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# Optimal procurement auction for a buyer with downward sloping demand: more simple economics

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## Abstract

A buyer with downward sloping demand faces a number of unit-supply sellers. The paper characterizes optimal auctions in this setting. For the symmetric case, a uniform auction (with price equal to lowest rejected offer) is optimal when complemented with reserve prices for different quantities acquired. For asymmetric sellers, the optimal distortions are familiar. The problem is similar to the third-degree discriminating monopsonist problem, just as in the unit (flat) demand case (Bulow-Roberts, 1989), and when the number of sellers (and the demand) grows their outcomes approach at the speed of the law of large numbers.

*JEL Classification:* D44, D42.

*Keywords:* Auctions, Monopsony

# 1 Introduction

In this paper we analyze the optimal mechanism (procurement auction) for a buyer with downward sloping demand that faces a set of privately informed, capacity constrained sellers. We then analyze the relationship between this optimal procurement auction and the classical model of monopsony. Also, we study how this relationship gets tighter when the number of sellers is large (or their size is small).

Optimal auctions for a seller facing several potential buyers or, equivalently, a buyer facing several potential sellers, have been widely studied. For instance, assume a buyer faces  $N$  potential suppliers, each of whom can provide a unit of some good at some privately known cost. Assume the sellers have private, independent costs, and the buyer's valuation of each of (at most)  $K \leq N$  units is some given constant. Then the surplus maximizing (incentive compatible) procurement mechanism or auction consists of acquiring one unit from the  $K$  sellers whose "virtual cost" (some transformation of their cost involving the idiosyncratic probability distributions of these costs) is lowest, provided they are below the buyer's unit valuation. In the symmetric case, where all sellers' costs are i.i.d., this is equivalent to buying from each of the  $K$  lowest cost sellers whose cost is below some reserve price (see Myerson 1981, and Engelbrecht-Wiggans 1988). For this same case, a simple  $K^{th} + 1$  price auction implements this optimal mechanism. In this auction, sellers bid simultaneously, but there is a highest acceptable price. Then (up to) the  $K$  lowest bidders supply one unit at a price equal to either the highest unsuccessful bid or the reserve price, whichever is lower. Another way to describe this auction mechanism is as follows: the buyer announces a (flat) demand curve, then sellers make their price offers, which are ordered from lowest to highest to construct a supply function. Then the price and

the quantity are determined by crossing these two curves (with price equal to the highest equilibrium price, if there are more than one).

This interpretation of the optimal auction as a market outcome is not arbitrary. Indeed, Bulow and Roberts (1989) have shown that the problem of designing an optimal auction for a buyer (seller in their case) is parallel to the (third-degree discriminating) monopsony (monopoly in their case) pricing problem.<sup>1</sup> Indeed, we can interpret probability as quantity, and represent each seller as a separate market, where the inverse supply function at each quantity below 1 (capacity) is the inverse of the cost distribution evaluated at a quantile equal to that quantity. Then the optimal reserve price for each seller is the price at which marginal expenditure in the corresponding market crosses the (flat) demand of the buyer. When demand is flat, the monopsonist faces separated markets.<sup>2</sup> When sellers are symmetric (markets are identical), of course, third-degree discrimination would be equivalent to non discrimination.<sup>3</sup>

The first thing we do in this paper is to extend this analysis to allow for

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<sup>1</sup>Bulow and Roberts carry out their analysis in a monopoly setting. We choose to analyze the monopsony case. In an auction setting, it is perhaps easier to picture decreasing willingness to pay than increasing opportunity cost for additional units. Nevertheless, it should be clear that both their analysis and the present paper apply to both cases. Thus, we will translate Bulow and Roberts' results to the monopsony case, when we refer to their results later.

<sup>2</sup>In the auction case, however there is a linkage through the fact that the buyer's demand function becomes vertical above some quantity. Thus, when this "capacity constraint" binds, i.e., when there are more sellers with costs below the reserve price than units the buyer demands, then the buyer buys from sellers in ascending order of their marginal expenditure.

<sup>3</sup>The parallel between markets models and auctions has been investigated beyond the monopoly case. See for example McAfee (1993), Peters (1997), and Burguet and Sakovics (1999).

downward sloping demand functions. Thus, we assume that the (marginal) buyer's willingness to pay for the  $k^{th}$  unit may be decreasing in  $k$ . We start by analyzing the equilibrium of a generalization of the  $K^{th} + 1$  price auction. In this generalized auction, the buyer announces a vector of reserve prices. This can be thought of as a demand (step) function. Then sellers bid simultaneously. The bids are ordered from lowest to highest, and this supply function is crossed to the announced demand function to determine price and quantity. (Again, when there is more than one equilibrium price the price is the highest of them.) Given a demand function, sellers have as a (weakly) dominant strategy to bid their true cost. Then the buyer's optimal reserve prices (steps of her demand function) are obtained by equating the "virtual cost" (cost plus inverse of the hazard rate) with the buyer's true willingness to pay for additional units.

