Centre de Referència en Economia Analítica

Barcelona Economics Working Paper Series

Working Paper nº 153

Bargained stable allocations in assignment markets

Marina Núñez and Carles Rafels

December, 2004

Bargained stable allocations in assignment markets

Marina Núñez and Carles Rafels *

Department of Actuarial, Financial and Economic Mathematics, CREB and CREA

University of Barcelona, Av.Diagonal, 690, E-08034 Barcelona, Spain

e-mail: mnunez@ub.edu; crafels@ub.edu

28th December 2004

Barcelona Economics WP nº 153

Abstract

For each assignment market, an associated bargaining problem is defined and some bargaining solutions to this problem are analyzed. For a particular choice of the disagreement point, the Nash solution and the Kalai–Smorodinsky solution coincide and give the midpoint between the buyers-optimal core allocation and the sellers-optimal core allocation, and thus they belong to the core. Moreover, under the assumption that all agents in the market are active, the subset of core allocations that can be obtained as a Kalai-Smorodinsky solution, from some suitable disagreement point, is characterized as the set of stable allocations where each agent is paid at least half of his maximum core payoff. All allocations in this last set can also be obtained by a negotiation procedure *à la* Nash.

Keywords: assignment game, core, bargaining problem, Nash solution, Kalai-Smorodinsky solution

JEL: C71, C78

*We are indebted to J.M. Izquierdo, J. Martínez de Albéniz, H. Peters and W. Thomson for their comments and suggestions.

Institutional support from research grants BEC 2002-00642 and SGR2001-0029 is gratefully acknowledged

1 Introduction

The aim of this paper is to make a connection between cooperative games in coalitional form and cooperative bargaining theory, for a particular situation which is that of the bilateral assignment markets.

In a bilateral assignment market a product that comes in indivisible units is exchanged for money, and each participant either supplies or demands exactly one unit. The units need not be alike and the same unit may have different values for different participants. From these valuations, a matrix can be defined which reflects the profit that can be obtained by each buyer-seller pair if they trade.

Assuming that side payments are allowed, Shapley and Shubik (1972) define the assignment game as a cooperative model for this bilateral market. In this model, utility is identified with money, and so an allocation is a vector of incomes such that the sum of payoffs of each optimally matched pair equals the output of that pairing. Moreover, an allocation is stable if no agent can match a different partner in such a way that the output of this pairing exceeds the sum of their payoffs. They prove that the core of the assignment game coincides with the set of stable allocations and is always nonempty.

The necessity of an analysis of the bargaining possibilities inherent in an assignment market is already highlighted in the work of Shapley and Shubik. As far as we know, the first work in this direction is due to Rochford (1984), where some core allocations are selected by means of classical cooperative bargaining theory. In Rochford's work, the optimal matching is assumed to be given exogenously; matched pairs then engage in a pairwise bargaining process which is solved symmetrically, after defining threats based on the outside opportunities given the current payoff to other pairs. Some sort of "complete information" is thus assumed in the market. Notice that in this two–person context the Nash solution (Nash, 1950), the Kalai-Smorodinsky solution (Kalai and Smorodinsky, 1975) and all other symmetric solutions coincide. It is precisely in the determination of the threats where the interaction among the whole set of players enters the model once again. Since the threats are understood as the bargainers' rational expectations of the consequences of not reaching an agreement, given a tentative stable allocation, the threat of an agent is the most he could obtain by matching a different partner and paying him according to this tentative allocation.

Rochford then defines a set of equilibria (the symmetrically pairwise-bargained allocations or SPB allocations) which are stable under rebargaining, and shows that this set coincides with the intersection of the kernel (a well-known set solution for transferable utility cooperative games defined by Davis and Maschler, 1965) and the core of the assignment game. After Granot (1995) and Driessen (1998), we know that the kernel of an assignment game is a subset of the core and thus the set of SPB allocations coincides with the kernel of the assignment game.

Crawford and Rochford (1986) define a symmetrically pairwise-bargained equilibrium in which threats are defined recursively by means of a reduced game and, once threats and a bargaining solution are specified, agents have fixed preferences for matches. Thus, in this model, the equilibrium matching is defined endogenously, but there is no guarantee in general that the bargained solution will be in the core of the assignment game, nor even that it will be efficient.

The model in Bennett (1988) can be seen as an extension of Rochford's model, when the optimal matching is not fixed beforehand and the bargaining process, although still pairwise, is not necessarily symmetric. The bargained solutions in this model lead to core allocations and, conversely, all core allocations can be reached by suitable bargained equilibria.

Finally, Kamecke (1989) presents a non-cooperative game in which the agents construct a matching and pairwise-bargained allocations which also remain in the core, while Moldovanu (1990) generalizes the work of Rochford to assignment games without transferable utility. In this last model, threats are computed by means of reduced games and the negotiation is also symmetric and pairwise.

All these papers have in common that they apply the cooperative bargaining theory to the assignment market but always under the assumption that negotiation takes place between pairs of agents, while the remaining agents appear in the definition of the threats.

In the present paper, we approach the assignment market from the point of view of pure cooperative bargaining theory, as has been already done for some other cooperative situations such as the bankruptcy problem (Dagan and Volij, 1993). This approach differs from those used in the literature cited above in that we consider that the bargaining process takes place in the whole set of agents. First we define a feasible set and a disagreement outcome. Once this is done, we focus our attention on the Nash and the Kalai-Smorodinsky solutions, in order to select some stable allocations supported by some of these bargaining solutions.

Our feasible set does not depend on a fixed optimal matching but takes into account all the optimal matchings present in the market and, since side-payments are allowed, all distributions which could be supported by any optimal matching are considered. The origin is taken as a first disagreement point but, later on, other disagreement points are taken into consideration. In Section 2 an example illustrates how this bargaining problem is built, while Section 3 recalls the main facts about the cooperative model of assignment games.

The bargaining problem associated to an assignment market is defined in Section 4. For those assignment markets with only one optimal matching, we prove that the Nash solution of the defined bargaining problem coincides with the Kalai–Smorodinsky solution. A particular choice of the disagreement point is introduced, which is based on the minimum expectations of each agent under cooperation. For this bargaining problem to be well defined we require that all agents in the market be active. In this case, the coincidence between Nash and Kalai-Smorodinsky solutions is recovered regardless of the number of optimal matchings. An expression is obtained for this solution which shows that it always selects a stable payoff which coincides with the τ–value of the underlying assignment game. Thus, the τ–value now appears as the result of a negotiation process between all traders.

Other core allocations may be reached as a Nash solution or a Kalai-Smorodinsky solution of this bargaining problem. In Section 5 we characterize the subset of core allocations that can be reached as a Kalai-Smorodinsky solution, for a suitable choice of the disagreement point. All allocations in this set, which we name the half–optimal core section, can also be obtained by a negotiation procedure *à la* Nash. Once again, if we want the above selected disagreement points to be admissible, we must restrict ourselves to assignment problems where all agents are active. In Section 6 we show that similar results can be obtained for markets with some nonactive agents.

2 An example

To illustrate how we will build this bargaining problem, let us consider the following example from Shapley and Shubik (1972). Let $M = \{1, 2, 3\}$ be the set of buyers (rows), $M' = \{4, 5, 6\}$ be the set of sellers (columns) and let

$$A = \begin{pmatrix} 5 & 8 & 2 \\ 7 & 9 & 6 \\ 2 & 3 & 0 \end{pmatrix}$$

be the assignment matrix, where a_{ij} is the joint profit of the pair $(i, j) \in M \times M'$. Notice there is only one optimal matching, which is $\mu = \{(1, 5), (2, 6), (3, 4)\}$, and the optimal profit is 16. Instead of assuming that each optimally matched pair engages in a bargaining process to divide their joint profit a_{ij}, let us assume that the whole set of agents enters a negotiation process to allocate the optimal profit. Let us take the origin as a disagreement point, which means that if the agents do not reach an agreement they receive nothing, and then consider the bargaining problem $(S_A, 0)$, where the feasible set is

$$S_A = \left\{ x \in \mathbb{R}^6_+ \,|x_1 + x_5 \le 8,\ x_2 + x_6 \le 6,\ x_3 + x_4 \le 2 \right\} .$$

This means that the set of outcomes they can agree on is determined by the amount each agent can afford by the optimal matching. The Nash solution to this problem is $N(S_A, 0) = (4, 3, 1; 1, 4, 3)$ and, after computing the ideal point, $a = (8, 6, 2; 2, 8, 6)$, it follows that the Kalai-Smorodinsky solution to this problem coincides with the Nash solution.

Nevertheless, the allocation $(4,3,1;1,4,3)$ is not stable, since $x_2 + x_4 = 4 < 7 = a_{24}$. This is an important drawback for a cooperative solution to an assignment market, where the existence of stable allocations is well known.

From a cooperative point of view, the starting point for negotiation may be other than the origin. Each agent k can compute his marginal contribution b_k^A to the market. The marginal contribution of a player k is the difference between the total profit of the market and the maximum profit that could be attained if agent k withdrew from the market. For instance, if buyer 1 were not present, the assignment matrix would be reduced to $\begin{pmatrix} 7 & 9 & 6 \\ 2 & 3 & 0 \end{pmatrix}$ and the optimal profit would be 11. Thus the marginal contribution of agent 1 is $b_1^A = 16 - 11 = 5$. Similarly, $b_2^A = 6$, $b_3^A = 1$, $b_4^A = 2$, $b_5^A = 5$ and $b_6^A = 1$. The marginal contribution of an agent is known to be the maximum payoff he can obtain in any stable allocation of the assignment market.

Now, a vector of threats *à la* Rochford, or a reference point for negotiation, can be defined where each agent calculates how much he could obtain in another partnership after his potential partner is paid his marginal contribution. This defines the minimal rights vector $\underline{d}^A$, where $\underline{d}_i^A = \max_{j \in M'} \{a_{ij} - b_j^A\}$, for all $i \in M$, and similarly for sellers. In the example above, the minimal rights vector is $\underline{d}^A = (3,5,0;1,3,0)$. In the bargaining problem $(S_A, \underline{d}^A)$, the Nash and Kalai-Smorodinsky solutions coincide again, $N(S_A, \underline{d}^A) = KS(S_A, \underline{d}^A) = (4, 5.5, 0.5; 1.5, 4, 0.5)$, and now this allocation belongs to the core of the assignment game.

A question that arises quite naturally at this point is this: which other core allocations can be achieved as Kalai-Smorodinsky solutions or Nash solutions of the assignment problem, for some choice of the disagreement point?

We will prove that, if we restrict ourselves to those disagreement points in between the origin and the minimal rights vector, $0 \leq d \leq \underline{d}^A$, the subset of core allocations that can be achieved as a Kalai-Smorodinsky solution of the assignment problem is always nonempty and consists of those core allocations where each agent gets at least one half of his marginal contribution.

In our example, this subset of core allocations coincides with the segment $[A,B]$, where $A=(3.5,5.5,0.5;1.5,4.5,0.5)$ and $B=(4.5,5.5,0.5;1.5,3.5,0.5)$. In Figure 1, we reproduce the core of this assignment game as in Shapley and Shubik (1972), and represent the above segment.

To see that the segment $[A,B]$ is the subset of core allocations where each agent is paid at least one half of his marginal contribution, recall first that the marginal contributions of the agents are $b^A=(5,6,1;2,5,1)$. Then, if x belongs to the core and each agent is paid at least one half of his marginal contribution, $x_1 \geq 2.5$, $x_2 \geq 3$, $x_3 \geq 0.5$, $x_4 \geq 1$, $x_5 \geq 2.5$ and $x_6 \geq 0.5$ must hold. Looking at the picture of the core, we see that all but two of these constraints are satisfied by all core allocations. It is enough to check that the six extreme core allocations satisfy $x_1 \geq 2.5$, $x_2 \geq 3$, $x_4 \geq 1$, and $x_5 \geq 2.5$.

