}$ and an un-weighted and un-directed network $\mathbf{g}$, with its corresponding graph representation.

Let's eliminate some player i from the current population. Suppose that for each possible value $v \in \{\underline{\sigma}, \overline{\sigma}\}$ for the cross effects, there exists at least two different pairs of players (i,j) and (i',j'), differing two-by-two, such that $\sigma_{ij} = \sigma_{i'j'} = v$. This is a mild requirement that guarantees that the values of β, γ and λ in the decomposition (2) of Σ do not change for any such single player removal. The adjacency matrix becomes $\mathbf{G}^{-i}$, obtained from $\mathbf{G}$ by setting to zero all of its ith row and column coefficients. The resulting network is $\mathbf{g}^{-i}$.[17]

Similarly, denote by Σ^{-i} the matrix that results from removing the ith row and column from Σ.

The planner's problem is to reduce $x^*(\Sigma)$ optimally by picking the adequate player from the population.[18] Formally, she solves $\max\{x^*(\Sigma) - x^*(\Sigma^{-i}) \mid i = 1, ..., n\}$, equivalent to:

$$\min\{x^*(\Sigma^{-i}) \mid i = 1, ..., n\} \tag{7}$$

This is a finite optimization problem, that admits at least one solution.

Let i^* be a solution to (7). We call player i^* the *key player*. Removing i^* from the initial network $\mathbf{g}$ has the highest overall impact on the aggregate equilibrium level. We provide a simple

[15] From the monotonicity of the largest eigenvalue with the coefficients of the matrix in Theorem I*, page 600 in Debreu and Hernstein (1953).

[16] Bollobás and Riordan (2003) contains a mathematical analysis of the relative network disruption effects of a topological attack versus random failures in large networks. See also Albert *et al.* (2000) for a numerical analysis for the case of the World Wide Web.

[17] If the primitive of our model is the bilinear expression for the payoffs in (1), the key player analysis applies to matrix of cross effects Σ symmetric and for which the σ_{ij}s only take two possible values, that is, $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$ with $\underline{\sigma} \leq 0$. In this case, $\mathbf{G}$ is a $(0,1)-$matrix. If, instead, the primitive of our model is the expression for the payoffs in (3), the key player analysis carries over to any symmetric adjacency matrix $\mathbf{G}$ with $0 \leq g_{ij} \leq 1$.

[18] Corollary 1 below considers the symmetric case where the planner wishes to increase $x^*(\mathbf{g}, \lambda^*)$ optimally.

and direct geometric characterization of the key player.

Definition 2 *Consider a network* $\mathbf{g}$ *with adjacency* $n-$*square matrix* $\mathbf{G}$ *and a scalar* a *such that* $\mathbf{M}(\mathbf{g},a)=[\mathbf{I}-a\mathbf{G}]^{-1}$ *is well-defined and non-negative. The inter-centrality of player* i *of parameter* a *in* $\mathbf{g}$ *is:*

$$c_i(\mathbf{g},a)=\frac{b_i^2(\mathbf{g},a)}{m_{ii}(\mathbf{g},a)}.$$

The Bonacich centrality of player i counts the number of paths in $\mathbf{g}$ stemming from i, while the inter-centrality computes the total number of such paths that hit i at some time. It is the sum of i's Bonacich centrality and i's contribution to every other player Bonacich centrality. Holding $b_i(\mathbf{g},\lambda^*)$ fixed, $c_i(\mathbf{g},\lambda^*)$ decreases with the proportion m_{ii}/b_i of i's Bonacich centrality due to self-loops.

Theorem 3 *If* $\beta>\lambda\mu_1(\mathbf{G})$, *the key player* i^* *that solves* $\min\{x^*(\Sigma^{-i})\mid i=1,...,n\}$ *is the one with the highest inter-centrality measure of parameter* λ^* *in* $\mathbf{g}$, *that is,* $c_{i^*}(\mathbf{g},\lambda^*)\geq c_i(\mathbf{g},\lambda^*)$, *for all* $i=1,...,n$.[19]

Proof. We first prove a useful result.

