# Centre de Referència en Economia Analítica

## Fair and Efficient Student Placement with Couples

**Bettina Klaus and Flip Klijn**

# Fair and Efficient Student Placement with Couples[*]

Bettina Klaus[†]        Flip Klijn[‡]

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### Abstract

We study situations of allocating positions to students based on priorities. An example
is the assignment of medical students to hospital residencies on the basis of entrance exams.
For markets without couples, *e.g.*, for undergraduate student placement, acyclicity is a nec-
essary and sufficient condition for the existence of a fair and efficient placement mechanism
(Ergin, 2002). We show that in the presence of couples acyclicity is still necessary, but
not sufficient. A second necessary condition is priority-togetherness of couples. A priority
structure that satisfies both necessary conditions is called *pt*-acyclic.

For student placement problems where all quotas are equal to one we characterize *pt*-
acyclicity and show that it is a sufficient condition for the existence of a fair and efficient
placement mechanism. If in addition to *pt*-acyclicity we require reallocation- and vacancy-
fairness for couples, the so-called dictator-bidictator placement mechanism is the unique fair
and efficient placement mechanism.

Finally, for general student placement problems, we show that *pt*-acyclicity may not
be sufficient for the existence of a fair and efficient placement mechanism. We identify a
sufficient condition such that the so-called sequential placement mechanism produces a fair
and efficient allocation.

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*Keywords:* student placement, fairness, efficiency, couples, acyclic priority structure.

## 1   Introduction

We consider so-called house allocation or student placement problems. A *student placement
problem* is determined by a set of position types, the number of available positions – the quota –
of each type, and the students' strict preferences over position types (*e.g.*, a position type could
be a house or a position at a university or firm) and remaining unassigned. A *(student) placement
mechanism* assigns to any given student placement problem an allocation of the position types
to the students such that every student receives at most one position and quotas are binding.
In contrast to a pure house allocation problem, where an assignment is made on the basis of
students' preferences over position types alone,[1] we assume that in a student placement problem

[1]Sometimes it is also assumed that exactly one position of each type is available. Some recent articles on house
allocation problems are Ehlers (2002), Ehlers *et al.* (2002), and Ehlers and Klaus (2003,2006a,b).

additional information is available.[2] For instance, college admissions of undergraduate students are often based on rankings obtained from one or several entrance exams. Then students who achieved higher test scores in the entrance exam of a certain college have higher priority for admission at that college than students with lower test scores. This situation can be described as a strict priority ranking of individuals for each position type. We call this collection of strict priority rankings a priority structure. A placement mechanism violates the priority of student $i$ for position $x$ if there exists a preference profile under which student $i$ envies student $j$ who obtains $x$ even though $i$ has a higher priority for $x$ than $j$. A placement mechanism is fair if it never violates the specified priorities. Ergin (2002, Theorem 1) shows that a fair placement mechanism is efficient if and only if the priority structure is acyclic.

We also consider the student placement problem, but have in mind situations where students may not only care about their own position, but also about the position of their partner. The instance of couples may not be a prominent feature in undergraduate admissions, but it definitely is an issue for many couples that search for their first professional position in the same labor market, for instance medical students that look for residencies. We show that in the presence of couples acyclicity is still a necessary condition for a placement mechanism to be fair and efficient, but it is not sufficient (Theorem 4.1). In order for a fair and efficient placement mechanism to exist, in addition to acyclicity the priority structure has to satisfy priority-togetherness of couples (Theorem 4.2), where loosely speaking, priority-togetherness of a couple means that the members of the couple are ranked "close enough together" in the priority ranking of all position types. A priority structure is $pt$-acyclic if it satisfies acyclicity and priority-togetherness.

For student placement problems where all quotas are equal to one we characterize $pt$-acyclicity (Lemma 5.1) and show that it is a sufficient condition for the existence of a fair and efficient placement mechanism (Theorem 5.1). If in addition to $pt$-acyclicity we require reallocation- and vacancy-fairness for couples, the so-called dictator-bidictator placement mechanism is the unique fair and efficient placement mechanism (Theorem 5.2).

Finally, for general student placement problems, we show that $pt$-acyclicity may not be sufficient for the existence of a fair and efficient placement mechanism (Examples 5.4, 5.5, and 5.6). We identify a sufficient condition such that the so-called sequential placement mechanism produces a fair and efficient allocation (Theorem 5.3).

The paper is organized as follows. In Section 2, we introduce student placement problems with couples. In Section 3, we introduce efficiency, fairness, and acyclicity. In Section 4, we study necessary conditions for the compatibility of fairness and efficiency. Section 5 is devoted to sufficient conditions for the compatibility of fairness and efficiency. In Subsection 5.1 we focus on placement problems when all quotas are equal to one. In Subsection 5.2 we conclude with an extension of some of the results from Subsection 5.1 to general student placement problems. The proofs of all our results are relegated to the Appendix.

## 2  Student Placement with Couples

Let $N = \{1, \dots, n\}$ denote the set of students. We assume that $N$ can be partitioned into a set of couples $C = \{c_1, \dots, c_k\}$ such that for all $i \in \{1, \dots, k\}$, $c_i = (2i - 1, 2i)$ and a set of single students $S = \{2k + 1, \dots, n\}$.[3] Hence for a market with three couples and four single students, $N = \{1, \dots, 10\}$ is composed of $C = \{c_1 = (1, 2), c_2 = (3, 4), c_3 = (5, 6)\}$ and $S = \{7, 8, 9, 10\}$. If $C = \emptyset$, then our model coincides, for instance, with the model analyzed by Ergin (2002).

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[2]See, for instance, Balinski and Sönmez (1999), Ergin (2002), and Kesten (2006).

[3]Note that all results can be straightforwardly extended to a model that includes triplets, quadruplets, *etc.*

Let $X$ denote the finite set of position types that the students apply to. For any $x \in X$, let $q_x \geq 1$ denote the number of available positions, the *quota*, of position type $x$. Let $q \equiv (q_x)_{x \in X}$. Note that we simply refer to "position $x$" when we mean one of the $q_x$ positions of position type $x$. Let $0$ denote the null position, which does not belong to $X$ ("receiving the null position" means "not receiving any position"). Since the null position is freely available, we set $q_0 \equiv \infty$.

Each single student $s \in S$ has an individual strict, transitive, and complete preference relation $R_s$ over $X \cup \{0\}$. Given $x, y \in X \cup \{0\}$, $x\, P_s\, y$ means that student $s$ strictly prefers $x$ to $y$. Let $\mathcal{RS}$ denote the set of all linear orders over $X \cup \{0\}$. In other words, $\mathcal{RS}$ equals the set of single students' preference relations. Let $\mathcal{RS}^S$ denote the set of all *(preference) profiles* $R_S = (R_s)_{s \in S}$ such that for all $s \in S$, $R_s \in \mathcal{RS}$.

Each couple $c \in C$ has an individual strict, transitive, and complete preference relation $R_c$ over all ordered pairs of feasible position type assignments $\mathcal{X} \equiv [(X \cup \{0\}) \times (X \cup \{0\})] \setminus \{(x,x) : x \in X \text{ and } q_x = 1\}$. To simplify notation, whenever we denote a position type assignment $(x, y)$ we always additionally assume that $(x, y)$ is feasible, *i.e.*, $(x, y) \in \mathcal{X}$. This also means that all statements we make with respect to position type assignments only apply if the position type assignments in question are feasible. Given $c = (i, j)$, $(x, y)$, and $(x', y')$, $(x, y)\, P_c\, (x', y')$ means that couple $c$ strictly prefers $(x, y)$, where $i$ is matched to $x$ and $j$ is matched to $y$, to $(x', y')$, where $i$ is matched to $x'$ and $j$ is matched to $y'$. Let $\mathcal{RC}$ denote the set of all linear orders over $\mathcal{X}$. In other words, $\mathcal{RC}$ equals the set of couples' preference relations. Let $\mathcal{RC}^C$ denote the set of all *(preference) profiles* $R_C = (R_c)_{c \in C}$ such that for all $c \in C$, $R_c \in \mathcal{RC}$.

We next introduce a subdomain of $\mathcal{RC}$ that allows us to relate a couple's preference relation to "individual preferences" of each member in a couple in a consistent way. Loosely speaking, this is the case when the unilateral improvement of one partner's position is considered beneficial for the couple as well. Couple $c = (i, j) \in C$ has *responsive* preferences if there exist preferences $R_i, R_j \in \mathcal{RS}$ such that for all $x, y, z \in X \cup \{0\}$, $[x\, P_i\, y$ implies $(x, z)\, P_c\, (y, z)]$ and $[x\, P_j\, y$ implies $(z, x)\, P_c\, (z, y)]$.[4] If these *associated individual preferences* $R_i$ and $R_j$ exist, then they are unique. Note that if a couple $c = (i, j)$ has responsive preferences, then one can easily derive the associated individual preferences $R_i$ and $R_j$. Let $\mathcal{RRC} \subset \mathcal{RC}$ equal the set of couples' possible responsive preference relations. Let $\mathcal{RRC}^C$ denote the set of all *responsive (preference) profiles* $R_C = (R_c)_{c \in C}$ such that for all $c \in C$, $R_c \in \mathcal{RRC}$. For notational convenience we define $\mathcal{R} \equiv \mathcal{RC}^C \times \mathcal{RS}^S$ and $\mathcal{RR} \equiv \mathcal{RRC}^C \times \mathcal{RS}^S$.

Let $x \in X$. We call a linear order $\succ_x$ over $N$ a *priority ordering for position type $x$*. A *priority structure* is a profile $\succ = (\succ_x)_{x \in X}$ specifying for each position type a priority ordering.

Since the set of students $N$, the set of position types $X$, their quotas $q$, and the priority structure $\succ$ are fixed, we denote a *(student) placement problem* by the students' preferences $R = (R_C, R_S) \in \mathcal{R}$. We assume that the null position is available in any placement problem.

When allocating positions each student either receives a "real position in $X$" or the null position $0$. The null position can be assigned to several students without any restriction, but for all other positions the associated quota is binding as an upper bound. Formally, an *allocation for $R$* is a list $\alpha = (\alpha_i)_{i \in N}$ such that for all $i \in N$, $\alpha_i \in X \cup \{0\}$, and for all $x \in X$, $|\{i \in N : \alpha_i = x\}| \leq q_x$. Thus, an allocation is by definition feasible. Note that not all available positions need to be assigned. Given $i \in N$, we call $\alpha_i$ the *allotment of student $i$ at $\alpha$*. Given $c = (i, j) \in C$, we call $\alpha_c \equiv (\alpha_i, \alpha_j)$ the *allotment of couple $c$ at $\alpha$*.

A *(student) placement mechanism* is a function $\varphi$ that assigns to each placement problem $R \in \mathcal{R}$ an allocation $\varphi(R)$.

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[4]Klaus *et al.* (2004) and Klaus and Klijn (2005,2006) use the same notion of responsiveness in the context of two-sided matching. In Remark 5.2 we compare the role of responsiveness in two-sided matching and student placement.

# 3   Efficiency, Fairness, and Acyclicity

We are interested in student placement mechanisms that choose (Pareto) efficient allocations.

**Definition 3.1** *Efficiency*
A placement mechanism $\varphi$ is *efficient* if for all $R \in \mathcal{R}$, there is no allocation $\alpha = (\alpha_i)_{i \in N}$ for $R$ such that for all $p \in S \cup C$, $\alpha_p \, R_p \, \varphi_p(R)$ and for some $q \in S \cup C$, $\alpha_q \, P_q \, \varphi_q(R)$.                $\triangle$

Next, we formulate the idea that an allocation may "violate the priority structure." First, we do so for single students.

An allocation violates the priority of a single student $i \in S$ if there exists a position $x$ such that student $i$ has a higher priority for $x$ than one of the students assigned to it and student $i$ prefers to switch to position $x$.

**Definition 3.2** *Fairness for Single Students* (Balinski and Sönmez, 1999)
Given a placement problem $R \in \mathcal{R}$, $i \in S$, and a priority structure $\succ$, an allocation $\alpha$ for $R$ *violates the priority of single student $i$* if there exist $x \in X$ and $k \in N\backslash\{i\}$ such that $\alpha_k = x$, $i \succ_x k$, and $x \, P_i \, \alpha_i$. A placement mechanism $\varphi$ is *fair for single students* if for all $R \in \mathcal{R}$, $\varphi(R)$ does not violate the priority of any single student.                $\triangle$

Next, we list several cases that describe when an allocation violates the priority of a couple $c = (i,j) \in C$. First, (a) an allocation violates the priority of couple $c$ if there exists a position $x \in X$ such that one member of the couple, *e.g.*, student $i$, has a higher priority for $x$ than one of the students whose allotment is $x$ and who is not $i$'s partner and couple $c$ prefers that $i$ switches to position $x$ while $j$ either keeps his/her allotment or switches to the null position.

Second, (b) an allocation violates the priority of couple $c$ if there exists a position $x \in X$ such that it is the allotment of one member of the couple, *e.g.*, student $j$, the other member of the couple $i$ has a higher priority for $x$ than $j$, and couple $c$ prefers that $i$ switches to position $x$ and $j$ switches to the null position.

Third, (c) an allocation violates the priority of couple $c$ if there exist positions $x, y \in X$ (possibly of the same type) and two students $k, l \in N$ (possibly $k = j$ or $l = i$) such that $k$'s allotment is $x$, $l$'s allotment is $y$, $i$ has a higher priority for $x$ than $k$, $j$ has a higher priority for $y$ than $l$, and couple $c$ prefers that $i$ switches to $x$ and $j$ switches to $y$.[5]

**Definition 3.3** *Fairness for Couples*
Given a placement problem $R \in \mathcal{R}$, $(i,j) \in C$, and a priority structure $\succ$, an allocation $\alpha$ for $R$ *violates the priority of couple $(i,j)$* if

**(a.1)** there exist $x \in X$ and $k \in N\backslash\{i,j\}$ such that $\alpha_k = x$, $i \succ_x k$, and
$[(x, \alpha_j) \, P_{(i,j)} \, (\alpha_i, \alpha_j)$ or $(x, 0) \, P_{(i,j)} \, (\alpha_i, \alpha_j)]$, or

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[5]The main idea of a fair allocation is that no single student (no couple) can justifiably appeal for another position (assignment of positions) that would make him/her (it) better off. For single students it is clear that only a position that is ranked higher than the allotment will make him/her better off. A couple, however, has various possibilities to improve its allotment: one or both partners could change their positions, one partner could change his/her position while the other partner switches to the null position at the same time, or one partner changes his/her position while the other partner receives his/her previous position. At this moment we only consider violations of a couple's priorities that would lead to a better allotment for the couple because one or both partners receive positions for which they have a higher priority and possibly one partner chooses the null position. Since students do not have any property rights over positions, passing on a position to a partner or adopting a vacant position may not be allowed for a couple and therefore violations that give better allotments conditioned on such transactions are not considered in our basic fairness concept, but later in two extra fairness conditions for couples (Definitions 5.4 and 5.5).

**(a.2)** there exist $x \in X$ and $k \in N \setminus \{i,j\}$ such that $\alpha_k = x$, $j \succ_x k$, and
$[(\alpha_i, x) \, P_{(i,j)} \, (\alpha_i, \alpha_j)$ or $(0, x) \, P_{(i,j)} \, (\alpha_i, \alpha_j)]$, or

**(b.1)** there exists $x \in X$ such that $\alpha_j = x$, $i \succ_x j$, and $(x, 0) \, P_{(i,j)} \, (\alpha_i, x)$, or

**(b.2)** there exists $x \in X$ such that $\alpha_i = x$, $j \succ_x i$, and $(0, x) \, P_{(i,j)} \, (x, \alpha_j)$, or

**(c)** there exist $x, y \in X$ and $k, l \in N$, $k \neq l$, such that $\alpha_k = x$, $\alpha_l = y$, $i \succ_x k$, $j \succ_y l$, and $(x, y) \, P_{(i,j)} \, (\alpha_i, \alpha_j)$.

A placement mechanism $\varphi$ is *fair for couples* if for all $R \in \mathcal{R}$, $\varphi(R)$ does not violate the priority of any couple. △

**Definition 3.4 *Fairness***
A placement mechanism is *fair* if it is fair for single students and couples. △

We next introduce an acyclicity condition, due to Ergin (2002), that turns out to be crucial for the existence of a fair and efficient placement mechanism.

