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Locating Public Facilities by Majority: Stability, Consistency and Group Formation

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Abstract. We consider the following allocation problem: A fixed number of public facilities must be located on a line. Society is composed of N agents, who must be allocated to one and only one of these facilities. Agents have single peaked preferences over the possible location of the facilities they are assigned to, and do not care about the location of the rest of facilities. There is no congestion. We show that there exist social choice correspondences that choose locations and assign agents to them in such a way that: (1) these decisions are Condorcet winners whenever one exists, (2) the majority of the users of each facility supports the choice of its location, and (3) no agent wishes to become a user of another facility, even if that could induce a change of its present location by majority voting.

Key words: Social Choice Correspondences, Condorcet Rules, Stability, Simpson Rule.

1. Introduction.

Collective choices often consist of a constellation of smaller decisions, each of which affects different groups of citizens in different degrees. For instance, when the government chooses a budget, some of its items (like the salaries of public servants, if I am one) may be very important to me, while I may be quite indifferent regarding others (like, say, how much of the agricultural subsidies go to olive producers and how much goes to wheat farmers). But, of course, other citizens may feel intensely about what I do not care, and little about my major concerns.

In this paper we examine the connections between "global" decisions, describing all components of a social policy and its effects upon all citizens, and "partial" decisions, which are components of a global decision which only affect a part of the population. Specifically, we are concerned about the consistency of decisions of these two types. We'd like to identify procedures and situations when the decisions, if taken globally with the participation of all citizens, would be perceived by the members of each group of interest as providing the same partial choice that they would have made in isolation. Conversely, we'd like to check how and when it would be possible to let each group of interest make a partial decision on its own, and have the aggregate decision be the same one that all citizens would have taken if they had jointly participated in choosing the overall outcome.

This consistency issue is compounded by an additional and subtle fact, one that we feel deserves attention on its own: it is the global decisions which often induce the formation of interest groups, as much as the interest groups can contribute to form the global decisions. In the process of discussing the issue of consistency between the large and the small, we shall also provide a theory of the endogenous formation of interest groups.

Our referent as a decision-making procedure is simple majority voting. It is well known that simple majority comparisons can lead to cyclical social preferences, and that majority (Condorcet) winners may not exist for some profiles of individual preferences. This is why, when required, we'll consider extensions of the simple majority rule, which still recommend definite choices when there are no Condorcet winners. The use of majority is so compelling, when it leads to definite results, and the practical use of extended majority rules is so widespread, that we consider it completely natural to concentrate on this class of methods.

To get a better feeling for our approach, consider the following example of a decision problem which does not admit consistent majoritarian choices. The example is interesting in its own, and it also suggests that the questions we want to address are only manageable in contexts that exhibit sufficient structure.

Twenty six agents must choose a delegation of three representatives out of five candidates (x, y, z, r, w), over which they have preferences represented in the following table,

$agents$	1, ..., 6	7, .., 13	14, .., 17	18, .., 21	22, .., 26
$preferences$	x	r	w	z	y
$from$	y	x	r	w	z
$better$	z	y	x	r	w
to	r	w	z	y	r
$worse$	w	z	y	x	x

If they use a consistent majoritarian rule, each one of the chosen delegates should be a Condorcet winner for the set of voters that he represents. Who represents whom can be specified in several ways. For example, we could assume that voters only get the chance to communicate with one delegate, and that this is the one we call his "representative". But here we concentrate on the case where, once the delegation is chosen, each voter identifies as his representative the one delegate that he likes most. There are ten possible delegations, xyz, xyw, xyr, xzw, xrw, xzr, yzw, yzr, yrw, zrw. If xyz was chosen, the sets of agents that would feel represented by x, y and z would be respectively, $U(x) = \{1, ..., 17\}$, $U(y) = \{22, .., 26\}$, $U(z) = \{18, .., 21\}$. But then, the Condorcet winner for the voters in $U(x)$ is r. If yzr was chosen, $U(y) = \{1, .., 6, 22, .., 26\}$, $U(z) = \{18, .., 21\}$, $U(r) = \{7, .., 17\}$. But then, the Condorcet winner for voters in $U(y)$ is x.

The reader may check that a similar inconsistency will appear with any of the remaining possible delegations. This proves that, in the case we just described, no social choice rule can meet our desiderata.

Since general results seem out of reach, we devote this paper to analyze consistency in connection to a specific class of economic problems related to the location of facilities and the provision of local public services. This problem is not only chosen for its tractability, though. It is of substantial interest of its own, and in connection to the inherited theory of local public goods. Specifically, we concentrate on problems involving the location of a fixed number of facilities, k, along a linear territory. Society is composed of N agents, who must be allocated to one and only one of these facilities. Agents have single peaked preferences over the possible location of the facilities they are assigned to, and do not care about the location of the rest of facilities. There is no congestion: the location of their facility is the only concern of agents, not the set of other agents with whom they share it.

In the context we just described, the location of all facilities is the basic global decision, which must be accompanied by a partition of the individuals into groups. Since the distribution of agents across facilities may be voluntary or imposed, one may think of a variety of assignments: for each possible one, the agent cares for the facility she gets. Hence, all people who care for the same facility become an interest group, induced by the particular distribution of citizens to facilities. The particular location of their common facility is the partial aspect of the global decision about which they have an actual interest.

Suppose, then, that all agents together voted by majority to make a global decision regarding the location of all facilities. In order to evaluate their options, these voters would need to know the consequences of each location decision on the particular facility that they would be able to enjoy. A particularly attractive assumption is that agents were allowed to select their preferred location. What can we say about the resulting allocations of agents to facilities, and about the degree of satisfaction of the induced interest groups?

One first difficulty on our way comes from the fact that there may not be a majority winner among the different global decisions. This is because, in spite of our assumption that preferences over individual locations are single-peaked, the preferences of voters over global decisions are more complex and can still lead to cycles for some preference profiles. We shall cope with this difficulty later on, but for the sake of the argument assume for now that a majoritarian global decision exists. When this is the case, there would exist allocations of agents to facilities having the property that no agent prefers any location to the one she is assigned (no envy). Now, with agents located in this way, what would happen if they were allowed to vote for the location that the majority of their group would prefer? Would this location coincide with the one where their facility is actually in? If so, we say that for simple majority the global decision and its partial consequences are consistent. Moreover, having introduced the notion that agents may vote on the preferred location of facilities, we can examine the consequences of some additional strategic behavior on their part. Rather than taking locations as data, they may want to join a group of users in the hope of influencing the location in their favor. This will not happen under global decisions whose envy-free agents could find no incentive to join another group and get the location for this group become better for them. We say that global decisions with this property are Nash-stable under simple majority. We also show that if a global decision is the winner by simple majority, it induces a partition of agents which is Nash-stable. Moreover, we prove that Nash stable decisions exist, even when global majoritarian decisions do not.