Lemma 1 *Let* $a>0$ *such that* $\mathbf{M}(\mathbf{g},a)=[\mathbf{I}-a\mathbf{G}]^{-1}$ *is well-defined and non-negative. Then,* $m_{ij}(\mathbf{g},a)m_{ik}(\mathbf{g},a)=m_{ii}(\mathbf{g},a)\left[m_{jk}(\mathbf{g},a)-m_{jk}(\mathbf{g}^{-i},a)\right]$.

Proof of the Lemma. First note that the symmetry of Σ implies that $m_{jk}(\mathbf{g},a)=m_{kj}(\mathbf{g},a)$, for all j,k and $\mathbf{g}$. We have:

$$\begin{aligned}m_{ii}(\mathbf{g},a)\left[m_{jk}(\mathbf{g},a)-m_{jk}(\mathbf{g}^{-i},a)\right] &= \sum_{p=1}^{+\infty}a^p\sum_{\substack{r+s=p\\ r\geq 0,s\geq 1}}g_{ii}^{[r]}\left(g_{jk}^{[s]}-g_{j(i^0)k}^{[s]}\right)=\sum_{p=1}^{+\infty}a^p\sum_{\substack{r+s=p\\ r\geq 0,s\geq 2}}g_{ii}^{[r]}g_{j(i)k}^{[s]}\\ &= \sum_{p=1}^{+\infty}a^p\sum_{\substack{r'+s'=p\\ r'\geq 1,s'\geq 1}}g_{ji}^{[r']}g_{ik}^{[s']}=m_{ji}(\mathbf{g},a)m_{ik}(\mathbf{g},a),\end{aligned}$$

where $g_{j(i^0)k}^{[s]}$ (resp. $g_{j(i)k}^{[s]}$) is the compound weight of length$-s$ paths from j to k not containing i (resp. containing i), and $g_{ii}^{[0]}=1$. *Q.E.D.*

First, note that $\mu_1(\mathbf{G})\geq\mu_1(\mathbf{G}^{-i})$. Therefore, if $\mathbf{M}(\mathbf{g},\lambda^*)$ is well-defined and non-negative (as implied by the condition in Theorem 1), so is $\mathbf{M}(\mathbf{g}^{-i},\lambda^*)$, for all $i=1,...,n$.

With $\alpha>0$, $x^*(\Sigma^{-i})$ increases in $b(\mathbf{g}^{-i},\lambda^*)$, and (7) is equivalent to $\min\{b(\mathbf{g}^{-i},\lambda^*)\mid i=1,...,n\}$. Define:

$$b_{ji}(\mathbf{g},\lambda^*)=b_j(\mathbf{g},\lambda^*)-b_j(\mathbf{g}^{-i},\lambda^*),\text{ for all } j\neq i.$$

[19] Note that there may be more than just one key player.

This is the contribution of i to j's Bonacich centrality in $\mathbf{g}$. Summing over all $j \neq i$, and adding $b_i(\mathbf{g}, \lambda^*)$ to both sides gives:

$$b(\mathbf{g}, \lambda^*) - b(\mathbf{g}^{-i}, \lambda^*) = b_i(\mathbf{g}, \lambda^*) + \sum_{j \neq i} b_{ji}(\mathbf{g}, \lambda^*) \equiv d_i(\mathbf{g}, \lambda^*).$$

The solution of (7) is i^* such that $d_{i^*}(\mathbf{g}, \lambda^*) \geq d_i(\mathbf{g}, \lambda^*)$, for all $i = 1, ..., n$. We have:

$$d_i(\mathbf{g}, \lambda^*) = b_i(\mathbf{g}, \lambda^*) + \sum_{j \neq i} \left[b_j(\mathbf{g}, \lambda^*) - b_j(\mathbf{g}^{-i}, \lambda^*) \right] = b_i(\mathbf{g}, \lambda^*) + \sum_{j \neq i} \sum_{k=1}^{n} \left[m_{jk}(\mathbf{g}, \lambda^*) - m_{jk}(\mathbf{g}^{-i}, \lambda^*) \right].$$