Next we characterize optimal auctions allowing for asymmetries, and show that the generalized  $K^{th} + 1$  price auction is indeed an optimal auction for the symmetric case. For the asymmetric case, the buyer computes seller-specific reserve price vectors (demand curves). For each seller, these vectors are obtained as in our  $K^{th} + 1$  price auction. Thus, if sellers have different cost distributions (and therefore hazard rates), their reserve price vectors will indeed be different. Then acquisitions are determined in sequence following the order of virtual cost: if  $k$  units have been assigned already (to the  $k$  sellers with lowest "virtual cost") then the  $k^{th} + 1$  unit will be acquired from the seller with  $k^{th} + 1$  lowest "virtual cost", if this is lower than the buyer's valuation for the  $k^{th} + 1$  unit. I.e., provided the cost of the seller is below the  $k^{th} + 1$  reserve price in his vector.

Both for the symmetric and the asymmetric cases, we then provide a market interpretation of these results that follows the lines of Bulow and Roberts

(1989). For the monopsonist, the main difference introduced by a downward sloping demand function is that markets are not separated even if the buyer can third-degree discriminate. Nevertheless, the optimal pricing rule for the monopsonist satisfies that, given the quantities traded in other markets, the marginal expenditure in any particular market is equal to the buyer's marginal willingness to pay. This is how the optimal auction determines trade, when again we interpret probability as quantity. Just as in the monopsony pricing problem, the main change with respect to the flat demand case is that markets are not separated, and then the trade possibilities cannot be determined without reference to other markets (sellers).

The last goal of this paper is to investigate an additional question about the relationship between the monopsony and the optimal auction models. This is whether the former can be thought of as a limiting case (for sufficiently large numbers of traders) of the latter. In other words, whether the fact that the single buyer confronts privately informed sellers becomes of no relevance when the number of sellers (that is, their size relative to the market) is sufficiently large. Or put in other terms, whether as aggregate uncertainty vanishes, we recover the monopsony pricing model, and then setting a price is just as good for the buyer as designing the best of mechanisms.<sup>4</sup> We show that this is indeed the case. Further, we investigate the speed of convergence. Prices, quantities, and consumer surplus all converge at the speed of the law of large numbers.

The paper is structured as follows. The model is described in the next section. In Section 3 we study the  $K^{th} + 1$  price auction for the symmetric

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<sup>4</sup>We ask this question in the symmetric seller case. If the number of "types" of sellers remains bounded as the number of sellers gets large, the parallel between the (third-degree discriminating) monopsonist and the optimal auction extends easily.

case. In Section 4 we extend the results on revenue equivalence and optimal auctions for the downward sloping case. Then Section 5 interprets these results in the light of the monopsony pricing model. Section 6 analyzes the convergence properties when the number of sellers gets large. Section 7 concludes.

## 2 The model

Each of  $N$  sellers can supply a unit of a homogeneous, indivisible good. Supplier  $i$ 's unit cost,  $c_i$ , is the realization of a random variable distributed with positive density  $f_i$  and c.d.f.  $F_i$  in  $[0, 1]$ . Costs are therefore independent and identically distributed, and the realization  $c_i$  is seller  $i$ 's private information, although  $F_i$  is common knowledge. Sellers face a unique buyer with a demand function that can be described by the ordered vector of valuations for additional units,  $\mathbf{v} = (v_1, v_2, \dots, v_M)$ , where  $v_i \geq v_j$  for all  $i < j$ ,  $v_i \in [0, 1]$ , and  $M \leq N$ . All agents are risk neutral. Also, we assume regularity of  $F_i$ . That is, we assume that

$$J_i(c) = c + \frac{F_i(c)}{f_i(c)}$$

is strictly increasing for all  $i$ .

