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Lars Ehlers and Bettina Klaus

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Abstract

We study the assignment of indivisible objects with quotas (universities, jobs, or offices) to a set of agents (students, job applicants, or professors). Each agent receives at most one object and monetary compensations are not possible. We characterize efficient priority rules by *efficiency*, *strategy-proofness*, and *reallocation-consistency*. Such a rule respects an acyclic priority structure and the allocations are determined using the deferred acceptance algorithm.

JEL Classification: D63, D70

Keywords: acyclic priority structures, indivisible objects.

1 Introduction

We study a basic indivisible-objects model with a finite number of object types and a finite quota of available objects of each type. Examples are the determination of access to education, allocation of graduate housing, offices, or tasks. Agents have strict preferences over object types and remaining unassigned. An assignment is an allocation of the objects to the agents such

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†Département de Sciences Économiques and CIREQ, Université de Montréal, Montréal, Québec H3C 3J7, Canada; e-mail: lars.ehlers@umontreal.ca. L. Ehlers gratefully acknowledges financial support from the SSHRC (Canada) and the FCAR (Québec).

‡Departament d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain; e-mail: bettina.klaus@uab.es. B. Klaus's research is supported by a Ramón y Cajal contract of the Spanish *Ministerio de Ciencia y Tecnología*. Furthermore, her work is partially supported by Research Grant BEC2002-02130 from the Spanish *Ministerio de Ciencia y Tecnología* and by the Barcelona Economics Program of CREA.

that every agent receives at most one object and quotas are binding. A rule associates an assignment to each preference profile. When the quota of each object type is one, this problem is known as house allocation. A number of recent papers studied the house allocation problem (for example, Abdulkadiroğlu and Sönmez (1999), Svensson (1999), Pápai (2000), Ergin (2000), Bogomolnaia and Moulin (2001), Ehlers (2002), Ehlers and Klaus (2002), Ehlers, Klaus, and Pápai (2002) and Kesten (2003a,b)).[1]

Typically, in house allocation it is assumed that agents' preferences over objects are strict. Therefore, any two available objects are non-identical. However, there are many real life assignment problems where the quota of some object types is greater than one. For instance, in university choice each university has a number of slots and students report rankings over universities only (*i.e.*, every student is indifferent among any two slots at the same university). Usually a ranking of the students is obtained through an objective test such as an entry exam at a university. Then students who achieved higher test scores than others have higher priority in that university. This situation can be recorded as a strict priority ranking of individuals for each object type where $i \succ_a j$ means "i has higher priority for object type a than j." A priority structure is a collection specifying for each object type a strict priority ranking. A rule violates the priority of agent i for object a if there is a preference profile under which agent i envies agent j who obtains a even though i has a higher priority for a than j. A rule respects a priority structure if it never violates the specified priorities. Gale and Shapley's (1962) students-proposing deferred acceptance algorithm is the so-called "best" (*efficient*) rule respecting a given priority structure. This means that any assignment, which does not violate any priority of any agent, is Pareto-dominated by the assignment calculated by the deferred acceptance algorithm. Balinski and Sönmez (1999), Ergin (2002), and Abdulkadiroğlu and Sönmez (2003) recently studied the students-proposing deferred acceptance algorithm in university choice and school choice. They convincingly argued that a priority structure is obtained through an objective test at each university (however, these tests may contain different questions and thus, may yield different priority rankings) and is not subject to manipulation. In other words, in university choice the priority structure is fixed. Furthermore, Ergin's (2002, Theorem 1) main result demonstrates that for the best rule respecting a fixed priority structure, *efficiency*, *group strategy-proofness*[2],

[1] This list is not exhaustive.

[2] *Group strategy-proofness* means that no group of agents can profit by joint misrepresentation of their preferences such that all members of the group weakly gain and one member of the group strictly gains.

consistency[3], and the *acyclicity*[4] of the priority structure are all equivalent.

In Balinski and Sönmez (1999), Ergin (2002), and Abdulkadiroğlu and Sönmez (2003) a priority structure is exogenously fixed. We ignore this assumption and allow for all rules. We say that a rule is a *priority rule* if there exists an endogenously given priority structure such that this rule chooses the same allocations that the deferred acceptance algorithm finds using that priority structure. Our main result is that a rule satisfies *efficiency*, *strategy-proofness*, and *reallocation-consistency*[5] if and only if it is an *efficient* priority rule. In other words, any rule satisfying our combination of axioms is a best rule for an endogenously given acyclic priority structure. Our paper complements the above mentioned papers in the sense that even if we ignore the priority structure and allow for any rule in university choice problems, then our three axioms bring us back to a best rule respecting an endogenously fixed acyclic priority structure. Furthermore, since Ergin's (2002, Theorem 1) result remains unchanged when *consistency* is replaced by *reallocation-consistency* (*i.e.*, for the best rule respecting a fixed priority structure, *efficiency*, *group strategy-proofness*, *reallocation-consistency* and *acyclicity* of the priority structure are all equivalent), our characterization can be considered to be the "converse" of his main result.

The paper is organized as follows. In Section 2 we introduce the model and our axioms. In Section 3 we define priority rules and present the characterization of efficient priority rules. Section 4 contains some concluding remarks.

2 Object Allocation with Quotas

Let N denote the finite set of agents. Let A denote the finite set of indivisible object types. Given object type $a \in A$, let $q_a \geq 1$ denote the number of available objects, or *quota*, of type a. Let $q \equiv (q_a)_{a \in A}$. Let 0 represent the null object. Not receiving any object is called "receiving the null object." The null object does not belong to A and is available in any economy.