Using Lemma 1, this becomes:

$$d_i(\mathbf{g}, \lambda^*) = b_i(\mathbf{g}, \lambda^*) + \sum_{j \neq i} \sum_{k=1}^{n} \frac{m_{ij}(\mathbf{g}, a) m_{ik}(\mathbf{g}, a)}{m_{ii}(\mathbf{g}, a)} = b_i(\mathbf{g}, \lambda^*) \left[1 + \sum_{j \neq i} \frac{m_{ij}(\mathbf{g}, a)}{m_{ii}(\mathbf{g}, a)} \right] = \frac{b_i^2(\mathbf{g}, \lambda^*)}{m_{ii}(\mathbf{g}, \lambda^*)}.$$

∎

Example 1 *Consider the network* $\mathbf{g}$ *in Figure 2 with eleven players.*

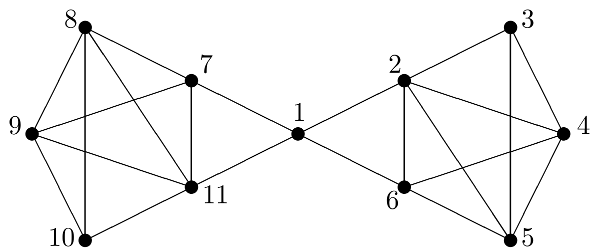


Figure 2

There are three different locations in this network: player 1, players 2, 6, 7 and 11, and players 3, 4, 5, 8, 9, and 10. Type-1 and type-3 players have four direct links, while type -2 players have five. Player 1 bridges together two fully intra-connected communities of five players each. By removing player 1, the network is maximally disrupted. By removing a type-2 player, we get a network with the lowest total number of links.

Table 1 computes the Bonacich and inter-centrality measures for different values of the decay factor a. A superscript star identifies the highest column value.[20]

[20] Here, the highest value for λ^* compatible with our definition of centrality measures is $\frac{2}{3+\sqrt{41}} \simeq 0.213$.

a	0.1		0.2	
Player Type	b_i	c_i	b_i	c_i
1	1.75	2.92	8.33	41.67*
2	1.88*	3.28*	9.17*	40.33
3	1.72	2.79	7.78	32.67

Table 1

Type-2 players have the highest Bonacich centrality. They have the highest number of direct connections and are directly connected to the bridge player 1, who gives them access to a wide span of indirect connections. When a is low, they are also the key players. When a is high, though, the most active players are not the key players anymore. Now, indirect effects matter, and eliminating player 1 has the highest joint direct and indirect effect on aggregate outcome.

Corollary 2 *If $\beta > \lambda\mu_1(\mathbf{G})$, the key player i^* that solves $\max\{x^*(\Sigma^{-i}) \mid i = 1, ..., n\}$ is the one with the lowest inter-centrality measure of parameter λ^* in $\mathbf{g}$, that is, $c_{i^*}(\mathbf{g}, \lambda^*) \leq c_i(\mathbf{g}, \lambda^*)$, for all $i = 1, ..., n$.*

Remark 4 *When Σ is not symmetric, Theorem 3 and Corollary 1 still hold where the intercentrality measure is now given by $\widetilde{c}_i(\mathbf{g}, a) = b_i(\mathbf{g}, a)(\sum_{j=1}^{n} m_{ji}(\mathbf{g}, a))/m_{ii}(\mathbf{g}, a)$.*

5 Applications

In this section, we propose three different applications of the previous results

5.1 Crime networks

There are n criminals, each exerting a level of crime x_i that results from a trade off between the costs and benefits of criminal activities. The expected utility of criminal i is:

$$u_i(\mathbf{x}, \mathbf{r}) = y_i(\mathbf{x}) - p_i(\mathbf{x}, \mathbf{r})f, \tag{8}$$

where $y_i(\mathbf{x})$ are the proceeds, $p_i(\mathbf{x}, \mathbf{r})$ the apprehension probability, and f the corresponding fine. Following Ballester *et al.* (2004) and Calvó-Armengol and Zenou (2004), the cost of committing crime $p_i(\mathbf{x}, \mathbf{r})f$ increases with x_i, as the apprehension probability increases with one's involvement in crime, hitherto, with one's exposure to deterrence.