## 3 Auction for the symmetric case

For the moment, assume sellers are symmetric. That is,  $F_i = F$  for all  $i$  (and  $J_i = J$  for all  $i$ ). Consider a simple auction format, where the buyer organizes the transaction as follows. She announces a demand schedule, that is, an ordered list  $\mathbf{r} = (r_1, r_2, \dots, r_M)$ , and asks for simultaneous bids by all sellers. Sellers' bids  $\mathbf{b} = (b_1, b_2, \dots, b_N)$  are ordered from lowest to highest,

and then this supply schedule is crossed to the demand schedule  $\mathbf{r}$ . That is, let  $k$  be such that  $b_k \leq r_k$ , and  $b_{k+1} > r_{k+1}$ , where  $r_{M+1} \equiv 0$ . Then  $k$  units are traded at a price  $\min\{r_k, b_{k+1}\}$ . In other words, trade would be Walrasian for the demand and supply functions  $\mathbf{r}$  and  $\mathbf{b}$ , respectively.

Notice that given this pricing rule, a seller's bid may affect whether the seller is scheduled to produce or not, but it never affects the price that this seller receives. This implies the following preliminary result.

**Lemma 1** *For each seller, bidding true cost is a (weakly) dominant strategy.*

Thus, the supply function will be the true supply function, for any demand function  $\mathbf{r}$  that the buyer announces. Given that sellers will bid their true costs, we can write the buyer's expected surplus for a given demand function  $\mathbf{r}$  as follows:

$$E\pi(\mathbf{r}; \mathbf{v}) = \sum_{k=1}^M \left\{ \left( \sum_{i=1}^k v_i - k \cdot r_k \right) \binom{N}{k} F(r_k)^k [1 - F(r_k)]^{N-k} + \int_{r_{k+1}}^{r_k} \left( \sum_{i=1}^k v_i - k \cdot c \right) \binom{N-1}{k} N f(c) F(c)^k [1 - F(c)]^{N-k-1} dc \right\}.$$

Then the buyer's optimal choice of  $\mathbf{r}$  maximizes the above expression.

**Proposition 2** *The optimal demand function for the buyer  $\mathbf{r}^*(\mathbf{v})$  is given by  $J(r_k) = v_k$ .*

**Proof.** Taking derivative of  $E\pi(\mathbf{r}; \mathbf{v})$  with respect to  $r_k$ , we have the first order condition

$$\begin{aligned} & \binom{N}{k} F(r_k)^{k-1} [1 - F(r_k)]^{N-k} \left\{ -k \frac{F(r_k)}{f(r_k)} + k \left( \sum_{i=1}^k v_i - k \cdot r_k \right) \right. \\ & \left. - \left( \sum_{i=1}^{k-1} v_i - (k-1) r_k \right) \frac{\binom{N-1}{k-1}}{\binom{N}{k}} N \right\} = 0, \end{aligned}$$



and the result follows since

$$\frac{\binom{N-1}{k-1}}{\binom{N}{k}}N = k.$$

■

Notice that  $r_k(\mathbf{v})$  depends only on  $v_k$ . Also notice that  $\mathbf{r}^*(\mathbf{v})$  is decreasing, since  $J$ , and therefore  $J^{-1}$ , is strictly increasing. That is,  $\mathbf{r}^*(\mathbf{v})$  is a demand function for any  $\mathbf{v}$ . Also notice that the demand function for the buyer is somehow independent of the size of the market. In this sense, if we replicate the market by, say, splitting the size of units in half (and multiplying the number of suppliers by two), the equilibrium demand function would still be the same.

Proposition 2 generalizes well-known results for the single-unit or flat-demand cases. There a single reserve price is optimal for any amount purchased. Here, however, the willingness to pay for additional units is decreasing with the size of the purchase. Accordingly, higher amounts are purchased only if that can be done at lower prices.

## 4 Optimal Auctions

We next consider the problem of designing an optimal selling mechanism for the above problem. We return to the general, asymmetric setting. Denote by  $\mathbf{c} = (c_1, c_2, \dots, c_N)$ . Also, denote by  $\mathbf{F}$ , and  $\mathbf{f}$  the distribution and density of  $\mathbf{c}$  and by  $F_{-i}$  and  $f_{-i}$  the marginal distribution and density of  $\mathbf{c}_{-i}$ .