Each agent $i \in N$ is equipped with a strict preference relation R_i over $A \cup \{0\}$. In other words, R_i is a linear order[6] over $A \cup \{0\}$. Given $x, y \in$

[3]We discuss *consistency* in Section 2. To be precise, in his characterization Ergin (2002, Theorem 1) requires *consistency* to hold for the so-called extended best rule.

[4]A formal definition of *acyclicity* is given in Section 3.

[5]*Reallocation-consistency* requires that when a set of agents leaves with their allotments, their assignments should remain unchanged when applying the same rule to the reallocation problem that consists of these agents and their allotments.

[6]A linear order is a complete, reflexive, transitive, and antisymmetric binary relation.

$A \cup \{0\}$, $x\, P_i\, y$ means that agent i strictly prefers x to y under R_i. Let $\mathcal{R}$ denote the set of all linear orders over $A \cup \{0\}$. Let $\mathcal{R}^N$ denote the set of all *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. Given $R \in \mathcal{R}^N$ and $M \subseteq N$, let R_M denote the restriction of R to M. We also use the notation $R_{-i} = R_{N \setminus \{i\}}$. For example, $(\bar{R}_i, R_{-i})$ denotes the profile obtained from R by replacing R_i by $\bar{R}_i$.

An economy consists of a set of agents $N' \subseteq N$, their preferences $R' \in \mathcal{R}^{N'}$, and a vector of quotas $q' = (q'_a)_{a \in A}$ such that for all $a \in A$, $q_a \geq q'_a \geq 0$. We suppress the set of agents and denote this economy by (R', q').

When allocating objects each agent either receives an object of type $a \in A$ or the null object. The null object can be assigned to several agents without any restriction, but for all other objects the associated quota is binding. Formally, given an economy (R', q'), an *allocation for* (R', q') is a list $\alpha = (\alpha_i)_{i \in N'}$ such that for all $i \in N'$, $\alpha_i \in A \cup \{0\}$, and for all $a \in A$, $|\{i \in N' : \alpha_i = a\}| \leq q'_a$. Note that not all available objects need to be assigned. Given $i \in N'$, we call α_i the *allotment* of agent i at α. An *unrestricted (allocation) rule* is a function φ that assigns to each economy (R', q') an allocation $\varphi(R', q')$.

We are only interested in economies where all agents are present and all objects are available with quotas q and all economies that result as reallocation problems from those economies. Therefore, we restrict any unrestricted rule to these economies. The set of *admissible* economies with agent set N is $\mathcal{E}^N = \{(R, q) : R \in \mathcal{R}^N\}$.

Given an unrestricted rule φ, we consider situations where, departing from an economy in $\mathcal{E}^N$, some agents may want to reallocate the objects assigned to them under φ. Given $R \in \mathcal{R}^N$ and $N' \subsetneq N$, let $r^{\varphi}_{N'}(R, q)$ denote the reallocation problem that the agents N' face after having left the economy (R, q) with their allotments at $\varphi(R, q)$. Formally, $r^{\varphi}_{N'}(R, q)$ denotes the economy $(R_{N'}, q')$ where $q'_a = |\{i \in N' : \varphi_i(R, q) = a\}|$ for all $a \in A$. Note that in a reallocation problem there are at most as many objects available as agents are present. Given an unrestricted rule φ, the set of *admissible* economies (or reallocation problems) with agent set $N' \subsetneq N$ is $\mathcal{E}^{N'}_{\varphi} = \{r^{\varphi}_{N'}(R, q) : R \in \mathcal{R}^N\}$. Slightly abusing notation, we write $\mathcal{E}^N_{\varphi}$ instead of $\mathcal{E}^N$.

Starting from an unrestricted rule φ, we consider the restriction of φ to all its admissible economies. Again slightly abusing notation, we use the same symbols for the restricted and the unrestricted rule. An *(allocation) rule* is a function φ that assigns to all $N' \subseteq N$ and all admissible economies $(R', q') \in \mathcal{E}^{N'}_{\varphi}$ an allocation $\varphi(R', q')$. Note that different unrestricted rules may induce the same rule.

Next, we introduce our main properties for rules. First, a rule chooses only (Pareto) efficient allocations.

Efficiency: For all $R \in \mathcal{R}^N$, there is no allocation $\alpha = (\alpha_i)_{i \in N}$ for (R, q) such that for all $i \in N$, $\alpha_i \; R_i \; \varphi_i(R, q)$, and for some $j \in N$, $\alpha_j \; P_j \; \varphi_j(R, q)$.

Second, no agent ever benefits from misrepresenting his preference relation.

Strategy-Proofness: For all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $\varphi_i(R, q) R_i$ $\varphi_i((R'_i, R_{-i}), q)$.

Note that as in Ergin (2002) we only require *efficiency* and *strategy-proofness* when all agents belonging to N are present and all objects are available with their maximal quotas in the economy.

Our last property is a stability condition for the allocations chosen by the rule when all agents are present. Suppose after the objects have been allocated, some agents decide to reallocate their allotments among themselves. What if the *same* rule is applied to the "reallocation problem"? The rule is "unstable" if its assignment to the agents in the reallocation problem differs from its original allotments to them. Here we are only interested in reallocation problems that are derived from economies in which all agents are present and all objects are available with quotas q.

Reallocation-Consistency: For all $R \in \mathcal{R}^N$, all $N' \subsetneq N$, and all $i \in N'$, $\varphi_i(R, q) = \varphi_i(r^{\varphi}_{N'}(R, q))$.