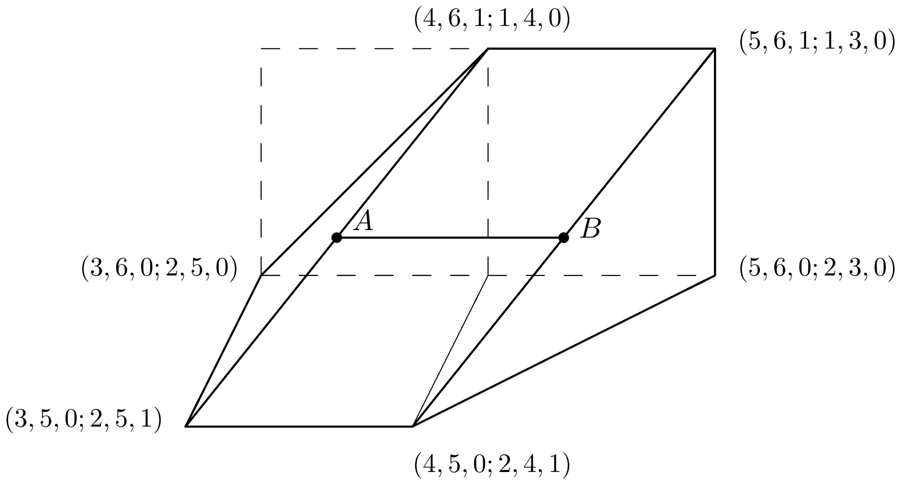

Figure 1:

We now need to establish which core allocations x satisfy the two additional constraints $x_3 \geq 0.5$ and $x_6 \geq 0.5$. Since the joint core payoff of all optimally matched pairs is fixed, from $x_3+x_4=a_{34}=2$ and $x_3 \geq 0.5$ we get $x_4 \leq 1.5$, and from $x_2+x_6=a_{26}=6$ and

$x_6 \geq 0.5$ it follows that $x_2 \leq 5.5$. Now, an allocation in the core ($x_2 + x_4 \geq 7$) and with the constraints $x_2 \leq 5.5$ and $x_4 \leq 1.5$ must satisfy $x_2 = 5.5$ and $x_4 = 1.5$ which, making use again of the core constraints $x_3 + x_4 = a_{34} = 2$ and $x_2 + x_6 = a_{26} = 6$, implies $x_3 = 0.5$ and $x_6 = 0.5$.

Therefore, $x = (x_1, 5.5, 0.5; 1.5, x_5, 0.5)$, where, because of the core constraints, $x_1 + x_4 \geq 5$, $x_1 + x_6 \geq 2$, $x_2 + x_5 \geq 9$, $x_3 + x_5 \geq 3$ and $x_1 + x_5 = 8$. All these imply that $3.5 \leq x_1 \leq 4.5$ while $x_5 = 8 - x_1$, and thus x is any point in the segment $[A, B]$.

In this case, this segment is also the set of core allocations that can be achieved as a Nash solution to the assignment problem.

The question now is to establish the extent to which what we have observed in this example depends on the fact that it has only one optimal matching. When several optimal matchings exist we could choose one of them and proceed as before. However, since there seems to be no reason to discriminate between them, we have opted to include in the feasible set the constraints imposed by all optimal matchings. In the example above, if we raise a_{25} from 9 to 11, we get an assignment market with two optimal matchings, which are $\mu_1 = \{(1,5),(2,6),(3,4)\}$ and $\mu_2 = \{(1,4),(2,5),(3,6)\}$. Then, the feasible set would be

$$S_A = \left\{ x \in \mathbb{R}^6_+ \;\middle|\; \begin{array}{ll} x_1 + x_5 \leq 8\,, & x_1 + x_4 \leq 5 \\ x_2 + x_6 \leq 6\,, & x_2 + x_5 \leq 11 \\ x_3 + x_4 \leq 2\,, & x_3 + x_6 \leq 0 \end{array} \right\}$$

The aim of this paper is to analyze this bargaining problem and to determine which stable allocations of the assignment market can be obtained as a bargained solution, be it a Nash or a Kalai-Smorodinsky solution.

3 Preliminaries: the assignment game

If M and M' are, respectively, the sets of rows and columns of a non-negative matrix A, usually representing agents in a two–sided market, we will denote by n the cardinality of

$M \cup M'$, $n = m + m'$, where m and m' are the cardinalities of M and M' respectively. *The assignment problem* (M, M', A) involves looking for an optimal matching between the two sides of the market. Then, a *matching* (or assignment) for A is a subset μ of $M \times M'$ such that each $k \in M \cup M'$ belongs at most to one pair in μ. We will denote by $\mathcal{M}(A)$ or $\mathcal{M}(M, M')$ the set of matchings of A. The matrix entry a_{ij} represents the profit obtained by the mixed–pair (i, j) when matched together. We say *a matching* μ *is optimal* if it is maximal with respect to inclusion and for all $\mu' \in \mathcal{M}(M, M')$, $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$, and will denote by $\mathcal{M}^*(A)$ the set of optimal matchings. Cooperative game theory can be used to analyze how to allocate the profit obtained by an optimal matching between the agents.

Assignment games were introduced by Shapley and Shubik (1972) as a cooperative model for a two–sided market with transferable utility. Given an assignment problem (M, M', A), where M is the set of buyers and M' is the set of sellers, the player set is $M \cup M'$, and the characteristic function will be denoted by w_A. The profits of mixed–pair coalitions, $\{i, j\}$ where $i \in M$ and $j \in M'$, are given by the non-negative matrix A: $w_A(\{i, j\}) = a_{ij} \geq 0$. This matrix A also determines the worth of any other coalition $S \cup T$, where $S \subseteq M$ and $T \subseteq M'$, in the following way: $w_A(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S, T)\}$, $\mathcal{M}(S, T)$ being the set of matchings between S and T. It will be assumed as usual that a coalition formed only by sellers or only by buyers has worth zero. For all $i \in M$ optimally matched by μ, we will denote by $\mu(i)$ the agent $j \in M'$ such that $(i, j) \in \mu$. Similarly, i could be denoted by $\mu^{-1}(j)$. Moreover, we say a buyer $i \in M$ is not assigned by μ if $(i, j) \notin \mu$ for all $j \in M'$ (and similarly for sellers).

Shapley and Shubik proved that the core of the assignment game $(M \cup M', w_A)$ is non-empty and coincides with the set of stable outcomes. This means that the core can be represented in terms of any optimal matching μ of $M \cup M'$ by

$$(1) \quad C(w_A) = \left\{ (u,v) \in \mathbb{R}^M \times \mathbb{R}^{M'} \;\middle|\; \begin{array}{l} u_i \geq 0, \text{ for all } i \in M\,;\ v_j \geq 0, \text{ for all } j \in M'\,, \\ u_i + v_j = a_{ij} \text{ if } (i,j) \in \mu\,, \\ u_i + v_j \geq a_{ij} \text{ if } (i,j) \notin \mu\,, \\ u_i = 0 \text{ if } i \text{ not assigned by } \mu\,, \\ v_j = 0 \text{ if } j \text{ not assigned by } \mu \end{array} \right\}.$$

Moreover, the core has a lattice structure with two special extreme core allocations: the *buyers-optimal core allocation*, $(\overline{u}^A, \underline{v}^A)$, where each buyer attains his maximum core payoff, and the *sellers-optimal core allocation*, $(\underline{u}^A, \overline{v}^A)$, where each seller does. Notice that when agents on one side of the market obtain their maximum core payoff, the agents on the opposite side obtain their minimum core payoff, as the joint payoff of an optimally matched pair is fixed: $u_i + v_j = a_{ij}$ for all $(u,v) \in C(w_A)$ if $(i,j) \in \mu$.

From Demange (1982) and Leonard (1983) we know that the maximum core payoff of any player coincides with his marginal contribution:

$$(2) \qquad \overline{u}_i^A = w_A(N) - w_A(N \setminus \{i\}) \text{ and } \overline{v}_j^A = w_A(N) - w_A(N \setminus \{j\})\,.$$

Let us denote by $b^A \in \mathbb{R}^M \times \mathbb{R}^{M'}$ the vector of marginal contributions. From (2) and the description of the core (1) the minimum core payoff of buyer i is

$$(3) \qquad \underline{u}_i^A = a_{i\mu(i)} - b_{\mu(i)}^A \text{ for all } \mu \in \mathcal{M}^*(A)\,,$$

while the minimum core payoff of seller j is

$$(4) \qquad \underline{v}_j^A = a_{\mu^{-1}(j)j} - b_{\mu^{-1}(j)}^A \text{ for all } \mu \in \mathcal{M}^*(A)\,.$$

4 The assignment problem as a bargaining problem

The abstract formulation of a bargaining problem is as follows (see for instance Thomson, 1994). Let $N = \{1, 2, \ldots, n\}$ be a set of agents. A *n–person bargaining problem* is a pair (S, d) where S is a convex, bounded and closed subset of $\mathbb{R}^n$ and d a point of S such that there exists $x \in S$ with $x_i > d_i$ for all $i \in N$. Then, S is the set of all feasible utility allocations the agents may reach after a bargaining process. If there is no agreement, each agent $i \in N$ obtains the level of utility d_i; this is why d is usually called *the disagreement point*. More generally (see for instance Chun and Thomson, 1992) point d can be understood as a reference point from which the agents find it natural to measure their utility gains in order to evaluate a proposed compromise. Often attention is focused on the sets S that are *d–comprehensive*, that is to say, if $x \in S$ and $x \geq y \geq d$, then $y \in S$.

A bargaining solution is a function that assigns to each bargaining problem (S, d) an element of S. Two well-known examples are the Nash bargaining solution (Nash, 1950) and the Kalai–Smorodinsky solution (Kalai and Smorodinsky, 1975). *The Nash solution*, $N(S, d)$, is the point where the function $\prod_{i=1}^{n}(x_i - d_i)$ attains its maximum in $\{x \in S \mid x_i \geq d_i\}$.

The Kalai–Smorodinsky solution, $KS(S, d)$, allocates the utilities that exceed the disagreement point in a way proportional to each agent's expectations. For all $i \in N$, let $a_i(S, d)$ be the maximum utility level agent i can attain subject to the constraint that no agent should receive less than his coordinate in the disagreement point, that is to say $a_i(S, d) = \max\{x_i \mid x \in S, x \geq d\}$. Then $KS(S, d)$ is the maximal point of S on the segment with extreme points d and the *ideal point* $a(S, d)$.

An element $x \in S$ is *Pareto optimal* (PO) if there does not exist $x' \in S$, $x \neq x'$, such that $x' \geq x$, which means $x'_k \geq x_k$ for all $k \in N$. An element $x \in S$ is *weak Pareto optimal* (WPO) if there does not exist $x' \in S$ such that $x' > x$, which means $x'_k > x_k$ for all $k \in N$. The Nash solution to a bargaining problem is always PO, while the Kalai–Smorodinsky solution is only WPO in general.

We now define a bargaining problem associated to each assignment problem (M, M', A). Let us then propose a feasible set S_A and a disagreement point.

The set S_A is defined taking into account only monetary transfers between pairs that are optimally matched by some $\mu \in \mathcal{M}^*(A)$:

$$(5) \quad S_A = \left\{ x \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'} \;\middle|\; \begin{array}{l} x_i + x_j \le a_{ij} \text{ for all } (i,j) \in M \times M' \text{ such that} \\ \exists \mu \in \mathcal{M}^*(A) \text{ and } (i,j) \in \mu \end{array} \right\}.$$

Recall that $\mathbb{R}_+^M$ denotes the set of nonnegative real vectors indexed by the set M (and similarly for $\mathbb{R}_+^{M'}$). This definition of the feasible set means that the agents are bargaining over what they can afford by any optimal matching.

Notice that S_A is convex, closed and 0–comprehensive, but not bounded in general. Then, in order to have a well-defined bargaining problem, our feasible set must also be bounded, and this is true if and only if every agent is matched at least by some optimal matching. If only one optimal matching exists, S_A is bounded if and only if A is a square matrix.

The set of Pareto optimal elements in S_A is

$$(6) \quad PO(S_A) = \left\{ x \in S_A \;\middle|\; \begin{array}{l} \text{for all } k \in M \text{, } \exists \mu \in \mathcal{M}^*(A) \text{ and } x_k + x_{\mu(k)} = a_{k\mu(k)} \text{,} \\ \text{for all } k \in M' \text{, } \exists \mu \in \mathcal{M}^*(A) \text{ and } x_{\mu^{-1}(k)} + x_k = a_{\mu^{-1}(k)k} \end{array} \right\}.$$

It is straightforward to notice that the core of the assignment game $(M \cup M', w_A)$ is contained in the feasible set S_A. Therefore, when computing a solution for a bargaining problem with feasible set S_A we may analyze if it belongs to the core. Notice that because of the definition of the feasible set, there are WPO elements which are not PO.