Also, and consistent with standard criminology theories, criminals improve illegal practice through interactions with their direct criminal mates.[21] Formally, criminals are connected through

[21] See, e.g., Sutherland (1947).

a friendship network $\mathbf{r}$, where $r_{ij} = 1$ when i and j are directly related to each other. For instance, let:

$$\begin{cases} y_i(\mathbf{x}) = x_i \left[1 - \eta \sum_{j=1}^{n} x_j\right] \\ p_i(\mathbf{x}, \mathbf{r}) = p_0 x_i \left[1 - \nu \sum_{j=1}^{n} r_{ij} x_j\right] \end{cases}.$$

The expected utility then becomes:

$$u_i(\mathbf{x}, \mathbf{r}) = (1 - \pi) x_i - \eta \sum_{j=1}^{n} x_i x_j + \pi\nu \sum_{j=1}^{n} r_{ij} x_i x_j, \tag{9}$$

where $\pi = p_0 f$ is the marginal expected punishment cost for an isolated criminal, and $-\eta < 0$ captures a congestion in the crime market. The utility function (9) coincides with the expression in (3) with $\alpha = 1 - \pi$, $\beta = \gamma = \eta$, $\lambda = \pi\nu$ and $\mathbf{g} = \mathbf{r}$. When $\pi\nu\mu_1(\mathbf{r}) < \eta$, the unique Nash equilibrium of the crime game with payoffs (9) is:

$$\mathbf{x}^* = \frac{1-\pi}{\eta} \frac{1}{1 + b(\mathbf{r}, \pi\frac{\nu}{\eta})} \mathbf{b}(\mathbf{r}, \pi\frac{\nu}{\eta}).$$

Here, the key player policy in Theorem 3 has both a direct and an indirect effect on crime reduction. On the contrary, a standard deterrence policy (an increase in π) has a positive direct impact on crime reduction, but a negative indirect effect, as criminals now counter the extra deterrence they face by strengthening their network interactions.

5.2 R&D collaboration networks

Consider a standard Cournot game with n (ex ante) identical firms, each of them choosing the quantity q_i. As in Goyal and Moraga-González (2001) and Goyal and Joshi (2003), firms can form bilateral agreements to jointly invest in cost-reducing R&D activities. We set $c_{ij} = 1$ when firms i and j set up a collaboration link. Firm i's marginal cost is $\lambda_0 - \lambda \sum_{j \neq i} r_{ij} q_j$. Here, $\lambda_0 > 0$, represents the marginal cost of an isolated firm, while $\lambda > 0$ is the cost reduction induced by each link it forms. With a linear inverse demand, the profit function of firm i is:

$$u_i(\mathbf{x}, \mathbf{r}) = \left[\phi - \sum_{j=1}^{n} q_j\right] q_i - \left[\lambda_0 - \lambda \sum_{j \neq i} r_{ij} q_j\right] q_i = (\phi - \lambda_0)\, q_i - \sum_{j=1}^{n} q_i q_j + \lambda \sum_{j \neq i} r_{ij} q_i q_j. \tag{10}$$

Again, this objective function is a particular case of (3), where $\alpha = \phi - \lambda_0 > 0$, $\beta = \gamma = 1$ and $\mathbf{g} = \mathbf{r}$. Using Corollary 1, we conclude that the Cournot game with payoffs (10) has a unique Nash equilibrium in pure strategies:

$$\mathbf{q}^* = \frac{\phi - \lambda_0}{1 + b(\mathbf{r}, \lambda)} \mathbf{b}(\mathbf{r}, \lambda),$$

when $1 > \lambda\sqrt{g+n-1}$. In particular, Theorem 2 implies that the overall industry output increases when the network of collaboration links expands, irrespective of this network geometry and the number of additional links. For the case of a linear inverse demand curve, this generalizes the findings in Goyal and Moraga-González (2001) and Goyal and Joshi (2003), where monotonicity of industry output is established for the case of regular collaboration networks, where each firm forms the same number of bilateral agreements. For such regular networks, links are added as multiples of n, as all firms' connections are increased simultaneously.