**Definition 3.5 *Cycles and Acyclicity***
Given a priority structure $\succ$, a *cycle* is constituted of distinct $x, y \in X$ and $i, j, k \in N$ such that the following two conditions are satisfied:
*cycle condition* $\quad i \succ_x j \succ_x k$ and $k \succ_y i$ and
*c-scarcity condition* $\quad$ there exist disjoint sets $N_x, N_y \subseteq N \setminus \{i, j, k\}$ (possibly $N_x = \emptyset$ or $N_y = \emptyset$) such that $N_x \subseteq \{l \in N : l \succ_x j\}$, $N_y \subseteq \{l \in N : l \succ_y i\}$, $|N_x| = q_x - 1$, and $|N_y| = q_y - 1$.
A priority structure is *acyclic* if no cycles exist. △

If quotas are all equal to 1, then the cycle condition is sufficient to establish the existence of a cycle. For other quotas, the c-scarcity condition limits the definition of a cycle to cases where there indeed exist placement problems in $\mathcal{R}$ such that students $i$, $j$, and $k$ compete for position types $x$ and $y$ (in the absence of this competition, *e.g.*, because the quotas do in fact not limit the access of the students to positions of type $x$ and $y$, a cycle will not lead to the violation of efficiency or the given priorities – see Ergin, 2002, for further discussion).

In the sequel we will sometimes write "D$x$" instead of "Definition $x$".

## 4 Fairness and Efficiency for Student Placement with Couples: Necessary Conditions

In the absence of couples, Ergin (2002) shows that the acyclicity of the priority structure is a necessary and sufficient condition for a placement mechanism to be fair and efficient. He shows that under the assumption of an acyclic priority structure, the unique efficient placement mechanism that is fair can be found using a simple adaptation of the so-called deferred acceptance algorithm (Gale and Shapley, 1962). Ergin's (2002) Theorem 1 implies the following result.

**Corollary 4.1 (Ergin, 2002)** *Let $C = \emptyset$.*
*A fair and efficient placement mechanism $\varphi$ exists (and is unique) if and only if $\succ$ is acyclic.*

With our first result, we show that for placement problems with couples acyclicity is still a necessary, but not a sufficient condition for the existence of a fair and efficient placement mechanism.

**Theorem 4.1  *Necessity of Acyclicity***
**(a)** *A placement mechanism $\varphi$ is fair and efficient only if $\succ$ is acyclic.*
**(b)** *The acyclicity of $\succ$ is not a sufficient condition for the existence of a fair and efficient placement mechanism $\varphi$.*

Note that in the proof of Theorem 4.1 (Appendix) we use only responsive preferences for couples. Hence, Theorem 4.1 remains valid if we restrict couples' preferences to be responsive.

**Remark 4.1**  In order to prove Theorem 4.1 (a) we assume that acyclicity is violated and derive a contradiction to fairness and efficiency. For any violation of acyclicity that only involves single students, the proof of Theorem 4.1 (a) essentially equals the proof of Corollary 4.1 (Ergin 2002, Theorem 1). However, extending the proof to situations where couples are involved in a violation of acyclicity requires substantial extra work. The key then is to define adequate individual preferences not only for single students, but also for the members of all couples and, in addition, corresponding couples' preferences.

Next, in order for fairness and efficiency to be compatible we need an additional condition on the priority structure: members of a couple have to be "close enough together" in the priority structure. In the following definition we formalize what we mean by "close enough together."

**Definition 4.1  *Priority-Separation, Priority-Togetherness, and pt-Acyclicity***
Let $c = (i, j) \in C$. Given a priority structure $\succ$, couple $c$ is *priority-separated* if there exists $x \in X$ and $k \in N$ such that the following two conditions are satisfied:
*weak priority-separation* **(i)** if $i \succ_x j$, then $i \succ_x k \succ_x j$, and **(ii)** if $j \succ_x i$, then $j \succ_x k \succ_x i$, and
*ps-scarcity condition*    there exists a set $\bar{N}_x \subseteq N\setminus\{i, j, k\}$ (possibly $\bar{N}_x = \emptyset$) such that
**(i)** if $i \succ_x j$, then $\bar{N}_x \subseteq \{l \in N : l \succ_x j\}$ and $|\bar{N}_x| = q_x - 1$, and
**(ii)** if $j \succ_x i$, then $\bar{N}_x \subseteq \{l \in N : l \succ_x i\}$ and $|\bar{N}_x| = q_x - 1$.
A priority structure satisfies *priority-togetherness (of couples)* if no couple is priority-separated.
A priority structure is *pt-acyclic* if it is acyclic and satisfies priority-togetherness.                 △

If quotas are all equal to 1, then weak priority-separation is sufficient to define priority-separation of couples. For other quotas, the *ps*-scarcity condition limits the definition of priority-separation to cases where there indeed exist placement problems in $\mathcal{R}$ such that couple $c$ and student $k$ compete for position type $x$ (in the absence of this competition, *e.g.*, because the quotas do in fact not limit the access of the students to position type $x$, a priority-separation will not lead to the violation of efficiency or the given priorities).

**Theorem 4.2  *Necessity of pt-Acyclicity***
*A placement mechanism $\varphi$ is fair and efficient only if $\succ$ is pt-acyclic.*

Note that in the proof of Theorem 4.2 (Appendix) we use only responsive preferences for couples. Hence, Theorem 4.2 remains valid if we restrict couples' preferences to be responsive.

# 5   Fairness and Efficiency for Student Placement with Couples: Sufficiency and Uniqueness

Our next goal is to introduce a fair and efficient placement mechanism for *pt*-acyclic priority structures. First we show that any priority structure defines in a natural way a partition of the students. As will turn out later, the structure of the partition induced by a *pt*-acyclic priority structure makes it possible to define fair and efficient placement mechanisms.

Given position type $x \in X$ and a subset $N' \subseteq N$ of students, let $\text{top}(x, N')$ denote the student with the highest priority for position type $x$ among the students in $N'$, *i.e.*, $i = \text{top}(\succ_x, N')$ if and only if $i \in N'$ and for all $j \in N'\backslash\{i\}$, $i \succ_x j$. Given a subset $N' \subseteq N$ of students, let $\text{top}(N')$ denote the set of students with the highest priority for some position type among the students in $N'$, *i.e.*, $\text{top}(N') \equiv \bigcup_{x \in X} \text{top}(\succ_x, N')$. Using this notation, we define a partition of the set of all students $N$ that is induced by (not necessarily *pt*-acyclic) priority structure $\succ$: $S_1$ denotes the set of students who have the highest priority for some position type. $S_2$ denotes the set of students in the remaining set of students $N\backslash S_1$ who now have the highest priority for some position type, *etc.* Given $S_k$ we can interpret the index $k$ as the *level of priority*: students in $S_1$ have the highest possible priority, students in $S_2$ have the next highest priority, *etc.*

**Definition 5.1 *Partition $\mathcal{S}$ Induced by $\succ$***
Let $S_1 \equiv \text{top}(N)$ and for $k > 1$, if $N\backslash\bigcup_{l=1}^{k-1} S_l \neq \emptyset$, then $S_k \equiv \text{top}\left(N\backslash\bigcup_{l=1}^{k-1} S_l\right) \neq \emptyset$. Hence, there exists $p \geq 1$, $p < \infty$, such that for all $k \in \{1, \ldots, p\}$, $S_k \neq \emptyset$, and $N\backslash\left[\bigcup_{l=1}^{p} S_l\right] = \emptyset$. By $\mathcal{S} \equiv \{S_1, \ldots, S_p\}$, we define the *partition (of $N$) induced by $\succ$*, *i.e.*, $N = \bigcup_{l=1}^{p} S_l$ and for $k, k' \in \{1, \ldots, p\}$, $k \neq k'$, $S_k \cap S_{k'} = \emptyset$. △

## 5.1 Sufficiency and Uniqueness when All Quotas are Equal to One

Throughout this subsection we assume that for all $x \in X$, $q_x = 1$ and refer to position $x$ instead of position type $x$. For this special situation, *pt*-acyclicity is equivalent to the following.

**Definition 5.2 *pt-Acyclicity when all Quotas are Equal to One***
A priority structure $\succ$ is *pt*-acyclic if it has neither cycles nor weakly priority-separated couples:
*no cycles*   there exist no $i, j, k \in N$ and $x, y \in X$ such that $i \succ_x j \succ_x k$ and $k \succ_y i$ and
*no weak priority-separation*   there exist no $(i, j) \in C$, $k \in N$, and $x \in X$ such that $i \succ_x k \succ_x j$ or $j \succ_x k \succ_x i$. △

The next lemma describes the implications that *pt*-acyclicity of a priority structure has on its induced partition. First, (ia) we observe that each set of the partition either contains one or two students, that is, at most two students share the same level of priority. Second, (ib) if two students $i, j$ share the same level of priority, then they are neighbors in the priority structure, *i.e.*, for all positions, either $i$ is ranked just after $j$ or $j$ is ranked just after $i$. Finally, members of a couple $c = (i, j)$ either (iia) share the same level of priority or (iib and iic) they do not share the level of priority with any other student(s), but they have subsequent levels of priority.

**Lemma 5.1** *Priority structure $\succ$ is pt-acyclic if and only if partition $\mathcal{S} = \{S_1, \ldots, S_p\}$ has the following properties.*
**(i)** *for all $k \in \{1, \ldots, p\}$,*
   **(a)** $|S_k| \leq 2$,
   **(b)** *if $S_k = \{i, j\}$, $i \neq j$, then for any $x \in X$,*
   $\left[\; i = \text{top}\left(x, N\backslash\cup_{l=1}^{k-1} S_l\right) \;\text{implies}\; j = \text{top}\left(x, N\backslash\left[\cup_{l=1}^{k-1} S_l \cup \{i\}\right]\right)\;\right]$ *and*
**(ii)** *for all $c = (i, j) \in C$, if for $k \in \{1, \ldots, p\}$, $S_k \cap \{i, j\} \neq \emptyset$, then*
   **(a)** $S_k = \{i, j\}$, *or*
   **(b)** $S_k \cup S_{k-1} = \{i, j\}$, *or*
   **(c)** $S_k \cup S_{k+1} = \{i, j\}$.

In the proof of Lemma 5.1 (Appendix) we show that condition (i) characterizes the acyclicity of priority structure $\succ$, and that given acyclicity, condition (ii) characterizes priority-togetherness. The following direct implications of condition (i) of Lemma 5.1 turn out to be useful later on.

**Lemma 5.2** *Let $\mathcal{S}$ be the partition induced by $\succ$ and assume $\mathcal{S}$ satisfies (i) in Lemma 5.1.*
**(a)** *Let $x \in X$ and $i, \bar{i}, i' \in N$ be such that $i \succ_x \bar{i} \succ_x i'$. Furthermore, let $k, k' \in \{1, \ldots, p\}$ be such that $i \in S_k$ and $i' \in S_{k'}$. Then, $k < k'$.*
**(b)** *If $i \in S_k$, $i' \in S_{k'}$, $k, k' \in \{1, \ldots, p\}$, and $k < k'$, then for all $x \in X$, $i \succ_x i'$.*

We can now use partition $\mathcal{S}$ to define the *dictator-bidictator deferred acceptance placement mechanism*, the DB *placement mechanism* for short. Informally, the DB placement mechanism works as follows. Determine who has the highest level of priority. If a single student has the highest priority, then she receives her most preferred position. If two single students share the highest priority, then either they receive their most preferred positions or, in case of a conflict, the priority decides who gets his/her most preferred position and who gets his/her second most preferred position. If a member of a couple has the highest level of priority or a couple shares the highest level of priority, then the couple receives its most preferred position assignment. Repeat this step with the remaining students and positions, *i.e.*, determine who has the next highest level of priority, *etc.*

Let $Y \subseteq X$ be a set of positions. Let $s \in S$ and $R_s \in \mathcal{RS}$. Then $\text{top}(R_s, Y)$ denotes student $s$'s most preferred position in $Y \cup \{0\}$, *i.e.*, $y = \text{top}(R_s, Y) \in Y \cup \{0\}$ if and only if for all $x \in Y \cup \{0\}$, $x \neq y$, $y \, P_s \, x$. Let $c \in C$ and $R_c \in \mathcal{RC}$. Then $\text{top}(R_c, Y)$ denotes couple $c$'s most preferred position assignment in $\mathcal{Y} \equiv [(Y \cup \{0\}) \times (Y \cup \{0\})] \backslash \{(y, y) : y \in Y\}$, *i.e.*, $(y, y') = \text{top}(R_c, Y) \in \mathcal{Y}$ if and only if for all $(x, x') \in \mathcal{Y}$, $(x, x') \neq (y, y')$, $(y, y') \, P_c \, (x, x')$.

**Definition 5.3** *The Dictator-Bidictator (DB) Placement Mechanism $\varphi^{\mathcal{S}}$*
Let $\succ$ be a *pt*-acyclic priority structure that induces partition $\mathcal{S} = \{S_1, \ldots, S_p\}$. Given $R \in \mathcal{R}$, we calculate $\varphi^{\mathcal{S}}(R)$ as follows. Set $k = 1$ and $X_1 \equiv X$. As long as $k \leq p$, do **Step $k$:**
**(a)** $S_k = \{s\}$ and $s \in S$: $\varphi_s^{\mathcal{S}}(R) \equiv \text{top}(R_s, X_k)$, $X_{k+1} \equiv X_k \backslash \{\varphi_s^{\mathcal{S}}(R)\}$, and set $k = k + 1$.
If a single student has the highest priority for all remaining real positions in $X_k$, she chooses her favorite position in $X_k \cup \{0\}$. The set of unassigned real positions equals $X_{k+1}$.
**(b)** $S_k = \{s\}$, $s \notin S$, and $S_k \cup S_{k+1} = \{s, s'\}$ such that $c = (s, s') \in C$ or $c = (s', s) \in C$: $\varphi_c^{\mathcal{S}}(R) \equiv \text{top}(R_c, X_k)$, $X_{k+2} \equiv X_k \backslash \{\varphi_s^{\mathcal{S}}(R), \varphi_{s'}^{\mathcal{S}}(R)\}$, and set $k = k + 2$.
If the member of a couple has the highest priority for all remaining real positions in $X_k$, then the couple chooses its most favorite position assignment in $\mathcal{X}_k$. The set of unassigned real positions equals $X_{k+2}$.
**(c)** $S_k = \{s, s'\}$ and $s, s' \in S$:
**(c.1)** if $\text{top}(R_s, X_k) \neq \text{top}(R_{s'}, X_k)$ or $\text{top}(R_s, X_k) = \text{top}(R_{s'}, X_k) = 0$, $\varphi_s^{\mathcal{S}}(R) \equiv \text{top}(R_s, X_k)$, $\varphi_{s'}^{\mathcal{S}}(R) \equiv \text{top}(R_{s'}, X_k)$, $X_{k+1} \equiv X_k \backslash \{\varphi_s^{\mathcal{S}}(R), \varphi_{s'}^{\mathcal{S}}(R)\}$, and set $k = k + 1$.
**(c.2)** if $\text{top}(R_s, X_k) = \text{top}(R_{s'}, X_k) \equiv \hat{x} \in X$ and $s \succ_{\hat{x}} s'$, $\varphi_s^{\mathcal{S}}(R) \equiv \text{top}(R_s, X_k)$, $\varphi_{s'}^{\mathcal{S}}(R) \equiv \text{top}(R_{s'}, X_k \backslash \{\varphi_s^{\mathcal{S}}(R)\})$, $X_{k+1} \equiv X_k \backslash \{\varphi_s^{\mathcal{S}}(R), \varphi_{s'}^{\mathcal{S}}(R)\}$, and set $k = k + 1$.
**(c.3)** if $\text{top}(R_s, X_k) = \text{top}(R_{s'}, X_k) \equiv \hat{x} \in X$ and $s' \succ_{\hat{x}} s$, $\varphi_{s'}^{\mathcal{S}}(R) \equiv \text{top}(R_{s'}, X_k)$, $\varphi_s^{\mathcal{S}}(R) \equiv \text{top}(R_s, X_k \backslash \{\varphi_{s'}^{\mathcal{S}}(R)\})$, $X_{k+1} \equiv X_k \backslash \{\varphi_s^{\mathcal{S}}(R), \varphi_{s'}^{\mathcal{S}}(R)\}$, and set $k = k + 1$.
If two single students share the highest priorities for all remaining real positions in $X_k$, then there are two possibilities. First, if feasible, both of them choose their favorite position in $X_k \cup \{0\}$; see (c.1). Second, if both of them prefer the same real position in $X_k$, then the student with the higher priority receives it and the remaining student is assigned his/her second best position in $X_k \cup \{0\}$; see (c.2) and (c.3). The set of unassigned real positions equals $X_{k+1}$.
**(d)** $S_k = \{s, s'\}$ and $c = (s, s') \in C$ or $c = (s', s) \in C$: $\varphi_c^{\mathcal{S}}(R) \equiv \text{top}(R_c, X_k)$, $X_{k+1} \equiv X_k \backslash \{\varphi_s^{\mathcal{S}}(R), \varphi_{s'}^{\mathcal{S}}(R)\}$, and set $k = k + 1$.
If the two members of a couple together have the highest priority for all remaining real positions in $X_k$, then the couple chooses its most favorite position assignment in $\mathcal{X}_k$. The set of unassigned real

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positions equals $X_{k+1}$.