In order to have a well-defined bargaining problem, we must take as a disagreement point any vector $d \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ such that

$$(7) \quad 0 \le d \in S_A \text{ and there exists } x \in S_A \text{ such that } x_k > d_k \text{ for all } k \in M \cup M'.$$

This assumption on the disagreement point avoids degenerated cases where only some of the agents can gain from the agreement. A vector $d \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ for which condition (7) holds will be called an *admissible disagreement point.*

In order to guarantee the existence of some admissible disagreement point, we need that $a_{ij} > 0$ for all $(i,j) \in \mu$ and all $\mu \in \mathcal{M}^*(A)$. If this condition holds in our assignment matrix, then at least the origin is admissible.

If d is an admissible disagreement point, then (S_A, d) is a *bargaining assignment problem* and we will denote by $N(S_A, d)$ and $KS(S_A, d)$ the Nash and Kalai–Smorodinsky solutions to this problem.

To illustrate all these definitions, let us consider the following example.

Example 1 Let us consider $M = \{1, 2, 3\}$, $M' = \{4, 5, 6\}$ and the assignment matrix

$$A = \begin{pmatrix} 8 & 9 & 5 \\ 1 & 4 & 2 \\ 7 & 0 & 6 \end{pmatrix}.$$

Notice that A has two optimal matchings which are $\mu_1 = \{(1, 4), (2, 5), (3, 6)\}$ and $\mu_2 = \{(1, 5), (2, 6), (3, 4)\}$.

Let us take the origin as a disagreement point, which means that, if the agents do not reach an agreement, they receive nothing, and then consider the bargaining problem $(S_A, 0)$, where the feasible set is

$$S_A = \left\{ x \in \mathbb{R}^6_+ \;\middle|\; \begin{array}{ll} x_1 + x_4 \leq 8\,, & x_1 + x_5 \leq 9 \\ x_2 + x_5 \leq 4\,, & x_2 + x_6 \leq 2 \\ x_3 + x_6 \leq 6\,, & x_3 + x_4 \leq 7 \end{array} \right\}.$$

When computing the Nash solution to this problem we obtain

$$N(S_A, 0) = (5.5166, 0.8453, 4.5166; 2.4834, 3.155, 1.1547)\,,$$

which does not belong to the core of the underlying assignment game, since $x_1 + x_5 < 9$.

On the other hand, the maximum feasible payoff to player 1 is $a_1 = \max_{x \in S_A} x_1 = \min\{a_{14}, a_{15}\} = 8$ and proceeding in the same way with the remaining agents we obtain the ideal point of $(S_A, 0)$ which is $a = (8, 2, 6; 7, 4, 2)$. Then the Kalai-Smorodinsky solution is the maximal feasible point in the segment $\lambda(8, 2, 6; 7, 4, 2)$, with $0 \leq \lambda \leq 1$, and this is $KS(S_A, 0) = (4, 1, 3; 3.5, 2, 1)$. This shows that the Nash and Kalai-Smorodinsky solutions need not coincide in the bargaining assignment problem and, moreover, the Kalai-

Smorodinsky solution need not be PO, just WPO. To check this last remark notice that only the constraint $x_2 + x_6 \leq 2$ is tight at $KS(S_A, 0)$.

Nevertheless, the next theorem shows that, when A has only one optimal matching, and for any selection of the admissible disagreement point, it turns out that the two solutions coincide. Notice that, generically, matrix A will have only one optimal matching. In other words, by performing an infinitesimal variation of some matrix entry, any matrix A will satisfy the assumptions of the following theorem.

Theorem 2 *Let* (M, M', A) *be an assignment problem with the same number of buyers as sellers and with only one optimal matching* μ, *and let* $d \in S_A$ *be an admissible disagreement point. Then, for all* $(i, j) \in \mu$,

$$\begin{aligned} N_i(S_A, d) &= KS_i(S_A, d) = \frac{a_{ij} - d_j + d_i}{2} \quad \textit{and} \\ N_j(S_A, d) &= KS_j(S_A, d) = \frac{a_{ij} - d_i + d_j}{2}. \end{aligned}$$

Proof: Notice first that the maximum of $\prod_{i \in M}(u_i - d_i) \prod_{j \in M'}(v_j - d_j)$ over $\{(u, v) \in S_A \mid (u, v) \geq d)\}$ is attained in a point that strictly dominates the disagreement point, as such points exist in our feasible set. Because the Nash solution is PO and taking into account that there is only one optimal matching, from equation (6), $u_i + v_j = a_{ij}$ for all $(i, j) \in \mu$ and all $(u, v) \in PO(S_A)$, we get $N(S_A, d) = (u^*, v^*)$ where u^* is the maximum of $h(u) = \prod_{i \in M}(-u_i^2 + (a_{ij} - d_j + d_i)u_i - d_i(a_{ij} - d_j))$ on the domain $d_i \leq u_i \leq a_{ij} - d_j$, where $(i, j) \in \mu$, and $v_j^* = a_{ij} - u_i^*$.

After some straightforward computations, for all $(i, j) \in \mu$, $u_i^* = \frac{a_{ij} - d_j + d_i}{2}$ and then $v_j^* = \frac{a_{ij} - d_i + d_j}{2}$. Notice that, since $d_i + d_j \leq a_{ij}$ for all $(i, j) \in \mu$, then $(u^*, v^*) \in S_A$.

To compute the Kalai–Smorodinsky solution, notice that, if $(i, j) \in \mu$, then the ideal point payoff for player i is $a_i(S_A, d) = \max\{x_i \mid x \in S_A,\, x \geq d\} = a_{ij} - d_j$ while for player j it is $a_j(S_A, d) = \max\{x_j \mid x \in S_A,\, x \geq d\} = a_{ij} - d_i$. Then, the midpoint between the disagreement point and the ideal point is, for all $(i,j)\in\mu$,

$$\left(\frac{1}{2}d+\frac{1}{2}a\right)_i=\frac{a_{ij}+d_i-d_j}{2}\quad\text{and}\quad\left(\frac{1}{2}d+\frac{1}{2}a\right)_j=\frac{a_{ij}+d_j-d_i}{2}\,.$$

This point belongs to the feasible set and is Pareto optimal which implies it is the maximal point of S_A in the segment $[d,a]$, and thus coincides with the Kalai–Smorodinsky solution of the problem (S_A,d). □

The formulae in Theorem 2 can be used to compute the Nash (and the Kalai-Smorodinsky) solution of the introductory problem $(S_A,0)$ in section 2.

We now propose a particular disagreement point for negotiation in an assignment market. Since these markets are known to have a nonempty core, agents would expect to finish the negotiation process in a stable allocation and no agent would accept a cooperative solution which allocated him less than his minimum core payoff. Let us then introduce the *payoff vector of minimal rights*, $\underline{d}^A=(\underline{u}^A,\underline{v}^A)\in\mathbb{R}^M_+\times\mathbb{R}^{M'}_+$. The minimal rights payoff vector can be obtained as a threat point *à la* Rochford, since the minimal rights payoff of an agent is the maximum he can obtain from the trade with any agent in the opposite side of the market after conceding him his marginal contribution:

$$\underline{d}_i^A=\max_{j\in M'}\{a_{ij}-b_j^A\}\,,\ \text{for all } i\in M \ \text{and}\ \underline{d}_j^A=\max_{i\in M}\{a_{ij}-b_i^A\}\,,\ \text{for all } j\in M'\,.$$

To see this, recall from (3) that, for all $i\in M$ and all $\mu\in\mathcal{M}^*(A)$, $\underline{u}_i=a_{ij}-\overline{v}_j$ if $(i,j)\in\mu$, while $\underline{u}_i\geq a_{ij'}-\overline{v}_{j'}$ for all $j'\in M'$ since $(\underline{u},\overline{v})$ is a core allocation.

This reference point $\underline{d}^A$ can also be endogenously generated from a particular set of claims by a procedure inspired in Herrero (1998). Assume each buyer $i\in M$ claims his marginal contribution b_i^A, since it is known to be achieved by a stable allocation of the assignment market, and then concedes to his optimally matched seller $\mu(i)\in M'$ the difference $a_{i\mu(i)}-b_i^A$, since such a concession is compatible with his claim. If the sellers proceed in the same way, by these concessions we obtain the minimal rights payoff vector, regardless of the optimal matching taken into account to make the above concessions.

In order to check that $(S_A, \underline{d}^A)$ is a well-defined bargaining problem, notice first that $\underline{d}^A$ belongs to the feasible set S_A: for any $\mu \in \mathcal{M}^*(A)$ and for all $(i,j) \in \mu$, $\underline{u}_i^A + \underline{v}_j^A \leq \underline{u}_i^A + \overline{v}_j^A = a_{ij}$, since $(\underline{u}^A, \overline{v}^A) \in C(w_A)$. But $\underline{d}^A$ may not be an admissible disagreement point for S_A. Notice that there may exist agents that cannot obtain a profit from negotiation, since their marginal contribution equals their minimal rights payoff.

Definition 3 *Let (M, M', A) be an assignment problem. An agent $k \in M \cup M'$ is active if and only if $\underline{d}_k^A < b_k^A$ (i.e. $\underline{u}_k^A < \overline{u}_k^A$ if $k \in M$ and $\underline{v}_k^A < \overline{v}_k^A$ if $k \in M'$).*

All those agents k which are not optimally matched by some optimal matching $\mu \in \mathcal{M}^*(A)$ are nonactive, since $\underline{d}_k^A = b_k^A = 0$. But there may also exist some nonactive agents which are matched in all $\mu \in \mathcal{M}^*(A)$. To see this, let us consider $M = \{1,2,3\}$, $M' = \{4,5,6\}$ and matrix

$$A_3 = \begin{pmatrix} 9 & 8 & 6 \\ 8 & 8 & 5 \\ 3 & 2 & 0 \end{pmatrix}, \tag{8}$$

and notice that there exist two optimal matchings, placed one in each diagonal. The vector of marginal contributions, $b^A = (b_1^A, b_2^A, b_3^A; b_4^A, b_5^A, b_6^A) = (6,6,0;3,3,0)$, is easily obtained. Then, by means of one of the optimal matchings, we get $\underline{d}_1^A = a_{14} - b_4^A = 6$, $\underline{d}_2^A = a_{25} - b_5^A = 5$, $\underline{d}_3^A = a_{36} - b_6^A = 0$, $\underline{d}_4^A = a_{14} - b_1^A = 3$, $\underline{d}_5^A = a_{25} - b_2^A = 2$ and $\underline{d}_6^A = a_{36} - b_3^A = 0$. By comparing the vector of marginal contributions and the payoff vector of minimal rights, $\underline{d}^A = (6,5,0;3,2,0)$ we conclude that there exists only one active pair, formed by buyer 2 and seller 5. Notice also that agents 1, 3, 4 and 6 are nonactive although each of them is optimally paired in all optimal matchings of the market.

It is easy to determine which agents are active even in those assignment markets with many agents. Once we have an optimal matching, and thus the efficiency level $w_A(M \cup M')$, the buyers-optimal core allocation $(\overline{u}^A, \underline{v}^A)$, is the solution of the linear problem

$$\begin{array}{ll} \max & \sum_{i\in M} x_i \\ \text{subject to} & x_i + x_j \geq a_{ij}\,, \text{ for all } (i,j)\in M\times M' \\ & \sum_{k\in M\cup M'} x_k = w_A(M\cup M')\,, \\ & x_i \geq 0 \text{ for all } i\in M\,,\ x_j\geq 0 \text{ for all } j \in M'\,, \end{array}$$

since all buyers attain their maximum core payoff, which is their marginal contribution, in the same core allocation. A similar linear program leads to the sellers-optimal core allocation $(\underline{u}^A, \overline{v}^A)$.