At first glance one may think that *reallocation-consistency* is equivalent to the "generic" consistency property for this model defined as follows: Suppose a group of agents leave with their allotments. Then the reduced economy consists of the remaining agents and the remaining resources (the allotments of the remaining agents *and* all unassigned objects). A rule is *consistent* if the allotments to the remaining agents do not change when the rule is applied to the reduced economy.[7] In a reduced economy there may be some unassigned objects in addition to the remaining agents' allotment – an incidence that cannot occur in a reallocation problem where agents can only reallocate their allotments among themselves.[8] In models where always all resources are assigned, both properties are indeed equivalent.

[7]**Consistency:** For all $R \in \mathcal{R}^N$, all $N' \subsetneq N$, and all $i \in N'$, $\varphi_i(R, q) = \varphi_i(R_{N'}, \bar{q})$ where $\bar{q}_a = q_a - |\{j \in N \backslash N' : \varphi_j(R, q) = a\}|$ for all $a \in A$.

[8]Ergin (2000) studies *consistency* for the house allocation problem in various combinations with *efficiency*, *converse consistency*, *neutrality*, and *anonymity*.

When considering *efficiency*, *strategy-proofness*, and *reallocation-consistency* in their present form, we can only derive conclusions for economies with agent set N and full quotas q and for all reallocation problems that are induced from these economies by the unrestricted rule. We do not require *strategy-proofness* for reallocation problems since agents revealed their preferences before reallocation and it is not possible for them to change them. For instance, our axioms do not impose any requirements on any economy in which not all agents are present and more objects are available than agents. This is why we restricted the domain of a rule to all its admissible economies. In the same vein as Ergin (2002) we require that when all agents are present, then all objects are available with quotas q. For example, for a serial dictatorship where agent 1 is the first dictator, it is not meaningful to consider (sub)economies as reallocation problems in which agent 1 is present but the quota of his favorite object type is 0. Such economies simply do not arise as reallocation problems for this rule (and our axioms do not impose any requirements on them).

3 Priority Rules

Given $a \in A$, let $\succ_a$ denote a linear order over N. We call $\succ_a$ a *priority ordering for object type* a. A *priority structure* is a profile $\succ = (\succ_a)_{a \in A}$ specifying for each object type a priority ordering. Given $N' \subseteq N$, an economy (R', q'), $i \in N'$, $a \in A$, and a priority structure $\succ$, an allocation α for (R', q') *violates the priority of* i *for* a if there exists $j \in N'$ such that $\alpha_j = a$, $i \succ_a j$, and $a \, P_i \, \alpha_i$ (*i.e.*, i has higher priority for object type a than j but j receives a and i envies j). A rule φ *respects a priority structure* $\succ$ if for all $N' \subseteq N$ and all $(R', q') \in \mathcal{E}_{\varphi}^{N'}$, $\varphi(R', q')$ does not violate the priority of any agent for any object type.[9]

Given a priority structure $\succ$ and $R \in \mathcal{R}^N$, Balinski and Sönmez (1999, Theorem 2) show that the students-proposing deferred acceptance (DA) algorithm applied to $\succ$ and (R, q) yields the best allocation among all allocations which do not violate the priority of any agent for any object type. In other words, if an allocation respects $\succ$ at te economy (R, q), then it is Pareto-dominated by the allocation calculated by the DA-algorithm. Let $f^{\succ}$ denote the *deferred acceptance rule with priority structure* $\succ$. We also call $f^{\succ}$ the best rule respecting the priority structure $\succ$. Note that under $f^{\succ}$ the agents propose to object types and, using $\succ_a$, object type a rejects

[9] Ergin (2002) uses the expression "a rule adapts to a priority structure" instead of "a rule respects a priority structure".

agents once the quota is filled. Formally, given $N' \subseteq N$ and an economy (R', q') with agent set N', the allocation $f^{\succ}(R', q')$ is determined as follows:

- At the first step every agent in N' "proposes" to his favorite object type in $A \cup \{0\}$. For each object type a, the q'_a applicants who have the highest priority under $\succ_a$ (all if there are fewer than q'_a) are placed on the waiting list of a, and the others are rejected.

- At the lth step every newly rejected agent proposes to his next best object type in $A \cup \{0\}$. For each object type a, the q'_a applicants who have the highest priority under $\succ_a$ (all if there are fewer than q'_a) among the new applicants and those on the waiting list are placed on the new waiting list and the others are rejected.

- The algorithm terminates when every agent belongs to a waiting list. Then object $a \in A$ is assigned to the agents on the waiting list of a.

Note that any agent who proposes to the null object is immediately accepted. Although the DA-algorithm calculates for each economy the best allocation among the allocations that respect the priority structure, the deferred acceptance rule may not be *efficient*.[10] Ergin (2002) identifies a necessary and sufficient condition on a priority structure such that the deferred acceptance rule yields an *efficient* allocation for all economies with agent set N.

Given a priority structure $\succ$, a *cycle* is constituted of distinct $a, b \in A$ and $i, j, k \in N$ such that the following are satisfied

(C) **Cycle condition:** $i \succ_a j \succ_a k$ and $k \succ_b i$ and

(S) **Scarcity condition:** there exist (possibly empty) disjoint sets $N_a, N_b \subseteq N \backslash \{i, j, k\}$ such that $N_a \subseteq \{l \in N : l \succ_a j\}$, $N_b \subseteq \{l \in N : l \succ_b i\}$, $|N_a| = q_a - 1$, and $|N_b| = q_b - 1$.

A priority structure is *acyclic* if no cycles exist.

If quotas are all equal to 1, then the scarcity condition is automatically satisfied. For other quotas, the scarcity condition limits the definition of a cycle to cases where there indeed exist economies in $\mathcal{E}^N$ such that agents i, j, and k actually compete for objects a and b (in the absence of this competition, *e.g.*, because the quotas in fact do not limit the access of the agents to objects a and b, a cycle will not lead to the violation of either *efficiency* or the given priorities – see Ergin (2002) for further discussion).