If there always existed majority (Condorcet) winners in the contest between global decisions, the consistency question would have a clear-cut and simple answer. Choose that global location and let agents use the one they prefer. This alone guarantees that all interest groups would be enjoying the location that they alone would choose by majority, and that no agent would consider joining any other group in the hope of altering the global decision by influencing any particular location. But the question is left what to choose when no Condorcet winner exists among the global decisions. Can we extend the majoritarian rule while preserving its nice consistency and stability properties? Our major result gives a positive answer to this question. There exist social choice correspondences which extend the majority rule and satisfy consistency and Nash stability.

This is in contrast with some of the conclusions of our preceding paper Self-Selection Consistent Functions (2002). There we consider this consistency issue for social choice functions, rather than correspondences, as we do here, and we provide a general characterization of functions whose global choices induce envy free allocations of agents to individual facilities. Unfortunately, we also conclude there that overall consistent majoritarian choices are not possible within the framework of social choice functions. In the present paper we allow for multi valued choices for some profiles. Thanks to that, the negative result of our preceding paper becomes a positive one.

Our assumption that agents only care about the facility they actually use, approaches our model to a class of coalition formation games, called hedonic games, which have received the attention of different authors (Banerjee, Konishi and Sonmez (1998) and Bogomolnaia and Jackson (1998) among others). In hedonic games, individuals have preferences over the coalitions that they can be part of, presumably because they get some benefit from their association with the rest of agents in the coalition. In our case, agents meet because they share a facility, and in our interpretation they draw no direct advantage from being together. But sharing the facility is tantamount to the wish of being together, and in this respect the two types of models are quite similar.

We want to emphasize, however, that our model has more structure and a more specific interpretation than the general framework of hedonic games. This allows us to have model-specific definitions and results. In particular our results are quite positive, thanks to the characteristics of the situations we model.

A recent paper by Milchtaich and Winter (2002) is closely related to ours in several ways. It will be useful to comment on the analogies and on the differences between the two papers. M&W analyze the stability of coalitions in a model of group formation where agents are driven by their attraction to people who are similar to them. They prove the existence of stable partitions in a one-dimensional model where the total number of groups to be formed is bounded above, and agents strive to minimize either the average distance among group members or their distance to the average member.

In our model, group formation is induced by the common use of a facility, rather than by an inherent desire to become associated with similar people. Yet, whenever agents are free to choose their preferred facility, this indirectly results in their association with individuals of similar tastes. Hence, our model is close to that of M&W, and more in general, to the literature on hedonic games. In fact, M&W take a step toward our formulation, when they suggest that a good interpretation of their model comes from assuming that agents choose one location. Indeed, this is what our agents do, once the facilities are located. We add some more richness to the analysis by explicitly allowing agents to play a role in determining the location of the facilities among which they may then choose from.

The rest of the paper is organized as follows. In Section 2, we present our model in detail. In Section 3 we discuss the properties that Condorcet winners have whenever they exist. Section 4 studies the existence of Condorcet rules that select internally consistent and Nash stable decisions. Finally, in Section 5 we conclude.

2. The Model.

The model we present is essentially the one considered in Barberà and Beviá (2002). The major difference is that here we model social decision rules as correspondences, while in the previous paper we considered the case of functions.

We consider problems that involve any finite set of agents. Agents are identified with elements in N, the set of natural numbers. Let S be the class of all finite subsets of N. Elements of S, denoted as $S, S', ...$, stand for particular societies, whose cardinality is denoted by $|S|$, $|S'|$, etc.

We now describe the decisions that societies can face. These are determined by the number and the position of relevant locations, and by the sets of agents who are allocated to each location.

A natural number $k \in \mathsf{N}$ will stand for the number of locations. Then, given

$S \in \mathcal{S}$ and $k \in \mathbb{N}$, an $S/k-$*decision* is a $k-$tuple of pairs $d = (x_h, S_h)_{h=1}^{k}$, where $x_h \in \mathbb{R}$, and $(S_1, .., S_k)$ is a partition of S. We interpret each x_h as a location and S_h as the set of agents who is assigned to the location x_h. Notice that some elements in the partition may be empty. This will be the case, necessarily, if $k > |S|$. We call $d_L = (x_1, .., x_k)$ the vector of *locations*, and $d_A = (S_1, .., S_k)$ the vector of *assignments*.

Let $D(S, k)$ be the set of $S/k-$decisions, $D(k) = \bigcup_{S \in \mathcal{S}} D(S, k)$ the set of $k-$decisions, and $D = \bigcup_{k \in \mathbb{N}} D(k)$ the set of decisions. For each agent $j \in \mathbb{N}$, the set of $k-$decisions which concern j is $D_j(k) = \bigcup_{\{S \in \mathcal{S} | j \in S\}} D(S, k)$ and the *set of decisions that concern* j is $D_j = \bigcup_{k \in \mathbb{N}} D_j(k)$.

Agents are assumed to have complete, reflexive, transitive preferences over decisions which concern them. That is, *agent* $i's$ *preferences* are defined on D_i, and thus, rank any pair of S/k and $S'/k'-$decisions provided that $i \in S \cap S'$. Denote by $\succsim_i$ the preferences of agent i on D_i.

We shall assume all along that preferences are *singleton-based*. Informally, this means that agents' rankings of decisions only depend on the location they are assigned to, not on the rest of locations or on the assignment of other agents to locations. This assumption is compatible with our interpretation that agents can only use the good provided at one location, and that this is a public good subject to no congestion. Formally, a preference $\succsim_i$ on D_i is singleton-based if there is a preference $\bar{\succsim}_i$ on $\mathbb{R}$ such that for all d, $d' \in D_i$, $d \succsim_i d'$ if and only if $x(i, d) \bar{\succsim}_i x(i, d')$, where $x(i, d)$ denotes the location to which agent i is assigned under the decision d.

In all that follows, we shall assume that for all $i \in N$, $\succsim_i$ is singleton-based, and in addition, that the order $\bar{\succsim}_i$ is *single-peaked*. That is: for each $\bar{\succsim}_i$, there is an alternative $p(i)$ which is the unique best element for $\bar{\succsim}_i$; moreover, for all

x, y, if $p(i) \geq x > y$, then $x \bar{\succ}_i y$, and if $y > x \geq p(i)$, then $x \bar{\succ}_i y$. Under the assumption that $\succsim_i$ is singleton-based, there is a one to one relation between preferences on decisions, $\succsim_i$, and preferences on locations, $\bar{\succsim}_i$. Thus, from now on we will not make any distinction between the two.