Lemma 4 *Let (M, M', A) be an assignment problem. The following statements are equivalent:*

1. *The feasible set S_A defined in (5) is bounded and the minimal rights payoff vector, $\underline{d}^A = (\underline{u}^A, \underline{v}^A)$, is an admissible disagreement point for S_A.*
2. *All agents in $M \cup M'$ are active.*

Proof: If $\underline{d}^A$ is admissible, there exists $x \in S_A$ such that $x_k > \underline{d}^A_k$ for all $k \in M \cup M'$. Since S_A is bounded, all agents are optimally matched by some $\mu \in \mathcal{M}^*(A)$. For all $i \in M$, take $\mu \in \mathcal{M}^*(A)$ such that $(i,j) \in \mu$ for some $j \in M'$. From $x_i + x_j \leq a_{ij}$, $x_i > \underline{d}^A_i$ and $x_j > \underline{d}^A_j$, we get $\underline{d}^A_i < x_i \leq a_{ij} - x_j < a_{ij} - \underline{d}^A_j = b^A_i$. The same argument applies to $j \in M'$.

If all agents are active, there cannot be unmatched players, since they receive zero payoff in all stable allocations. Then, the feasible set is bounded. Moreover, $x = \frac{1}{2}(\underline{u}^A, \overline{v}^A) + \frac{1}{2}(\overline{u}^A, \underline{v}^A)$ belongs to S_A, since it is a core allocation, and $x_i > \underline{u}^A_i = \underline{d}^A_i$ for all $i \in M$, while $x_j > \underline{v}^A_j = \underline{d}^A_j$ for all $j \in M'$. □

For this proposed disagreement point, the related bargaining problem $(S_A, \underline{d}^A)$ will be named $\underline{d}^A$*–bargaining assignment problem.* We will show that the Nash solution to this problem coincides with the Kalai-Smorodinsky solution and is always a core allocation of the underlying assignment game.

Theorem 5 *Let* (M, M', A) *be an assignment problem where all agents are active. The Nash and Kalai-Smorodinsky solutions of* $(S_A, \underline{d}^A)$ *satisfy*

$$N_i(S_A, \underline{d}^A) = KS_i(S_A, \underline{d}^A) = \tfrac{\underline{u}_i^A + \overline{u}_i^A}{2}\,, \quad \textit{for all } i \in M\,,$$
$$N_j(S_A, \underline{d}^A) = KS_j(S_A, \underline{d}^A) = \tfrac{\underline{v}_j^A + \overline{v}_j^A}{2}\,, \quad \textit{for all } j \in M'\,,$$

and are core allocations.

Proof: Let us fix $\mu \in \mathcal{M}^*(A)$ and consider the auxiliary bargaining problem $(S_A^\mu, \underline{d}^A)$ where

$$S_A^\mu = \{x \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'} \mid x_i + x_j \le a_{ij} \text{ for all } (i,j) \in \mu\}\,.$$

Notice that since all agents are active both the problems $(S_A, \underline{d}^A)$ and $(S_A^\mu, \underline{d}^A)$ are well defined, and whenever A has only one optimal matching, $S_A^\mu = S_A$.

Now the same proof of Theorem 2 applies to the problem $(S_A^\mu, \underline{d}^A)$ to see that

$$N_i(S_A^\mu, \underline{d}^A) = KS_i(S_A^\mu, \underline{d}^A) = \tfrac{a_{i\mu(i)} - \underline{d}_{\mu(i)}^A + \underline{d}_i^A}{2} = \tfrac{\underline{u}_i^A + \overline{u}_i^A}{2}\,, \quad \text{for all } i \in M\,,$$
$$N_j(S_A^\mu, \underline{d}^A) = KS_j(S_A^\mu, \underline{d}^A) = \tfrac{a_{\mu^{-1}(j)j} - \underline{d}_{\mu^{-1}(j)}^A + \underline{d}_j^A}{2} = \tfrac{\underline{v}_j^A + \overline{v}_j^A}{2}\,, \quad \text{for all } j \in M'\,,$$

where the last equalities in each row follow taking into account that $(\overline{u}^A, \underline{v}^A)$ and $(\underline{u}^A, \overline{v}^A)$ are core allocations and thus $\overline{u}_i^A + \underline{v}_j^A = \underline{u}_i^A + \overline{v}_j^A = a_{ij}$ for all $(i,j) \in \mu$. Moreover, from the convexity of the core it follows that $N(S_A^\mu, \underline{d}^A) \in C(w_A) \subseteq S_A$.

Now, since $S_A^\mu \supseteq S_A$ and $N(S_A^\mu, \underline{d}^A) \in S_A$, from the independence of irrelevant alternatives of the Nash solution, we get $N(S_A, \underline{d}^A) = N(S_A^\mu, \underline{d}^A)$.

It only remains to prove that this is also the Kalai-Smorodinsky solution of $(S_A, \underline{d}^A)$. This follows easily since the ideal point for $(S_A, \underline{d}^A)$ is $a = (\overline{u}^A, \overline{v}^A)$ and $N(S_A, \underline{d}^A) = \frac{1}{2}a + \frac{1}{2}\underline{d}^A$ is Pareto optimal. □

The above theorem can be used to obtain the Nash (and the Kalai-Smorodinsky) solution to the introductory problem $(S_A, \underline{d}^A)$ in section 2.

The midpoint between the buyers-optimal and the sellers-optimal core allocations had already been proposed by Thompson (1981) as a point solution for the assignment game, with the name of *fair solution*. More recently, it has been proved in Núñez and Rafels

(2002) that this point coincides with the τ–value of the assignment game. The τ–value is a well-known point solution for cooperative TU games (Tijs, 1981). Now, as an immediate consequence of Theorem 5, we obtain that the Nash solution of the $\underline{d}^A$-bargaining assignment problem coincides not only with the Kalai–Smorodinsky solution, but also with the τ–value of the underlying assignment game.

5 Bargained stable allocations

Which other stable allocations of an assignment market can be selected that are supported by a bargaining procedure? This is a natural question once we have seen that there exists at least one, the τ-value.

Let us assume that each agent in the market (M, M', A) claims at least one half of his or her maximum core payoff, and let us look for stable allocations that concede these claims.

Definition 6 *Let $(M \cup M', w_A)$ be an assignment game where all agents are active. The half–optimal core of w_A is*

$$\mathcal{HC}(w_A) = \left\{ (u,v) \in C(w_A) \;\middle|\; \begin{array}{l} u_i \geq \frac{\overline{u}_i^A}{2} \textit{ for all } i \in M \\ v_j \geq \frac{\overline{v}_j^A}{2} \textit{ for all } j \in M' \end{array} \right\}.$$

See Section 6 for the definition of the half optimal core when there are nonactive agents in the market.

Notice that $\mathcal{HC}(w_A)$ is a convex and compact subset of the core. Moreover, since $\tau(w_A)_i = \frac{1}{2}\overline{u}_i^A + \frac{1}{2}\underline{u}_i^A \geq \frac{1}{2}\overline{u}_i^A$, for all $i \in M$, and $\tau(w_A)_j = \frac{1}{2}\overline{v}_j^A + \frac{1}{2}\underline{v}_j^A \geq \frac{1}{2}\overline{v}_j^A$ for all $j \in M'$, the half-optimal core always contains the τ-value, and this implies that $\mathcal{HC}(w_A)$ is always nonempty. From Definition 6 it also follows that all assignment games with the same core have the same half–optimal core.

In Example 7 below the half-optimal core reduces to the τ-value. This shows that if all agents claimed more than one half of their maximum core payoffs, those claims could not be satisfied inside the core. Then, one half of the maximum core payoff is the most we can guarantee for all agents at the same time without losing stability.

Before looking at the following examples, let us recall that once we fix an optimal matching $\mu \in \mathcal{M}^*(A)$, the core of the assignment game is determined by its projection on the space of payoffs to one side of the market, let us say the buyers' side. We name this projection the *u–core* of the assignment game, and denote it by $C(w_A)_u$. Then, if we take into account that $\overline{u}_i^A = a_{i\mu(i)} - \underline{d}_{\mu(i)}^A$ and $\overline{v}_{\mu(i)}^A = a_{i\mu(i)} - \underline{d}_i^A$, and also the core constraint $u_i + v_{\mu(i)} = a_{i\mu(i)}$, the projection of $\mathcal{HC}(w_A)$ turns out to be the section of $C(w_A)_u$ with a hypercube:

$$(9) \quad \mathcal{HC}(w_A)_u = \left\{ u \in C(w_A)_u \,\middle|\, \frac{a_{i\mu(i)} - \underline{d}_{\mu(i)}^A}{2} \leq u_i \leq \frac{a_{i\mu(i)} + \underline{d}_i^A}{2} \text{ for all } i \in M \right\}.$$

It can be seen from the above expression that the projection of $\mathcal{HC}(w_A)$ on the space of the buyers' payoffs is a 45^o–degree lattice (see Quint, 1991) and thus it turns out to be the u–core of another assignment game. Therefore, it is possible to find an assignment matrix B such that $\mathcal{HC}(w_A)_u = C(w_B)_u$. Since the extreme core allocations of an assignment game have been characterized in Núñez and Rafels (2003), a description of the extreme points of $\mathcal{HC}(w_A)$ could be given.

Example 7 Let us consider the assignment market defined by $A = \begin{pmatrix} 6 & 7 \\ 1 & 3 \end{pmatrix}$. Notice that A has only one optimal matching, which is $\mu = \{(1,3),(2,4)\}$. The core of the underlying assignment game is

$$C(w_A) = convex\{(4,0;2,3),(5,0;1,3),(6,1;0,2),(6,2;0,1)\}.$$

Since $(\overline{u}^A, \overline{v}^A) = (6,2;2,3)$, $\mathcal{HC}(w_A)$ is the subset of core allocations such that $3 \leq u_1 \leq 5$ and $1 \leq u_2 \leq 1.5$. These constraints determine, in the space of the buyers' payoffs, a rectangle which only contains one core element, as shown in Figure 2.

In this example both the nucleolus and the kernel of the assignment game (the set of SPB allocations in Rochford, 1984) are in the relative interior of the core; therefore neither is included in the half-optimal core.

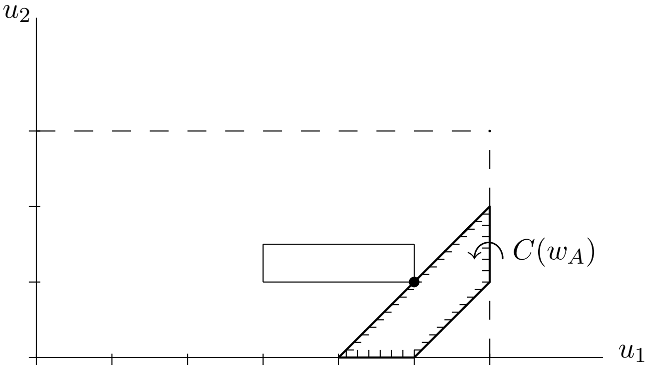

Figure 2:

In the next example, an assignment market is given with a half-optimal core not reduced to a point.

Example 8 Let us consider the assignment market defined by $A = \begin{pmatrix} 6 & 4 \\ 1 & 3 \end{pmatrix}$ which has only one optimal matching while the core is

$$C(w_A) = convex\{(1,0;5,3),(4,3;2,0),(5,0;1,3),(6,1;0,2),(6,3;0,0)\},$$

as shown in Figure 3.

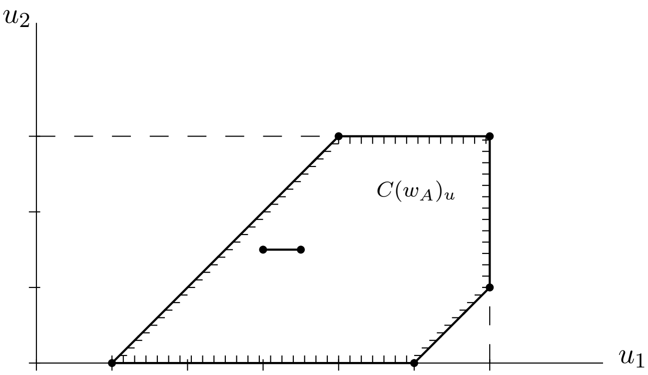

Figure 3:

The buyers-optimal core allocation is then $(6,3;0,0)$ while the sellers-optimal core allocation is $(1,0;5,3)$. Then, the set $\mathcal{HC}(w_A)$ consists of those core allocations such that $3 \leq u_1 \leq 3.5$ and $1.5 \leq u_2 \leq 1.5$, and this is the segment with extreme points (3, 1.5; 3, 1.5) and (3.5, 1.5; 2.5, 1.5)$= \tau(w_A)$.