[10] See Roth and Sotomayor (1990, Example 2.31).

Proposition 1 (Ergin, 2002, Theorem 1). *Let $\succ$ be a priority structure. Then $f^{\succ}$ is efficient if and only if $\succ$ is acyclic.*

We say that a rule φ is a *priority rule* if there exists a priority structure $\succ$ such that $\varphi = f^{\succ}$. We call a rule φ an *efficient priority rule* if there exists an acyclic priority structure $\succ$ such that $\varphi = f^{\succ}$.

While Ergin (2002) focuses on the best rule respecting an exogenously given priority structure, we consider all rules. Our main result shows that if a rule satisfies *efficiency*, *strategy-proofness*, and *reallocation-consistency*, then it is a best rule for an endogenously given acyclic priority structure.

Theorem 1. *Efficient priority rules are the only rules satisfying efficiency, strategy-proofness, and reallocation-consistency.*

Remark 1. As the careful reader may check, Ergin's result (2002, Theorem 1) remains unchanged when *consistency* is replaced by *reallocation-consistency*, *i.e.*, for the best rule respecting a fixed priority structure, *efficiency*, *group strategy-proofness*, *reallocation-consistency* and *acyclicity* of the priority structure are all equivalent.[11] Hence, our characterization establishes a converse of Ergin's Theorem 1.

Remark 2. Theorem 1 is the first characterization in the indivisible-objects model with quotas when *objective* indifferences and *all* rules are allowed. All other papers rule out objective indifferences (*i.e.*, $q_a = 1$ for all $a \in A$) or consider the (students-proposing) DA-algorithm for a fixed priority structure.

Before proving Theorem 1 we establish the independence of the axioms. The rule that assigns the null object to all agents for all admissible economies satisfies *strategy-proofness* and *reallocation-consistency*, but not *efficiency*.

Let $\succ$ denote the priority structure such that for all $a \in A$, $1 \succ_a 2 \succ_a \cdots \succ_a |N|$. Let $\succ'$ denote the priority structure such that for all $a \in A$, $2 \succ'_a 3 \succ'_a \cdots \succ'_a |N| \succ'_a 1$. Given $b \in A$, let φ^b be the rule such that for all $N' \subseteq N$ and all $(R', q') \in \mathcal{E}^{N'}_{\varphi^b}$, (i) if $1 \in N'$ and $b \, P'_1 \, 0$, then $\varphi^b(R', q') \equiv f^{\succ}(R', q')$ and (ii) otherwise $\varphi^b(R', q') \equiv f^{\succ'}(R', q')$. Then φ^b satisfies *efficiency* and *reallocation-consistency*, but not *strategy-proofness*.

[11] The proof is identical to Ergin's (2002) proof. It is easy to see that his proof "(i) Efficiency $\Rightarrow$ (iii) Consistency" shows "(i) Efficiency $\Rightarrow$ (iii) Reallocation-consistency" and "(iii) Consistency $\Rightarrow$ (iii) Acyclicity of $\succ$" shows "(iii) Reallocation-consistency $\Rightarrow$ (iii) Acyclicity of $\succ$". Thus, by Ergin (2002, Theorem 1), *efficiency*, *group strategy-proofness*, *consistency*, *reallocation-consistency* and *acyclicity* of the priority structure are all equivalent for the best rule respecting a fixed priority structure.

Given $\succ$ and $\succ'$ as above, define φ as follows: (i) for all $R \in \mathcal{R}^N$, $\varphi(R,q) \equiv f^{\succ}(R,q)$ and (ii) for all $N' \subsetneq N$ and all $(R',q') \in \mathcal{E}_{\varphi}^{N'}$, $\varphi(R',q') \equiv f^{\succ'}(R',q')$. Then φ satisfies *efficiency* and *strategy-proofness*, but not *reallocation-consistency*.

Proof of Theorem 1

Let φ be an *efficient* priority rule. Then there exists an acyclic priority structure $\succ$ such that $\varphi = f^{\succ}$. Since any deferred acceptance rule is *strategy-proof* it follows that φ is *strategy-proof*. To show *reallocation-consistency*, let $R \in \mathcal{R}^N$. Because $\succ$ is acyclic, $f^{\succ}$ is *efficient*. Thus, for all $a \in A$, if $|\{i \in N : f_i^{\succ}(R,q) = a\}| < q_a$, then for all $i \in N$, $\varphi_i(R,q)\; R_i\; a$. For all $a \in A$, let $q'_a \equiv |\{i \in N : \varphi_i(R,q) = a\}|$. When calculating $f^{\succ}(R,q)$ the waiting list of any object type contains at any step at most q'_a applicants. Thus, applying the DA-algorithm to (R,q') yields $f^{\succ}(R,q)$. Hence,

$$f^{\succ}(R,q') = f^{\succ}(R,q). \tag{1}$$

By definition of q', at $f^{\succ}(R,q')$ all objects are assigned. Because $\succ$ is acyclic, $f^{\succ}$ is consistent (Ergin, 2002, Theorem 1). Now for all $N' \subsetneq N$ and all $i \in N'$,

$$\varphi_i(R,q) = f_i^{\succ}(R,q) = f_i^{\succ}(R,q') = f_i^{\succ}(r_{N'}^{\varphi}(R,q)) = \varphi_i(r_{N'}^{\varphi}(R,q)),$$

where the first and the last equality follow from $\varphi = f^{\succ}$, the second from (1), and the third from the facts that at $f^{\succ}(R,q')$ all objects are assigned and $f^{\succ}$ is *consistent*.[12] Hence, φ satisfies *reallocation-consistency*.