Using the Revelation Principle, a trading mechanism or "auction" in this setting can be characterized by some lists of functions,  $\{X_i, P_i\}_{i=1,2,\dots,N}$ , where  $X_i : [0, 1]^N \rightarrow \{0, 1\}$ , and  $P_i : [0, 1]^N \rightarrow R$ .  $X_i(\mathbf{c})$  should be interpreted as the probability that seller  $i$  sells one unit if the cost realizations are  $\mathbf{c}$ . In general, we could consider random allocation, so that this value

could take any real number between 0 and 1. However, as usual, nothing is gained with this generalization.  $P_i(\mathbf{c})$  should be interpreted as the (expected, if you wish) payment that seller  $i$  receives if the cost realizations are  $\mathbf{c}$ . Once the mechanism is given, we can define a new collection of functions,  $\{x_i, p_i\}_{i=1,2,\dots,N}$ , where  $x_i : [0, 1] \rightarrow [0, 1]$ , with  $x_i(c_i) = E_{\mathbf{c}_{-i}} [X_i(\mathbf{c}_{-i}, c_i)]$ , and  $p_i : [0, 1] \rightarrow R$ , with  $p_i(c_i) = E_{\mathbf{c}_{-i}} [P_i(\mathbf{c}_{-i}, c_i)]$ . A necessary condition for the mechanism  $\{X_i, P_i\}$  to be incentive compatible is that<sup>5</sup>

$$p'_i - x'_i c_i = 0,$$

which, by integrating, implies the well known expression of revenue equivalence

$$p_i(c_i) = c_i x_i(c_i) + \int_{c_i}^1 x_i(z) dz, + R_i(1),$$

where  $R_i(1) = p_i(1) - 1x_i(1)$  are the rents for seller  $i$  with valuation 1 (which in the optimal mechanism will be set to zero). As could be expected, two mechanisms that assign the purchases in the same way (and leave zero rents to a seller with the highest possible realization of cost) will also result in the same expected payment for the buyer and the same expected (conditional on their cost) rents for the sellers. Using the above result, we can substitute the incentive compatibility (revenue equivalence) equation on the buyer's objective function (the expected *consumer surplus*),

$$\int_{\mathbf{c}} \left( \sum_i X_i(\mathbf{c}) - \sum_{k=1} v_k - \sum_i P_i(\mathbf{c}) \right) dF(\mathbf{c}) - \sum_i R_i(1),$$

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<sup>5</sup>We assume that the functions are differentiable. As usual, this is only to save in notation and space.

so that this objective function can be written as

$$\int_{\mathbf{c}} \left( \sum_{k=1}^{\sum_i X_i(\mathbf{c})} v_k - \sum_i \left( c_i + \frac{F_i(c_i)}{f_i(c_i)} \right) X_i(\mathbf{c}) \right) dF(\mathbf{c}) - \sum_i R_i(1).$$

By simply inspecting this expression, we observe that, for any realization of  $\mathbf{c}$ , once decided the number of units to be bought,  $\sum_i X_i(\mathbf{c})$ , the optimal choice for the buyer is to buy (make  $X_i(\mathbf{c}) = 1$ ) from the sellers with lowest values of  $J_i(c_i) = c_i + \frac{F_i(c_i)}{f_i(c_i)}$ . This is the well known result for unit auctions. But here the number of units bought is decided by "equating" the marginal willingness to pay,  $v_k$ , with the corresponding  $k^{th}$  order statistic of the  $J_i(c_i)$ 's realizations! Any mechanism that induces this allocation (with zero rents for sellers with cost 1) is an optimal mechanism.

Notice that in the symmetric case (when  $J(c)$  is a monotone function), the  $k^{th}$  order statistic of the  $J_i(c_i)$ 's corresponds with the  $k^{th}$  order statistic of the  $c_i$ 's. Thus, the auction we presented in Section 3 is indeed an optimal mechanism for the buyer. When sellers are asymmetric, however, the optimal auction discriminates in favor of (ex-ante) weaker sellers, just as the optimal auction for the single object case. We next turn to the economic interpretation of our results.

## 5 Simple economics

What is the relationship between the above one buyer optimal auction and the classical monopsony pricing model? One difference is that in our model the supply function is random. The position of the supply curve at the  $k^{th}$  unit is the realization of the  $\frac{k}{N}$  quantile of the sample of components of the random variable with distribution  $\mathbf{F}$ . This is a random variable itself. The

second difference is that our buyer does not fix the transaction price, but rather the demand to confront with the (ex-post realized) supply.