We ask now if there are allocations in the half-optimal core, other than the τ-value, that can be supported by a bargaining procedure *à la* Nash or *à la* Kalai-Smorodinsky. Of course by choosing disagreement points close enough to the Pareto boundary, almost all core elements can be achieved; but most of these disagreement points do not seem a sensible starting point for a negotiation.

As in the previous section, we restrict ourselves to those assignment problems where $\underline{d}^A$ is an admissible disagreement point, which is equivalent to saying that all agents are active.

We now consider the following set of disagreement points, which is the 0-comprehensive closure of $\{\underline{d}^A\}$:

$$\mathcal{D}_A = \{d \in \mathbb{R}^M \times \mathbb{R}^{M'} \mid 0 \leq d_k \leq \underline{d}^A_k \text{ for all } k \in M \cup M'\}\,.$$

Notice that, since we assume $\underline{d}^A$ is admissible, all points in $\mathcal{D}_A$ are also admissible for S_A. Moreover, the points in $\mathcal{D}_A$ seem to be an acceptable starting point for negotiating a cooperative solution, since they are Pareto-dominated by all core allocations.

The set $\mathcal{D}_A$ may reduce to the point $\underline{d}^A$. This happens when the matrix is *dominant diagonal* (Solymosi and Raghavan, 2001), as this means that $\underline{d}^A = 0$. Then $\mathcal{D}_A = \{0\}$ and, by Theorem 5, $N(S_A, 0) = KS(S_A, 0) = \tau(w_A) \in C(w_A)$.

We look for core allocations that can be obtained as a Nash solution or a Kalai-Smorodinsky solution of a bargaining problem with feasible set S_A and disagreement point in $\mathcal{D}_A$. Let us denote by $\mathcal{N}(S_A, \mathcal{D}_A)$ and $\mathcal{KS}(S_A, \mathcal{D}_A)$ these two sets. Our first result (Theorem 9) states that all allocations in the half-optimal core can be reached as a Nash solution. To prove this theorem, we will need an auxiliary bargaining problem, already used in the proof of Theorem 5.

Given an assignment problem (M, M', A) and an optimal matching $\mu \in \mathcal{M}^*(A)$, we may consider the auxiliary bargaining problem (S^μ_A, d), and name it the μ–bargaining assignment problem, where

$$S^\mu_A = \{x \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+ \mid x_i + x_j \leq a_{ij} \text{ for all } (i,j) \in \mu\}$$

and $d \in \mathcal{D}_A$. Notice that, since all agents are active, this is a well-defined bargaining problem.

Let us also consider the Nash solution to this problem, for all disagreement point in $\mathcal{D}_A$:

$$\begin{array}{rcl} \mathcal{N}_{S^\mu_A} : \mathcal{D}_A & \longrightarrow & \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+ \\ d & \mapsto & \mathcal{N}_{S^\mu_A}(d) = N(S^\mu_A, d)\,. \end{array}$$

The proof used in Theorem 2 can be applied to the problem (S^μ_A, d) to show that, for $(i,j) \in \mu$,

$$\begin{array}{rcl} N_i(S^\mu_A, d) & = & KS_i(S^\mu_A, d) = \frac{a_{ij}-d_j+d_i}{2} \text{ and} \\ N_j(S^\mu_A, d) & = & KS_j(S^\mu_A, d) = \frac{a_{ij}-d_i+d_j}{2}\,, \end{array} \tag{10}$$

and thus $\mathcal{N}_{S^\mu_A}$ preserves the convex combination of disagreement points:[1]

$$\mathcal{N}_{S^\mu_A}(\lambda d + (1-\lambda)d') = \lambda \mathcal{N}_{S^\mu_A}(d) + (1-\lambda)\mathcal{N}_{S^\mu_A}(d') \quad \text{for all } 0 \leq \lambda \leq 1. \tag{11}$$

We will first determine the subset of core allocations which can be obtained as a Nash solution to S^μ_A, that is to say, $\mathcal{N}(S^\mu_A, \mathcal{D}_A) \cap C(w_A)$.

Theorem 9 *Let (M, M', A) be an assignment problem where all agents are active. Then,*

1. $\mathcal{HC}(w_A) = \mathcal{N}(S^\mu_A, \mathcal{D}_A) \cap C(w_A)$,

2. $\mathcal{HC}(w_A) \subseteq \mathcal{N}(S_A, \mathcal{D}_A) \cap C(w_A)$, *for all* $\mu \in \mathcal{M}^*(A)$.

Proof: Notice that statement 2) follows quite straightforwardly from statement 1). If $x \in \mathcal{HC}(w_A)$, then, by 1), for all $\mu \in \mathcal{M}^*(A)$ there exists $d \in \mathcal{D}_A$ such that $x = N(S^\mu_A, d)$. Since $x \in S_A \subseteq S^\mu_A$, by independence of irrelevant alternatives of the Nash solution, $N(S_A, d) = N(S^\mu_A, d)$, which implies $x \in \mathcal{N}(S_A, \mathcal{D}_A) \cap C(w_A)$.

Let us then prove statement 1). Notice first that, since both the core and the Nash solution are Pareto optimal, $C(w_A)$ and $\mathcal{N}(S^\mu_A, \mathcal{D}_A)$ are determined by their projection to the space of payoffs of one side of the market, $C(w_A)_u = \{u \in \mathbb{R}^m \mid$ there exists $v \in \mathbb{R}^{m'}$ and $(u,v) \in$

[1]This will no longer be true when we consider the Nash solution to the problem S_A.

$C(w_A)\}$ and $\mathcal{N}(S_A^\mu,\mathcal{D}_A)_u=\{u\in\mathbb{R}^m \mid$ there exists $v\in\mathbb{R}^{m'}$ and $(u,v)\in\mathcal{N}(S_A^\mu,\mathcal{D}_A)\}$. Let us prove now that

$$\mathcal{N}(S_A^\mu,\mathcal{D}_A)_u=\prod_{i\in M}\left[\frac{a_{i\mu(i)}-\underline{d}^A_{\mu(i)}}{2},\ \frac{a_{i\mu(i)}+\underline{d}^A_i}{2}\right].$$

Since $\mathcal{D}_A$ is a convex and bounded set of $\mathbb{R}^M_+\times\mathbb{R}^{M'}_+$, if $d\in\mathcal{D}_A$, then $d=\sum_{k=1}^r\lambda_k x^k$ with $\sum_{k=1}^r\lambda_k=1$ and $\lambda_k\geq 0$ for all $k\in\{1,2,\ldots,r\}$, where $x^1,x^2,\ldots,x^r$ are the extreme points of $\mathcal{D}_A$. Notice that, by the definition of $\mathcal{D}_A$, the l-coordinate of any extreme point is either 0 or $\underline{d}^A_l$. When we compute the Nash solution taking each one of these extreme points as disagreement points, from (10), we obtain $\mathcal{N}_{S_A^\mu}(x^k)_i=\frac{a_{i\mu(i)}-x^k_{\mu(i)}+x^k_i}{2}$, for all $i\in M$. Since $x^k_l\in\{0,\underline{d}^A_l\}$ for all $k\in\{1,2,\ldots,r\}$ and all $l\in M\cup M'$, it follows trivially that

$$\min_{i=1,2,\ldots,r}\mathcal{N}_{S_A^\mu}(x^k)_i=\frac{a_{i\mu(i)}-\underline{d}^A_{\mu(i)}}{2}\ \text{ and }\ \max_{i=1,2,\ldots,r}\mathcal{N}_{S_A^\mu}(x^k)_i=\frac{a_{i\mu(i)}+\underline{d}^A_i}{2}.$$

Then, from (11), we obtain that for all $i\in M$,

$$\mathcal{N}_{S_A^\mu}(d)_i=\sum_{k=1}^r\lambda_k\mathcal{N}_{S_A^\mu}(x^k)_i\in\left[\frac{a_{i\mu(i)}-\underline{d}^A_{\mu(i)}}{2},\frac{a_{i\mu(i)}+\underline{d}^A_i}{2}\right].$$

Conversely, if $x\in\prod_{i\in M}\left[\frac{a_{i\mu(i)}-\underline{d}^A_{\mu(i)}}{2},\frac{a_{i\mu(i)}+\underline{d}^A_i}{2}\right]$, then for all $i\in M$ there exists $0\leq\lambda_i\leq 1$ such that $x_i=\lambda_i\left(\frac{a_{i\mu(i)}-\underline{d}^A_{\mu(i)}}{2}\right)+(1-\lambda_i)\left(\frac{a_{i\mu(i)}+\underline{d}^A_i}{2}\right)$. Define $\tilde{d}\in\mathbb{R}^M_+\times\mathbb{R}^{M'}_+$ by $\tilde{d}_i=(1-\lambda_i)\underline{d}^A_i$ for all $i\in M$ and $\tilde{d}_{\mu(i)}=\lambda_i\underline{d}^A_{\mu(i)}$ for all $\mu(i)\in M'$. Then, $\tilde{d}\in\mathcal{D}_A$ holds and $N(S_A^\mu,\tilde{d})_i=x_i$, for all $i\in M$, which proves $x\in\mathcal{N}(S_A^\mu,\mathcal{D}_A)_u$. □

Notice that when the matrix A has only one optimal matching, then the two sets in statement 2) of Theorem 9 coincide. In general, as shown in the following example, the inclusion may be strict.

Example 10 Let us consider the assignment matrix $A=\begin{pmatrix}18 & 26\\ 10 & 18\end{pmatrix}$. It is not difficult to see that the core reduces to a segment, with extreme points $(\overline{u}^A,\underline{v}^A)=(18,10;0,8)$ and $(\underline{u}^A,\overline{v}^A)=(8,0;10,18)$. Notice also that $\mathcal{HC}(w_A)$ reduces to only one point which is $(13,5;5,13)$; it is therefore the τ-value. Since $\underline{d}^A=(8,0;0,8)$, we have $\mathcal{D}_A=\{(d_1,0,0,d_4)\mid$

$0 \leq d_1 \leq 8\,,\, 0 \leq d_4 \leq 8\}$. For the disagreement point $d = (8,0,0,0) \in \mathcal{D}_A$ we obtain $N(S_A, d) = (14, 6; 4, 12) = B$, while for $d = (0,0,0,8)$ we get $N(S_A, d) = (12, 4; 6, 14) = A$; and both are core allocations. In fact, for this example it can be proved that $\mathcal{N}(S_A, \mathcal{D}_A) \cap C(w_A)$ is the segment with extreme points A and B, as shown in Figure 4.

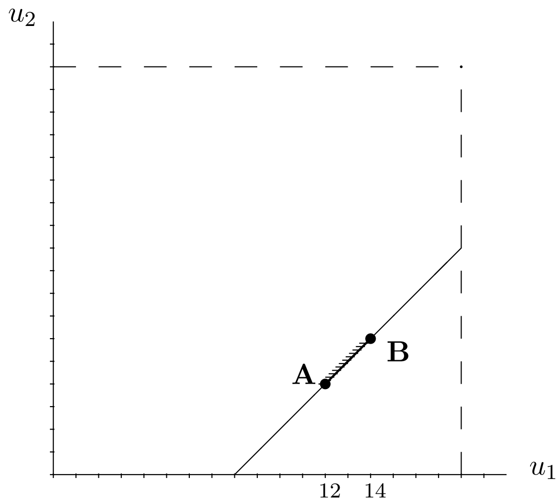

Figure 4:

The main result in this section will state that the half–optimal core of the assignment game is the set of core allocations which can be obtained as a Kalai–Smorodinsky solution of the problem (S_A, d) with disagreement point in $\mathcal{D}_A$. Some preliminary work is needed before reaching this result.