The allocation $\varphi^{\mathcal{S}}(R)$ is obtained after at most $p$ steps. $\triangle$

**Example 5.1** *An Application of the DB Placement Mechanism*

Let $N = \{1, \ldots, 8\}$ be such that $C = \{c_1 = (1,2), c_2 = (3,4)\}$ and $S = \{5,6,7,8\}$; and $X = \{x_1, \ldots, x_7\}$ with quota 1 for each position type. We depict the *pt*-acyclic priority structure $\succ$ and preference profile $R \in \mathcal{R}$ in the two tables below.

| $\succ_{x_1}$ | $\succ_{x_2}$ | $\succ_{x_3}$ | $\succ_{x_4}$ | $\succ_{x_5}$ | $\succ_{x_6}$ | $\succ_{x_7}$ |
|---|---|---|---|---|---|---|
| 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 2 | 1 | 1 | 2 | 1 | 2 | 2 |
| 1 | 2 | 2 | 1 | 2 | 1 | 1 |
| 7 | 5 | 7 | 5 | 5 | 5 | 7 |
| 5 | 7 | 5 | 7 | 7 | 7 | 5 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 |

and

| $R_{c_1}$ | $R_{c_2}$ | $R_5$ | $R_6$ | $R_7$ | $R_8$ |
|---|---|---|---|---|---|
| $(x_3, x_1)$ | $(x_2, x_1)$ | $x_1$ | $x_3$ | $x_1$ | $x_1$ |
| $(x_1, x_3)$ | $(x_4, x_5)$ | $x_2$ | $x_1$ | $x_2$ | $x_2$ |
| $(x_2, x_1)$ | $(x_6, x_5)$ | $x_3$ | $x_2$ | $x_3$ | $x_3$ |
| $(x_1, x_2)$ | $(x_7, x_5)$ | $x_4$ | $x_4$ | $x_4$ | $x_4$ |
| $(x_3, x_2)$ | $(0, x_5)$ | $x_5$ | $x_7$ | $x_5$ | $x_5$ |
| $(x_2, x_3)$ | $(x_5, x_7)$ | $x_6$ | $0$ | $x_6$ | $x_6$ |
| $(x_3, 0)$ | $(x_4, x_7)$ | $x_7$ | $x_5$ | $x_7$ | $x_7$ |
| $(0, x_3)$ | $(x_6, x_7)$ | $0$ | $x_6$ | $0$ | $0$ |
| $(x_1, 0)$ | $(0, x_7)$ | | | | |
| $(0, x_1)$ | $(0, 0)$ | | | | |
| $\cdots$ | $\cdots$ | | | | |

In the table denoting priority structure $\succ$, students with higher priority for a position are denoted above students with lower priorities, *e.g.*, $6 \succ_{x_1} 2 \succ_{x_1} 1 \succ_{x_1} 7 \succ_{x_1} 5 \succ_{x_1} 4 \succ_{x_1} 3 \succ_{x_1} 8$. In the table denoting preference profile $R$, position assignments that are more preferred are denoted above less preferred position assignments, *e.g.*, $(x_3, x_1) \, R_{c_1} \, (x_1, x_3) \, R_{c_1} \, (x_2, x_1) \, R_{c_1} \, (x_1, x_2) \, R_{c_1}$ $(x_3, x_2) \, R_{c_1} \ldots$ or $x_1 \, R_5 \, x_2 \, R_5 \, x_3 \, R_5 \, x_4 \, R_5 \, x_5 \, R_5 \ldots$.

Priority structure $\succ$ induces partition $S = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ with $S_1 = \{6\}$, $S_2 = \{1, 2\}$, $S_3 = \{5, 7\}$, $S_4 = \{4\}$, $S_5 = \{3\}$, and $S_6 = \{8\}$. Note that $S_4 \cup S_5 = \{3, 4\}$ consists of couple $c_2$. We now calculate $\varphi^{\mathcal{S}}(R)$. Recall that $X_1 = X$.

**Step 1:** $S_1 = \{6\}$ and $6 \in S$: $\varphi^{\mathcal{S}}_6(R) = x_3$ and $X_2 = \{x_1, x_2, x_4, x_5, x_6, x_7\}$.

**Step 2:** $S_2 = \{1, 2\}$ and $(1, 2) \in C$: $\varphi^{\mathcal{S}}_{c_1}(R) = (x_2, x_1)$ and $X_3 = \{x_4, x_5, x_6, x_7\}$.

**Step 3:** $S_3 = \{5, 7\}$ and $5, 7 \in S$: $\varphi^{\mathcal{S}}_5(R) = x_4$, $\varphi^{\mathcal{S}}_7(R) = x_5$, and $X_4 = \{x_6, x_7\}$.

**Step 4:** $S_4 = \{4\}$, $S_5 = \{3\}$, and $(3, 4) \in C$: $\varphi^{\mathcal{S}}_{c_2}(R) = (x_6, x_7)$ and $X_6 = \emptyset$.

**Step 6:** $S_6 = \{8\}$ and $8 \in S$: $\varphi^{\mathcal{S}}_8(R) = 0$ and $X_7 = \emptyset$.

Hence, $\varphi^{\mathcal{S}}(R) = (x_2, x_1, x_6, x_7, x_4, x_3, x_5, 0)$. $\diamond$

Our next result is that for *pt*-acyclic priority structures the DB placement mechanism is fair and efficient.

**Theorem 5.1** *Quotas Equal to One: Sufficiency of pt-Acyclicity*

*Let $\succ$ be a pt-acyclic priority structure. Then, the DB placement mechanism $\varphi^{\mathcal{S}}$ is fair and efficient.*

**Remark 5.1** Theorem 5.1 is implied by one of our later results for general placement problems (Theorem 5.3). In the proof of Theorem 5.3 we use a result from Ergin (2002). However, a self-contained proof of Theorem 5.1 is available from any of the authors upon request.

The following two examples show exactly why *pt*-acyclicity is not strong enough to obtain the DB Placement Mechanism as the unique fair and efficient mechanism.

**Example 5.2 *Fairness, Efficiency, and Reallocation***
Let $N = \{1, 2, 3\}$ be such that $C = \{c = (1, 2)\}$ and $S = \{3\}$; and $X = \{x, y, z\}$ with quota 1 for each position type. Let $\succ$ be such that for all $x' \in X$, $1 \succ_{x'} 2 \succ_{x'} 3$. Consider $R_c \in \mathcal{RRC}$ and $R_3 \in \mathcal{RS}$ such that $(x, y) P_c (y, z) P_c \ldots$ and $x P_3 y P_3 z P_3 0$, where the "tail" of $R_c$, denoted by "$\ldots$", can be any fixed linear order of the remaining position assignments such that couple $c$'s preferences are responsive. Let $R = (R_c, R_3)$. It is easy to check that $S_i = \{i\}$ and hence $\varphi^S(R) = (x, y, z)$. However, allocation $\alpha = (y, z, x)$ is also fair and efficient.

Suppose that we allow couples to reallocate positions among themselves. Then, we could argue that allocation $\alpha$ "violates the priority of couple $c$ after reallocation" since student 1 can pass on position $y$ to his/her partner and then complain that his/her priority for position $x$ is violated because $1 \succ_x 3$. Note that by passing position $y$ to student 2, student 3's priority is not violated because $2 \succ_y 3$ (since student 2 is student 1's partner, $1 \succ_y 2$ is not relevant). ◇

**Example 5.3 *Fairness, Efficiency, and Vacancies***
Let $N = \{1, 2, 3\}$ be such that $C = \{c = (1, 2)\}$ and $S = \{3\}$; and $X = \{x, y, z, q\}$ with quota 1 for each position type. Let $\succ$ be such that for all $x' \in X$, $1 \succ_{x'} 2 \succ_{x'} 3$. Consider $R_c \in \mathcal{RC}$ and $R_3 \in \mathcal{RS}$ such that $(x, y) P_c (z, q) P_c \ldots$ and $x P_3 y P_3 z P_3 q P_3 0$, where the "tail" of $R_c$, denoted by "$\ldots$", can be any fixed linear order of the remaining position assignments. Let $R = (R_c, R_3)$. It is easy to check that $S_i = \{i\}$ and hence $\varphi^S(R) = (x, y, z)$. However, allocation $\alpha = (z, q, x)$ is also fair and efficient.

Suppose that we allow students to occupy a vacant position. Then, we could argue that allocation $\alpha$ "violates the priority of couple $c$ when taking into account vacancies" since student 2 can claim position $y$ and then his/her partner can complain that his/her priority for position $x$ is violated because $1 \succ_x 3$. Note that student 2's occupation of position $y$ does not violate student 3's priority. ◇

**Definition 5.4 *Reallocation-Fairness for Couples***
Given a placement problem $R \in \mathcal{R}$, $i \in N$, and a priority structure $\succ$, an allocation $\alpha$ for $R$ *violates the priority of couple* $(i, j) \in C$ *after reallocation* if
**(r.1)** $\alpha_i \in X$ and there exist $x \in X$ and $k \in N \backslash \{i, j\}$ such that $\alpha_k = x$, $i \succ_x k$, and $(x, \alpha_i) P_{(i,j)} (\alpha_i, \alpha_j)$ or
**(r.2)** $\alpha_j \in X$ and there exist $x \in X$ and $k \in N \backslash \{i, j\}$ such that $\alpha_k = x$, $j \succ_x k$, and $(\alpha_j, x) P_{(i,j)} (\alpha_i, \alpha_j)$.
A placement mechanism $\varphi$ is *reallocation-fair (for couples)* if for all $R \in \mathcal{R}$, $\varphi(R)$ does not violate the priority of any couple after reallocation. △

**Definition 5.5 *Vacancy-Fairness for Couples***
Given a placement problem $R \in \mathcal{R}$, $i \in N$, and a priority structure $\succ$, an allocation $\alpha$ for $R$ *violates the priority of couple* $(i, j) \in C$ *when taking vacancies into account* if
**(v.1)** there exist $x \in X$, $k \in N \backslash \{i, j\}$, and $v \in X \backslash \{\alpha_l : l \in N\}$ such that $\alpha_k = x$, $i \succ_x k$, and $(x, v) P_{(i,j)} (\alpha_i, \alpha_j)$ or
**(v.2)** there exist $x \in X$, $k \in N \backslash \{i, j\}$, and $v \in X \backslash \{\alpha_l : l \in N\}$ such that $\alpha_k = x$, $j \succ_x k$, and $(v, x) P_{(i,j)} (\alpha_i, \alpha_j)$.
A placement mechanism $\varphi$ is *vacancy-fair (for couples)* if for all $R \in \mathcal{R}$, $\varphi(R)$ does not violate the priority of any couple when taking vacancies into account. △

It is easy to check that the DB placement mechanism is reallocation- and vacancy-fair (for *pt*-acyclic priority structures). Given a *pt*-acyclic priority structure a reallocation- and vacancy-fair placement mechanism is fair and efficient if and only if it is the DB placement mechanism.

**Theorem 5.2** *Quotas Equal to One: Uniqueness of the DB Placement Mechanism*
*Let $\succ$ be a pt-acyclic priority structure. Let $\varphi$ be a reallocation- and vacancy-fair placement mechanism. Then, $\varphi$ is fair and efficient if and only if $\varphi = \varphi^{\mathcal{S}}$.*

Note that the proof (Appendix) of Theorem 5.2 remains the same for the domain of responsive preferences. Hence, Theorem 5.2 remains valid if we restrict couples' preferences to be responsive.

In Example 5.2 we exhibit a placement problem with responsive preferences such that there is an allocation that violates the priority of a couple after reallocation. In Example 5.3 we cannot exhibit a placement problem with responsive preferences such that there is an allocation that violates the priority of a couple when taking vacancies into account. The following result explains why.

**Proposition 5.1** *Let $\varphi$ be a placement mechanism on the domain of responsive preferences $\mathcal{RR}$. If $\varphi$ is fair and efficient (on $\mathcal{RR}$), then $\varphi$ is also vacancy-fair (on $\mathcal{RR}$).*

**Corollary 5.1** *Let $\succ$ be a pt-acyclic priority structure. Let $\varphi$ be a reallocation-fair placement mechanism on the domain of responsive preferences $\mathcal{RR}$. Then, $\varphi$ is fair and efficient (on $\mathcal{RR}$) if and only if $\varphi = \varphi^{\mathcal{S}}$ (on $\mathcal{RR}$).*

## 5.2  Sufficiency: The General Case

First, we demonstrate with three examples that if quotas are not all equal to one, then *pt*-acyclicity may not be sufficient for the existence of a fair and efficient placement mechanism. Note that all three examples are minimal in the sense that the number of students and couples are as small as possible and still present a non-trivial placement problem.

**Example 5.4** *pt-Acyclic Priorities and No Fair and Efficient Allocation I*
Let $N = \{1, 2, 3\}$ be such that $C = \{c_1 = (1, 2)\}$ and $S = \{3\}$; and $X = \{x, y\}$ with $q_x = 1$ and $q_y = 2$. Let $\succ$ be such that $1 \succ_x 2 \succ_x 3$ and $1 \succ_y 3 \succ_y 2$. Note that $\succ$ is pt-acyclic. Consider $R_{c_1} \in \mathcal{RC}$ and $R_3 \in \mathcal{RS}$ such that $(y, y) P_{c_1}(x, y) P_{c_1}(0, 0) P_{c_1} \ldots$ and $y P_3 0 P_3 x$, where the "tail" of $R_{c_1}$, denoted by "…", can be any fixed linear order of the remaining position type assignments. The only efficient allocations for placement problem $R = (R_{c_1}, R_3)$ are $\alpha = ((y, y), 0)$ and $\beta = ((x, y), y)$. Since $3 \succ_y 2$ and $y P_3 0$, $\alpha$ violates the priority of student 3 (D3.2). Furthermore, since $1 \succ_y 3$ and $(y, y) P_{c_1}(x, y)$, $\beta$ violates the priority of couple $c_1$ (D3.3, a.1).  ◇

**Example 5.5** *pt-Acyclic Priorities and No Fair and Efficient Allocation II*
Let $N = \{1, 2, 3, 4\}$ be such that $C = \{c_1 = (1, 2), c_2 = (3, 4)\}$; and $X = \{x, y\}$ with $q_x = 3$ and $q_y = 1$. Let $\succ$ be such that $1 \succ_x 3 \succ_x 2 \succ_x 4$ and $2 \succ_y 1 \succ_y 4 \succ_y 3$. Note that $\succ$ is pt-acyclic. Consider $R_{c_1}, R_{c_2} \in \mathcal{RC}$ such that $(x, x) P_{c_1}(y, x) P_{c_1}(0, 0) P_{c_1} \ldots$ and $(x, x) P_{c_2}(y, x) P_{c_2}(0, 0) P_{c_2} \ldots$, where the "tails" of $R_{c_1}$ and $R_{c_2}$, denoted by "…", can be any fixed linear order of the remaining position type assignments. The only efficient allocations for this placement problem $R = (R_{c_1}, R_{c_2})$ are $\alpha = ((x, x), (y, x))$ and $\beta = ((y, x), (x, x))$. Since $3 \succ_x 2$ and $(x, x) P_{c_2}(y, x)$, $\alpha$ violates the priority of couple $c_2$ (D3.3, a.1). Furthermore, since $1 \succ_x 3$ and $(x, x) P_{c_1}(y, x)$, $\beta$ violates the priority of couple $c_1$ (D3.3, a.1).  ◇

Examples 5.4 and 5.5 suggest that the reason for the incompatibility of fairness and efficiency is due to the fact that a couple is weakly priority-separated (D4.1). The following example demonstrates that it is not sufficient to exclude weak priority-separation of all couples to guarantee the existence of a fair and efficient allocation.