Conversely, let φ be a rule satisfying *efficiency*, *strategy-proofness*, and *reallocation-consistency*. First we construct for each object type a priority ordering. Second we show that the constructed priority structure is acyclic. Third we prove that φ and the best rule respecting the constructed priority structure coincide.

Given $x \in A \cup \{0\}$, fix $R^x \in \mathcal{R}^N$ such that for all $i \in N$ and all $y \in (A \cup \{0\}) \setminus \{x\}$, $x\, R_i^x\, 0\, R_i^x\, y$. Given $a \in A$, we define $\succ_a$ inductively as follows:

Step 1: For all $i,j \in N$, (a) if $\varphi_i(R^a,q) = a \neq \varphi_j(R^a,q)$, then $i \succ_a j$, and (b) if $\varphi_i(R^a,q) = a = \varphi_j(R^a,q)$ and $i < j$, then $i \succ_a j$.

[12] By *consistency*, for all $i \in N'$, $f_i^{\succ}(R,q') = f_i^{\succ}(R_{N'},\bar{q})$ where $\bar{q}_a = q'_a - |\{j \in N \backslash N' : f_j^{\succ}(R,q') = a\}|$ for all $a \in A$. Note that $r_{N'}^{\varphi}(R,q) = (R_{N'},\hat{q})$ where $\hat{q}_a = |\{i \in N' : f_i^{\succ}(R,q) = a\}|$ for all $a \in A$. So, by construction, for all $a \in A$, $\bar{q}_a = \hat{q}_a$. Thus, $(R_{N'},\bar{q}) = r_{N'}^{\varphi}(R,q)$ and $f_i^{\succ}(R,q') = f_i^{\succ}(r_{N'}^{\varphi}(R,q))$.

If $q_a \geq |N|-1$, then for all distinct $i,j \in N$ we have $i \succ_a j$ or $j \succ_a i$ and $\succ_a$ is completely defined. If $q_a < |N|-1$, then it is possible that for distinct $i,j \in N$, $\varphi_i(R^a,q) = 0 = \varphi_j(R^a,q)$. To define $\succ_a$ in these cases, we extend the definition inductively.

Step 2: Suppose $q_a < |N|-1$. Because $q_a \geq 1$, there exists $l_1 \in N$ such that for all $i \in N\backslash\{l_1\}$, $l_1 \succ_a i$. Then for all $i,j \in N\backslash\{l_1\}$, if $\varphi_i((R^0_{l_1}, R^a_{-l_1}),q) = a \neq \varphi_j((R^0_{l_1}, R^a_{-l_1}),q)$, then $i \succ_a j$. If $q_a \geq |N|-2$, then for all distinct $i,j \in N$ we have $i \succ_a j$ or $j \succ_a i$.

Step 3: Suppose $q_a < |N|-2$. Because $q_a \geq 1$, there exists $l_2 \in N\backslash\{l_1\}$ such that for all $i \in N\backslash\{l_1,l_2\}$, $l_2 \succ_a i$. Then for all $i,j \in N\backslash\{l_1,l_2\}$, if $\varphi_i((R^0_{\{l_1,l_2\}}, R^a_{N\backslash\{l_1,l_2\}}),q) = a \neq \varphi_j((R^0_{\{l_1,l_2\}}, R^a_{N\backslash\{l_1,l_2\}}),q)$, then $i \succ_a j$; *etc.*

After at most $n-1$ inductive steps (if $q_a = 1$), $\succ_a$ is completely defined, *i.e.*, for any distinct $i,j \in N$ we have $i \succ_a j$ or $j \succ_a i$.

Lemma 1. *$\succ_a$ is a well-defined linear order.*

Proof. First we show that $\succ_a$ is well-defined. Suppose that for some $i,j \in N$ we have both $i \succ_a j$ and $j \succ_a i$. Obviously, $i \succ_a j$ and $j \succ_a i$ cannot be defined in the same inductive step. Thus, in particular, $q_a < |N|-1$. Without loss of generality, let $i \succ_a j$ be defined first.

Because $j \succ_a i$ there is some $t \in \{1,\ldots,|N|-1\}$ such that for $L_t = \{l_1,\ldots,l_t\}$ we have $i,j \in N\backslash L_t$ and $\varphi_j((R^0_{L_t}, R^a_{N\backslash L_t}),q) = a \neq \varphi_i((R^0_{L_t}, R^a_{N\backslash L_t}),q)$. By *efficiency*, $\varphi_i((R^0_{L_t}, R^a_{N\backslash L_t}),q) = 0$. Let q^a denote the profile of quotas such that $q^a_a = 1$ and for all $b \in A\backslash\{a\}$, $q^a_b = 0$. Then $r^{\varphi}_{\{i,j\}}((R^0_{L_t}, R^a_{N\backslash L_t}),q) = (R^a_{\{i,j\}}, q^a)$. By *reallocation-consistency*,

$$\varphi_j(R^a_{\{i,j\}}, q^a) = a. \tag{2}$$

Because $i \succ_a j$ is defined before $j \succ_a i$, either

(a) there exists $L \subsetneq L_t$ such that $i,j \in N\backslash L$ and $\varphi_i((R^0_L, R^a_{N\backslash L}),q) = a \neq \varphi_j((R^0_L, R^a_{N\backslash L}),q)$ or

(b) $\varphi_i(R^a,q) = a = \varphi_j(R^a,q)$ and $i < j$ ((b) in Step 1).