Lemma 11 *Let (M, M', A) be an assignment problem.*

1. *If $i, i' \in M$ are such that there exist $\mu_1, \mu_2 \in \mathcal{M}^*(A)$ and $\mu_1(i) = \mu_2(i') = j \in M'$, then $a_{ij} - \underline{d}_i^A = a_{i'j} - \underline{d}_{i'}^A$.*

2. *If $j, j' \in M'$ are such that there exist $\mu_1, \mu_2 \in \mathcal{M}^*(A)$ and $\mu_1^{-1}(j) = \mu_2^{-1}(j') = i \in M$, then $a_{ij} - \underline{d}_j^A = a_{ij'} - \underline{d}_{j'}^A$.*

Proof: Just notice that, since both pairs (i,j) and (i',j) are optimally matched and $(\underline{u}^A, \overline{v}^A)$ is a core element, $a_{ij} - \underline{u}_i^A = \overline{v}_j^A = a_{i'j} - \underline{u}_{i'}^A$. □

Let us now define a binary relation R on the set of buyers M: for all $i, i' \in M$, iRi' if and only if there exist μ, $\mu' \in \mathcal{M}^*(A)$ such that $\mu(i) = \mu'(i')$. Since we assume all agents are active, the number of buyers equals the number of sellers and every agent is optimally matched. Then, this relation is reflexive and symmetric, but not transitive. We thus consider its transitive closure $\bar{R}$: for all i, $i' \in M$, $i\bar{R}i'$ if and only if there exist $i_1, i_2, \ldots, i_k \in M$ such that iRi_1, i_1Ri_2, …, i_kRi'. The binary relation $\bar{R}$ is an equivalence relation in M and we will denote by I_1, I_2, …, I_r its equivalence classes.

Similarly, two binary relations R' and $\bar{R}'$ can be defined in M'. For all j, $j' \in M'$, $jR'j'$ if and only if there exist μ, $\mu' \in \mathcal{M}^*(A)$ such that $\mu^{-1}(j) = \mu'^{-1}(j')$, and $j\bar{R}'j'$ if and only if there exist j_1, $j_2, \ldots, j_k \in M'$ and $jR'j_1$, $j_1R'j_2$, …, $j_kR'j'$. $\bar{R}$ is an equivalence relation in M' and we will denote its equivalence classes by J_1, $J_2, \ldots, J_s$.

Lemma 12 states that each equivalence class in M is mapped to the same equivalence class in M' by all optimal matchings. The proof will be found in the appendix.

Lemma 12 *Let (M, M', A) be an assignment problem where all agents are active, and let $\{I_k\}_{k=1}^r$ be the equivalence classes of $\bar{R}$, while $\{J_k\}_{k=1}^s$ are the equivalence classes of $\bar{R}'$. Then, for all $p \in \{1, 2, \ldots, r\}$ there exists $q \in \{1, 2, \ldots, s\}$ such that*

$$\mu(I_p) = J_q\,, \text{ for all } \mu \in \mathcal{M}^*(A)\,.$$

Recall that, from statement 1) in Theorem 9, all allocations in $\mathcal{HC}(w_A)$ can be obtained as the Nash solution of any auxiliary μ-bargaining assignment problem, for some disagreement point in $\mathcal{D}_A$. The next proposition states that, given an allocation in the half-optimal core, the same disagreement point can be chosen for all S_A^μ. Moreover, the proof of the proposition gives a constructive method to obtain such a common disagreement point.

Proposition 13 *Let (M, M', A) be an assignment problem where all agents are active. For all $x \in \mathcal{HC}(w_A)$ there exists $d \in \mathcal{D}_A$ such that $x = N(S_A^\mu, d)$ for all $\mu \in \mathcal{M}^*(A)$.*

Proof: Let us take $x \in \mathcal{HC}(w_A)$. By Definition 6 and statement 1) in Theorem 9, for all $\mu \in \mathcal{M}^*(A)$ there exists $d \in \mathcal{D}_A$ such that $x = N(S_A^\mu, d)$. Let us denote by $\mathcal{D}_{A,x}^\mu$ the set of disagreement points in $\mathcal{D}_A$ which lead to x after a μ–bargaining procedure. That is to say, $\mathcal{D}^{\mu}_{A,x} = \{d \in \mathcal{D}_A \mid N(S^{\mu}_A, d) = x\}$. Notice that, for all $x \in \mathcal{HC}(w_A)$ and all $\mu \in \mathcal{M}^*(A)$, $\mathcal{D}^{\mu}_{A,x} \neq \emptyset$.

If $x \in \mathcal{HC}(w_A)$ and $d \in \mathcal{D}^{\mu}_{A,x}$ for some $\mu \in \mathcal{M}^*(A)$, then, from (10), we get $d_i = d_{\mu(i)} - 2x_{\mu(i)} + a_{i\mu(i)}$, for all $i \in M$. Since $0 \leq d_i \leq \underline{d}^A_i$ and $0 \leq d_{\mu(i)} \leq \underline{d}^A_{\mu(i)}$, for all $i \in M$, we deduce

$$\mathcal{D}^{\mu}_{A,x} = \left\{ d \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+ \;\middle|\; \begin{array}{c} \max\{0, a_{i\mu(i)} - 2x_{\mu(i)}\} \leq d_i \leq \min\{\underline{d}^A_i, \underline{d}^A_{\mu(i)} - 2x_{\mu(i)} + a_{i\mu(i)}\}, \\ d_{\mu(i)} = d_i + 2x_{\mu(i)} - a_{i\mu(i)}, \text{ for all } i \in M \end{array} \right\}. \tag{12}$$

A necessary condition for the existence of a common disagreement point is that, for all $\mu \in \mathcal{M}^*(A)$, the intersection of the intervals above, one for each $i \in M$, be nonempty. For all $i \in M$, take $\mu_1, \mu_2 \in \mathcal{M}^*(A)$ such that

$$\max\{0, a_{i\mu_1(i)} - 2x_{\mu_1(i)}\} = \max_{\mu \in \mathcal{M}^*(A)} \{\max\{0, a_{i\mu(i)} - 2x_{\mu(i)}\}\}$$

and

$$\min\{\underline{d}^A_i, \underline{d}^A_{\mu_2(i)} - 2x_{\mu_2(i)} + a_{i\mu_2(i)}\} = \min_{\mu \in \mathcal{M}^*(A)} \{\min\{\underline{d}^A_i, \underline{d}^A_{\mu(i)} - 2x_{\mu(i)} + a_{i\mu(i)}\}\}.$$

Notice that $\underline{d}^A_i \geq 0$, and also $\underline{d}^A_i \geq a_{i\mu_1(i)} - 2x_{\mu_1(i)}$, which follows from the fact that, since $x \in \mathcal{HC}(w_A)$, $2x_{\mu_1(i)} \geq \overline{v}^A_{\mu_1(i)}$.

Moreover, taking into account that $x_i + x_{\mu_2(i)} = a_{i\mu_2(i)}$, we can write $\underline{d}^A_{\mu_2(i)} - 2x_{\mu_2(i)} + a_{i\mu_2(i)} = \underline{d}^A_{\mu_2(i)} - a_{i\mu_2(i)} + 2x_i = \underline{d}^A_{\mu_2(i)} - (\overline{u}^A_i + \underline{d}^A_{\mu_2(i)}) + 2x_i = -\overline{u}^A_i + 2x_i \geq 0$, where the last inequality follows from $x \in \mathcal{HC}(w_A)$.

Finally, from $x_i + x_{\mu_1(i)} = a_{i\mu_1(i)}$ and $x_i + x_{\mu_2(i)} = a_{i\mu_2(i)}$, we obtain that the inequality $a_{i\mu_1(i)} - 2x_{\mu_1(i)} \leq \underline{d}^A_{\mu_2(i)} - 2x_{\mu_2(i)} + a_{i\mu_2(i)}$ is equivalent to $-a_{i\mu_1(i)} \leq \underline{d}^A_{\mu_2(i)} - a_{i\mu_2(i)} = -\overline{u}^A_i$, which holds trivially.

The above considerations prove that

$$\max\{0, a_{i\mu_1(i)} - 2x_{\mu_1(i)}\} \leq \min\{\underline{d}^A_i, \underline{d}^A_{\mu_2(i)} - 2x_{\mu_2(i)} + a_{i\mu_2(i)}\},$$

for all $i \in M$.

We are now ready for the choice of a disagreement point $d \in \mathbb{R}^M \times \mathbb{R}^{M'}$, making use of the partition $\{I_k\}_{k=1}^r$ of M given by the equivalence classes of $\bar{R}$, and the partition $\{J_k\}_{k=1}^r$ of M' given by the equivalence classes of $\bar{R}'$. Because of Lemma 12, let us assume, without loss of generality, that the above classes have been ordered in such a way that $\mu(I_k) = J_k$ for all $\mu \in \mathcal{M}^*(A)$ and all $k \in \{1, 2, \dots, r\}$.

For all $k \in \{1, 2, \dots, r\}$ take $i_k \in I_k$ such that $\underline{d}_{i_k}^A = \min_{i \in I_k} \underline{d}_i^A$ and choose d_{i_k} such that

$$\max_{\mu \in \mathcal{M}^*(A)} \{0, a_{i_k \mu(i_k)} - 2x_{\mu(i_k)}\} \leq d_{i_k} \leq \min_{\mu \in \mathcal{M}^*(A)} \{\underline{d}_{i_k}^A, \underline{d}_{\mu(i_k)}^A - 2x_{\mu(i_k)} + a_{i_k \mu(i_k)}\}.$$

Now, for all $i \in I_k$, define

$$d_i := d_{i_k} + \underline{d}_i^A - \underline{d}_{i_k}^A$$

and notice that this implies that

$$d_i - d_{i'} = \underline{d}_i^A - \underline{d}_{i'}^A \quad \text{for all} \quad i,\, i' \in I_k. \tag{13}$$

Choose also any $\mu^* \in \mathcal{M}^*(A)$ and define $d_{\mu^*(i)} := 2x_{\mu^*(i)} + d_i - a_{i\mu^*(i)}$, for all $i \in M$.

Now it only remains to prove that (i) $x = N(S_A^\mu, d)$ for all $\mu \in \mathcal{M}^*(A)$, and (ii) $d \in \mathcal{D}_A$.

(i) By (10), it is enough to prove that, for all $i \in M$, $x_{\mu(i)} = \frac{a_{i\mu(i)} - d_i + d_{\mu(i)}}{2}$ for all $\mu \in \mathcal{M}^*(A)$, since then $x_i = \frac{a_{i\mu(i)} - d_{\mu(i)} + d_i}{2}$ follows from $x \in C(w_A)$.

For all $i \in M$ and all $\mu \in \mathcal{M}^*(A)$, there exists $i' \in M$ such that $\mu(i) = \mu^*(i')$, where μ^* is the optimal matching previously fixed. Then $i,\, i' \in I_k$ for some $k \in \{1, 2, \dots, r\}$ and, by (13) and Lemma 11 we have

$$\begin{aligned} d_{\mu(i)} &= d_{\mu^*(i')} = 2x_{\mu^*(i')} + d_{i'} - a_{i'\mu^*(i')} = \\ &= 2x_{\mu(i)} + (d_i - \underline{d}_i^A + \underline{d}_{i'}^A) - (a_{i\mu(i)} + \underline{d}_{i'}^A - \underline{d}_i^A) = 2x_{\mu(i)} + d_i - a_{i\mu(i)}. \end{aligned} \tag{14}$$

(ii) From the definition of i_k and d_{i_k} we know that, for all $k \in \{1, 2, \dots, r\}$, $0 \leq d_{i_k} \leq \underline{d}_{i_k}^A$ and $\underline{d}_{i_k}^A \leq \underline{d}_i^A$ for all $i \in I_k$. Then, for all $k \in \{1, 2, \dots, r\}$ and all $i \in I_k$,

$$0 \leq d_{i_k} \leq d_i = d_{i_k} + \underline{d}_i^A - \underline{d}_{i_k}^A \leq \underline{d}_i^A.$$

It only remains to prove that similar inequalities hold for $d_{\mu(i)}$, for all $i \in M$.