Given $S \in \mathcal{S}$, *preference profiles for* S are $|S|$-tuples of preferences, and we denote them by $P_S, P'_S, ...$

We denote by $\mathcal{P}$ the set of all preferences described above, and by $\mathcal{P}^S$ the set of preference profiles for S satisfying those requirements.

A *collective choice correspondence* will select a set of $k-$decisions, for each given k, on the basis of the preferences of agents in coalition S, for any coalition $S \in \mathcal{S}$. Formally,

Definition 1. *A collective choice correspondence is a correspondence* $\varphi : \bigcup_{S \in \mathcal{S}} \mathcal{P}^S \times \mathbb{N} \rightarrow D$ *such that, for all* $S \in \mathcal{S}$, $P_S \in \mathcal{P}^S$ *and* $k \in \mathbb{N}$, $\varphi(P_S, k) \subset D(S, k)$.

We have defined $S/k-$decisions to represent both the location of facilities and the assignment of agents to locations. Given that, a collective choice correspondence simultaneously determines both aspects of the decision. Although this is a valid description of the final result of the allocation process, it is not contradictory with the more vivid two-stage interpretation that is suggested in the introduction. Namely, one can think that the final $S/k-$decision is arrived at because agents first determine locations and then allocate themselves voluntarily to one of these locations. We shall return to this interpretation later on.

3. Condorcet Winners and their Properties.

We now define the classical notions of efficiency and of Condorcet winner as they apply to our specific model.

Definition 2. *An $S/k-$decision d is efficient if there is no $S/k-$decision d' such that $d' \succsim_i d$ for every agent $i \in S$ and $d' \succ_j d$ for some $j \in S$.*

Notice that efficiency in our model imposes two requirements. One is that all locations should be in between the peaks of the agents who use them (otherwise, there would be an obvious Pareto improvement, since preferences are single peaked). The other requirement, once the locations are fixed, is that all agents should be assigned to the location they most prefer (otherwise, bringing them to their preferred location would be a Pareto improvement). This last requirement implies that efficient decisions are envy-free in the following sense:

Definition 3. *An $S/k-$decision $d = ((x_h, S_h))_{h=1}^k \in D(S,k)$ is* envy-free *if for all $i \in S$, $x(i,d) \succsim_i x_h$ for all $x_h \in d_L$.*

Definition 4. *An $S/k-$decision $d \in D(S,k)$ is a Condorcet winner for P_S and k if $|\{i \in S \mid d \succ_i d'\}| \geq |\{i \in S \mid d' \succ_i d\}|$ for all $d' \in D(S,k)$. Given $S \in \mathsf{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$, let $CW(P_S,k)$ be the set of $S/k-$decisions that are Condorcet winners for P_S and k.*

When a Condorcet winner exists, it is the outcome that would result from a majority vote among all possible $S/k-$decisions. Alternatively, we can justify our interest in Condorcet winners as a test for the solidity of collective decisions. Indeed, the choice of a Condorcet winner guarantees that no majority can challenge the selected outcome.

In our model, Condorcet winners have nice properties whenever they exist.

Let us illustrate the class of properties that decisions and rules can be expected to satisfy by considering an example.

Example 1. *A society with 16 agents must locate two facilities and assign each of the agents to one of the two. Agents have Euclidean preferences on the line*[1]*, with peaks as follows: For* $i \in \{1,2\}$, $p(i) = i$, $p(3) = 4.5$, $p(4) = 5$, $p(5) = 6$, *for all* $i \in \{6,..,10\}$, $p(i) = 8$, $p(11) = 10.1$, $p(12) = 11$, $p(13) = p(14) = 13$, $p(15) = p(16) = 14$. *In this society, decisions are expressed by two pairs* $((x_1, S_1), (x_2, S_2))$, *where* x_1 *and* x_2 *are the locations of the public facilities and* S_1, S_2 *are the set of agents who are assigned to these facilities, respectively. Figure 1 provides a sketchy graphical representation of the agents' preferences and it shows a particular decision* $d = ((8, \{1, ..., 11\}), (13, \{12, .., 16\}))$. *With Euclidean preferences it is enough to represent the peaks of the agents on the line. Each circle over a point of the peaks contains a number, indicating what agent has a peak at that point. A decision* $d = ((x_1, S_1), (x_2, S_2))$ *will be represented by the labels* x_1, x_2 *at the appropriate points of the line, and by squares containing the agents in* S_1 *and* S_2 *respectively.*

If the members of this society were to choose one decision by majority voting, which one (s) would they select? There may be several possible choices, but they should all be Condorcet winners: that is, decisions which would not loose by majority against any other one. In our example, the decision that we have represented, $d = ((x_1, S_1), (x_2, S_2))$ *is a Condorcet winner.*

[1] Euclidean preferences have their best element at some point t, and can be represented by the utility function $u(x) = -|t - x|$. They are a special case among the single-peaked.

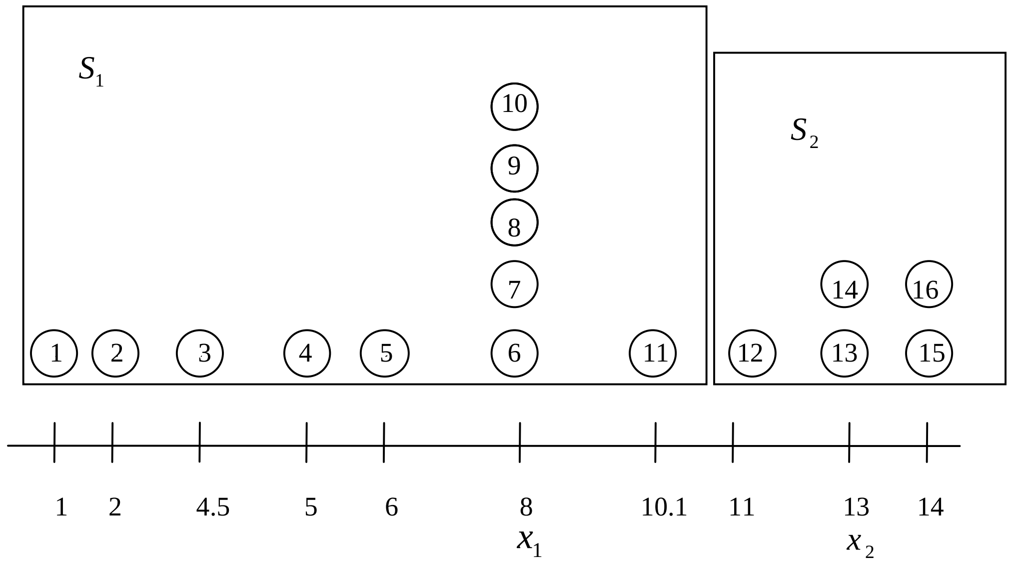

figure 1

The fact that d is a Condorcet winner has consequences. Specifically, it guarantees that it satisfies a number of nice properties, which we shall here describe informally, and then define precisely.