5.3 Conformism and social norms

There are n players whose well-being depends on their behavior compared to that of their reference group. More precisely, each player chooses an action $x_i \geq 0$ and loses utility when failing to conform to the social norm of his reference group, equal to the average action of its members. This framework encompasses a variety of issues where conformism is the driving force for individual behavior.[22] Here, contrarily to previous models, we allow for the reference group, and its induced social norm, to vary with the friendship and community ties of each player.

Formally, when i and j are friends we set $f_{ij} = 1$. Let also $f_{ii} = 0$ for all i. This collection of links constitutes a network $\mathbf{f}$. Player i has $f_i = \sum_{j=1}^{n} f_{ij}$ direct links in $\mathbf{f}$, whose average action is $\overline{x}_i = \sum_{j=1}^{n} f_{ij} x_j / f_i$. This is the social norm of player i. We assume that $f_i > 0$, for all i.

Consider the following utility function, with $\xi, \alpha, \theta, d > 0$:

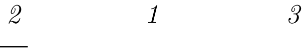
$$u_i(\mathbf{x}, \mathbf{f}) = \xi + \alpha x_i - \theta x_i^2 - d(x_i - \overline{x}_i)^2. \tag{11}$$

In words, non-conformist behavior entails a quadratic utility loss. When $\mathbf{f}$ is the complete network with self-loops, this is equation (5) in Akerlof (1997), page 1009. We have:

$$\frac{\partial^2 u_i(\mathbf{x}, \mathbf{f})}{\partial x_i \partial x_j} = \begin{cases} -2(\theta + d), \text{ when } i = j \\ 0, \text{ when } i \neq j \text{ and } f_{ij} = 0 \\ 2d/f_i > 0, \text{ when } i \neq j \text{ and } f_{ij} = 1 \end{cases}.$$

This utility function (11) thus coincides with (3) with $\beta = 2(\theta + d)$, $\gamma = 0$, $\lambda = 2d$ and $g_{ij} = f_{ij}/f_i$. Note that $\mathbf{g}$ is a row-normalization of the initial friendship network $\mathbf{f}$, as illustrated in the following example, where $\mathbf{F}$ and $\mathbf{G}$ are the adjacency matrices of, respectively, $\mathbf{f}$ and $\mathbf{g}$.

Example 2 *Consider the following friendship network* $\mathbf{f}$*:*



[22] Different issues have been explored in the literature. For example, and to name a few, (*i*) peer pressures and partnerships, when individuals are penalized for working less than the group norm (Kandel and Lazear 1992), (*ii*) religion, when the benefits of praying increase with the number of participants (Iannaccone 1992, Berman 2000), (*iii*) social status and social distance, when deviations from the social norm imply a loss of reputation and status (Akerlof 1980 and 1997 and Bernheim 1994, among others).

Then,

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{G} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

If $\theta > (n-2)\,d$,[23] this conformity game with payoffs (11) has a unique Nash equilibrium in pure strategies given by:

$$\mathbf{x}^* = \frac{\alpha}{2(\theta + d)}\mathbf{b}\left(\mathbf{g}, \frac{d}{\theta + d}\right).$$

6 Discussion and extensions

We discuss a number of possible extensions of this work.

First, our analysis is restricted to linear-quadratic utility functions that capture linear externalities in players' actions. First order conditions for interior equilibria then produce a system of linear equations that leads to the Bonacich-Nash linkage. Suppose, instead, that externalities are non-linear, and that utility functions $\mathbf{u}$ are C^2. Let $\Sigma(\mathbf{x}^*)$ be the (symmetric) Jacobian of $\nabla\mathbf{u}$ evaluated at an interior Nash equilibrium $\mathbf{x}^* > \mathbf{0}$. Decompose $\Sigma(\mathbf{x}^*)$ as in (2). Then, by a simple continuity argument, the first order approximation of $\mathbf{x}^*$ corresponds to the Bonacich centrality vector for this decomposition.