Notice first that, for all $j, j' \in M'$ such that $jR'j'$, we know there exists $i \in M$ and $\mu_1, \mu_2 \in \mathcal{M}^*(A)$ such that $\mu_1^{-1}(j) = \mu_2^{-1}(j') = i$ and, once proved that $x = N(S_A^{\mu}, d)$ for all $\mu \in \mathcal{M}^*(A)$, this means that

$$x_i = \frac{a_{i\mu_1(i)} - d_{\mu_1(i)} + d_i}{2} = \frac{a_{i\mu_2(i)} - d_{\mu_2(i)} + d_i}{2}$$

and, by Lemma 11, this implies

$$d_{\mu_2(i)} = d_{\mu_1(i)} + \underline{d}^A_{\mu_2(i)} - \underline{d}^A_{\mu_1(i)} \,. \tag{15}$$

It is easy to prove, and it is left to the reader, that the same equality holds for all $j, j' \in J_k$, for all $k \in \{1, 2, \ldots, r\}$:

$$d_{j'} = d_j + \underline{d}^A_{j'} - \underline{d}^A_j \,. \tag{16}$$

On the other hand, from the definition of d_{i_k}, and also by (14), it follows that $0 \leq d_{\mu(i_k)} \leq \underline{d}^A_{\mu(i_k)}$ for all $\mu \in \mathcal{M}^*(A)$. Then, for all $i \in I_k$, $i \neq i_k$, we have $\mu(i), \mu(i_k) \in \mu(I_k) = J_k$ and, by equation (16), we can write $d_{\mu(i)} = d_{\mu(i_k)} + \underline{d}^A_{\mu(i)} - \underline{d}^A_{\mu(i_k)} \leq \underline{d}^A_{\mu(i)}$.

Now, for all $k \in \{1, 2, \ldots, r\}$, if there exists $i' \in I_k$ such that $d_{\mu(i')} = \underline{d}^A_{\mu(i')}$, then, from (15), $d_{\mu(i)} = \underline{d}^A_{\mu(i)} \geq 0$, for all $i \in I_k$. If there exists $i' \in I_k$ such that $d_{i'} = \underline{d}^A_{i'}$, then $d_i = \underline{d}^A_i$ also holds for all $i \in I_k$, and consequently

$$\begin{aligned} d_{\mu(i)} &= 2x_{\mu(i)} + \underline{d}^A_i - a_{i\mu(i)} = \\ &= 2x_{\mu(i)} + \underline{d}^A_i - (\underline{d}^A_i + \overline{v}^A_{\mu(i)}) = 2x_{\mu(i)} - \overline{v}^A_{\mu(i)} \geq 0 \,, \end{aligned}$$

for all $i \in I_k$, where the last inequality follows from $x \in \mathcal{HC}(w_A)$.

If $d_l < \underline{d}_l$ for all $l \in I_k \cup J_k$, take $\varepsilon = \min_{l \in I_k \cup J_k} (\underline{d}^A_l - d_l) > 0$ and define a new disagreement point $d' \in \mathbb{R}^M \times \mathbb{R}^{M'}$ by

$$d'_l = \begin{cases} d_l + \varepsilon & \text{for all } l \in I_k \cup J_k \,, \\ d_l & \text{for all } l \in (M \cup M') \setminus (I_k \cup J_k) \,. \end{cases}$$

It is straightforward to check that $x = N(S_A^{\mu}, d')$ for all $\mu \in \mathcal{M}^*(A)$. Moreover, $d'_l \geq d_l \geq 0$ for all $l \in I_k$, and ε has been taken in such a way that $d'_l \leq \underline{d}^A_l$ for all $l \in I_k \cup J_k$.

Finally, there now exists $l_0 \in I_k \cup J_k$ such that $d'_{l_0} = \underline{d}'^A_{l_0}$ and, by an argument already used above, this means that $d'_l \geq 0$ for all $l \in J_k$. $\square$

We are now ready to prove that the half-optimal core can be characterized as those core allocations that can be achieved as the Kalai-Smorodinsky solution of a bargaining problem with feasible set S_A and disagreement point in $\mathcal{D}_A$. Notice first that when A has only one optimal matching μ, $S_A = S_A^\mu$ and by (10) and statement 1) in Theorem 9 the following statement is already known to be true.

Theorem 14 *Let (M, M', A) be an assignment problem where all agents are active. Then,*

$$\mathcal{HC}(w_A) = \mathcal{KS}(S_A, \mathcal{D}_A) \cap C(w_A).$$

PROOF: To prove $\mathcal{HC}(w_A) \supseteq \mathcal{KS}(S_A, \mathcal{D}_A) \cap C(w_A)$, recall that $x \in \mathcal{KS}(S_A, \mathcal{D}_A)$ implies $x = d + \lambda(a - d)$ for some $d \in \mathcal{D}_A$, a being the ideal point of the problem (S_A, d), and $\lambda \in [0, 1]$ maximal such that $x \in S_A$. For all $i \in M$,

$$a_i = \min_{\mu \in \mathcal{M}^*(A)} \{a_{i\mu(i)} - d_{\mu(i)}\} \geq \min_{\mu \in \mathcal{M}^*(A)} \{a_{i\mu(i)} - \underline{d}^A_{\mu(i)}\} = \overline{u}^A_i. \tag{17}$$

Similarly, if $j \in M'$, $a_j \geq \overline{v}^A_j$.

From weak Pareto optimality of the Kalai-Smorodinsky solution, there exists $\mu \in \mathcal{M}^*(A)$ and $(i, j) \in \mu$ such that $x_i + x_j = a_{ij}$. This means that $\lambda = \frac{a_{ij} - d_i - d_j}{a_i + a_j - d_i - d_j}$. Notice that λ is well defined since all agents are active and then $d_i \leq \underline{d}^A_i < \overline{u}_i \leq a_i$ and $d_j \leq \underline{d}^A_j < \overline{v}_j \leq a_j$. Moreover, from (17) follows $a_i = \min_{\mu \in \mathcal{M}^*(A)} \{a_{i\mu(i)} - d_{\mu(i)}\} \leq a_{ij} - d_j$ and similarly $a_j \leq a_{ij} - d_i$ which implies $\lambda = \frac{a_{ij} - d_i - d_j}{a_i + a_j - d_i - d_j} \geq \frac{1}{2}$. Then, for all $i \in M$ and all $j \in M'$, taking into account $a_i \geq \overline{u}^A_i$, $a_j \geq \overline{v}^A_j$, $d_i \geq 0$ and $d_j \geq 0$, we have

$$x_i \geq d_i + \frac{1}{2}(a_i - d_i) = \frac{a_i + d_i}{2} \geq \frac{\overline{u}^A_i}{2} \text{ and } x_j \geq d_j + \frac{1}{2}(a_j - d_j) = \frac{a_j + d_j}{2} \geq \frac{\overline{v}^A_j}{2}$$

which means, from definition 6, that $x \in \mathcal{HC}(w_A)$.

The converse inclusion remains to be proved. Since Proposition 13 means that $\bigcap_{\mu \in \mathcal{M}^*(A)} \mathcal{D}^\mu_{A,x} \neq \emptyset$, for all $x \in \mathcal{HC}(w_A)$, let us see that for all $x \in \mathcal{HC}(w_A)$ and all $d \in \bigcap_{\mu \in \mathcal{M}^*(A)} \mathcal{D}^\mu_{A,x}$, we have $KS(S_A, d) = x$.

Take then $x \in \mathcal{HC}(w_A)$ and $d \in \bigcap_{\mu \in \mathcal{M}^*(A)} \mathcal{D}^{\mu}_{A,x}$. For all $i \in M$ and all $\mu \in \mathcal{M}^*(A)$, $\frac{a_{i\mu(i)} - d_{\mu(i)} + d_i}{2} = x_i$, while for all $j \in M'$, $\frac{a_{\mu^{-1}(j)j} - d_{\mu^{-1}(j)} + d_j}{2} = x_j$ holds. Then, when we compute the ideal point, for all $i \in M$ we obtain

$$a_i = \min_{\mu \in \mathcal{M}^*(A)} \{a_{i\mu(i)} - d_{\mu(i)}\} = 2x_i - d_i$$

while for all $j \in M'$,

$$a_j = \min_{\mu \in \mathcal{M}^*(A)} \{a_{\mu^{-1}(j)j} - d_{\mu^{-1}(j)}\} = 2x_j - d_j \, .$$

Now, $a = 2x - d$, which means that $x = \frac{a+d}{2}$ is the midpoint between the ideal point and the disagreement point. Moreover, x is Pareto optimal in S_A, since $x \in \mathcal{HC}(w_A) \subseteq C(w_A)$, and thus x coincides with the Kalai-Smorodinsky solution of the problem (S_A, d). $\square$

In Section 2 we have computed the half-optimal core of an assignment game found in Shapley and Shubik (1972). By the above theorem, this is also the set of core allocations that can be obtained as a Kalai-Smorodinsky solution to the related bargaining problem.

Notice to conclude that the half-optimal core is a subset of stable allocations supported by a bargaining procedure and always containing a single-valued cooperative game solution, the τ-value, just as the kernel (Davis and Maschler, 1965) is the set of symmetrically pairwise bargained allocations (Rochford, 1984) and always contains the nucleolus.

In both Example 7 and Example 8 it is not difficult to check that the kernel does not meet the half-optimal core. Since the assignment market of each one of these examples has only one optimal matching, we know that $\mathcal{HC}(w_A) = \mathcal{N}(S_A, \mathcal{D}_A) \cap C(w_A) = \mathcal{KS}(S_A, \mathcal{D}_{\mathcal{A}}) \cap C(w_A)$. Then, no allocation in the kernel of those markets can be obtained by a Nash or a Kalai-Smorodinsky bargaining procedure, with a disagreement point dominated by the minimal rights payoff vector. Thus, the half-optimal core is a different way in which classical bargaining theory provides a nonempty selection of stable allocations in an assignment market.

6 Assignment problems with nonactive agents

We will now show that the analysis in the previous sections can be easily extended to those assignment problems that do not satisfy the assumption made in this paper. Let us describe the procedure when there are some nonactive pairs. Recall from Definition 3 that an agent $k \in M \cup M'$ is nonactive if and only if $\underline{d}_k^A = b_k^A$, and if such agents exist then, by Lemma 4, the bargaining problem $(S_A, \underline{d}^A)$ is not well defined.

Now, given an assignment problem (M, M', A), let us write $M = M_1 \cup M_2$ and $M' = M_1' \cup M_2'$ where M_1 is the subset of active buyers and M_1' is the subset of active sellers. And let A_1 be the submatrix of A with row and columns corresponding to active agents. Notice that if agent i is optimally matched by μ to agent j then $\overline{u}_i^A + \underline{v}_j^A = a_{ij} = \underline{u}_i^A + \overline{v}_j^A$ which implies $\overline{v}_j^A - \underline{v}_j^A = \overline{u}_i^A - \underline{u}_i^A$ and, as a consequence, i is active if and only if j is also active. Then, the cardinality of M_1 and M_1' is the same.

For all $\mu \in \mathcal{M}^*(A)$, the restriction of μ to $M_1 \times M_1'$ is also an optimal matching for $M_1 \times M_1'$. And conversely, every optimal matching for $M_1 \times M_1'$ can be completed to obtain an optimal matching of $M \times M'$.

Let us define in M the binary relation $\bar{R}$ (see page 26), adding the condition that if $i \in M$ is never optimally matched then $i\bar{R}i$, in order to make $\bar{R}$ reflexive in M, and let us similarly make $\bar{R}'$ reflexive in M'. Then, each equivalence class of $\bar{R}$ and $\bar{R}'$ consists of either only active agents or only nonactive agents.

Nonactive agents $k \in M_2 \cup M_2'$ will be constrained to receive the payoff $\underline{d}_k^A$ and the negotiation will take place between the agents in $M_1 \cup M_1'$. So let us assume that there exists at least one active pair. However, we should not restrict our attention to the matrix entries a_{ij} where $(i,j) \in M_1 \times M_1'$, since some active agents may have trading possibilities with some nonactive agents that should be taken into account when defining the feasible set.