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**Example 5.6** *pt-**Acyclic Priorities and No Fair and Efficient Allocation III***
Let $N = \{1, 2, 3, 4\}$ be such that $C = \{c_1 = (1, 2), c_2 = (3, 4)\}$; and $X = \{x, y\}$ with $q_x = q_y = 2$. Let $\succ$ be such that $1 \succ_x 2 \succ_x 3 \succ_x 4$ and $3 \succ_y 4 \succ_y 1 \succ_y 2$. Note that $\succ$ is *pt*-acyclic and no couple is weakly priority-separated. Consider $R_{c_1}, R_{c_2} \in \mathcal{RC}$ such that $(y, y) \, P_{c_1} \, (x, x) \, P_{c_1} \, (x, 0) \, P_{c_1} \, (0, 0) \, P_{c_1} \ldots$ and $(x, y) \, P_{c_2} \, (x, 0) \, P_{c_2} \, (0, 0) \, P_{c_2} \ldots$, where the "tails" of $R_{c_1}$ and $R_{c_2}$, denoted by "...", can be any fixed linear order of the remaining position type assignments. The only efficient allocations for this placement problem $R = (R_{c_1}, R_{c_2})$ are $\alpha = ((y, y), (x, 0))$ and $\beta = ((x, 0), (x, y))$. Since $4 \succ_y 1$ and $(x, y) \, P_{c_2} \, (x, 0)$, $\alpha$ violates the priority of couple $c_2$ (D3.3, a.2). Furthermore, since $2 \succ_x 3$ and $(x, x) \, P_{c_1} \, (x, 0)$, $\beta$ violates the priority of couple $c_1$ (D3.3, a.2). ◇

Next, we strengthen *pt*-acyclicity in order to prevent incompatibilities as described in Examples 5.4, 5.5, and 5.6. We impose a restriction on *pt*-acyclic priority structures that will allow us to extend the DB placement mechanism to the general case with arbitrary quotas. For that purpose, we introduce the notion of strong acyclicity of the partition induced by a priority structure $\succ$. First, (i) members of a couple $c = (i, j)$ either share the same level of priority or they do not share the level of priority with any other student(s), but they have subsequent levels of priority. Second, (ii) if student $i$ has a higher level of priority than student $j$, then student $i$ has a higher priority for all position types than student $j$.

**Definition 5.6** *Strongly pt-Acyclic Partition $\mathcal{S}$ Induced by $\succ$*
Let $\succ$ be a *pt*-acyclic priority structure and $\mathcal{S} \equiv \{S_1, \ldots, S_p\}$ be the *partition (of $N$) induced by $\succ$*. Partition $\mathcal{S}$ is *strongly pt-acyclic* if
**(i)** for each couple $c = (i, j)$ there is $k \in \{1, \ldots, p\}$ with $S_k = \{i, j\}$ or $S_k \cup S_{k+1} = \{i, j\}$, and
**(ii)** for all $k, k' \in \{1, \ldots, p\}$ such that $k < k'$: if $i \in S_l$ and $j \in S_{k'}$, then for all $x \in X$, $i \succ_x j$. △

In Example 5.4 the partition induced by $\succ$ equals $S_1 = \{1\}$ and $S_2 = \{2, 3\}$, violating (i) in Definition 5.6. In Example 5.5 the partition induced by $\succ$ equals $S_1 = \{1, 2\}$ and $S_2 = \{3, 4\}$, violating (ii) in Definition 5.6. In Example 5.6 the partition induced by $\succ$ equals $S_1 = \{1, 3\}$ and $S_2 = \{2, 4\}$, violating (i) and (ii) in Definition 5.6.

Examples 5.4 and 5.5 demonstrate that conditions (i) and (ii) in the definition of a strong *pt*-acyclic partition are not vacuous. Lemmas 5.1 and 5.2, however, show that when all quotas are equal to one, any *pt*-acyclic priority structure induces a strongly *pt*-acyclic partition.

We will use the following notation to discuss implications of a strongly *pt*-acyclic partition. Let $c = (i, j) \in C$, $c' = (i', j') \in C$, and $x \in X$. Denote $c \succ_x c'$ if and only if $[i \succ_x i', i \succ_x j', j \succ_x i'$, and $j \succ_x j']$. Furthermore, denote $c \succ c'$ if and only if for all $x \in X$, $c \succ_x c'$. The following direct implications of a strongly *pt*-acyclic partition $\mathcal{S}$ turn out to be useful later on.

**Lemma 5.3** *Implications of a Strongly pt-Acyclic Partition*
*Let $\succ$ be a pt-acyclic priority structure that induces a strongly pt-acyclic partition $\mathcal{S} = \{S_1, \ldots, S_p\}$. Then,*
**(a)** *no couple is weakly priority-separated (D4.1);*
**(b)** *all couples have "the same sequence of priorities for all position types," i.e., for $c, c' \in C$, either $c \succ c'$ or $c' \succ c$. We assume without loss of generality that $c_1 \succ c_2 \succ \ldots \succ c_k$;*
**(c)** *for all $k \in \{1, \ldots, p\}$, either $S_k \subseteq S$ or there exists $c = (i, j) \in C$ such that $S_k = \{i, j\}$, $S_{k-1} \cup S_k = \{i, j\}$, or $S_k \cup S_{k+1} = \{i, j\}$.*

Before we introduce the so-called sequential (deferred acceptance) placement mechanism, we introduce the concept of a reduced placement problem and the deferred acceptance algorithm for single students.

**Definition 5.7** *Reduced Placement Problems and the DA Algorithm* $\tilde{\varphi}$

Let $S' \subseteq S$ and $q \geq q' \geq 0$. For any $R \in \mathcal{R}$, define $R_{S'} \equiv (R_i)_{i \in S'}$. Then, given placement problem $R \in \mathcal{R}$, we denote by $(R_{S'}, q')$ the *reduced placement problem* where position type quotas have been reduced to $q'$ and only students in $S'$ apply for positions.

For any reduced placement problem $(R_{S'}, q')$ we determine allocation $\tilde{\varphi}(R_{S'}, q')$ using the deferred acceptance (DA) algorithm of Gale and Shapley (1962):

At the first step every student in $N'$ "proposes" to his/her favorite position type in $X \cup \{0\}$. For each position type $x$, the $q'_x$ applicants who have the highest priority under $\succ_x$ (none if $q'_x = 0$ and all if there are fewer than $q'_x$) are placed on the waiting list of $x$, and the others are rejected. Every student that applies for the null position is placed on its "waiting list."

At any subsequent step every newly rejected student proposes to his/her next best position type in $X \cup \{0\}$. For each position type $x$, the $q'_x$ applicants who have the highest priority under $\succ_x$ (none if $q'_x = 0$ and all if there are fewer than $q'_x$) among the new applicants and those on the waiting list are placed on the new waiting list and the others are rejected. Every student that applies for the null position is placed on its "waiting list."

The algorithm terminates when every student belongs to a waiting list. Then positions of type $x \in X \cup \{0\}$ are assigned to the students on the waiting list of $x$. △

We can now use the strongly $pt$-acyclic partition $\mathcal{S}$ and the deferred acceptance algorithm for reduced placement problems to define the *sequential deferred acceptance placement mechanism*, *sequential placement mechanism* for short. Informally, it works as follows. Determine who has the highest level of priority. If a set of single students has the highest priority, then they receive their position types according to the DA algorithm. If a member of a couple has the highest level of priority or a couple shares the highest level of priority, then the couple receives its most preferred position assignment. Repeat this step with the remaining students and positions, *i.e.*, determine who has the next highest level of priority, *etc.*

Given a reduced quota vector $q'$ such that $q \geq q' \geq 0$, a position type $x \in X$ *is available* if $q'_x \geq 1$. By $X(q')$ we denote the set of all available position types at $q'$. Let $s \in S$ and $R_s \in \mathcal{RS}$. Then $\text{top}(R_s, q')$ denotes the student's most preferred available position type in $X(q') \cup \{0\}$, *i.e.*, $x = \text{top}(R_s, q')$ if and only if for all $y \in X(q') \cup \{0\}$, $y \neq x$, $x P_s y$. Let $c \in C$ and $R_c \in \mathcal{RC}$. Then $\text{top}(R_c, q')$ denotes the couple's most preferred available position type assignment in $\mathcal{X}(q') \equiv [(X(q') \cup \{0\}) \times (X(q') \cup \{0\})] \backslash \{(x, x) : q'_x = 1\}$, *i.e.*, $(x, x') = \text{top}(R_c, q') \in \mathcal{X}(q')$ if and only if for all $(y, y') \in \mathcal{X}(q')$, $(y, y') \neq (x, x')$, $(x, x') P_c (y, y')$.

Given a set of students $S' \subseteq S$ with allotments $\varphi_{S'}(R) \equiv (\varphi_i(R))_{i \in S'}$, assume that $q'$ is a (reduced) quota vector for which the students' allotments $\varphi_{S'}(R)$ are feasible, *i.e.*, for all $x \in X$, $|\{s \in S' : \varphi_s(R) = x\}| \leq q'_x \leq q_x$. Then, for all $x \in X$ we define $q'_x \backslash \varphi_{S'}(R) \equiv q'_x - |\{s \in S' : \varphi_s(R) = x\}|$ and $q' \backslash \varphi_{S'}(R) \equiv (q'_x \backslash \varphi_{S'}(R))_{x \in X}$. Hence, $q' \backslash \varphi_{S'}(R)$ denotes the reduced quota vector obtained by removing the students' allotments $\varphi_{S'}(R)$ from the placement problem.

Given a couple $c = (i, j) \in C$ with allotment $\varphi_c(R) = (\varphi_i(R), \varphi_j(R))$, assume that $q'$ is a (reduced) quota vector for which the couple's allotment $\varphi_c(R)$ is feasible, *i.e.*, for all $x \in X$, $|\{k \in \{i, j\} : \varphi_k(R) = x\}| \leq q'_x \leq q_x$. Then, for all $x \in X$ we define $q'_x \backslash \varphi_c(R) \equiv q'_x - |\{k \in \{i, j\} : \varphi_k(R) = x\}|$ and $q' \backslash \varphi_c(R) \equiv (q'_x \backslash \varphi_c(R))_{x \in X}$. Hence, $q' \backslash \varphi_c(R)$ denotes the reduced quota vector obtained by removing the couple's allotment $\varphi_c(R)$ from the placement problem.

**Definition 5.8** *The Sequential Placement Mechanism* $\varphi^{\mathcal{S}}$

Let $\succ$ be a $pt$-acyclic priority structure that induces a strongly $pt$-acyclic partition $\mathcal{S} = \{S_1, \ldots, S_p\}$. Given $R \in \mathcal{R}$, we calculate $\varphi^{\mathcal{S}}(R)$ as follows. Set $k = 1$, $N^1 \equiv N$, and $q^1 \equiv q$. As long as $k \leq p$, do **Step** *k*:

**(a)** $S_k \subseteq S$: $\varphi^{\mathcal{S}}_{S_k}(R) \equiv \tilde{\varphi}(R_{S_k}, q^k)$, $q^{k+1} \equiv q^k \backslash \varphi^{\mathcal{S}}_{S_k}(R)$, $N^{k+1} \equiv N^k \backslash S_k$, and set $k = k + 1$.

If a set of single students $S_k$ share the highest priority for all remaining available position types given by $q^k$, then the students' assignments are determined by applying the DA algorithm to the reduced placement problem where they are the only applicants. After the students' assignment, the set of remaining available positions is described by $q^{k+1}$.

**(b)** $S_k = \{s\}$, $s \notin S$, and $S_k \cup S_{k+1} = \{s, s'\}$ such that $c = (s, s') \in C$ or $c = (s', s) \in C$: $\varphi_c^{\mathcal{S}}(R) \equiv \text{top}(R_c, q^k)$, $q^{k+2} \equiv q^k \backslash \varphi_c^{\mathcal{S}}(R)$, $N^{k+2} \equiv N^k \backslash \{i, j\}$, and set $k = k + 2$.

If a member of a couple has the highest priority for all remaining available position types given by $q^k$, then the couple chooses its best pair of positions in $\mathcal{X}(q^k)$. After the couple's assignment, the set of remaining available positions is described by $q^{k+2}$.

**(c)** $S_k = \{s, s'\}$ and $c = (s, s') \in C$ or $c = (s', s) \in C$: $\varphi_c^{\mathcal{S}}(R) \equiv \text{top}(R_c, q^k)$, $q^{k+1} \equiv q^k \backslash \varphi_c^{\mathcal{S}}(R)$, $N^{k+1} \equiv N^k \backslash \{i, j\}$, and set $k = k + 1$.

If the members of a couple together have the highest priority for all remaining available position types given by $q^k$, then the couple chooses its best pair of positions in $\mathcal{X}(q^k)$. After the couple's assignment, the set of remaining available positions is described by $q^{k+1}$.

The allocation $\varphi^{\mathcal{S}}(R)$ is obtained after at most $p$ steps. $\triangle$

Note that when all quotas are equal to one, the sequential placement mechanism and the DB placement mechanism are identical.

**Example 5.7 *An Application of the Sequential Placement Mechanism***
Let $N = \{1, \ldots, 8\}$ be such that $C = \{c_1 = (1, 2), c_2 = (3, 4)\}$ and $S = \{5, 6, 7, 8\}$; and $X = \{x, y, z\}$ with $q_x = q_y = q_z = 3$. We depict the *pt*-acyclic priority structure $\succ$ and preference profile $R \in \mathcal{R}$ in the two tables below.

| $\succ_x$ | $\succ_y$ | $\succ_z$ |
|---|---|---|
| 5 | 6 | 7 |
| 6 | 7 | 5 |
| 7 | 5 | 6 |
| 1 | 2 | 1 |
| 2 | 1 | 2 |
| 3 | 3 | 3 |
| 4 | 4 | 4 |
| 8 | 8 | 8 |

and

| $R_{c_1}$ | $R_{c_2}$ | $R_5$ | $R_6$ | $R_7$ | $R_8$ |
|---|---|---|---|---|---|
| $(x, x)$ | $(x, x)$ | $y$ | $x$ | $x$ | $x$ |
| $(z, x)$ | $(z, x)$ | $x$ | $y$ | $y$ | $z$ |
| $(x, z)$ | $(x, z)$ | $z$ | $z$ | $z$ | $0$ |
| $\cdots$ | $(z, z)$ | $0$ | $0$ | $0$ | $y$ |
| | $\cdots$ | | | | |

Priority structure $\succ$ induces the strongly *pt*-acyclic partition $S = \{S_1, S_2, S_3, S_4, S_5\}$ with $S_1 = \{5, 6, 7\}$, $S_2 = \{1, 2\}$, $S_3 = \{3\}$, $S_4 = \{4\}$, and $S_5 = \{8\}$. Note that $S_4 \cup S_5 = \{3, 4\}$ consists of couple $c_2$. We now calculate $\varphi^{\mathcal{S}}(R)$. Recall that $N^1 = N$ and $q^1 \equiv q = (q_x, q_y, q_z) = (3, 3, 3)$.

**Step 1:** $S_1 = \{5, 6, 7\} \subseteq S$: $\varphi_5^{\mathcal{S}}(R) = y$, $\varphi_6^{\mathcal{S}}(R) = x$, $\varphi_7^{\mathcal{S}}(R) = x$, $N^2 = \{1, 2, 3, 4, 8\}$, and $q^2 = (q_x^2, q_y^2, q_z^2) = (1, 2, 3)$.