In a celebrated and influential paper, Bulow and Roberts (1989) have shown that despite these differences, these two problems are somehow homomorphic for the case of flat demand. Indeed, the problem faced by the single buyer in designing an optimal auction is parallel to the pricing decision of a third-degree discriminating monopsonist where the supply in each of the  $N$  markets (one for each seller) is given by the cost distribution of the corresponding seller. That is, what is probability is interpreted as quantity, when constructing a supply (marginal cost) function from a probability distribution on costs. There are two key points that make this parallel. First, the optimal auction incorporate seller-specific reserve prices (maximum costs), so that trade takes place with one particular seller only if the cost of this seller is below some value strictly lower than the (constant) unit valuation of the buyer. These reserve prices are set at the level where willingness to pay intersects marginal expenditure for that seller/market, just as in the third-degree discriminating monopsonist. Second, the buyer buys from sellers whose realized marginal expenditure are lowest. This second feature is the result of a vertical portion of the demand (if  $M > N$  it is irrelevant) that may introduce some link between otherwise separated markets.

It is convenient to briefly explain the parallel between prices and reserve prices. Consider a symmetric case, in which case one (but not the only one) optimal mechanism is to conduct an auction and setting the price equal to the minimum rejected bid or the (common) reserve price, whichever is lower. Reserve prices are not prices, since sellers will in general receive a higher payment, conditional on selling. However, reserve prices are the prices when (and only when) the marginal seller faces no competition from other sellers.

That is, when given the bids of other sellers, this particular seller sells if and only if her bid is below the reserve price. In this case, markets are separated since "demand constraints" do not bind, and then the problem faced by the buyer with respect to each of the sellers is indeed that of a buyer that makes a take it or leave it offer. This problem is homomorphic to that of a monopsonist facing a supply schedule given by the distribution of cost of the seller.

The (lack of) separation of markets is the main difference that downward sloped demands introduce. Yet, the parallelism between the third-degree discriminating monopsonist pricing and the buyer's auction design problems can be extended to this case. When a monopsonist demand is downward sloping, her markets are not separated even if she can price discriminate.<sup>6</sup> Indeed, her residual demand curve depends on trade in other markets. Nevertheless, given the quantity traded in other markets, the solution for the monopsonist is to equate marginal expenditure in any given market to her residual demand. Next we show that this is how reserve prices are determined in the optimal mechanism characterized in the previous section.

Indeed, given the (indirect) supply function for a particular market  $i$ ,

$$P_i^s(q_i) = F_i^{-1}(q_i),$$

we can invert it to obtain  $Q_i^s = F_i(P_i^s)$ . The monopsonist's optimal choice of trade  $q_i$ , given the quantity traded in the rest of markets  $Q_{-i}$ , results from equating her willingness to pay (residual demand)  $v_{Q_{-i}+q_i}$  with the marginal expenditure,

$$\frac{d(P_i^s(q_i) \cdot q_i)}{dq_i} = \frac{dP_i^s(q_i)}{dq_i}q_i + P_i^s(q_i),$$

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<sup>6</sup>In the monopoly case, markets are not separated when the marginal cost is increasing.

where

$$\frac{dP_i^s(q_i)}{dq_i} = \frac{1}{f_i(P_i^s(q_i))}.$$

That is, the monopsonist's optimal choice is given by

$$v_{Q_{-i}+q_i} = \frac{q_i}{f_i(P_i^s(q_i))} + P_i^s(q_i),$$

which in terms of  $P^s$ , and recalling that  $P_i^s(q_i) = F_i^{-1}(q_i)$ , can be written as

$$v_{Q_{-i}+q_i} = \frac{F_i(P_i^s)}{f_i(P_i^s)} + P_i^s = J_i(P_i^s).$$

This is precisely the way the optimal mechanism characterized in the previous section sets reserve prices. Indeed, if the quantity traded with other sellers is  $Q_{-i} = k - 1$ , then the buyer will consider buying from seller  $i$  only if her cost (supply) is below  $J_i^{-1}(v_k)$ . When indeed the cost realizations for exactly  $k - 1$  other sellers  $j$ 's are such that their respective costs are below  $J_j^{-1}(v_k)$ , the buyer will acquire  $k$  units if (and only if) seller  $i$ 's cost is below  $J_i^{-1}(v_k)$ .

The second, perhaps less interesting parallelism between optimal auctions and monopsonist pricing models that we have mentioned also extends to the downward demand case. Indeed, if more than  $k$  sellers have a cost below their respective value of  $J_j^{-1}(v_k)$ , but less than  $k + 1$  have a cost below  $J_j^{-1}(v_{k+1})$ , then  $k$  units are traded with the  $k$  sellers whose realized  $J_j(c_j)$  are the lowest. As we have just seen,  $J_j(c_j)$  represents the marginal expenditure in market  $j$ , and therefore this is the same criterion for the flat demand case.