(1) The decision d is Pareto efficient. Indeed, there is no other way to locate the decisions and to assign agents to them which all agents would unanimously support over d (with some being possibly indifferent).

(2) The decision d satisfies an attractive property of internal consistency. Notice that d identifies the set S_i *as that of the users of the facility located at* x_i*. Suppose that agents in* S_i *were in a position to reconsider, just by themselves, whether* x_i *is indeed the best location for the facility that they are the only users of. If these agents were to vote by majority on that issue,* x_i *would indeed be chosen, since it is the median of the peaks of agents in* S_i*, each of which has single peaked preferences. Global decisions give rise to a partition of society, and internal con-*

sistency means that the members inside each partition agree, by majority, on the locations of their own facility.

(3) The decision d satisfies nice properties of no-envy and Nash stability, both related to the question whether agents who are assigned to a certain facility will have incentives to accept this assignment, or else would prefer to join a different set of users. Condorcet winners in our model are envy-free. This means that no agent who is assigned to one facility would prefer to use any of the other facilities which are made available to other agents as part of the same decision. It is easy to see that, if the initial allocation is efficient, and ours is, then it is envy-free. Indeed, this is due to the absence of congestion in the consumption of the public facilities. Therefore, no agent will have incentives to change facilities under an efficient decision, unless the agent can foresee that joining another group may induce a favorable change in the location of the relevant facilities. This leads us to the notion of Nash stability, involving the comparison of any allocation with others that may have resulted if agents had the right to be allocated as they wished. In order to define Nash stability we need to be specific on how one agent could modify an initial allocation by joining a group different than the one she was originally assigned to. In our example, we may check that, under the sustained hypothesis that the facility assigned to each group is a majority winner for its members, no agent could benefit from joining a different group than the one they are initially assigned to[2]*. The fact that decision d satisfies these interesting properties is not an accident. Rather, we shall prove that it is a consequence of the fact that this decision is a Condorcet winner among global decisions of the*

[2] Group Nash Stability is a stronger requirement than Nash stability, that will not always be possible to respect in our context, but would also hold for Condorcet winners. We discuss this in the Appendix.

form $((x_1, S_1), (x_2, S_2))$[3].

Let us now define formally the properties discussed in Example 1, and prove that whenever a global Condorcet winner exists, it satisfies all of them.

Definition 5. *An $S/k-$decision $d = ((x_h, S_h))_{h=1}^k \in D(S,k)$ is* internally consistent *if $d_h = (x_h, S_h) \in CW(P_{S_h}, 1)$ for all h such that $S_h \neq \emptyset$.*

An interesting implication of choosing a global Condorcet winner is that this guarantees the internal consistency and efficiency of our choices.

Proposition 1. *Given $S \in \mathsf{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathsf{N}$, if an $S/k-$decision $d = ((x_h, S_h))_{h=1}^k \in CW(P_S, k)$, then d is internally consistent and efficient.*

Proof. Internal consistency is proven in Barberà and Beviá (2002). For efficiency, notice first that, since a global Condorcet winner is internally consistent, each of the locations is a Condorcet winner among their users, therefore the locations are in between the peaks of the agents who use them. It remains to prove that the decision is envy free. Suppose that d is not envy free, that is, there is an agent i in one of the groups that would prefer the location assigned to other group than the one he is assigned to. Without loss of generality suppose that $i \in S_j$, and $x_h \succ_i x_j$. Let d' be such that $d'_l = d_l$ for all $l \notin \{j, h\}$, $d'_h = (x_h, S_h \cup \{i\})$, $d'_j = (x_j, S_j \backslash \{i\})$. All the agents, except i, are indifferent between d and d'. But

[3] In this and all other examples, we shall assume that agents have euclidean preferences. One consequence of this is that, all groups in partitions satisfying our requirements will be connected. But this is just for these special preferences, and it should not be expected in the more general case of single peaked preferences, which are the ones we assume in the paper. The reader may check that our formal proofs never assume connectedness of groups, as it need not hold in the general framework.

agent i strictly prefers d'. Therefore, decision d' beats d in majority comparisons, which contradicts the fact that $d \in CW(P_S, k)$. ■

We now turn to Nash stability, which occurs when agents, after envisaging the possibility of joining another group, and considering that the object assigned to this new group may be reallocated accordingly, consider that they should remain at their original location. Formally,

Definition 6. *An $S/k-$decision $d = ((x_h, S_h))_{h=1}^{k} \in D(S, k)$ is* Nash stable *if for all h, j, with $h \neq j$ and for all $i \in S_h$, there is $\overline{x} \in CW(P_{S_j \cup \{i\}}, 1)$ such that $x(i, d) <_i \overline{x}$.*

Our definition of Nash stability requires agents to compare the sets of decisions resulting from assigning themselves to their initially preferred location, with those that would arise if they declared themselves members of another group of users, and had an influence on the final locations. Since we are working with correspondences, agents may thus be led to compare sets of decisions, while at present we have only defined their preferences over single decisions. Extending preferences from single alternatives to sets is always a delicate exercise, that has consequences (see Barberà, Bossert and Pattanaik (2004)). What we choose to do here is to consider a weak extension of the preferences, whose consequences are that agents will only engage in deviations from one group to another when the gains from such moves are unequivocal. Because of this, Nash stability is easier to reach under our extension to set comparisons than it would be under stronger extensions. Notice, however, that even with our present definition, group-Nash stability may not be guaranteed (see Appendix). Indeed, we are imposing a meaningful restriction.

Let us briefly comment on interpretations. No envy is a natural stability requirement if we interpret that the locations of the different facilities are final.

Then, the only possible changes in allocations are changes in the sets of agents who get service at each location. No envy guarantees that agents are at their best location among these fixed ones. Nash stability is a stronger requirement, which makes sense when we think that the decisions are at an interim stage, and that the location of facilities can still be changed by the vote of those who claim to have an interest on them. Then, a Nash stable allocation is one where no agent would want to express interest for joining any group other than the one he is assigned to.

Our next proposition shows that global Condorcet winners are Nash stable.