**Step 2:** $S_2 = \{1, 2\}$ and $c_1 = (1, 2) \in C$: $\varphi_{c_1}^{\mathcal{S}}(R) = (z, x)$, $N^3 = \{3, 4, 8\}$, and $q^3 = (q_x^3, q_y^3, q_z^3) = (0, 2, 2)$.

**Step 3:** $S_3 \cup S_4 = \{3, 4\}$ and $c_2 = (3, 4) \in C$: $\varphi_{c_2}^{\mathcal{S}}(R) = (z, z)$, $N^5 = \{8\}$, and $q^5 = (q_x^5, q_y^5, q_z^5) = (0, 2, 0)$.

**Step 5:** $S_5 = \{8\} \subseteq S$: $\varphi_8^{\mathcal{S}}(R) = 0$.

Hence, $\varphi^{\mathcal{S}}(R) = (z, x, z, z, y, x, x, 0)$. $\diamond$

Our next result is that for *pt*-acyclic priority structures that induce strongly *pt*-acyclic partitions the sequential placement mechanism is fair and efficient.

**Theorem 5.3** *General Quotas: Sufficiency of a Strongly pt-Acyclic Partition*
*Let $\succ$ be a pt-acyclic priority structure that induces a strongly pt-acyclic partition. Then, the sequential placement mechanism $\varphi^{\mathcal{S}}$ is fair and efficient.*

**Remark 5.2** *Responsiveness in Student Placement and Two-Sided Matching*
The student placement model we consider and two-sided matching markets (see Roth and Sotomayor, 1990) are closely related. In our context of student placement, we consider students' preferences and a priority structure as inputs and focus on the properties fairness and efficiency. In contrast, a two-sided matching problem also consists of students' preferences, but priorities over position types are now replaced by the institutions that offer these position types (*e.g.*, universities, firms, or hospitals) and their preferences over students or sets of students. An important property for two-sided matching is stability: loosely speaking, an outcome or matching is stable if there are no students (couples) and no institutions that are not matched with each other, but in fact would prefer to be. A key result (among many other results) for two-sided matching problems is that under the appropriate substitutability or responsiveness condition stable matchings always exist; see for instance Roth (1985, many-to-one matching without money), Kelso and Crawford (1982, many-to-one matching with money), Alkan and Gale (2003, many-to-many schedule matching), Klaus and Klijn (2005, many-to-one matching with couples), and Hatfield and Milgrom (2005, two-sided matching with contracts). More specifically, whenever couples with non-responsive preferences are present in a two-sided matching market, a stable outcome may not exist; see Roth (1984) and Klaus and Klijn (2005). Klaus and Klijn (2005,2006) show that indeed responsiveness of couples' preferences is often a necessary condition for their results.

Because of the important role that responsiveness plays in two-sided matching markets, one may wonder if one could obtain stronger results for the student placement problems with couples if couples' preferences are restricted to always be responsive. However, since all proofs are designed in such a way that they also apply if couples' preferences are responsive, requiring that all couples' preferences are responsive will not change any of our results.

**An Open Problem:** *Necessity and Uniqueness*
We have shown that *pt*-acyclicity of the priority structure is a necessary and, in combination with strong *pt*-acyclicity of the induced partition, a sufficient condition for the existence of a fair and efficient placement mechanism. The determination of further necessary conditions is an open problem. Consequently, we were also not able to address the question of uniqueness for general placement problems.

# Appendix: Proofs

**Proof of Theorem 4.1.**
**(a)** Assume $\succ$ violates acyclicity. Thus, there exists a cycle, *i.e.*, there exist distinct $x, y \in X$ and $i, j, k \in N$ such that the following two conditions are satisfied:
*cycle condition* $\quad i \succ_x j \succ_x k$ and $k \succ_y i$ and
*c-scarcity condition* $\quad$ there exist disjoint sets $N_x, N_y \subseteq N \setminus \{i, j, k\}$ (possibly $N_x = \emptyset$ or $N_y = \emptyset$) such that $N_x \subseteq \{l \in N : l \succ_x j\}$, $N_y \subseteq \{l \in N : l \succ_y i\}$, $|N_x| = q_x - 1$, and $|N_y| = q_y - 1$.
We now construct a preference profile $R \in \mathcal{R}$. After that we complete the proof by showing that no fair and efficient mechanism exists.

First, we specify preferences in $\mathcal{RS}$ for *all* students $s \in N$. Let $N_0 \equiv N \setminus [\{i, j, k\} \cup N_x \cup N_y]$.
**(SP) Students' Preferences:** Let $s \in N$ and $R_s \in \mathcal{RS}$ be such that
$\quad$ if $s \in N_0$, then $0 \ P_s \ldots$ ,

if $s \in N_x \cup \{j\}$, then $x \, P_s \, 0 \, P_s \ldots$,

if $s \in N_y$, then $y \, P_s \, 0 \, P_s \ldots$,

if $s = i$, then $y \, P_s \, x \, P_s \, 0 \, P_s \ldots$, and

if $s = k$, then $x \, P_s \, y \, P_s \, 0 \, P_s \ldots$,

where the "tail" of any of the above preference relations, denoted by "$\ldots$", can be any fixed linear order of the remaining position types.

Second, using the above specification of preferences for students, we specify responsive preferences for all possible couples.

**(CP) Couples' Preferences:** Let $c = (l, m) \in C$. We specify $R_c \in \mathcal{RRC}$ such that couple $c$'s preferences are responsive with respect to associated individual preferences $R_l$ and $R_m$ that are as above. Let $R_c \in \mathcal{RRC}$ be such that[6]

if $l, m \in N_0$, then $(0, 0) \, P_c \ldots$,

if $l \in N_0$ and $m \in N_x \cup \{j\}$, then $(0, x) \, P_c \, (0, 0) \, P_c \ldots$,

if $l \in N_0$ and $m \in N_y$, then $(0, y) \, P_c \, (0, 0) \, P_c \ldots$,

if $l \in N_0$ and $m = i$, then $(0, y) \, P_c \, (0, x) \, P_c \, (0, 0) \, P_c \ldots$,

if $l \in N_0$ and $m = k$, then $(0, x) \, P_c \, (0, y) \, P_c \, (0, 0) \, P_c \ldots$,

if $l \in N_x \cup \{j\}$ and $m \in N_0$, then $(x, 0) \, P_c \, (0, 0) \, P_c \ldots$,

if $l \in N_x$ and $m \in N_x \cup \{j\}$, then $(x, x) \, P_c \, (x, 0) \, P_c \, (0, x) \, P_c \, (0, 0) \, P_c \ldots$,

if $l = j$ and $m \in N_x$, then $(x, x) \, P_c \, (0, x) \, P_c \, (x, 0) \, P_c \, (0, 0) \, P_c \ldots$,

if $l \in N_x \cup \{j\}$ and $m \in N_y$, then $(x, y) \, P_c \, (0, y) \, P_c \, (x, 0) \, P_c \, (0, 0) \, P_c \ldots$,

if $l \in N_x$ and $m = i$, then $(x, y) \, P_c \, (x, x) \, P_c \, (x, 0) \, P_c \, (0, y) \, P_c \, (0, x) \, P_c \, (0, 0) \, P_c \ldots$,

if $l = j$ and $m = i$, then $(x, y) \, P_c \, (0, y) \, P_c \, (x, x) \, P_c \, (0, x) \, P_c \, (x, 0) \, P_c \, (0, 0) \, P_c \ldots$,

if $l \in N_x \cup \{j\}$ and $m = k$, then $(x, x) \, P_c \, (x, y) \, P_c \, (x, 0) \, P_c \, (0, x) \, P_c \, (0, y) \, P_c \, (0, 0) \, P_c \ldots$,

if $l \in N_y$ and $m \in N_0$, then $(y, 0) \, P_c \, (0, 0) \, P_c \ldots$,

if $l \in N_y$ and $m \in N_x \cup \{j\}$, then $(y, x) \, P_c \, (y, 0) \, P_c \, (0, x) \, P_c \, (0, 0) \, P_c \ldots$,

if $l, m \in N_y$, then $(y, y) \, P_c \, (y, 0) \, P_c \, (0, y) \, P_c \, (0, 0) \, P_c \ldots$,

if $l \in N_y$ and $m = i$, then $(y, y) \, P_c \, (y, x) \, P_c \, (y, 0) \, P_c \, (0, y) \, P_c \, (0, x) \, P_c \, (0, 0) \, P_c \ldots$,

if $l \in N_y$ and $m = k$, then $(y, x) \, P_c \, (y, y) \, P_c \, (y, 0) \, P_c \, (0, x) \, P_c \, (0, y) \, P_c \, (0, 0) \, P_c \ldots$,

if $l = i$ and $m \in N_0$, then $(y, 0) \, P_c \, (x, 0) \, P_c \, (0, 0) \, P_c \ldots$,

if $l = i$ and $m \in N_x$, then $(y, x) \, P_c \, (x, x) \, P_c \, (0, x) \, P_c \, (y, 0) \, P_c \, (x, 0) \, P_c \, (0, 0) \, P_c \ldots$,

if $l = i$ and $m = j$, then $(y, x) \, P_c \, (y, 0) \, P_c \, (x, x) \, P_c \, (x, 0) \, P_c \, (0, x) \, P_c \, (0, 0) \, P_c \ldots$,

if $l = i$ and $m \in N_y$, then $(y, y) \, P_c \, (x, y) \, P_c \, (0, y) \, P_c \, (y, 0) \, P_c \, (x, 0) \, P_c \, (0, 0) \, P_c \ldots$,

if $l = i$ and $m = k$,

   then $(y, x) \, P_c \, (y, y) \, P_c \, (x, x) \, P_c \, (x, y) \, P_c \, (y, 0) \, P_c \, (x, 0) \, P_c \, (0, x) \, P_c \, (0, y) \, P_c \, (0, 0) \, P_c \ldots$,

if $l = k$ and $m \in N_0$, then $(x, 0) \, P_c \, (y, 0) \, P_c \, (0, 0) \, P_c \ldots$,

if $l = k$ and $m \in N_x \cup \{j\}$, then $(x, x) \, P_c \, (y, x) \, P_c \, (0, x) \, P_c \, (x, 0) \, P_c \, (y, 0) \, P_c \, (0, 0) \, P_c \ldots$,

if $l = k$ and $m \in N_y$, then $(x, y) \, P_c \, (y, y) \, P_c \, (0, y) \, P_c \, (x, 0) \, P_c \, (y, 0) \, P_c \, (0, 0) \, P_c \ldots$, and

if $l = k$ and $m = i$,

   then $(x, y) \, P_c \, (y, y) \, P_c \, (x, x) \, P_c \, (y, x) \, P_c \, (0, y) \, P_c \, (0, x) \, P_c \, (x, 0) \, P_c \, (y, 0) \, P_c \, (0, 0) \, P_c \ldots$,

where the "tail" of any of the above preference relations, denoted by "$\ldots$", can be any fixed linear order of the remaining position type assignments that complies with the responsiveness requirement induced by the associated individual preferences.

Finally, we define $R \in \mathcal{R}$ such that single students have preferences as specified in (SP) and couples have preferences as specified in (CP). We now complete the proof by showing that there is no fair and efficient mechanism $\varphi$. Suppose to the contrary there is such a mechanism.

**Step 1:** We prove that for all $l \in N_0$, $\varphi_l(R) = 0$.

---

[6]If $q_x = 1$, then delete $(x, x)$ from all couples' preferences specified below. If $q_y = 1$, then delete $(y, y)$ from all couples' preferences specified below.

Let $l \in N_0 \cap S$. By efficiency, $\varphi_l(R) = 0$. Let $l \in N_0 \backslash S$ and assume that $\varphi_l(R) \neq 0$ and $c = (l, m)$ (for $c = (m, l)$ interchange the roles of $l$ and $m$). Recall that the couple's preferences $R_c$ are responsive with respect to $R_l, R_m \in \mathcal{RS}$. Thus, $0\, P_l\, \varphi_l(R)$ implies $(0, \varphi_m(R))\, P_c\, (\varphi_l(R), \varphi_m(R))$. Since the null position is freely available, this contradicts efficiency.

**Step 2:** We prove that for all $l \in N_x$, $\varphi_l(R) = x$.

Let $l \in N_x$ and assume that $\varphi_l(R) \neq x$. Then, by efficiency, the definition of single students' preferences, and responsiveness of couples' preferences, $\varphi_l(R) = 0$ and all position types $x$ have to be assigned to students in $[N_x \cup \{i, j, k\}]\backslash\{l\}$. Hence, there exist (at least) two distinct students in $\{i, j, k\}$ whose allotment equals $x$. Thus,

$$\varphi_j(R) = x \text{ or } \varphi_k(R) = x. \tag{1}$$

Let $l \in N_x \cap S$. Since $l \succ_x j \succ_x k$, (1) violates student $l$'s priority (D3.2). Let $l \in N_x \backslash S$ and $c = (l, m)$ (for $c = (m, l)$ interchange the roles of $l$ and $m$). If $m \in N \backslash \{j, k\}$, then by responsiveness of $R_c$, $(x, \varphi_m(R))\, P_c\, (\varphi_l(R), \varphi_m(R))$. (Note that $(x, \varphi_m(R)) \in \mathcal{X}$ since $q_x \geq 2$ as there are at least two distinct students in $\{i, j, k\}$ whose allotment equals $x$.) Since $l \succ_x j \succ_x k$, (1) violates couple $c$'s priority (D3.3, a.1).

If $m \in \{j, k\}$ and $\varphi_m(R) = x$, then by our definition of $R_c$, $(x, 0)\, P_c\, (0, x) = (\varphi_l(R), \varphi_m(R))$. Since $l \succ_x m$, (1) violates couple $c$'s priority (D3.3, b.1).

If $m \in \{j, k\}$ and $\varphi_m(R) \neq x$, then by responsiveness of $R_c$, $(x, \varphi_m(R))\, P_c\, (\varphi_l(R), \varphi_m(R))$. Let $\{m, m'\} = \{j, k\}$. Then, by (1), $\varphi_{m'}(R) = x$. Since $l \succ_x m'$, $\varphi_{m'}(R) = x$ violates couple $c$'s priority (D3.3, a.1).

**Step 3:** We prove that for all $l \in N_y$, $\varphi_l(R) = y$.

Let $l \in N_y$ and assume that $\varphi_l(R) \neq y$. Then, by efficiency, the definition of single students' preferences, and responsiveness of couples' preferences, $\varphi_l(R) = 0$ and all position types $y$ have to be assigned to students in $[N_y \cup \{i, k\}]\backslash\{l\}$. Hence, $\varphi_k(R) = y$ and

$$\varphi_i(R) = y. \tag{2}$$

Let $l \in N_y \cap S$. Since $l \succ_y i$, (2) violates student $l$'s priority (D3.2). Let $l \in N_y \backslash S$ and $c = (l, m)$ (for $c = (m, l)$ interchange the roles of $l$ and $m$). If $m \in N \backslash \{i\}$, then by responsiveness of $R_c$, $(y, \varphi_m(R))\, P_c\, (\varphi_l(R), \varphi_m(R))$. (Note that $(y, \varphi_m(R)) \in \mathcal{X}$ since $q_y \geq 2$ as $\varphi_k(R) = \varphi_i(R) = y$.) Since $l \succ_y i$, (2) violates couple $c$'s priority (D3.3, a.1).

If $m = i$, then by our definition of $R_c$, $(y, 0)\, P_c\, (0, y) = (\varphi_l(R), \varphi_m(R))$. Since $l \succ_y m$, (2) violates couple $c$'s priority (D3.3, b.1).

Note that now only one position of type $x$ and one position of type $y$ is "left to be assigned" to the students in $\{i, j, k\}$.