If **(a)**, then by *efficiency*, $\varphi_j((R^0_L, R^a_{N\backslash L}),q) = 0$. Then $r^{\varphi}_{\{i,j\}}((R^0_L, R^a_{N\backslash L}),q) = (R^a_{\{i,j\}}, q^a)$. By *reallocation-consistency*,

$$\varphi_i(R^a_{\{i,j\}}, q^a) = a. \tag{3}$$

By (2) and (3),

$$|\{k \in \{i,j\} : \varphi_k(R^a_{\{i,j\}}, q^a) = a\}| = |\{i,j\}| = 2 > 1 = q^a_a,$$

which contradicts the fact that $\varphi(R^a_{\{i,j\}}, q^a)$ is an allocation for $(R^a_{\{i,j\}}, q^a)$.

If **(b)**, then by *efficiency*, there exists $k \in N$ such that $\varphi_k(R^a, q) = 0$ and $\varphi_k((R^0_{L_t}, R^a_{N\backslash L_t}), q) = a$. Hence, by *reallocation-consistency* and by similar arguments as for (a), $\varphi_i(R^a_{\{i,k\}}, q^a) = a$ and $\varphi_k(R^a_{\{i,k\}}, q^a) = a$. Similarly as in (a) this yields a contradiction.

Completeness and transitivity of $\succ_a$ follow straightforwardly from the inductive definition. $\square$

Lemma 2. *The priority structure $\succ \equiv (\succ_a)_{a\in A}$ is acyclic.*

Proof. Suppose that $\succ$ contains a cycle. Then there are $a, b \in A$ and $i, j, k \in N$ such that (C) $i \succ_a j \succ_a k$ and $k \succ_b i$ and (S) there exist (possibly empty) disjoint sets $N_a, N_b \subseteq N\backslash\{i, j, k\}$ such that $N_a \subseteq \{l \in N : l \succ_a j\}$, $N_b \subseteq \{l \in N : l \succ_b i\}$, $|N_a| = q_a - 1$, and $|N_b| = q_b - 1$.

Let $R \in \mathcal{R}^N$ be such that

- for all $l \in N_a$, $R_l = R^a_l$,
- for all $l \in N_b$, $R_l = R^b_l$,
- for all $l \in N\backslash(N_a \cup N_b \cup \{i, j, k\})$, $R_l = R^0_l$,
- $R_j = R^a_j$ and for all $c \in A\backslash\{a, b\}$, $b\ P_i\ a\ P_i\ 0\ P_i\ c$ and $a\ P_k\ b\ P_k\ 0\ P_k\ c$.

We now calculate $\varphi(R, q)$. By *efficiency*, $|\{l \in N : \varphi_l(R, q) = a\}| = q_a$ and $|\{l \in N : \varphi_l(R, q) = b\}| = q_b$. Because $|N_a \cup N_b \cup \{i, j, k\}| = q_a + q_b + 1$, there exists $\hat{l} \in N_a \cup N_b \cup \{i, j, k\}$ such that $\varphi_{\hat{l}}(R, q) = 0$.

If $\hat{l} \in N_a \cup \{j\}$, then by *efficiency*, $\varphi_k(R, q) = a$. Thus, by *strategy-proofness*, $\varphi_k((R^a_k, R_{-k}), q) = a$ and for some $\tilde{l} \in N_a \cup \{j\}$, $\varphi_{\tilde{l}}((R^a_k, R_{-k}), q) = 0$. By *reallocation-consistency* and $r^{\varphi}_{\{k,\tilde{l}\}}((R^a_k, R_{-k}), q) = ((R^a_k, R^a_{\tilde{l}}), q^a)$,

$$\varphi_k((R^a_k, R^a_{\tilde{l}}), q^a) = a \text{ and } \varphi_{\tilde{l}}((R^a_k, R^a_{\tilde{l}}), q^a) = 0. \tag{4}$$

On the other hand, since by (C) $j \succ_a k$, $N_a \cup \{j\} \subseteq \{l \in N : l \succ_a k\}$. Thus, $\tilde{l} \succ_a k$ and by definition of $\succ_a$ either

(a) there exists $L \subseteq N$ such that $\tilde{l}, k \in N\backslash L$, $\varphi_{\tilde{l}}((R^0_L, R^a_{N\backslash L}), q) = a \neq \varphi_k((R^0_L, R^a_{N\backslash L}), q)$ or

(b) $\varphi_{\tilde{l}}(R^a, q) = a = \varphi_k(R^a, q)$ and $\tilde{l} < k$ ((b) in Step 1).

If **(a)**, then by *reallocation-consistency* and $r^{\varphi}_{\{k,\tilde{l}\}}((R^0_L, R^a_{N\setminus L}), q) = ((R^a_k, R^a_{\tilde{l}}), q^a)$,

$$\varphi_k((R^a_k, R^a_{\tilde{l}}), q^a) = 0 \text{ and } \varphi_{\tilde{l}}((R^a_k, R^a_{\tilde{l}}), q^a) = a. \tag{5}$$

Now (4) and (5) contradict the fact that $q^a_a = 1$.

If **(b)**, then, because $|N_a \cup \{j,k\}| = q_a + 1$, there exists $l' \in (N_a \cup \{j\}) \setminus \{\tilde{l}\}$ such that $\varphi_{l'}(R^a, q) = 0$. Thus, by the definition of $\succ_a$, $k \succ_a l'$. This contradicts $l' \in N_a \cup \{j\} \subseteq \{l \in N : l \succ_a k\}$.