Proposition 2. *Given $S \in \mathsf{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$ if an S/k-decision $d = ((x_h, S_h))_{h=1}^k \in CW(P_S, k)$, then d is Nash stable.*

Proof. Suppose that there is a global Condorcet winner which is not Nash stable. That is, there is $d \in CW(P_S, k)$, h, q, with $h \neq q$ and $i \in S_h$, such that $\overline{x} \succ_i x(i,d)$ for all $\overline{x} \in CW(P_{S_q \cup \{i\}}, 1)$. Without loss of generality suppose that $h = q + 1$. Let $[y_1, y_2] = CW(P_{S_q \cup \{i\}}, 1)$. Let d' be such that $d'_l = d_l$ for all $l \notin \{q, h\}$, $d'_q = (y_1, S_q \cup \{i\})$, $d'_h = (x_h, S_h \backslash \{i\})$. We'll prove that d' wins by majority over d, a contradiction to the initial assumption that d is a global Condorcet winner. Let us check that $|\{j \in S \mid d' \succ_j d\}| > |\{j \in S \mid d \succ_j d'\}|$. Notice that $\{j \in S \mid d' \succ_j d\} = \{j \in S_q \cup \{i\} \mid y_1 \succ_j x_q\}$, and since $y_1 \in CW(P_{S_q \cup \{i\}}, 1)$, $|\{j \in S_q \cup \{i\} \mid y_1 \succ_j x_q\}| \geq |\{j \in S_q \cup \{i\} \mid x_q \succ_j y_1\}|$. Hence, we only need to prove that this inequality is strict. Given that $d \in CW(P_S, k)$, d is envy-free, then $x(i,d) <_i x_q$, and since agent i prefers any element in $CW(P_{S_q \cup \{i\}}, 1)$ to $x(i,d)$, $x_q \notin CW(P_{S_q \cup \{i\}}, 1)$, which implies that $x_q < y_1$. Then $|\{j \in S_q \cup \{i\} \mid y_1 \succ_j x_q\}| > |\{j \in S_q \cup \{i\} \mid x_q \succ_j y_1\}|$, and trivially, $|\{j \in S_q \cup \{i\} \mid x_q \succ_j y_1\}| = |\{j \in S \mid d \succ_j d'\}|$, which contradicts the fact that $d \in CW(P_S, k)$. ■

We have shown that global Condorcet winners are efficient, envy-free, internally consistent and Nash stable decisions[4]. Hence, if it were always possible to choose Condorcet winners, this would define a rule guaranteeing the simultaneous satisfaction of all these desiderata. However, Condorcet winners do not always exist in our setup, as shown in the following example. The inexistence occurs despite our strong restrictions on preferences, which guarantee the existence of Condorcet winners when $k = 1$. Because of that, the design of attractive rules will require further attention.

Example 2. ***Condorcet winners may fail to exist.*** *A society with 14 agents must locate two facilities and assign each of the agents to one of the two. Agents have Euclidean preferences on the line, with peaks as follows:* $p(i) = i$, *for all* $i = 1, .., 4$ *and* $p(5) = 32$, $p(6) = 33$, $p(7) = 34$, $p(8) = 67$, $p(9) = 68$, $p(10) = 69$, $p(11) = 97$, $p(12) = 98$, $p(13) = 99$, $p(14) = 100$. *Let us see that* $CW(P_S, 2) = \emptyset$. *We already know that if* $d = (d_1, d_2) \in CW(P_S, 2)$, *then* $d_h \in CW(P_{S_h}, 1)$ *for* $h \in \{1, 2\}$ *and* d *should be an efficient* $S/2-$*decision. We represent in figure 2 the preferences of the agents and the unique decision,* $d = ((x_1, S_1), (x_2, S_2))$ *that satisfies these properties, and thus, the unique potential candidate. However* d *is not a Condorcet winner for* P_S *and 2 since it is defeated by majority by* $d' = ((y_1, S'_1), (y_2, S'_2))$ *(this decision is represented in figure 2 with dotted lines). Indeed,* $\{i \in S \mid d' \succ d\} = \{5, 6, 7, 8, 9, 10, 12, 13, 14\}$, *and* $\{i \in S \mid d \succ d'\} = \{1, 2, 3, 4, 11\}$.

[4] Actually, these global Condorcet winners, when they exist, also satisfy the stronger condition of group Nash stability (see the Appendix).

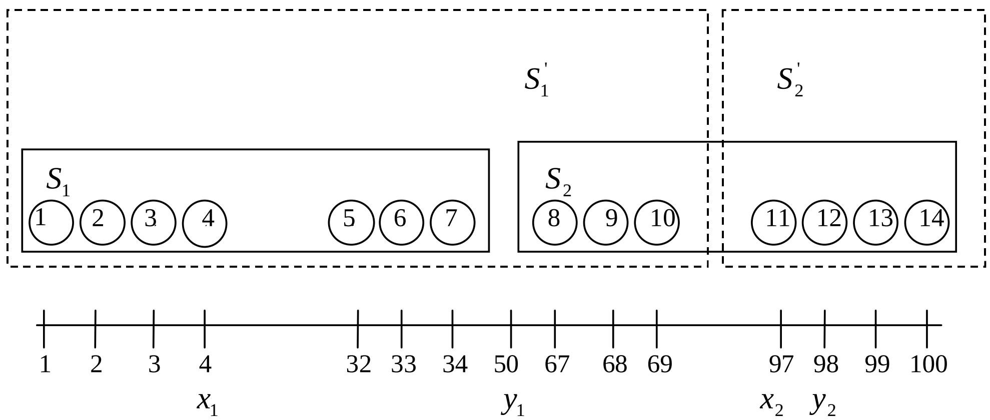

figure 2

In view of this example, we can not expected any collective choice correspondence to always select decisions which are Condorcet winners. We can demand, however, that they always do if such winners exist at all. Our analysis will concentrate on rules that satisfy this requirement, to be called Condorcet rules.

Definition 7. *A collective choice correspondence* φ is a Condorcet rule *if for all* $S \in \mathsf{S}$, $P_{\mathsf{S}} \in \mathcal{P}^{\mathsf{S}}$ *and* $k \in \mathsf{N}$ *such that* $CW(P_{\mathsf{S}}, k) \neq \emptyset$, $\varphi(P_{\mathsf{S}}, k) = CW(P_{\mathsf{S}}, k)$.

4. Internally Consistent, Nash-stable and Efficient Condorcet Rules.

The possibility that collective choices may fail to be Condorcet winners at some profiles leads to the following question: are there Condorcet rules which are internally consistent, Nash-stable and efficient?

As a first step toward an answer, we need to prove that there are always internally consistent, Nash stable and efficient decisions. This is the objective of the following proposition.