**Step 4:** We prove that $\varphi_j(R) = x$.

Suppose that $\varphi_j(R) \neq x$. Then, by efficiency, the definition of single students' preferences, and responsiveness of couples' preferences, $\varphi_j(R) = 0$, $\varphi_i(R) = y$, and $\varphi_k(R) = x$. Suppose $j \in S$. Since $j \succ_x k$, $\varphi_k(R) = x$ violates student $j$'s priority (D3.2). So suppose $j \in N \backslash S$ and let $c = (j, m)$ (for $c = (m, j)$ interchange the roles of $j$ and $m$). If $m \in N \backslash \{k\}$, then by responsiveness of $R_c$, $(x, \varphi_m(R))\, P_c\, (\varphi_j(R), \varphi_m(R))$. (Note that $(x, \varphi_m(R)) \in \mathcal{X}$. If not, then $q_x = 1$ and $\varphi_m(R) = x$, contradicting $m \neq k$.) Since $j \succ_x k$, $\varphi_k(R) = x$ violates couple $c$'s priority (D3.3, a.1).

If $m = k$, then by our definition of $R_c$, $(x, 0)\, P_c\, (0, x) = (\varphi_j(R), \varphi_m(R))$. Since $j \succ_x m$, $\varphi_k(R) = x$ violates couple $c$'s priority (D3.3, b.1).

Now only one position of type $y$ is "left to be assigned" to either student $i$ or $k$. Hence, by efficiency, the definition of single students' preferences, and responsiveness of couples' preferences, either $[\varphi_i(R) = y$ and $\varphi_k(R) = 0]$ or $[\varphi_i(R) = 0$ and $\varphi_k(R) = y]$.

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**Step 5:** We obtain a contradiction.

Suppose $i \in S$. Suppose $\varphi_i(R) = y$ and $\varphi_k(R) = 0$. Recall that $k \succ_y i$. If $k \in S$, $\varphi_i(R) = y$ violates student $k$'s priority (D3.2).

If $k \in N \backslash S$, then let $c = (k, m)$ (for $c = (m, k)$ interchange the roles of $k$ and $m$). By responsiveness of $R_c$, $(y, \varphi_m(R)) \, P_c \, (\varphi_k(R), \varphi_m(R))$. (Note that $(y, \varphi_m(R)) \in \mathcal{X}$. If not, then $q_y = 1$ and $\varphi_m(R) = y$, contradicting $m \neq i$.) Since $k \succ_y i$, $\varphi_i(R) = y$ violates couple $c$'s priority (D3.3, a.1). Hence, $\varphi_i(R) = 0$ and $\varphi_k(R) = y$.

Note that $x P_i \varphi_i(R)$. Since $i \succ_x j$, $\varphi_j(R) = x$ violates student $i$'s priority (D3.2). So suppose $i \in N \backslash S$ and let $c = (i, m)$ (for $c = (m, i)$ interchange the roles of $i$ and $m$). If $m = k$, then by $(y, 0) \, P_c \, (0, y)$ and efficiency, $\varphi_i(R) = y$ and $\varphi_k(R) = 0$. Thus, by the construction of preference relation $R_c$, $(x, y) \, P_c \, (\varphi_i(R), \varphi_k(R))$. Since $i \succ_x j$ and $k \succ_y i$, $\varphi_j(R) = x$ and $\varphi_i(R) = y$ violate couple $c$'s priority (D3.3, c). Hence, $m \neq k$. Suppose $\varphi_i(R) = y$ and $\varphi_k(R) = 0$. Recall that $k \succ_y i$. If $k \in S$, $\varphi_i(R) = y$ violates student $k$'s priority (D3.2).

If $k \in N \backslash S$, then let $\bar{c} = (k, \bar{m})$ (for $\bar{c} = (\bar{m}, k)$ interchange the roles of $k$ and $\bar{m}$). By responsiveness of $R_{\bar{c}}$, $(y, \varphi_{\bar{m}}(R)) \, P_{\bar{c}} \, (\varphi_k(R), \varphi_{\bar{m}}(R))$. (Note that $(y, \varphi_{\bar{m}}(R)) \in \mathcal{X}$. If not, then $q_y = 1$ and $\varphi_{\bar{m}}(R) = y$, contradicting $\bar{m} \neq i$.) Since $k \succ_y i$, $\varphi_i(R) = y$ violates couple $\bar{c}$'s priority (D3.3, a.1). Hence, $\varphi_i(R) = 0$ and $\varphi_k(R) = y$. If $m \in N \backslash \{j, k\}$, then by responsiveness of $R_c$, $(x, \varphi_m(R)) \, P_c \, (\varphi_i(R), \varphi_m(R))$. (Note that $(x, \varphi_m(R)) \in \mathcal{X}$. If not, then $q_x = 1$ and $\varphi_m(R) = x$, contradicting $m \neq j$.) Since $i \succ_x j$, $\varphi_j(R) = x$ violates couple $c$'s priority (D3.3, a.1).

If $m = j$, then by Step 4, $\varphi_c(R) = (0, x)$. Since $(x, 0) P_c (0, x) = \varphi_c(R)$, $\varphi(R)$ is not an efficient allocation. Alternatively, since $i \succ_x j$, $\varphi_j(R) = x$ violates couple $c$'s priority (D3.3, b.1).

**(b)** Let $N = \{1, 2, 3\}$ be such that $C = \{(1, 2)\}$ and $S = \{3\}$, $X = \{x\}$, and $q_x = 1$. Let $\succ$ be such that $1 \succ_x 3 \succ_x 2$. Since $|X| = 1$, $\succ$ is acyclic. Assume that $\varphi$ is fair and efficient. Consider $R = (R_c, R_3) \in \mathcal{R}$ such that $(0, x) \, P_c \, (x, 0) \, P_c \, (0, 0)$ and $x P_3 0$. Note that $R_c \in \mathcal{RRC}$. If $\varphi_3(R) = x$, then $\varphi_c(R) = (0, 0)$. Since $1 \succ_x 3$, $(x, 0) \, P_c \, (0, 0) = \varphi_c(R)$ violates couple $c$'s priority (D3.3, a.1). If $\varphi_3(R) = 0$, by efficiency, $\varphi_c(R) = (0, x)$. Since $3 \succ_x 2$, $x P_3 0 = \varphi_3(R)$ violates student 3's priority (D3.2). $\square$

**Proof of Theorem 4.2.** By Theorem 4.1 (a) we already know that acyclicity is necessary for the existence of a fair and efficient placement mechanism. So assume $\succ$ violates priority-togetherness, *i.e.*, there exists $\bar{c} = (i, j) \in C$ that is priority-separated. Without loss of generality there exist $x \in X$ and $k \in N$ such that

*priority-separation condition*   $i \succ_x k \succ_x j$ and

*ps-scarcity condition*   there exists a set $\bar{N}_x \subseteq N \backslash \{i, j, k\}$ (possibly $\bar{N}_x = \emptyset$) such that $\bar{N}_x \subseteq \{l \in N : l \succ_x j\}$ and $|\bar{N}_x| = q_x - 1$.

We now construct a preference profile $R \in \mathcal{R}$. After that we complete the proof by showing that no fair and efficient mechanism exists.

First, we specify preferences in $\mathcal{RS}$ for *all* students $s \in N$. Let $\bar{N}_0 \equiv N \backslash [\{i, j, k\} \cup \bar{N}_x]$.

**(SP) Students' Preferences:** Let $s \in N$ and $R_s \in \mathcal{RS}$ be such that

  if $s \in \bar{N}_0$, then $0 \, P_s \ldots$,

  if $s \in \bar{N}_x \cup \{i, j, k\}$, then $x \, P_s \, 0 \, P_s \ldots$,

where the "tail" of any of the above preference relations, denoted by "$\ldots$", can be any fixed linear order of the remaining position types.

Second, using the above specification of preferences for students, we specify responsive preferences for all possible couples.

**(CP) Couples' Preferences:** Let $c = (l, m) \in C \backslash \{\bar{c}\}$. We specify $R_c \in \mathcal{RRC}$ such that couple $c$'s preferences are responsive with respect to associated individual preferences $R_l$ and

$R_m$ as above. Without loss of generality, assume that $q_x > 1$. Let $R_c \in \mathcal{RRC}$ be such that[7]

    if $l, m \in \bar{N}_0$, then $(0,0) \, P_c \ldots$ ,

    if $l \in \bar{N}_0$ and $m \in \bar{N}_x \cup \{k\}$, then $(0,x) \, P_c \, (0,0) \, P_c \ldots$ ,

    if $l \in \bar{N}_x \cup \{k\}$ and $m \in \bar{N}_0$, then $(x,0) \, P_c \, (0,0) \, P_c \ldots$ ,

    if $l, m \in \bar{N}_x \cup \{k\}$, then $(x,x) \, P_c \, (0,x) \, P_c \, (x,0) \, P_c \, (0,0) \, P_c \ldots$ ,

and in addition

    $(x,x) \, P_{\bar{c}} \, (0,x) \, P_{\bar{c}} \, (x,0) \, P_{\bar{c}} \, (0,0) \, P_{\bar{c}} \ldots$ ,

where the "tail" of any of the above preference relations, denoted by "$\ldots$", can be any fixed linear order of the remaining position type assignments that complies with the responsiveness requirement induced by the associated individual preferences.

Finally, we define $R \in \mathcal{R}$ such that single students have preferences as specified in (SP) and couples have preferences as specified in (CP). We now complete the proof by showing that there is no fair and efficient mechanism $\varphi$. Suppose to the contrary there is such a mechanism.

**Step 1:** We prove that for all $l \in \bar{N}_0$, $\varphi_l(R) = 0$.

Let $l \in \bar{N}_0 \cap S$. By efficiency, $\varphi_l(R) = 0$. Let $l \in \bar{N}_0 \backslash S$ and assume that $\varphi_l(R) \neq 0$ and $c = (l, m)$ (for $c = (m, l)$ interchange the roles of $l$ and $m$). Recall that the couple's preferences $R_c$ are responsive with respect to $R_l \in \mathcal{RS}$ and $R_m \in \mathcal{RS}$. Thus, $0 \, P_l \, \varphi_l(R)$ implies $(0, \varphi_m(R)) \, P_c \, (\varphi_l(R), \varphi_m(R))$. Since the null position is freely available, this contradicts efficiency.

**Step 2:** We prove that for all $l \in \bar{N}_x \cup \{k\}$, $\varphi_l(R) = x$.

Let $l \in \bar{N}_x \cup \{k\}$ and assume that $\varphi_l(R) \neq x$. Then, by efficiency, the definition of single students' preferences, and responsiveness of couples' preferences, $\varphi_l(R) = 0$ and all position types $x$ have to be assigned to students in $[\bar{N}_x \cup \{i, j, k\}] \backslash \{l\}$. Hence,

$$\varphi_i(R) = x \text{ or } \varphi_j(R) = x. \tag{3}$$

Assume that $\varphi_j(R) = x$. Let $l \in \left[\bar{N}_x \cup \{k\}\right] \cap S$. Since $l \succ_x j$, $\varphi_j(R) = x$ violates student $l$'s priority (D3.2). Let $l \in \left[\bar{N}_x \cup \{k\}\right] \backslash S$ and $c = (l, m)$ (for $c = (m, l)$ interchange the roles of $l$ and $m$). Then, by responsiveness of $R_c$, $(x, \varphi_m(R)) \, P_c \, (\varphi_l(R), \varphi_m(R))$. (Note that $(x, \varphi_m(R)) \in \mathcal{X}$. If not, then $q_x = 1$ and $\varphi_m(R) = x$. Hence, $m = j$. But then $c = \bar{c}$, i.e., $l = i$. Hence, $i = l \in \bar{N}_x \cup \{k\}$, a contradiction.) Since $l \succ_x j$, $\varphi_j(R) = x$ violates couple $c$'s priority (D3.3, a.1). Hence, $\varphi_j(R) = 0$ and by (3), $\varphi_{\bar{c}}(R) = (x, 0)$. Since $(0, x) \, P_{\bar{c}} \, (x, 0) = \varphi_{\bar{c}}(R)$, $\varphi(R)$ is not an efficient allocation.

**Step 3:** We obtain a contradiction.

Step 2 implies that $\varphi_k(R) = x$ and $\varphi_{\bar{c}}(R) = (0, 0)$. Since $i \succ_x k$, $(x, 0) \, P_{\bar{c}} \, (0, 0) = \varphi_{\bar{c}}(R)$ violates couple $\bar{c}$'s priority (D3.3, a.1).     $\square$

**Proof of Lemma 5.1.** Recall that all quotas equal one. We prove Lemma 5.1 in four steps. Steps 1 and 2 show that $pt$-acyclicity implies conditions (i) and (ii) and Steps 3 and 4 show that conditions (i) and (ii) imply $pt$-acyclicity.

**Step 1:** "acyclicity $\Rightarrow$ (i)"

Let $\succ$ be acyclic and $k \in \{1, \ldots, p\}$. We prove that acyclicity implies (i.a) and (i.b) for $S_k$. Assume that (i.a) is violated for $S_k$. Suppose that $|S_k| > 2$, e.g., there exist distinct $i_1, i_2, i_3$ such that $S_k = \{i_1, i_2, i_3, \ldots\} \subseteq N \backslash \bigcup_{l=1}^{k-1} S_l$.[8] Since $i_1, i_2, i_3 \in S_k$, there exist distinct $x_1, x_2, x_3 \in X$ such that $i_1 = \text{top}(x_1, N \backslash \bigcup_{l=1}^{k-1} S_l)$, $i_2 = \text{top}(x_2, N \backslash \bigcup_{l=1}^{k-1} S_l)$, and $i_3 = \text{top}(x_3, N \backslash \bigcup_{l=1}^{k-1} S_l)$. Hence, $i_1 \succ_{x_1} i_2$, $i_1 \succ_{x_1} i_3$, $i_2 \succ_{x_2} i_1$, and $i_3 \succ_{x_3} i_1$. Note that either

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[7]If $q_x = 1$, then delete $(x,x)$ from all couples' preferences specified below.

[8]If $k = 1$, set $\bigcup_{l=1}^{k-1} S_l = \emptyset$.

$i_2 \succ_{x_1} i_3$ or $i_3 \succ_{x_1} i_2$. Thus, in contradiction to acyclicity, either $[i_1 \succ_{x_1} i_2 \succ_{x_1} i_3$ and $i_3 \succ_{x_3} i_1]$ or $[i_1 \succ_{x_1} i_3 \succ_{x_1} i_2$ and $i_2 \succ_{x_2} i_1]$ constitutes a cycle. Hence, (i.a) for $S_k$ is implied by acyclicity.

Assume that (i.b) is violated for $S_k$. Suppose $S_k = \{i, j\}$ and there exists $x \in X$ such that $i = \text{top}(x, N \backslash \bigcup_{l=1}^{k-1} S_l)$ and $j \neq \text{top}(x, N \backslash [\bigcup_{l=1}^{k-1} S_l \cup \{i\}])$. Thus, there exists $j' \in N \backslash [\bigcup_{l=1}^{k-1} S_l \cup \{i, j\}]$ such that $j' = \text{top}(x, N \backslash [\bigcup_{l=1}^{k-1} S_l \cup \{i\}])$. Hence, $i \succ_x j' \succ_x j$. Since $j \in S_k$, there exists $y \in X$ such that $j = \text{top}(y, N \backslash \bigcup_{l=1}^{k-1} S_l)$. Hence, $j \succ_y i$. Thus, in contradiction to acyclicity, $i \succ_x j' \succ_x j$ and $j \succ_y i$ constitutes a cycle. Hence, (i.b) for $S_k$ is implied by acyclicity.

**Step 2:** "$pt$-acyclicity $\Rightarrow$ (ii)"

Let $\succ$ be $pt$-acyclic. Hence, $\succ$ is acyclic and by Step 1, Lemma 5.1 (i) (and therefore Lemma 5.2) applies. Assume that there exists a couple $c = (i, j) \in C$ such that (ii) is violated. Then, there exists $k \in \{1, \ldots, p\}$ such that $[S_k = \{i, m\}$ for $m \in N \backslash \{i, j\}]$ or $[S_k = \{j, m\}$ for $m \in N \backslash \{i, j\}]$, or $[S_k = \{i\}$ and there exists $k' \in \{1, \ldots, p\} \backslash \{k-1, k, k+1\}$ such that $S_{k'} = \{j\}]$.