Second, Theorem 3 characterizes the key player when the planner's objective function is the aggregate group outcome $x^*(\Sigma)$. Suppose, instead, that the planner's objective is to maximize the welfare function $W^*(\Sigma) = \sum_{i=1}^n u_i(\mathbf{x}^*(\Sigma))$. Simple algebra gives $2W^*(\Sigma) = (\beta + \gamma)\sum_{i=1}^n x_i^*(\Sigma)^2$ and, when $\gamma = 0$, this becomes $2\beta W^*(\Sigma) = \alpha^2 \sum_{i=1}^n b_i(\mathbf{g}, \lambda^*)^2$. A geometric characterization of the key player is also possible in this case. The building block is provided by Lemma 1, which characterizes all the path changes in a network when a node is removed. The network index obtained in this case is less intuitive than the inter-centrality measure, as it now accounts for individual direct contributions to the aggregate outcome, indirect contributions, and the variance of the latter.

Third, Theorem 3 characterizes geometrically single player targets, but the inter-centrality measure can be generalized to a group index.[24] Note that the group target selection problem is not amenable to a sequential key player problem. For instance, the key group of size 2 in Example 1 when $a = 0.2$ is $\{2, 7\}$, rather than the sequential optimal pair $\{1, 2\}$.

Fourth, beyond the optimal player removal problem, the network policy analysis can also accommodate more general optimal targeted tax or subsidy problems. Consider a population of $n+1$ agents $i = 0, 1, ..., n$ and a matrix of cross effects Σ with associated network $\mathbf{g}$ in (2). Suppose that

[23] Here, we use the fact that $\mu_1(\mathbf{G}) \leq \mu_1(\mathbf{F}) \leq n - 1$.

[24] See Ballester *et al.* (2004).

the planner holds the outcome of player $i = 0$ to some fixed exogenous value $s \in \mathbb{R}$. The case $s > 0$ (resp. $s < 0$) is a subsidy (resp. tax), while $s = 0$ corresponds to the key player problem solved above. Players $i = 1, ..., n$ then play an $n-$player game with interior Bonacich-Nash equilibrium $\mathbf{x}^*_{-0}(\Sigma^{-0}, s)$. Denote by $\mathbf{g}_0$ the $n-$dimensional column vector with coordinates $g_{01}, ..., g_{0n}$ that keeps track of player 0's direct contacts in $\mathbf{g}$, let $\alpha^* = \alpha/\beta$ and $\gamma^* = \gamma/\beta$. Then, the total equilibrium population outcome is $s + x^*_{-0}(\Sigma^{-0}, s)$, where:

$$x^*_{-0}(\Sigma^{-0}, s) = \frac{1}{1 + \gamma^* b(\mathbf{g}^{-0}, \lambda^*)}[(\alpha^* - s)b(\mathbf{g}^{-0}, \lambda^*) + \lambda^* s b_{\mathbf{g}_0}(\mathbf{g}^{-0}, \lambda^*)].$$

Here, $b_{\mathbf{g}_0}(\mathbf{g}^{-0}, \lambda^*)$ is the aggregate weighted Bonacich centrality defined in Remark 1. Given an objective function related to the total population output $s + x^*_{-0}(\Sigma^{-0}, s)$, and a set of constraints, the planner's problem is to fix optimally the value of s and the target identity i. Holding s constant, the choice of the optimal target is a simple finite optimization problem. In particular, when $s = 0$ and the planner wants to minimize the overall output, the solution to this problem is $i^* \in \arg\max\{c_i(\mathbf{g}, \lambda^*) \mid i = 1, ..., n\}$.

Finally, the analysis so far deals with a fixed network. When $\mathbf{G}$ is a $(0, 1)-$matrix, we can easily endogenize the network with a two-stage game the following way. In the first stage, players decide simultaneously to stay in the network or to drop out of it (and get their outside option). This is modelled as a simple binary decision. In the second stage, the players that stay play the network game on the resulting network. Uniqueness of the second-stage Nash equilibrium and its closed-form expression crucially simplify the analysis of this two-stage game. See, e.g, Ballester *et al.* (2004) and Calvó-Armengol and Jackson (2004) for analysis along this vein.

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