Let us first fix some notation. The lower median of a finite collection K of real numbers is denoted by $lmed(K)$. It stands for the median when the cardinality of K is odd, and for the lowest value of the median if the cardinality is even[5].

Proposition 3. *For each* $S \in \mathsf{S}$*,* $P_{\mathsf{S}} \in \mathcal{P}^{\mathsf{S}}$ *and* $k \in \mathsf{N}$, *there exist decisions* $d \in D(P_{\mathsf{S}}, k)$ *that are internally consistent, Nash stable and efficient*[6]*.*

Proof. Let $S \in \mathsf{S}$, $P_{\mathsf{S}} \in \mathcal{P}^{\mathsf{S}}$ and $k \in \mathsf{N}$. If there are at most k different peaks, we are done. Hence, suppose that there are at least k different peaks. Let us order the agents by increasing order of their peaks. Let $S_h^1 = \{i \in S \mid p(i) = p(h)\}$ for all $h \in \{1, .., k-1\}$, and $S_k^1 = \{i \in S \mid p(i) > p(k-1)\}$. Let $x^1 = (x_h^1)_{h=1}^k$ be such that $x_h^1 = lmed(p(i))_{i \in S_h^1}$, for $h \in \{1, .., k\}$, and let $d^1 = (x_h^1, S_h^1)_{h=1}^k$. By construction, this decision is internally consistent. Notice that, $x_h^1 = p(h)$ for all $h \in \{1, .., k-1\}$. If for all $i \in S_k^1$, there is $\overline{x} \in CW(P_{S_{k-1}^1 \cup \{i\}}, 1)$ such that $x_k^1 \succeq_i \overline{x}$, this decision is Nash stable. Notice also that the decision is envy free because $x_{k-1}^1 \leq \overline{x} \leq x_k^1$ and since preferences are single peaked, for all $i \in S_k^1$, $x_k^1 \succeq_i x_{k-1}^1$. In this case we are done. Otherwise, let $I = \{i \in S_k^1 \mid \text{for all } \overline{x} \in CW(P_{S_{k-1}^1 \cup \{i\}}, 1),\ \overline{x} \succ_i x_k^1\}$, and let $S_{k-1} = S_{k-1}^1 \cup I$. Let $y_{k-1} = lmed(p(l))_{l \in S_{k-1}}$. First of all, notice that for all $i, j \in I$, $lmed(p(l))_{l \in S_{k-1}^1 \cup \{i\}} = lmed(p(l))_{l \in S_{k-1}^1 \cup \{j\}} \leq y_{k-1} < x_k^1$. Notice that we can have agents in S_k^1 which prefer x_k^1 to any point in $CW(P_{S_{k-1}^1 \cup \{i\}}, 1)$, but they have their peaks between x_{k-1}^1 and y_{k-1}, which implies that they prefer y_{k-1} to x_k^1. Let $S_k' \subset S_k^1$ be the set of those agents. For each $i \in S_k'$, let z_i be such that agent i is indifferent between x_k^1 and z_i. Let us order the agents in S_k' by increasing order of z_i. Then $[z_{i+1}, x_k^1] \subseteq [z_i, x_k^1]$. Take the

[5] We cannot simplify our analysis by assuming an odd number of voters because the nature of our questions require the size of the electorate to be variable and endogenously given.

[6] There are cases where requiring group Nash stability would be too strong. As we show in the Appendix, we can not strengthen Propositioin 3 to incluide group Nash stability.

first agent in this set. Then, $lmed(p(l))_{l\in S_{k-1}\cup\{1\}} <_1 y_{k-1}$, and since $y_{k-1} <_1 x_k^1$, $lmed(p(l))_{l\in S_{k-1}\cup\{1\}} \in [z_1, x_k^1]$. If $lmed(p(l))_{l\in S_{k-1}\cup\{1\}} \in [z_2, x_k^1]$, add agent 2 to $S_{k-1}\cup\{1\}$, and we get that $lmed(p(l))_{l\in S_{k-1}\cup\{1,2\}} <_2 lmed(p(l))_{l\in S_{k-1}\cup\{1\}})$, which will imply that $lmed(p(l))_{l\in S_{k-1}\cup\{1,2\}} \in [z_2, x_k^1]$. We keep adding agents from S'_k in the above defined order whenever $lmed(p(l))_{l\in S_{k-1}\cup\{1,2,..,i\}}) \in [z_{i+1}, x_2^1]$. Let S'^1_k be this subset of agents. Then, for all $i \in S'_k \backslash S'^1_k$, $lmed(p(l))_{l\in S_{k-1}\cup S'^1_k} \notin [z_i, x_k^1]$. Notice that all agents in $S'_k \backslash S'^1_k$ have their peaks between $lmed(p(l))_{l\in S_{k-1}\cup S'^1_k}$ and x_k^1. Once this process is completed, consider the following sets of agents: $S^2_{k-1} = S_{k-1} \cup S'^1_k$, $S^2_k = S^1_k \backslash \{I \cup S'^1_k\}$, and $S^2_h = S^1_h$ for all $h \in \{1, .., k-2\}$. Let $x^2_h = lmed((p(l))_{l\in S^2_h}$ for all $h \in \{1, .., k\}$. Notice first that for all $i \in I$, $lmed(p(l))_{l\in S^1_{k-1}\cup\{i\}} \leq x^2_{k-1} < x^1_k$ and therefore, $x^2_{k-1} <_i x^1_k$. Also, by construction, for all $l \in S'^1_k$, $x^2_{k-1} <_l x^1_k$. And since $x^2_k \geq x^1_k$, for all $l \in S^2_{k-1}$, $x^2_{k-1} <_l x^2_k$. Therefore, once the process has finished, no agent that has been moved in order to join a different group, wants to go back to his initial group. If the decision $(x^2_h, S^2_h)^k_{h=1}$ is Nash stable we are done. If not, we repeat the process. Notice that, if in step j we do not get a decision Nash stable decision, it is because for some h, some $i \in S^j_h$ and all $\bar{x} \in CW(P_{S^j_{h-1}\cup\{i\}}, 1)$, $\bar{x} \succ_i x^j_h$. Thus, if this is the case, in each step we add agents from S^j_h to S^j_{h-1} in the way described above. The process will end in a finite number of steps because there are a finite number of agents, at each step $x^j_h \geq x^{j-1}_h$ for all $h \in \{1, ..k\}$, and furthermore, $S^{j-1}_1 \subseteq S^j_1$ and $S^j_k \subseteq S^{j-1}_k$. ■

For each $S \in \mathsf{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathsf{N}$, let $D^{NC}(P_S, k)$ be the set of all internally consistent, Nash stable and efficient decisions. The above proposition tells us that this set is not empty. However, not all the decisions in this set are global Condorcet winners whenever they exist, as the following example shows.