First, assume that $S_k = \{i, m\}$ for $m \in N \backslash \{i, j\}$. Let $k' \in \{1, \ldots, p\}$ be such that $j \in S_{k'}$. If $k < k'$, then by Lemma 5.2 (b), for all $x \in X$, $i \succ_x j$ and $m \succ_x j$. Since $i \in S_k$, there exists $y \in X$ such that $i = \text{top}(y, N \backslash \bigcup_{l=1}^{k-1} S_l)$. Hence, $i \succ_y m$. Thus, $i \succ_y m \succ_y j$ and, in contradiction to priority-togetherness, couple $c$ is priority-separated. If $k > k'$, then by Lemma 5.2 (b), for all $x \in X$, $j \succ_x i$ and $j \succ_x m$. Since $m \in S_k$, there exists $y \in X$ such that $m = \text{top}(y, N \backslash \bigcup_{l=1}^{k-1} S_l)$. Hence, $m \succ_y i$. Thus, $j \succ_y m \succ_y i$, and, in contradiction to priority-togetherness, couple $c$ is priority-separated.

Second, assume $S_k = \{j, m\}$ for $m \in N \backslash \{i, j\}$. By interchanging the roles if $i$ and $j$ in the proof above we again obtain a contradiction to priority-togetherness.

Third, assume that $S_k = \{i\}$ and there exists $k' \in \{1, \ldots, p\} \backslash \{k-1, k, k+1\}$ such that $S_{k'} = \{j\}$. If $k < k'$, then $k < k+1 < k'$. Let $m \in S_{k+1}$. Then, by Lemma 5.2 (b), for all $x \in X$, $i \succ_x m \succ_x j$, which contradicts priority-togetherness. If $k > k'$, then $k > k-1 > k'$. Let $m \in S_{k-1}$. Then, by Lemma 5.2 (b), for all $x \in X$, $j \succ_x m \succ_x i$, which contradicts priority-togetherness.

**Step 3:** "(i) $\Rightarrow$ acyclicity"

Let $\mathcal{S}$ satisfy (i). Then, Lemma 5.2 applies as well. Let $i_1, i_2, i_3 \in N$ and $x \in X$ be such that $i_1 \succ_x i_2 \succ_x i_3$. Furthermore, let $k_1, k_3 \in \{1, \ldots, p\}$ be such that $i_1 \in S_{k_1}$ and $i_3 \in S_{k_3}$. By Lemma 5.2 (a), $k_1 < k_3$. Hence, by Lemma 5.2 (b), for all $y \in X$, $i_1 \succ_y i_3$ and no cycles exist.

**Step 4:** "(i) and (ii) $\Rightarrow$ priority-togetherness"

Let $\mathcal{S}$ satisfy (i) and (ii). Then, Lemma 5.2 applies as well. Assume that priority-togetherness is violated, i.e., there exists a couple $c = (i, j) \in C$, $m \in N$, and $x \in X$ such that either $i \succ_x m \succ_x j$ or $j \succ_x m \succ_x i$. Assume that $i \succ_x m \succ_x j$ (for $j \succ_x m \succ_x i$ interchange the roles of $i$ and $j$). Let $i \in S_k$ and $j \in S_{k'}$. Lemma 5.2 (a) and $i \succ_x m \succ_x j$ imply that $k < k'$. Hence, neither (ii.a) nor (ii.b) can be true. Thus, by (ii.c), $k' = k+1$. Hence, $S_k = \{i\}$ and $S_{k+1} = \{j\}$. Let $l$ be such that $m \in S_l$. Since $i \succ_x m \succ_x j$, from Lemma 5.2 (b) it follows that $k < l$ and $l < k+1$, a contradiction. $\square$

**Proof of Theorem 5.2.** Recall that all quotas equal one. Since $\varphi^{\mathcal{S}}$ is fair and efficient (Theorem 5.1) it suffices to show uniqueness. So, suppose that $\varphi$ is fair and efficient. Let $R \in \mathcal{R}$. We prove that $\varphi(R) = \varphi^{\mathcal{S}}(R)$. Let $k \in \{1, \ldots, p\}$ be such that Step $k$ is well-defined in Definition 5.3. Assume that for all $i \in \bigcup_{l<k} S_l$, $\varphi_i(R) = \varphi_i^{\mathcal{S}}(R)$ (this is vacuously true for $\bigcup_{l<1} S_l \equiv \emptyset$). We prove that for any student $i \in N$ that is assigned his/her allotment $\varphi_i^{\mathcal{S}}(R)$ in Step $k$ of Definition 5.3, it holds that $\varphi_i(R) = \varphi_i^{\mathcal{S}}(R)$. We distinguish among the following cases:

**(a)** $S_k = \{s\}$ and $s \in S$;

**(b)** $S_k = \{s\}$, $s \notin S$, and $S_k \cup S_{k+1} = \{s, s'\}$ such that $c = (s, s') \in C$ or $c = (s', s) \in C$;

**(c)** $S_k = \{s, s'\}$, $s, s' \in S$, and

**(c.1)** $\text{top}(R_s, X_k) \neq \text{top}(R_{s'}, X_k)$ or $\text{top}(R_s, X_k) = \text{top}(R_{s'}, X_k) = 0$;

**(c.2)** $\text{top}(R_s, X_k) = \text{top}(R_{s'}, X_k) \equiv \hat{x} \in X$ and $s \succ_{\hat{x}} s'$;

**(c.3)** $\text{top}(R_s, X_k) = \text{top}(R_{s'}, X_k) \equiv \hat{x} \in X$ and $s' \succ_{\hat{x}} s$;

**(d)** $S_k = \{s, s'\}$ and $c = (s, s') \in C$ or $c = (s', s) \in C$.

Given allocation $\varphi(R)$, $x \in X \cup \{0\}$, and $s \in N$, $(\varphi_{-s}(R), x)$ denotes the allocation obtained from $\varphi(R)$ by replacing $\varphi_s(R)$ by $x$, whenever this is feasible.

**(a)** Let $\hat{x} \equiv \text{top}(R_s, X_k) = \varphi_s^{\mathcal{S}}(R)$. Assume $\varphi_s(R) \neq \hat{x}$. If $\hat{x} = 0$ or $[\hat{x} \in X$ and there is no $i \in N$ with $\varphi_i(R) = \hat{x}]$, then allocation $\alpha \equiv (\varphi_{-i}(R), \hat{x})$ Pareto dominates $\varphi(R)$, i.e., $\alpha_s \, P_s \, \varphi_s(R)$ and for all $p \in (S \cup C) \setminus \{s\}$, $\alpha_p \, R_p \, \varphi_p(R)$, contradicting efficiency. So, $\hat{x} \in X$ and there is $i \in N \setminus \{s\}$ with $\varphi_i(R) = \hat{x}$. By the induction hypothesis, $i \in N \setminus \bigcup_{l=1}^{k} S_l$. By Lemma 5.2 (b), $s \succ_{\hat{x}} i$. Since $\varphi_s^{\mathcal{S}}(R) = \hat{x} \, P_s \, \varphi_s(R)$, allocation $\varphi(R)$ violates the priority of $s$ (D3.2); a contradiction. Hence, $\varphi_s(R) = \varphi_s^{\mathcal{S}}(R)$.

Given allocation $\varphi(R)$, $x, y \in X \cup \{0\}$, and $c \in C$, $(\varphi_{-c}(R), (x, y))$ denotes the allocation obtained from $\varphi(R)$ by replacing $\varphi_c(R)$ by $(x, y)$, whenever this is feasible.

**(b)** Assume without loss of generality that $c = (s, s') \in C$. Let $(\hat{x}, \hat{y}) = \text{top}(R_c, X_k) = \varphi_c^{\mathcal{S}}(R)$. Assume $\varphi_c(R) \neq (\hat{x}, \hat{y})$. Suppose $\varphi_c(R) = (\hat{y}, \hat{x})$. Then, allocation $\alpha \equiv (\varphi_{-c}(R), (\hat{x}, \hat{y}))$ Pareto dominates $\varphi(R)$, i.e., $\alpha_c \, P_c \, \varphi_c(R)$ and for all $p \in (S \cup C) \setminus \{c\}$, $\alpha_p \, R_p \, \varphi_p(R)$, contradicting efficiency.

Suppose $\varphi_c(R) = (x, \hat{y})$ for some $x \neq \hat{x}$. If $\hat{x} = 0$ or $[\hat{x} \in X$ and there is no $i \in N$ with $\varphi_i(R) = \hat{x}]$, then allocation $(\varphi_{-c}(R), (\hat{x}, \hat{y}))$ Pareto dominates $\varphi(R)$, contradicting efficiency. So, $\hat{x} \in X$ and there is $i \in N \setminus \{s, s'\}$ with $\varphi_i(R) = \hat{x}$. By the induction hypothesis, $i \in N \setminus \bigcup_{l=1}^{k+1} S_l$. By Lemma 5.2 (b), $s \succ_{\hat{x}} i$. Since $(\hat{x}, \hat{y}) \, P_c \, (x, \hat{y})$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, a.1); a contradiction.

Suppose $\varphi_c(R) = (\hat{x}, y)$ for some $y \neq \hat{y}$. If $\hat{y} = 0$ or $[\hat{y} \in X$ and there is no $i \in N$ with $\varphi_i(R) = \hat{y}]$, then allocation $(\varphi_{-c}(R), (\hat{x}, \hat{y}))$ Pareto dominates $\varphi(R)$, contradicting efficiency. So, $\hat{y} \in X$ and there is $i \in N \setminus \{s, s'\}$ with $\varphi_i(R) = \hat{y}$. By the induction hypothesis, $i \in N \setminus \bigcup_{l=1}^{k+1} S_l$. By Lemma 5.2 (b), $s' \succ_{\hat{y}} i$. Since $(\hat{x}, \hat{y}) \, P_c \, (\hat{x}, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, a.2); a contradiction.

Suppose $\varphi_c(R) = (x, \hat{x})$ for some $x \neq \hat{y}$. If $\hat{y} = 0$ or $[\hat{y} \in X$ and there is no $i \in N$ with $\varphi_i(R) = \hat{y}]$, then allocation $(\varphi_{-c}(R), (\hat{x}, \hat{y}))$ Pareto dominates $\varphi(R)$, contradicting efficiency. So, $\hat{y} \in X$ and there is $i \in N \setminus \{s, s'\}$ with $\varphi_i(R) = \hat{y}$. By the induction hypothesis, $i \in N \setminus \bigcup_{l=1}^{k+1} S_l$. By Lemma 5.2 (b), $s' \succ_{\hat{y}} i$. If $\hat{x} = 0$, then since $(\hat{x}, \hat{y}) \, P_c \, (x, \hat{x})$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, a.2); a contradiction. If $\hat{x} \in X$, then since $S_k = \{s\}$ and $S_{k+1} = \{s'\}$, $s \succ_{\hat{x}} s'$. Thus, since $(\hat{x}, \hat{y}) \, P_c \, (x, \hat{x})$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, c); a contradiction.

Suppose $\varphi_c(R) = (\hat{y}, y)$ for some $y \neq \hat{x}$. If $\hat{x} = 0$ or $[\hat{x} \in X$ and there is no $i \in N$ with $\varphi_i(R) = \hat{x}]$, then allocation $(\varphi_{-c}(R), (\hat{x}, \hat{y}))$ Pareto dominates $\varphi(R)$, contradicting efficiency. So, $\hat{x} \in X$ and there is $i \in N \setminus \{s, s'\}$ with $\varphi_i(R) = \hat{x}$. By the induction hypothesis, $i \in N \setminus \bigcup_{l=1}^{k+1} S_l$. By Lemma 5.2 (b), $s \succ_{\hat{x}} i$. If $\hat{y} = 0$, then since $(\hat{x}, \hat{y}) \, P_c \, (\hat{y}, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, a.1); a contradiction. If $\hat{y} \in X$, then since $(\hat{x}, \hat{y}) \, P_c \, (\hat{y}, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ after reallocation (D5.4, r.1); a contradiction.

Finally, suppose $\varphi_c(R) = (x, y)$ for some $x, y \notin \{\hat{x}, \hat{y}\}$. If $(\hat{x} = 0$ or $[\hat{x} \in X$ and there is no $i \in N \setminus \{s, s'\}$ with $\varphi_i(R) = \hat{x}])$ and $(\hat{y} = 0$ or $[\hat{y} \in X$ and there is no $i \in N \setminus \{s, s'\}$ with $\varphi_i(R) = \hat{y}])$, then allocation $(\varphi_{-c}(R), (\hat{x}, \hat{y}))$ Pareto dominates $\varphi(R)$, contradicting efficiency. So, there is $i \in N \setminus \{s, s'\}$ such that $[\hat{x} \in X$ and $\varphi_i(R) = \hat{x}]$ or $[\hat{y} \in X$ and $\varphi_i(R) = \hat{y}]$. By the induction hypothesis, $i \in N \setminus \bigcup_{l=1}^{k+1} S_l$. By Lemma 5.2 (b), [if $\hat{x} \in X$ and $\varphi_i(R) = \hat{x}$, then $s \succ_{\hat{x}} i$]

and [if $\hat{y} \in X$ and $\varphi_i(R) = \hat{y}$, then $s' \succ_{\hat{y}} i$].

If there is $i \in N\backslash\{s, s'\}$ such that $[\hat{x} \in X$ and $\varphi_i(R) = \hat{x}]$ and $\hat{y} = 0$, then, since $(\hat{x}, \hat{y}) \, P_c \, (x, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, a.1); a contradiction.

If there is $i \in N\backslash\{s, s'\}$ such that $[\hat{y} \in X$ and $\varphi_i(R) = \hat{y}]$ and $\hat{x} = 0$, then, since $(\hat{x}, \hat{y}) \, P_c \, (x, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, a.2); a contradiction.

If there is $i \in N\backslash\{s, s'\}$ such that $[\hat{x} \in X$ and $\varphi_i(R) = \hat{x}]$ and $\hat{y} \in X\backslash\{\varphi_l(R) : l \in N\}$, then, since $(\hat{x}, \hat{y}) \, P_c \, (x, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ taking into account vacancies (D5.5, v.1); a contradiction.

If there is $i \in N\backslash\{s, s'\}$ such that $[\hat{y} \in X$ and $\varphi_i(R) = \hat{y}]$ and $\hat{x} \in X\backslash\{\varphi_l(R) : l \in N\}$, then, since $(\hat{x}, \hat{y}) \, P_c \, (x, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ taking into account vacancies (D5.5, v.2); a contradiction.

If there are $i, j \in N\backslash\{s, s'\}$ such that $[\hat{x} \in X$ and $\varphi_i(R) = \hat{x}]$ and $[\hat{y} \in X$ and $\varphi_j(R) = \hat{y}]$, then $s \succ_{\hat{x}} i$ and $s' \succ_{\hat{y}} j$. Thus, since $(\hat{x}, \hat{y}) \, P_c \, (x, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, c); a contradiction. Hence, to summarize Case (b), $\varphi_c(R) = \varphi_c^{\mathcal{S}}(R)$.

Given allocation $\varphi(R)$, $x, y \in X \cup \{0\}$, and $i, j \in N$, $(\varphi_{-i,j}(R), x, y)$ denotes the allocation obtained from $\varphi(R)$ by replacing $\varphi_i(R)$ with $x$ and $\varphi_j(R)$ with $y$, whenever this is feasible.

**(c.1)** Let $\hat{x} = \text{top}(R_s, X_k)$ and $\hat{y} = \text{top}(R_{s'}, X_k)$. If $\hat{x} = \hat{y} = 0$ and $[\varphi_s(R) \neq 0$ or $\varphi_{s'}(R) \neq 0]$, then efficiency is violated. Hence we can assume that $\hat{x} \neq \hat{y}$ and $[\hat{x} \neq 0$ or $\hat{y} \neq 0]$. Assume $\varphi_s(R) \neq \hat{x}$ and $\varphi_{s'}(R) \neq \hat{x}$. If $\hat{x} = 0$ or $[\hat{x} \in X$ and there is no $i \in N$ with $\varphi_i(R) = \hat{x}]$, then allocation $(\varphi_{-s}(R), \hat{x})$ Pareto dominates $\varphi(R)$, contradicting efficiency of $\varphi$. So $\hat{x} \in X$ and there is $i \in N\backslash\{s, s'\}$ with $\varphi_i(R) = \hat{x}$. By the induction hypothesis, $i \in N\backslash\bigcup_{l=1}^{k} S_l$. By Lemma 5.2 (b), $s \succ_{\hat{x}} i$. Since $\hat{x} \, P_s \, \varphi_s(R)$, allocation $\varphi(R)$ violates the priority of $s$ (D3.2); a contradiction. Hence, $\varphi_s(R) = \hat{x}$ or $\varphi_{s'}(R) = \hat{x}$. Similarly, $\varphi_s(R) = \hat{y}$ or $\varphi_{s'}(R) = \hat{y}$. Suppose $\varphi_s(R) = \hat{y}$ and $\varphi_{s'}(R) = \hat{x}$. Then, allocation $(\varphi_{-s,s'}(R), \hat{x}, \hat{y})$ Pareto dominates $\varphi(R)$, contradicting efficiency. So, $\varphi_s(R) = \hat{x} = \varphi_s^{\mathcal{S}}(R)$ and $\varphi_{s'}(R) = \hat{y} = \varphi_{s'}^{\mathcal{S}}(R)$.