Let it be $c \in \mathbb{R}_+^{M_1} \times \mathbb{R}_+^{M_1'}$ where

$$c_i = \max_{j \in M_2'} \{0, a_{ij} - \underline{d}_j^A\} \text{ for all } i \in M_1$$

and

$$c_j = \max_{i \in M_2}\{0, a_{ij} - \underline{d}_i^A\} \text{ for all } j \in M_1'.$$

Then, we define the feasible set for the negotiation among the agents $M_1 \cup M_1'$ by

$$\tilde{S}_{A_1} = \left\{ x \in \mathbb{R}_+^{M_1} \times \mathbb{R}_+^{M_1'} \;\middle|\; \begin{array}{l} x_i + x_j \leq a_{ij} \text{ for all } (i,j) \in M_1 \times M_1', \text{ such that} \\ \text{there exists } \mu \in \mathcal{M}^*(A) \text{ and } (i,j) \in \mu, \\ x_i \geq c_i \text{ for all } i \in M_1, \\ x_j \geq c_j \text{ for all } j \in M_1', \end{array} \right\}$$

Notice that, for all $i \in M_1$, either $c_i = 0 \leq \underline{d}_i^A$ or $c_i = a_{ij} - \underline{d}_j^A$ for some $j \in M_2'$. In this case, since $\underline{d}_j^A = \overline{v}_j^A$ and $(\underline{u}^A, \overline{v}^A) \in C(w_A)$, we also have $c_i = a_{ij} - \overline{v}_j^A \leq \underline{d}_i^A$. A similar argument applied to the case $j \in M_1'$ gives that the restriction of the payoff vector of minimal rights to the set of active agents, $\underline{d}^A_{|M_1 \times M_1'}$, dominates vector c. As a consequence, both c and $\underline{d}^A_{|M_1 \times M_1'}$ belong to the feasible set $\tilde{S}_{A_1}$.

Moreover, $\tilde{S}_{A_1}$ is compact, convex and c–comprehensive (if $x \in \tilde{S}_{A_1}$ and $x \geq y \geq c$, then $y \in \tilde{S}_{A_1}$) and the restriction of $\underline{d}^A$ to $M_1 \cup M_1'$ is an admissible disagreement point for $\tilde{S}_{A_1}$. Notice also that, if all agents are active, the set $\tilde{S}_{A_1}$ coincides with the feasible set S_A of the previous sections.

If we take d as an admissible disagreement point for this new feasible set, and under the assumption that the restriction of A to $M_1 \times M_1'$, A_1, has only one optimal matching, the Nash and the Kalai–Smorodinsky solutions coincide and have the same expressions as those found in Theorem 2.

Regardless of the number of optimal matchings in $M_1 \times M_1'$, when we take the restriction of $\underline{d}^A$ to the subset of active agents as a disagreement point in $\tilde{S}_{A_1}$, the Nash and the Kalai-Smorodinsky solutions coincide and, when completed with the minimal rights payoff for the nonactive agents, they coincide with $\tau(w_A)$. This result extends Theorem 5 and shows that also when nonactive agents exist, the negotiation process may lead to a stable allocation. Therefore, it makes sense to ask what other stable allocations can be obtained in this way.

To answer this question, let us first select a set of admissible disagreement points. Notice that in this new problem it may happen that $0 \notin \tilde{S}_{A_1}$ and thus the origin may not be admissible as a disagreement point. If we consider disagreement points in the c-comprehensive closure of $\underline{d}^A$ we obtain the set $\mathcal{D}_{A_1} = \{d \in \mathbb{R}_+^{M_1} \times \mathbb{R}_+^{M_1'} \mid c_k \leq d_k \leq \underline{d}_k^A\}$.

We also extend the definition of half-optimal core and take it to be

$$\mathcal{HC}(w_A) = \left\{ (u,v) \in C(w_A) \;\middle|\; \begin{array}{ll} u_i \geq \frac{\overline{u}_i^A + c_i}{2}, & \text{for all } i \in M_1, \\ v_j \geq \frac{\overline{v}_j^A + c_j}{2}, & \text{for all } j \in M_1' \end{array} \right\},$$

since nonactive agents are constrained to receive their minimal rights payoff in any stable allocation. Notice that $\mathcal{HC}(w_A)$ always contains the τ-value and, therefore, it is a nonempty set.

On the other hand, we denote by $\widehat{\mathcal{KS}}(\tilde{S}_{A_1}, \mathcal{D}_{A_1})$ and $\widehat{\mathcal{N}}(\tilde{S}_{A_1}, \mathcal{D}_{A_1})$ respectively the set of Kalai-Smorodinsky or Nash solutions to the problem $(\tilde{S}_{A_1}, d)$, with disagreement point in $\mathcal{D}_{A_1}$, once they have been completed to a payoff vector to the whole set of agents $M \cup M'$, by giving each nonactive agent his minimal rights payoff.

Then, Theorems 9 and 14 are easily extended and we obtain that the half-optimal core of the assignment game $(M \cup M', w_A)$ coincides with the set of core allocations that can be reached as a Kalai-Smorodinsky solution of $(\tilde{S}_{A_1}, d)$ with disagreement point in $\mathcal{D}_{A_1}$. Moreover, all allocations in this set can also be reached as the Nash solution of $\tilde{S}_{A_1}$ with disagreement point in $\mathcal{D}_{A_1}$:

$$\mathcal{HC}(w_A) = \widehat{\mathcal{KS}}(\tilde{S}_{A_1}, \mathcal{D}_{A_1}) \cap C(w_A) \subseteq \widehat{\mathcal{N}}(\tilde{S}_{A_1}, \mathcal{D}_{A_1}) \cap C(w_A).$$

A Appendix

Proof of Lemma 11: Let us take an equivalence class I_p of $\bar{R}$ and choose any element $i_0 \in I_p$. Choose also an optimal matching $\mu^* \in \mathcal{M}^*(A)$ and take $q \in \{1, 2, \ldots, s\}$ such that $\mu^*(i_0) \in J_q$. Notice that for all other $\mu \in \mathcal{M}^*(A)$, $\mu(i_0) R' \mu^*(i_0)$ and thus $\mu(i_0) \in J_q$ for all $\mu \in \mathcal{M}^*(A)$.

We will first prove that $\mu(I_p) \subseteq J_q$ for all $\mu \in \mathcal{M}^*(A)$. Take $i \in I_p$, $i \neq i_0$ and let us see that $\mu(i) \in J_q$ for all $\mu \in \mathcal{M}^*(A)$. Since $i_0 \overline{R} i$, there exist $i_1, i_2, \ldots, i_k \in M$ such that $i_0 R i_1,$, $i_1 R i_2$, ..., $i_k R i$. From $i_0 R i_1$ follows that there exist $\mu_0, \mu_1 \in \mathcal{M}^*(A)$ with $\mu_0(i_0) = \mu_1(i_1)$ which implies $\mu_1(i_1) \in J_q$ and consequently, since $\mu_1(i_1) R' \mu(i_1)$ for all $\mu \in \mathcal{M}^*(A)$, we obtain $\mu(i_1) \in J_q$ for all $\mu \in \mathcal{M}^*(A)$. By repeating the same argument, we iteratively obtain that $\mu(i_2), \mu(i_3), \ldots, \mu(i_k)$ and $\mu(i)$ belong to J_q for all $\mu \in \mathcal{M}^*(A)$.

Let us check now that $J_q \subseteq \mu(I_p)$ for all $\mu \in \mathcal{M}^*(A)$. Take $j \in J_q$ and any $\mu \in \mathcal{M}^*(A)$. From $\mu(i_0) \in J_q$ it follows that there exist $j_1, j_2, \ldots, j_k \in M'$ such that $\mu(i_0) R' j_1$, $j_1 R' j_2, \ldots, j_k R' j$. Since $\mu(i_0) R' j_1$, there exist $i_1 \in M$ and $\mu_0, \mu_1 \in \mathcal{M}^*(A)$ such that $i_1 = \mu_0^{-1}(j_1) = \mu_1^{-1}(\mu(i_0))$. Then, $\mu_1(i_1) = \mu(i_0)$ which means that $i_1 R i_0$ and consequently $i_1 \in I_p$. Moreover, $\mu_0(i_1) = j_1 = \mu(\mu^{-1}(j_1))$, and this implies that $i_1 R \mu^{-1}(j_1)$, that is to say $\mu^{-1}(j_1) \in I_p$ or, equivalently, $j_1 \in \mu(I_p)$. By repeatedly applying the same argument we obtain $j_2, \ldots, j_k$ and j also belong to $\mu(I_p)$ for all $\mu \in \mathcal{M}^*(A)$. □

The above lemma shows that the number of classes in both partitions of M and M' is the same and all optimal matchings map each class in M onto the same class in M'. Then, an ordering can be taken in the set of buyers such that buyers in I_1 precede any other buyer in $I_2 \cup I_3 \cup \cdots \cup I_r$, and, for all $k \in \{2, \ldots, r-1\}$, buyers in I_k precede any buyer in $I_{k+1} \cup \cdots \cup I_r$. Similarly, an ordering can be taken in the set of sellers such that those sellers in $\mu(I_1)$ precede any other seller and, for all $k \in \{2, \ldots, r-1\}$, sellers in $\mu(I_k)$ precede any seller in $\mu(I_{k+1}) \cup \cdots \cup \mu(I_r)$. The assignment matrix corresponding to these orderings on the player set will be a partitioned matrix presenting all the optimal matchings in the diagonal blocks.

References

[1] Bennett, E., 1988. Consistent bargaining conjectures in marriage matching. J. Econ. Theory 45, 392–407.

[2] Chun, Y., Thomson, W., 1992. Bargaining problems with claims. Math. Soc. Sci., 24, 1, 19–33.

[3] Crawford, V.P., Rochford, S.C.,1986. Bargaining and competition in matching markets. Int. Econ. Rev. 27, 329–348.

[4] Dagan, N., Volij, O., 1993. The bankruptcy problem: A cooperative bargaining approach. Math. Soc. Sci. 26, 287–297.

[5] Davis, M., Maschler, M., 1965. The kernel of a cooperative game. Naval Research Logistics Quarterly 12, 223–259.

[6] Demange, G., 1982. Strategyproofness in the Assignment Market Game. Laboratoire d'Économétrie de l'École Polytechnique, Paris.

[7] Driessen, T.S.H., 1988. A note on the inclusion of the kernel in the core of the bilateral assignment game. Int. J. Game Theory, 27, 301–303.

[8] Granot, D., 1995. On a new bargaining set for cooperative games. Working paper n. 95-MSC-005. Faculty of Commerce and Business Administration, The University of British Columbia, Vancouver, B.C., Canada.

[9] Herrero, C., 1998. Endogenous reference points and the adjusted proportional solution for the bargaining problems with claims. Soc. Choice and Welfare 15, 113–119.

[10] Kalai, E., Smorodinsky, M., 1975. Other Solutions to Nash's Bargaining Problem. Econometrica 43, 513–518.

[11] Kamecke, U., 1989. Non-cooperative matching games. Int. J. Game Theory 18, 423–431.

[12] Leonard, H.B., 1983. Elicitation of Honest Preferences for the Assignment of Individuals to Positions. J. Polit. Economy 91, 461–479.

[13] Moldovanu, B., 1990. Stable bargained equilibria for assignment games without side payments. Int. J. Game Theory 19, 171–190.

[14] Nash, J.F., 1950. The Bargaining Problem. Econometrica 28, 155-162.

[15] Núñez, M., Rafels, C., 2002. The assignment game: the τ–value. Int. J. Game Theory 31, 411-422.

[16] Núñez, M., Rafels, C., 2003. Characterization of the extreme core allocations of the assignment game. Games Econ. Behav. 44, 311–331.

[17] Quint, T., 1991. Characterization of cores of assignment games. Int. J. Game Theory 19, 413–420.

[18] Rochford, S.C., 1984. Symmetrically pairwise-bargained allocations in an assignment market. J. Econ. Theory, 34, 262–281.

[19] Shapley, L.S., Shubik, M., 1972. The Assignment Game I: The Core. Int. J. Game Theory 1, 111–130.

[20] Solymosi, T., Raghavan, T.E.S., 2001. Assignment games with stable core. Int. J. Game Theory, 30, 177–185.

[21] Thomson, W., 1994. Cooperative Models of Bargaining. In: Aumann, R., Hart S. (Eds.), Handbook of Game Theory with Economic Applications, vol. II. North-Holland, Amsterdam, 1237–1284.

[22] Thompson, G.L., 1981. Auctions and Market Games. In: Aumann, R. (Ed.), Essays in Game Theory in honor of Oskar Morgenstern, 181–196.

[23] Tijs, S.H., 1981. Bounds for the core and the τ–value. In: Moeschlin, O., Pallaschke D. (Eds.), Game Theory and Mathematical Economics, North Holland Publishing Company, 123–132.