Example 3. *The society is the same as in Example 1. Consider the decision*

$d = ((y_1, S_1), (y_2, S_2))$ represented in figure 3.

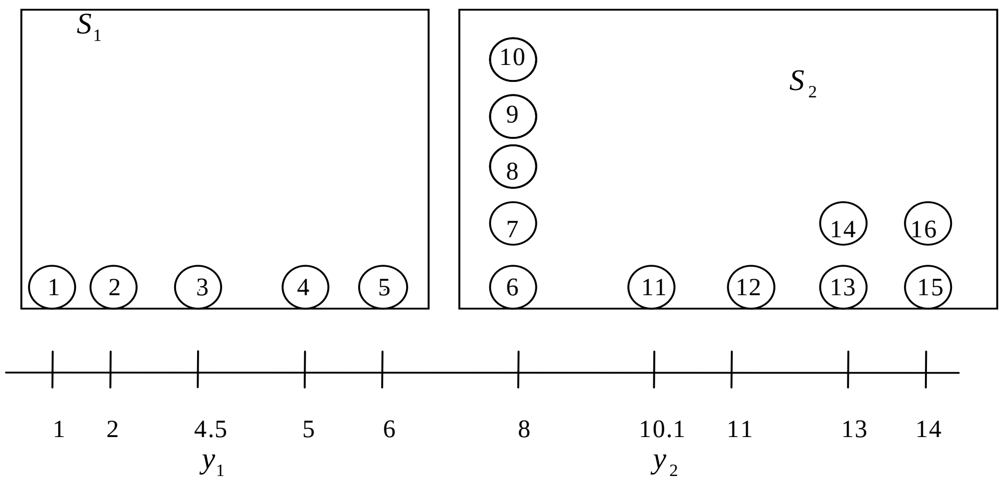

figure 3

The decision d is internally consistent, Nash stable and efficient. However, it is not a global Condorcet winner. Notice that the decision $d' = ((7, \{1, ..., 10\}), (11, \{11, ..., 16\})$ beats the decision d in majority comparisons.

Therefore, a Condorcet rule satisfying all the preceding properties should map into elements from de set $D^{NC}(S, k)$, but cannot simply select all of them for all profiles. Our next proposition will show that this delicate selection process is possible. We propose one specific selection procedure, and then comment on other similar methods that may lead to alternative rules also satisfying our properties.

Definition 8. *Given $S \in \mathsf{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathsf{N}$, for any $d, d' \in D(S, k)$, let $N(d, d') = |\{i \in S \mid d <_i d'\}|$. Given $d \in D(S, k)$, the Simpson score of d, denoted $SC(d)$, is the minimum of $N(d, d')$ over all $d' \in D(S, k)$, $d' \neq d$.*

An $S/\ k$-decision $d \in D^{\mathsf{NC}}(S,k)$ is a restricted Simpson winner *for P_{S} and k if $SC(d) \geq SC(d')$ for all $d' \in D^{\mathsf{NC}}(S,k)$. The* restricted Simpson correspondence*, SW, is defined so that for each $S \in \mathsf{S}$, $P_{\mathsf{S}} \in \mathcal{P}^{\mathsf{S}}$ and $k \in \mathsf{N}$,*

$$SW(P_{\mathsf{S}},k) = \{d \in D^{\mathsf{NC}}(S,k) \mid SC(d) \geq SC(d') \text{ for all } d' \in D^{\mathsf{NC}}(S,k)\}$$

In its usual version, the Simpson rule is an example of a Condorcet consistent rule. This is because the value of $N(d,d')$ for a Condorcet winner is always greater or equal than a majority, whereas alternatives which are not Condorcet loose by majority at least once, and cannot have a score reaching the majority size. Notice that the Simpson correspondence is usually defined as choosing from the same set of elements that are used to define the Simpson score. Our variation consists in forming the Simpson scores on the basis of pairwise comparisons on the whole set $D(S,k)$, but then choosing only among the elements of $D^{\mathsf{NC}}(S,k)$. This will prove useful in our case.

Proposition 4. *The restricted Simpson correspondence is a Condorcet rule that selects internally consistent, efficient and Nash stable decisions.*

Proof. Internal consistency, efficiency and Nash stability are satisfied because we always select elements in $D^{\mathsf{NC}}(S,k)$. It suffices to show that, whenever Condorcet winners exist, they are the chosen decisions. This follows from the fact that Condorcet winners, when there are, always belong to $D^{\mathsf{NC}}(S,k)$, as shown in Propositions 1 and 2. Therefore, restricting our maximization process to the smaller set $D^{\mathsf{NC}}(S,k)$ does not preclude us from still choosing them. ■

Remark 1. *The Simpson rule is one of the many procedures that have been suggested to extend the majoritarian principle, producing definite choices for all* *preference profiles, and choosing Condorcet alternatives whenever they exist. Another famous Condorcet extension rule is Copeland's, that is usually defined when the number of alternatives is finite. Then, each alternative is assigned one point for every win in a pairwise contest, minus one for every pairwise loss, and zero for any tie. Copeland's score for each alternative is the sum of the scores obtained from all pairwise contests, and Copeland's rule is the one that chooses the alternatives with highest Copeland scores. Notice that the definition of Simpson's score is not affected by the cardinality of the set of alternatives. The values of* $N(d, d')$ *will be finite as long as the number of voters is finite, and the only difference is that , when the number of alternatives is infinite, we take the max of* $N(d, d')$ *over an infinite set of contenders. Moreover, since the possible values of the total scores are finite, existence of an alternative with the maximum Simpson score is not an issue.*

By contrast, extending Copeland's rule involves new technicalities. A bit of measure theory is needed to extend the rule, and the existence of maximal elements needs discussion.

In principle, our arguments in the proof of Proposition 4 will stand for any Condorcet consistent rule that can be obtained by maximizing transitive relations taking maximal values at Condorcet alternatives, when these exist. However, and somewhat surprisingly, tournament theory is lacking suggestions of other rules that may naturally extend majority by assigning alternatives an ordering with the above characteristics. This, and the also surprising lack of characterizations for most of the tournament solution concepts, explain why we fall short of a full characterization of the rules under study, and must remain content with a constructive existence proof.

5. APPENDIX

We prove in this appendix that whenever a global Condorcet winner exists it satisfies group Nash stability. But, in the statement of Proposition 3 Nash stability cannot be strengthened to group Nash stability.