**(c.2)** Recall $\hat{x} = \text{top}(R_s, X_k)$. Let $\hat{y} = \text{top}(R_{s'}, X_k\backslash\{\hat{x}\})$. Note $\hat{x} \neq \hat{y}$. Assume $\varphi_s(R) \neq \hat{x}$ and $\varphi_{s'}(R) \neq \hat{x}$. If $\hat{x} = 0$ or $[\hat{x} \in X$ and there is no $i \in N$ with $\varphi_i(R) = \hat{x}]$, then allocation $(\varphi_{-s}(R), \hat{x})$ Pareto dominates $\varphi(R)$, contradicting efficiency. So, $\hat{x} \in X$ and there is $i \in N\backslash\{s, s'\}$ with $\varphi_i(R) = \hat{x}$. By Lemma 5.2 (b), $s \succ_{\hat{x}} i$. Since $\hat{x} \, P_s \, \varphi_s(R)$, allocation $\varphi(R)$ violates the priority of $s$ (D3.2); a contradiction. Hence, either $\varphi_s(R) = \hat{x}$ or $\varphi_{s'}(R) = \hat{x}$. Similarly, either $\varphi_s(R) = \hat{y}$ or $\varphi_{s'}(R) = \hat{y}$. Suppose $\varphi_s(R) = \hat{y}$ and $\varphi_{s'}(R) = \hat{x}$. Then, $s \succ_{\hat{x}} s'$ and $\hat{x} \, P_s \, \hat{y}$ imply that $\varphi(R)$ violates the priority of $s$ (D3.2); a contradiction. Hence, $\varphi_s(R) = \hat{x} = \varphi_s^{\mathcal{S}}(R)$ and $\varphi_{s'}(R) = \hat{y} = \varphi_{s'}^{\mathcal{S}}(R)$.

**(c.3)** An argument similar as in (c.2) shows that $\varphi_s(R) = \varphi_s^{\mathcal{S}}(R)$ and $\varphi_{s'}(R) = \varphi_{s'}^{\mathcal{S}}(R)$.

**(d)** Assume without loss of generality that $c = (s, s') \in C$. Let $(\hat{x}, \hat{y}) = \text{top}(R_c, X_k) = \varphi_c^{\mathcal{S}}(R)$. Assume $\varphi_c(R) \neq (\hat{x}, \hat{y})$. Suppose $\varphi_c(R) = (\hat{y}, \hat{x})$. Then, allocation $(\varphi_{-c}(R), (\hat{x}, \hat{y}))$ Pareto dominates $\varphi(R)$, contradicting efficiency.

Suppose $\varphi_c(R) = (x, \hat{y})$ for some $x \neq \hat{x}$. If $\hat{x} = 0$ or $[\hat{x} \in X$ and there is no $i \in N$ with $\varphi_i(R) = \hat{x}]$, then allocation $(\varphi_{-c}(R), (\hat{x}, \hat{y}))$ Pareto dominates $\varphi(R)$, contradicting efficiency. So, $\hat{x} \in X$ and there is $i \in N\backslash\{s, s'\}$ with $\varphi_i(R) = \hat{x}$. By the induction hypothesis, $i \in N\backslash\bigcup_{l=1}^{k} S_l$. By Lemma 5.2 (b), $s \succ_{\hat{x}} i$. Since $(\hat{x}, \hat{y}) \, P_c \, (x, \hat{y})$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, a.1); a contradiction.

Suppose $\varphi_c(R) = (\hat{x}, y)$ for some $y \neq \hat{y}$. If $\hat{y} = 0$ or $[\hat{y} \in X$ and there is no $i \in N$ with $\varphi_i(R) = \hat{y}]$, then allocation $(\varphi_{-c}(R), (\hat{x}, \hat{y}))$ Pareto dominates $\varphi(R)$, contradicting efficiency. So, $\hat{y} \in X$ and there is $i \in N\backslash\{s, s'\}$ with $\varphi_i(R) = \hat{y}$. By the induction hypothesis, $i \in N\backslash\bigcup_{l=1}^{k} S_l$.

By Lemma 5.2 (b), $s' \succ_{\hat{y}} i$. Since $(\hat{x}, \hat{y}) \, P_c \, (\hat{x}, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, a.2); a contradiction.

Suppose $\varphi_c(R) = (x, \hat{x})$ for some $x \neq \hat{y}$. If $\hat{y} = 0$ or $[\hat{y} \in X$ and there is no $i \in N$ with $\varphi_i(R) = \hat{y}]$, then allocation $(\varphi_{-c}(R), (\hat{x}, \hat{y}))$ Pareto dominates $\varphi(R)$, contradicting efficiency. So, $\hat{y} \in X$ and there is $i \in N \backslash \{s, s'\}$ with $\varphi_i(R) = \hat{y}$. By the induction hypothesis, $i \in N \backslash \bigcup_{l=1}^{k} S_l$. By Lemma 5.2 (b), $s' \succ_{\hat{y}} i$. If $\hat{x} = 0$, then since $(\hat{x}, \hat{y}) \, P_c \, (x, \hat{x})$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, a.2); a contradiction. If $\hat{x} \in X$, then since $(\hat{x}, \hat{y}) \, P_c \, (x, \hat{x})$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ after reallocation (D5.4, r.2); a contradiction.

Suppose $\varphi_c(R) = (\hat{y}, y)$ for some $y \neq \hat{x}$. If $\hat{x} = 0$ or $[\hat{x} \in X$ and there is no $i \in N$ with $\varphi_i(R) = \hat{x}]$, then allocation $(\varphi_{-c}(R), (\hat{x}, \hat{y}))$ Pareto dominates $\varphi(R)$, contradicting efficiency. So, $\hat{x} \in X$ and there is $i \in N \backslash \{s, s'\}$ with $\varphi_i(R) = \hat{x}$. By the induction hypothesis, $i \in N \backslash \bigcup_{l=1}^{k} S_l$. By Lemma 5.2 (b), $s \succ_{\hat{x}} i$. If $\hat{y} = 0$, then since $(\hat{x}, \hat{y}) \, P_c \, (\hat{y}, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, a.1); a contradiction. If $\hat{y} \in X$, then since $(\hat{x}, \hat{y}) \, P_c \, (\hat{y}, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ after reallocation (D5.4, r.1); a contradiction.

Finally, suppose $\varphi_c(R) = (x, y)$ for some $x, y \notin \{\hat{x}, \hat{y}\}$.

If $(\hat{x} = 0$ or $[\hat{x} \in X$ and there is no $i \in N \backslash \{s, s'\}$ with $\varphi_i(R) = \hat{x}])$ and $(\hat{y} = 0$ or $[\hat{y} \in X$ and there is no $i \in N \backslash \{s, s'\}$ with $\varphi_i(R) = \hat{y}])$, then allocation $(\varphi_{-c}(R), (\hat{x}, \hat{y}))$ Pareto dominates $\varphi(R)$, contradicting efficiency. So, there is $i \in N \backslash \{s, s'\}$ such that $[\hat{x} \in X$ and $\varphi_i(R) = \hat{x}]$ or $[\hat{y} \in X$ and $\varphi_i(R) = \hat{y}]$. By the induction hypothesis, $i \in N \backslash \bigcup_{l=1}^{k} S_l$. By Lemma 5.2 (b), [if $\hat{x} \in X$ and $\varphi_i(R) = \hat{x}$, then $s \succ_{\hat{x}} i$] and [if $\hat{y} \in X$ and $\varphi_i(R) = \hat{y}$, then $s' \succ_{\hat{y}} i$].

If there is $i \in N \backslash \{s, s'\}$ such that $[\hat{x} \in X$ and $\varphi_i(R) = \hat{x}]$ and $\hat{y} = 0$, then, since $(\hat{x}, \hat{y}) \, P_c \, (x, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, a.1); a contradiction.

If there is $i \in N \backslash \{s, s'\}$ such that $[\hat{y} \in X$ and $\varphi_i(R) = \hat{y}]$ and $\hat{x} = 0$, then, since $(\hat{x}, \hat{y}) \, P_c \, (x, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, a.2); a contradiction.

If there is $i \in N \backslash \{s, s'\}$ such that $[\hat{x} \in X$ and $\varphi_i(R) = \hat{x}]$ and $\hat{y} \in X \backslash \{\varphi_l(R) : l \in N\}$, then, since $(\hat{x}, \hat{y}) \, P_c \, (x, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ taking into account vacancies (D5.5, v.1); a contradiction.

If there is $i \in N \backslash \{s, s'\}$ such that $[\hat{y} \in X$ and $\varphi_i(R) = \hat{y}]$ and $\hat{x} \in X \backslash \{\varphi_l(R) : l \in N\}$, then, since $(\hat{x}, \hat{y}) \, P_c \, (x, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ taking into account vacancies (D5.5, v.2); a contradiction.

If there are $i, j \in N \backslash \{s, s'\}$ such that $[\hat{x} \in X$ and $\varphi_i(R) = \hat{x}]$ and $[\hat{y} \in X$ and $\varphi_j(R) = \hat{y}]$, then $s \succ_{\hat{x}} i$ and $s' \succ_{\hat{y}} j$. Thus, since $(\hat{x}, \hat{y}) \, P_c \, (x, y)$, or equivalently $\varphi_c^{\mathcal{S}}(R) \, P_c \, \varphi_c(R)$, allocation $\varphi(R)$ violates the priority of $c$ (D3.3, c); a contradiction.

Hence, to summarize Case (d), $\varphi_c(R) = \varphi_c^{\mathcal{S}}(R)$. $\qquad\qquad\square$

**Proof of Proposition 5.1.** Suppose that $\varphi$ is fair and efficient, but not vacancy-fair on $\mathcal{RR}$. Then there is $R \in \mathcal{RR}$ such that allocation $\varphi(R)$ violates the priority of $c$ taking into account vacancies (D5.5). Let $\alpha_l := \varphi_l(R)$ for all $l \in N$. Assume, without loss of generality, that Definition 5.5 (v.1) applies. Hence, there exist $x \in X$, $k \in N \backslash \{i, j\}$, and $v \in X \backslash \{\alpha_l : l \in N\}$ such that $\alpha_k = x$, $i \succ_x k$, and $(x, v) \, P_{(i,j)} \, (\alpha_i, \alpha_j)$. By responsiveness of couple $c$'s preferences associated individual preferences $R_i, R_j \in \mathcal{RS}$ exist for students $i$ and $j$ such that $x \, P_i \, \alpha_i$ or $v \, P_j \, \alpha_j$. If $x \, P_i \, \alpha_i$, then by responsiveness, $(x, \alpha_j) \, P_c \, (\alpha_i, \alpha_j)$; a violation of couple $c$'s priority (D3.3, a.1). So, $v \, P_j \, \alpha_j$. But then, by responsiveness, $(\alpha_i, v) \, P_c \, (\alpha_i, \alpha_j)$, contradicting efficiency of $\varphi$. $\qquad\qquad\square$

**Proof of Theorem 5.3.** Let $\succ$ be a *pt*-acyclic priority structure that induces the strongly *pt*-acyclic partition $\mathcal{S} = \{S_1, \ldots, S_p\}$. Let $R \in \mathcal{R}$. In each step of the sequential placement mechanism either a couple chooses its best pair of available positions or a set of students obtain efficient allotments through the deferred acceptance algorithm (because of the acyclicity the deferred acceptance algorithm allocation is an efficient allocation for the reduced placement problem, see Ergin, 2002, Theorem 1). Hence, there is an order $\sigma(R)$ of the students (*i.e.*, single students and couples) associated with the execution of the sequential placement mechanism such that $\varphi^{\mathcal{S}}(R)$ equals the allocation obtained by applying the serial dictatorship implied by $\sigma(R)$. Hence, $\varphi^{\mathcal{S}}(R)$ is efficient.[9]

It remains to prove that $\varphi^{\mathcal{S}}(R)$ does not violate the priority of any single student or any couple. Let $k \in \{1, \ldots, p\}$ be such that Step $k$ is well-defined in Definition 5.8. Assume that $\varphi^{\mathcal{S}}(R)$ does not violate the priority of any single student and any couple in $\bigcup_{l<k} S_l$ (this is vacuously true for $\bigcup_{l<1} S_l \equiv \emptyset$). We prove that $\varphi^{\mathcal{S}}(R)$ also does not violate the priority of any single student and any couple involved in Step $k$. We distinguish among the following cases:
**(a)** $S_k \subseteq S$;
**(b)** $S_k = \{s\}$, $s \notin S$, and $S_k \cup S_{k+1} = \{s, s'\}$ such that $c = (s, s') \in C$ or $c = (s', s) \in C$;
**(c)** $S_k = \{s, s'\}$ and $c = (s, s') \in C$ or $c = (s', s) \in C$.
**(a)** Let $s \in S_k$. By Definition 5.8, all position types that student $s$ likes better than $\varphi_s^{\mathcal{S}}(R)$ are assigned to students in $\bigcup_{l<k} S_l$ or to other students in $S_k$. Since all students in $\bigcup_{l<k} S_l$ have a higher priority for all position types (D5.6, ii), $\varphi^{\mathcal{S}}(R)$ does not violate the priority of $s$ because of any student in $\bigcup_{l<k} S_l$. Furthermore, since the allocation for students in $S_k$ is obtained by applying the DA algorithm, which always produces a fair allocation, to the reduced problem that only contains students in $S_k$, $\varphi^{\mathcal{S}}(R)$ does not violate the priority of $s$ because of any other student in $S_k$. Thus, $\varphi^{\mathcal{S}}(R)$ does not violate the priority of $s$.
**(b)** Without loss of generality assume that $c = (s, s') \in C$. Suppose $\varphi^{\mathcal{S}}(R)$ violates the priority of $c$. There are five subcases (see Definition 3.3) that we have to distinguish for this violation. However, we only consider the first subcase, Definition 3.3 (a.1), as the other subcases run analogously. There exist $x \in X$ and $\tilde{s} \in N \backslash \{s, s'\}$ such that $\varphi_{\tilde{s}}^{\mathcal{S}}(R) = x$, $s \succ_x \tilde{s}$, and $(x, \varphi_{s'}^{\mathcal{S}}(R)) \, P_c \, (\varphi_s^{\mathcal{S}}(R), \varphi_{s'}^{\mathcal{S}}(R))$ or $(x, 0) \, P_c \, (\varphi_s^{\mathcal{S}}(R), \varphi_{s'}^{\mathcal{S}}(R))$. From Definition 5.6, (ii), $s \succ_x \tilde{s}$, and $\tilde{s} \in N \backslash \{s, s'\}$ it follows that $(x, \varphi_{s'}^{\mathcal{S}}(R))$ and $(x, 0)$ were available options at Step $k$. Apparently, $c$ was not assigned its most favorite position assignment according to $q^k$, a contradiction to Definition 5.8. Hence, $\varphi^{\mathcal{S}}(R)$ does not violate the priority of $c$.
**(c)** A similar argument as in (b) shows that $\varphi^{\mathcal{S}}(R)$ does not violate the priority of couple $c$. □

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[9] In Example 5.7, for instance, $\sigma(R) = (5, 6, 7, (1, 2), (3, 4), 8)$ (in fact any sequence of students 5, 6, and 7 at the beginning of the serial dictatorship is possible for this specific example).

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