## 6 Large markets: the convergence properties of optimal auctions

Let us now return to the symmetric case, where third-degree discrimination coincides with no discrimination at all. In the previous sections we have

discussed how the "logic" of monopsony pricing is the same behind optimal auctions. Despite this parallel, the trade (quantity and price) realized in the optimal auction described in Section 3 does not coincide with the trade of a monopsonist facing the realized supply. The divergence is due to the fact that the buyer does not know where exactly lies the supply function.<sup>7</sup> As the number of sellers gets large, the uncertainty about where exactly lies the supply function should disappear. Thus we might suspect that the difference in trade between a model of monopsony pricing and the auction model should also disappear. This is what we do in this section. That is, we show that the monopsony pricing model, that shares the logic with the optimal auction, is itself a model of how information is aggregated in our auction model.

First, in order to simplify in notation and give a clear meaning to the convergence criterion, assume the demand function  $\mathbf{v}$  is the vector of order statistics from  $M$  independent realizations of some random variable with distribution  $G$  and density  $g$  in  $[0, 1]$ . Of course, since  $\mathbf{r}$  is announced, these realizations need not be private information for the buyer. Also, we do not lose further generality by assuming  $M = N$ . We will be interested in the convergence properties of the outcome of this game when the number of sellers and units demanded,  $N$ , gets large. (Alternatively, when the size of sellers gets small.)

For any  $q \in (0, 1)$  and integer  $N$ , let  $k_{q,N}$  be the largest integer such that  $k_{q,N} \leq qN$ . Then, we can define the  $q$ -quantile (ordering from largest to smallest) of the sample of size  $N$  of valuations,  $\mathbf{v}^N = (v_1, v_2, \dots, v_N)$ , as  $v_{k_{q,N}}$ .

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<sup>7</sup>The buyer, as opposed to the monopsonist in the pricing model, can in principle use a wider set of instruments, and not just a price. However, the mere asymmetry of information makes this different irrelevant.

This is a random variable. As  $N$  gets large,

$$N^{\frac{1}{2}} (v_{k_q, N} - G^{-1}(1 - q)) \xrightarrow{L} N \left( 0, \frac{q(1 - q)}{g(G^{-1}(1 - q))^2} \right).$$

Likewise, the  $q$ -quantile (in the usual ascending order) of the sample  $J(\mathbf{c}^N)$  of sellers' costs transformed by the monotone, continuous function  $J$ , defined in a similar way,  $J(c_{k_q, N})$ , is a random variable, and as  $N$  gets large,

$$N^{\frac{1}{2}} (J(c_{k_q, N}) - J(F^{-1}(q))) \xrightarrow{L} N \left( 0, \frac{q(1 - q)}{f(J(F^{-1}(q)))^2} \right).$$

Now, for a monopsonist market with "demand" given by  $Q^d = N(1 - G(P))$  and supply given by  $Q^s = NF(P)$ , the monopsonist maximizes surplus by buying an amount  $Q$  such that

$$G^{-1}\left(1 - \frac{Q}{N}\right) = J\left(F^{-1}\left(\frac{Q}{N}\right)\right).$$

Let  $q^*$  be the unique solution for  $\frac{Q}{N}$  in the equation above. From the convergence properties of quantiles presented above, it is immediate that the realized trade should get infinitely close to  $q^*N$  with probability infinitely close to 1, as  $N$  is large. The next proposition shows that this is indeed the case, and also the speed at which trade approaches this quantity.

**Proposition 3** *The per seller quantity traded in the optimal auction with  $N$  sellers (and  $N$ -demand),  $q_N^o$ , converges in probability to  $q^*$ . The sequence  $q_N^o$  is at most of order  $n^{\frac{1}{2}}$  in probability,  $O_p(n^{\frac{1}{2}})$ .*

**Proof.** Since both  $N^{\frac{1}{2}} (v_{k_q, N} - G^{-1}(1 - q))$  and  $N^{\frac{1}{2}} (J(c_{k_q, N}) - J(F^{-1}(q)))$  converge in distribution, they are both  $O_p(1)$  (i.e., are at most of order  $N^0$  in probability). Thus both  $(v_{k_q, N} - G^{-1}(1 - q))$  and  $(J(c_{k_q, N}) - J(F^{-1}(q)))$  are  $O_p(N^{-1/2})$ , and since  $G^{-1}(1 - q)$  and  $J(F^{-1}(q))$  are non stochastic, this means that both  $v_{k_q, N}$  and  $J(c_{k_q, N})$  are  $O_p(N^{-1/2})$ . Subtracting one sequence