Group Nash stability has the same flavor than Nash stability with the added possibility that agents might coordinate with others in the same group when deciding whether or not to change groups. Formally,

Definition 9. *An $S/k-$decision $d = ((x_h, S_h))_{h=1}^k \in D(S,k)$ is* group Nash stable *if for all h, j, with $h \neq j$ and for all $I \subset S_h$, there is $\overline{x} \in CW(P_{S_j \cup I}, 1)$ such that $x(i,d) <_i \overline{x}$ for all $i \in I$.*

Our next proposition shows that global Condorcet winners are group Nash stable.

Proposition 5. *Given $S \in \mathsf{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathsf{N}$ if an $S/k-$decision $d = ((x_h, S_h))_{h=1}^k \in CW(P_S, k)$, then d is group Nash stable.*

Proof. Suppose that there is a global Condorcet winner which is not group Nash stable. That is, there is $d \in CW(P_S,k)$, h, j, with $h \neq j$ and $I \subset S_h$, such that $\overline{x} \succ_i x(i,d)$ for all $\overline{x} \in CW(P_{S_j \cup I}, 1)$, for all $i \in I$. Without loss of generality suppose that $h = j+1$. Let $[y_1, y_2] = CW(P_{S_j \cup I}, 1)$. Let d' be such that $d'_l = d_l$ for all $l \notin \{j,h\}$, $d'_j = (y_1, S_j \cup I)$, $d'_h = (x_h, S_h \backslash I)$. We'll prove that d' wins by majority over d, a contradiction to the initial assumption that d is a Condorcet winner. Let us check that $|\{i \in S \mid d' \succ_i d\}| > |\{i \in S \mid d \succ_i d'\}|$. Notice that $\{i \in S \mid d' \succ_i d\} = \{i \in S_j \cup I \mid y_1 \succ_i x_j\}$, and since $y_1 \in CW(P_{S_j \cup I}, 1)$, $|\{i \in S_j \cup I \mid y_1 \succ_i x_j\}| \geq |\{i \in S_j \cup I \mid x_j \succ_i y_1\}|$. Hence, we only need to prove that this inequality is strict. Given that $d \in CW(P_S, k)$, d is envy-free, then

$x(i,d) <_i x_j$ for all $i \in I$, and since all the agents in I prefer any element in $CW(P_{S_j \cup I}, 1)$ to $x(i,d)$, $x_j \notin CW(P_{S_j \cup I}, 1)$, which implies that $x_j < y_1 \leq y$ for all $y \in CW(P_{S_j \cup I}, 1)$. Then $|\{i \in S_j \cup I \mid y_1 \succ_i x_j\}| > |\{i \in S_j \cup I \mid x_j \succ_i y_1\}|$, and trivially, $|\{i \in S_j \cup I \mid x_j \succ_i y_1\}| = |\{i \in S \mid d \succ_i d'\}|$, which contradicts the fact that $d \in CW(P_S, k)$. ■

When a global Condorcet winner does not exist, it may not always be possible to have internally consistent and group Nash stable decisions.

Example 4. ***Non existence of efficient, internally consistent and group Nash stable decisions.*** *Consider a society with 25 agents with euclidean preferences with the following peaks:* $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(i) = 12$ *for all* $i \in \{4,5\}$, $p(i) = 14$ *for all* $j \in \{6,7,8,9\}$ *and* $p(i) = 15$ *for all* $i \in \{10,...,25\}$. *In figure 4, we represent this society and the decision* $d = ((x_1, S_1), (x_2, S_2))$ *which is the unique decision that is internally stable and efficient.*

No rule satisfying efficiency and internal stability can be group Nash stable. To check that, notice that these rules will always select the median of groups, and thus, $S = \{6,7,8,9\}$ *will prefer to join* S_1 *and then get* $\bar{x}_1 = 14$. *Moreover, notice that a cycle arises, which also makes this new coalition (and the next) unstable. Indeed, under the assignment* $((14, \{1,2,3,6,7,8,9\}), (15, \{4,5,10,..,25\})$ *the group* $\{4,5\}$ *would like to join* $\{1,2,3,6,7,8,9\}$ *and get the facility located in* 12. *And the resulting allocation,* $((12, \{1,2,..,9\}), (15, \{10,..,25\})$ *will then induce* $\{6,..,9\}$ *to join* $\{10,..,25\}$ *in order to get allocated to 15, which they prefer to 12, then leaving* $\{1,2,3,4,5\}$ *with a facility in* 3, *for which now* $\{4,5\}$ *will want to depart, thus closing the cycle.*

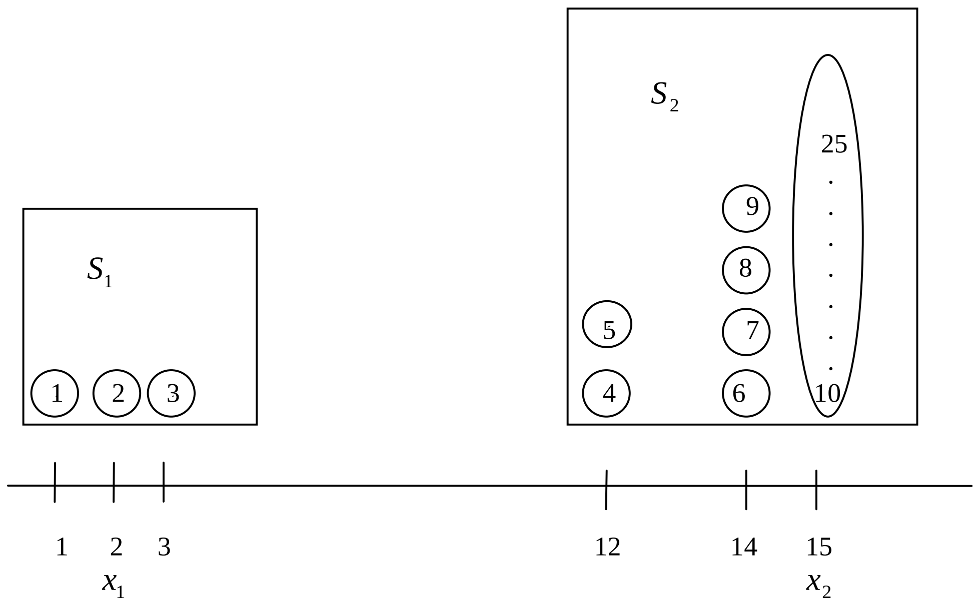

figure 4

The notion of group Nash stability that we are using deals with special deviations involving a group that is contained in a set S_{h} of the partition that joins another group S_{j} of the partition. It would be possible to consider more general deviations, for example, the group who deviates could come from different elements in the partition. But this will be a stronger requirement than the one we present, and it would not help, since we already obtain an impossibility result under the weaker requirement.

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