Recall that so far we have assumed that $\varphi_{\hat{l}}(R, q) = 0$ for $\hat{l} \in N_a \cup \{j\}$. If $\hat{l} \in N_b \cup \{k\}$, then by $\hat{l} \notin N_a \cup \{j\}$ we have for all $l \in N_a \cup \{j\}$, $\varphi_l(R, q) = a$. Thus, by *efficiency*, $\varphi_i(R, q) = b$. Then, by *strategy-proofness*, $\varphi_i((R^b_i, R_{-i}), q) = b$ and for some $\tilde{l} \in N \setminus \{i\}$, $\varphi_{\tilde{l}}((R^b_i, R_{-i}), q) = 0$. We have already shown that $\tilde{l} \in N_a \cup \{j\}$ yields a contradiction. Hence, $\tilde{l} \in N_b \cup \{k\}$. By *reallocation-consistency* and $r^{\varphi}_{\{i,\tilde{l}\}}((R^b_i, R_{-i}), q) = ((R^b_i, R^b_{\tilde{l}}), q^b)$,

$$\varphi_i((R^b_i, R^b_{\tilde{l}}), q^b) = b \text{ and } \varphi_{\tilde{l}}((R^b_i, R^b_{\tilde{l}}), q^b) = 0. \tag{6}$$

On the other hand, since by (C) $k \succ_b i$, $N_b \cup \{k\} \subseteq \{l \in N : l \succ_b i\}$. Thus, $\tilde{l} \succ_b i$. Now, similarly as before, we derive a contradiction using (6) and the definition of $\succ_b$.

Finally, if $\hat{l} = i$, then for all $l \in N_a \cup \{j\}$, $\varphi_l(R, q) = a$. In particular, $\varphi_j(R, q) = a$. By *strategy-proofness* and *efficiency*, $\varphi_i((R^a_i, R_{-i}), q) = 0$ and $\varphi_j((R^a_i, R_{-i}), q) = a$. By *reallocation-consistency* and $r^{\varphi}_{\{i,j\}}((R^a_i, R_{-i}), q) = ((R^a_i, R^a_j), q^a)$,

$$\varphi_j((R^a_i, R^a_j), q^a) = a \text{ and } \varphi_i((R^a_i, R^a_j), q^a) = 0. \tag{7}$$

On the other hand (C) $i \succ_a j$. Now, similarly as before, we derive a contradiction using (7) and the definition of $\succ_a$. □

Lemma 3. $\varphi = f^{\succ}$.

Proof. By Lemma 2, $\succ$ is acyclic. Thus, $f^{\succ}$ is *reallocation-consistent*. Because φ is *reallocation-consistent*, in showing $\varphi = f^{\succ}$ it suffices to show that for all $R \in \mathcal{R}^N$, $\varphi(R, q) = f^{\succ}(R, q)$. First we show the following claim.

Claim: If $\varphi \neq f^{\succ}$, then there exists $R \in \mathcal{R}^N$ such that $\varphi(R, q) \neq f^{\succ}(R, q)$ and for all $i \in N$, $\varphi_i(R, q) \neq f^{\succ}_i(R, q)$ implies $R_i \in \{R^x_i : x \in A\}$.

Proof of Claim: Suppose $\varphi \neq f^{\succ}$. Let $R \in \mathcal{R}^N$ be such that $\varphi(R, q) \neq f^{\succ}(R, q)$. Let $i \in N$ be such that $\varphi_i(R, q) \neq f^{\succ}_i(R, q)$ and $R_i \notin \{R^x_i :$

$x \in A\}$. Without loss of generality, suppose $\varphi_i(R,q)\ P_i\ f_i^{\succ}(R,q)$. By *efficiency*, $\varphi_i(R,q) \in A$. Let $\varphi_i(R,q) = a$. Because both φ and $f^{\succ}$ are *strategy-proof*, we have $\varphi_i((R_i^a, R_{-i}), q) = a$ and $f_i^{\succ}((R_i^a, R_{-i}), q) = 0$. Thus, $\varphi((R_i^a, R_{-i}), q) \neq f^{\succ}((R_i^a, R_{-i}), q)$ and i's preference relation belongs to $\{R_i^x : x \in A\}$. Continuing this procedure yields the desired profile specified in the Claim.

Suppose $\varphi \neq f^{\succ}$. Then by the Claim there exists $R \in \mathcal{R}^N$ such that $\varphi(R,q) \neq f^{\succ}(R,q)$ and for all $i \in N$, $\varphi_i(R,q) \neq f_i^{\succ}(R,q)$ implies $R_i \in \{R_i^x : x \in A\}$. Because $\succ$ is acyclic, $f^{\succ}$ is *efficient*. Thus, by $\varphi(R,q) \neq f^{\succ}(R,q)$ and *efficiency* of φ, there exists $j \in N$ such that $f_j^{\succ}(R,q)\, P_j\, \varphi_j(R,q)$. By *efficiency*, $f_j^{\succ}(R,q) \in A$. Let $f_j^{\succ}(R,q) = a$. By our choice of R and $\varphi_j(R,q) \neq f_j^{\succ}(R,q)$, we have $R_j = R_j^a$. Hence, $\varphi_j(R,q) = 0$ and by *efficiency*, $|\{i \in N : \varphi_i(R,q) = a\}| = q_a$. Thus, there exists $k \in N$ such that $\varphi_k(R,q) = a \neq f_k^{\succ}(R,q)$. But then again by our choice of R we have $R_k = R_k^a$ and $f_k^{\succ}(R,q) = 0$. Thus, $r_{\{j,k\}}^{\varphi}(R,q) = ((R_j^a, R_k^a), q^a)$. By $\varphi_k(R,q) = a$ and *reallocation-consistency*,

$$\varphi_k((R_j^a, R_k^a), q^a) = a. \tag{8}$$

On the other hand $f^{\succ}$ respects $\succ$. Thus, by $f_j^{\succ}(R,q) = a$, $f_k^{\succ}(R,q) = 0$, and $a\, P_k\, 0$, we have $j \succ_a k$. Hence, by definition of $\succ_a$ either

(a) there exists $L \subseteq N$ such that $j,k \in N \backslash L$, $\varphi_j((R_L^0, R_{N\backslash L}^a), q) = a \neq \varphi_k((R_L^0, R_{N\backslash L}^a), q)$ or

(b) $\varphi_j(R^a, q) = a = \varphi_k(R^a, q)$ and $j < k$ ((b) in Step 1).