from the other, we have that  $(v_{k_{q,N}} - J(c_{k_{q,N}}))$  converges in probability as well (and is  $O_p(N^{-1/2})$ ). That is, there exists a scalar  $b$  such that  $\forall \epsilon > 0$ ,  $\Pr[|v_{k_{q,N}} - J(c_{k_{q,N}}) - b| < \epsilon] \rightarrow 1$  as  $N \rightarrow \infty$ . Given the convergence in distribution of  $N^{\frac{1}{2}}(v_{k_{q,N}} - G^{-1}(1 - q))$  and  $N^{\frac{1}{2}}(J(c_{k_{q,N}}) - J(F^{-1}(q)))$ , this value  $b$  could be nothing but  $G^{-1}(1 - \frac{Q}{N}) - J(F^{-1}(\frac{Q}{N}))$ . Thus, the realized trade  $q_N^o$  converges in probability to  $q^*$ . This proves the first part of the proposition. Now, to prove that  $q_N^o$  is  $O_p(n^{-1/2})$ , we just prove by contradiction that  $N^{\frac{1}{2}}(q_N^o - q^*) \xrightarrow{P} 0$ . Assume otherwise, that is, assume  $\exists \epsilon > 0$  and  $\delta > 0$  such that  $\forall N' \exists N > N'$  such that  $\Pr[|N^{1/2}(q_N^o - q^*)| > \epsilon] > \delta$ . This implies that

$$\Pr[q_N^o > q^* + \frac{\epsilon}{N^{1/2}}] + \Pr[q_N^o < q^* - \frac{\epsilon}{N^{1/2}}] > \delta.$$

That is,

$$\begin{aligned} & \Pr[v_{k_{(q^* + \frac{\epsilon}{N^{1/2}}), N}} - J(c_{k_{(q^* + \frac{\epsilon}{N^{1/2}}), N}}) > 0] + \\ & \Pr[v_{k_{(q^* - \frac{\epsilon}{N^{1/2}}), N}} - J(c_{k_{(q^* - \frac{\epsilon}{N^{1/2}}), N}}) < 0] > \delta. \end{aligned}$$

To shorten notation, let us define  $\Delta(q) = G^{-1}(1 - q) - J(F^{-1}(q))$ . Remember that  $\Delta(q^*) = 0$ , and that the random variable  $N^{1/2}(v_{k_{q,N}} - J(c_{k_{q,N}})) \xrightarrow{L} N(N^{1/2}\Delta(q), \sigma^2(q))$ , where  $\sigma^2(q)$  is constant in  $N$ . Thus, for large enough  $N$ , the distribution of  $v_{k_{q,N}} - J(c_{k_{q,N}})$  is arbitrarily close to the normal distribution with a expectation  $\Delta(q)$  and variance  $N^{-1}\sigma^2(q)$ . Then,  $v_{k_{(q^* + \frac{\epsilon}{N^{1/2}}), N}} - J(c_{k_{(q^* + \frac{\epsilon}{N^{1/2}}), N}})$  is arbitrarily close to a normal distribution with density

$$f(x) = \frac{N^{1/2}}{\sigma(q^* - \frac{\epsilon}{N^{1/2}})\sqrt{2\pi}} \exp\left[-\frac{N(x - \Delta(q^* - \frac{\epsilon}{N^{1/2}}))^2}{2\sigma^2(q^* - \frac{\epsilon}{N^{1/2}})}\right],$$

which can be written

$$f(x) = \left[ \Phi\left(\Delta(q^*), N^{-1}\sigma^2(q^* - \frac{\epsilon}{N^{1/2}})\right) \right] \Theta(N, \epsilon),$$

where  $\Phi(\mu, \sigma^2)$  is the density of a normal random variable with expectation  $\mu$  and variance  $\sigma^2$  and

$$\Theta(N, \epsilon) = \exp \left[ -\frac{N}{2\sigma^2(q^* - \frac{\epsilon}{N^{1/2}})} \left\{ \left( \Delta(q^*) - \Delta(q^* - \frac{\epsilon}{N^{1/2}}) \right)^2 - 2 \left( \Delta(q^*) - \Delta(q^* - \frac{\epsilon}{N^{1/2}}) \right) (x - \Delta(q^*)) \right\} \right]$$

As  $N$  gets large, and applying L'Hôpital rule,  $\Theta(N, \epsilon)$  converges to

$$N^{1/2} \left( \Delta(q^*) - \Delta(q^* - \frac{\epsilon}{N^{1/2}}) \right) \rightarrow \epsilon \frac{d(\Delta(z))}{dz} \Big|_{z=q^* - \frac{\epsilon}{N^{1/2}}},$$

which is bounded away from zero (and independent of  $x$ ). Also,

$$N \left( \Delta(q^*) - \Delta(q^* - \frac{\epsilon}{N^{1/2}}) \right) (x - \Delta(q^*)) \rightarrow 0.$$