If **(a)**, then by *reallocation-consistency* and $r_{\{j,k\}}^{\varphi}((R_L^0, R_{N\backslash L}^a), q) = ((R_j^a, R_k^a), q^a)$,

$$\varphi_j((R_j^a, R_k^a), q^a) = a. \tag{9}$$

Now (8) and (9) contradict the fact that $q_a^a = 1$.

If **(b)**, then by *efficiency*, there must exist $l \in N$ such that $\varphi_l(R^a, q) = 0$ and $\varphi_l(R,q) = a$. Thus, $j \succ_a k \succ_a l$. If $f_l^{\succ}(R,q) = a$, then by $f_k^{\succ}(R,q) = 0$, $k \succ_a l$, and $R_k = R_k^a$, $f^{\succ}(R,q)$ does not respect $\succ$, a contradiction. Hence, $f_l^{\succ}(R,q) \neq \varphi_l(R,q)$ and by construction, $R_l = R_l^a$ and $f_l^{\succ}(R,q) = 0$. Thus, by *reallocation-consistency* and $r_{\{j,l\}}^{\varphi}(R,q) = ((R_j^a, R_l^a), q^a)$, $\varphi_l((R_j^a, R_l^a), q^a) = a$. Since $\varphi_l(R^a, q) = 0$ and $\varphi_j(R^a, q) = a$, $r_{\{j,l\}}^{\varphi}(R^a, q) = ((R_j^a, R_l^a), q^a)$. Thus, by *reallocation-consistency* for $r_{\{j,l\}}^{\varphi}(R^a, q)$, $\varphi_j((R_j^a, R_l^a), q^a) = a$. This and $\varphi_l((R_i^a, R_l^a), q^a) = a$ contradict the fact that $q_a^a = 1$. This finishes the proof. □

4 Concluding Remarks

We have shown that any rule satisfying *efficiency*, *strategy-proofness*, and *reallocation-consistency* is an efficient priority rule. For a mechanism designer, who wishes to implement a rule satisfying these properties, this means that he must choose an acyclic priority structure and determine the allocations using the deferred acceptance algorithm and the chosen priority structure.

Our formulation of the axioms is identical with the one by Ergin (2002) – *efficiency* and *strategy-proofness* are only required for all economies with agent set N and quotas q and *reallocation-consistency* only needs to hold for all reallocation problems arising from such economies. If we defined our axioms for all economies, then the characterized (unrestricted) rules are priority rules such that the associated priority structure does not satisfy the cycle condition (C). In other words the scarcity condition (S) becomes redundant. This is because when considering the full domain, economies are admissible in which each object type is available with quota one or zero. The same is true for Ergin's (2002, Theorem 1) main result. If all economies are considered, then for the best rule respecting a fixed priority structure, *efficiency*, *group strategy-proofness*, *reallocation-consistency*, *consistency*, and the priority structure not satisfying (C) are all equivalent.

References

Abdulkadiroğlu, A., and T. Sönmez (1999): House Allocations with Existing Tenants, *Journal of Economic Theory* 88, 233–260.

Abdulkadiroğlu, A., and T. Sönmez (2003): School Choice: A Mechanism Design Approach, *American Economic Review* 93, 729–747.

Balinski, M., and T. Sönmez (1999): A Tale of Two Mechanisms: Student Placement, *Journal of Economic Theory* 84, 73–94.

Bogomolnaia, A., and H. Moulin (2001): A New Solution to the Random Assignment Problem, *Journal of Economic Theory* 100, 295–328.

Ehlers, L. (2002): Coalitional Strategy-Proof House Allocation, *Journal of Economic Theory* 105, 298–317.

Ehlers, L., and B. Klaus (2002): Resource-Monotonic House Allocation, *International Journal of Game Theory*, forthcoming.

Ehlers, L., B. Klaus, and S. Pápai (2002): Strategy-Proofness and Population-Monotonicity for House Allocation Problems, *Journal of Mathematical Economics* 83(3), 329–339.

Ergin, H. İ. (2000): Consistency in House Allocation Problems, *Journal of Mathematical Economics* 34, 77–97.

Ergin, H. İ. (2002): Efficient Resource Allocation on the Basis of Priorities, *Econometrica* 70, 2489-2497.

Gale, D., and L. S. Shapley (1962): College Admissions and the Stability of Marriage, *American Mathematical Monthly* 69, 9–15.

Kesten, O. (2003a): On Two Competing Mechanisms, Working Paper.

Kesten, O. (2003b): Coalitional Strategy-Proofness and Resource Monotonicity for House Allocation Problems, Working Paper.

Pápai, S. (2000): Strategyproof Assignment by Hierarchical Exchange, *Econometrica* 68, 1403–1433.

Roth, A. E., and M. Sotomayor (1990): Two-Sided Matching: A Study in Game Theoretic Modeling and Analysis, London/New York: Cambridge University Press.

Svensson, L.-G. (1999): Strategy-Proof Allocation of Indivisible Goods, *Social Choice and Welfare* 16, 557–567.