Therefore,  $f(x)$  is (arbitrarily close to) proportional to the density of a normal random variable with expectation  $\Delta(q^*) = 0$  and a variance arbitrarily close to zero. Therefore  $\Pr[v_k_{(q^* + \frac{\epsilon}{N^{1/2}}), N} - J(c_k_{(q^* + \frac{\epsilon}{N^{1/2}}), N}) > 0]$  approaches zero as  $N$  converges to zero. Similarly,  $\Pr[v_k_{(q^* - \frac{\epsilon}{N^{1/2}}), N} - J(c_k_{(q^* - \frac{\epsilon}{N^{1/2}}), N}) > 0]$  converges to zero, and this contradiction shows that indeed,  $q_N^o$  is  $O_p(n^{\frac{1}{2}})$ . ■

**Corollary 4** *The per seller consumer surplus converges to that of a monopsonist facing supply  $Q^s = NF(P)$  and is  $O_p(n^{\frac{1}{2}})$ .*

**Proof.** For a given realization of  $\mathbf{v}^N$  and  $\mathbf{c}^N$ , the realized consumer surplus is simply

$$CS = \int_0^{Nq_N^o} (v_{k_{x,N}} - p_N^o) dx$$

where  $p_N^o$  is the realized price. Then, if we denote by  $CS^*$  the surplus of a monopsonist facing  $Q^s = NF(P)$ , we have

$$\frac{CS^* - CS}{N} = q^* (p^* - p_N^o) + \int_{Nq_N^o}^{Nq^*} \frac{(v_{k_{x,N}} - p^*)}{N} dx,$$

and the absolute value of the second term in the right hand side is bounded by  $|(q^* - q_N^o) + (q^* - q_N^o)(p^* - p_N^o)|$ , which converges to zero and is  $O_p(n^{\frac{1}{2}})$ , just as the first term. ■

The speed of convergence we have obtained is simply the speed of convergence in the law of large numbers. This is slow as compared, for instance, to the speed of convergence of prices and quantities in double auctions to those of the competitive equilibrium. The latter is  $O_p(n)$  (see Rustichini, Satterthwaite, and Williams, 1994). It is interesting to understand this difference. In a double auction, (many) buyers and sellers submit their bid-ask prices and an auctioneer determines trade and prices at which the market clears. Thus the only divergence between prices and quantities traded and those that correspond to the competitive equilibrium arises from misrepresentation (difference between bids/asks and opportunity costs/willingness to pay). As the number of agents grow, both the probability of influencing the price and the maximum amount of this effect converge to zero. That is, the slack both in manipulating the "order" in the demand/supply schedule and the "share" of the surplus that can be retained converge to zero. As a result, the misrepresentation converges to zero at an speed double than the one of the law of large numbers.

In this paper, there is no auctioneer to cross demand and supply. The divergence between the price and quantities traded and those that would be traded if the buyer knew the realized supply schedule all depend on the inability of the buyer to know the "order" of sellers with certainty. That is, to know where exactly lies the k-quantile of the supply function. This difficulty disappears only at a speed given by the law of large numbers. But no parallel convergence due to the relatively smaller size the agents occurs. The buyer remains large relative to the market as the market grows large.

## 7 Concluding remarks

A uniform-price auction is optimal for a buyer with downward sloping demand facing symmetric, unit-supply sellers. The buyer announces a vector of reserve prices, that determine a demand function. Given the sellers' bids, price and quantities are determined so that the market clears. The vector of reserve prices are determined by equating willingness to pay for each additional unit with "virtual cost", just as in the unit (or flat demand) case.

We have characterized the optimal mechanism for asymmetric sellers as well. Asymmetry between sellers implies that reserve prices are personalized for each seller. Acquisitions are then decided following the order of personalized "virtual costs".

We have discussed the economic interpretation of optimal auctions with downward sloping demand in line with that of Bulow and Roberts (1989). Here too, the problem for the buyer is similar to that of a third degree discriminating monopsonist. The difference with the unit (or flat) demand is that markets are not separated. Finally, we have obtained convergence of the optimal mechanism to the monopsony (price) solution. The speed of convergence of prices, quantities, and surplus sharing coincides with that of the law of large numbers.

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