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Partnership in Open Innovation

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Abstract

This paper aims at assessing the importance of the initial technological endowments when firms decide to create R&D agreements. We study a Bertrand duopoly where firms evaluate the returns of an agreement according to its length. A learning process allows us to depict a close connection between firms' technology and the possibility to achieve a positive outcome from creating an agreement. Moreover, as far as learning is modeled as an iterative process, a suitable set of initial conditions is the basic factor leading to successful ventures.

Keywords: Firm agreements, Learning, open innovation

JEL classification: D21, D43, L1

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1 Introduction.

There is recently a growing tendency of firms to engage in partnership agreements in R&D as means to increase their competitiveness. By an agreement, we mean a bilateral contract in which two firms (partners) agree in developing a common research project in order to improve their available technology. Harrigan (1986) stresses that firms engage in different types of R&D partnerships to exploit knowledge in new applications, to enter in new fields etc. Indeed, these ventures allow to share research costs, to save on assets, and to avoid to replicate laboratories and testing periods. Along the 20th century, the innovation paradigm was characterized by firms with large investments in R&D. Chesbrough (2003) refers to it as the *closed innovation model*. In contrast, recently there have appeared some crucial changing factors such as (i) an increasing cost of the R&D activities, (ii) a larger number and more mobile knowledge workers, and (iii) a higher availability of venture capital. As a consequence, firms have more tendency to develop and commercialize new ideas externally and internally by developing outside and in-house innovation activities such as licensing agreements and partnerships both with competitor firms and companies with complementary technologies. This flow of ideas and human capital among firms has given rise to a new *open innovation paradigm*. Naturally, not all industries migrate from the closed to the open innovation model. We could envisage a continuum from completely closed to completely open innovation pattern and locate different industrial activities therein.

In this paper, we intend to focus in the open innovation paradigm and study the impact of the initial technological firms' endowments at the moment of deciding to sign a particular kind of collaborative R&D agreements, known in the literature as non-equity contracts. Hagedoorn (2002) argues that not-equity contractual forms of R&D partnership, such as joint R&D pacts and joint development agreements, have become important kinds of interfirm collaboration. These collaborative agreements cover technology and R&D sharing between two or more companies in combination with joint development projects.

As an illustration, Segrestin (2005) explores the Renault-Nissan alliance as a new way to develop high risk innovative business opportunities involving the design of a new collective identity. Such (successful) alliance had to cope with coordination and cohesion issues in the form of a new managerial organization, and the appropriateness of existing legal frameworks to the new entity. All along the process, both manufacturers could refrain from collaboration if the threat of opportunism outweighted profit expectations.

These partnerships are different from the standard joint-venture between two partners, since they do not involve neither any monetary transition between the two counterparts nor any equity exchange. The economic literature devotes quite a lot of attention to this phenomenon from the empirical viewpoint. In the following subsection we summarize the most relevant evidence.

1.1 Empirical evidence

Empirical evidence stresses that there are two basic features distinguishing R&D agreements: the stategic choice of the partner and the lenght of the agreement.

UNCTAD's 1997 report stresses that cross border agreements between firms (including joint ventures, licensing, subcontracting, franchising, R&D agreements, and others) have become important complements to the traditional investment activity. Most of these agreements involve joint programs to share the high-tech R&D and innovation activities in order to reduce production costs. Hagedoorn and van Kranenburg (2003) attempt to quantify this phenomenon. They show that in the period 1960-1998 the evolution of the number of mergers and acquisitions is different from that of R&D partnerships. According to their database the number of totally new established R&D partnerships (joint ventures) was 3627 (1482) for the period 1980-1989, and it raised to 4743 (reduced to 791) for the period 1990-1998. Also, established partnerships in high-tech sectors was 2271 out of 3375 (1980-1989) and 3795 out of 4464 (1990-1998). The proliferation of partnerships and collaborations between transnational firms in OECD countries confirms this tendency. Most of those contracts involved firms from the European Union, Japan and United States, while developing countries recently are participating more and more in equity-based agreements. Moreover, it also seems that European firms tend to have a much higher share of international alliances than US and Japanese firms, i.e. in most of their agreements at least one partner is a European one.

Statistics confirm that the number of cross-border *strategic* R&D partnerships increased from 280 in 1991 to 430 in 1993. Most of them involved firms in developed countries that often are competitors in the same final good market (UNCTAD, 1997). Two reasons help to explain such a profile: (i) nowadays it becomes harder for individual firms to go on making the R&D and capital investments required to stay competitive, and (ii) firms usually face demands for more competing capital-intensive projects. Unfortunately, mergers and acquisitions proved to be insufficiently flexible to cope with changing partners and decreased product life cycles. Hence, strategic partnerships provide an easy access to complementary technologies, reduce costs and risks, and create synergies and spillovers.

Cainarca et al. (1992) establish that the highest propensity to sign firm agreements (both equity and non-equity) appear both in the early development and in the maturity of the life cycle, whereas they seem less attractive during the full development and in the decline phase. In the same spirit, Chesbrough (2003) and Zeller (2004) provides some examples of R&D agreement for firms operating in high-tech sectors and aiming at development new technologies to exploit for commercialization. Most of such agreements are signed with start-up firms. In particular, Zeller (2004) looks at innovation in the Swiss pharmaceutical corporations, their collaborations with biotechnology companies, and the intrafirm and extrafirm relations, knowledge and technology flows.

As argued by Narula and Hagedoorn (1999), when a firm engages in an agreement, it often foregoes higher short-term profits in the hope that the agreement will enhance its long-term market position. Large firms engage more in R&D alliances than smaller firms because, given their high failure rate, a minimum amount of resources are needed to guarantee the success of the alliance. Therefore, even if data may suggest that a great number of alliances involve small and medium businesses, at least one of the partners is a large firm with the resources to invest in the alliance.

We cite just a few examples.

Rycroft (2002) reports on the agreement between Hewlett-Packard (HP) and Philips for developing some products in medical care. Another example is provided by Sony and Philips to establish DVD technology standards. In addition, still in the European context, we may easily realize that most of the actual agreements involve the development and exploitation of the so-called new technologies (see Leamer and Storper, 2001).

Finally, focusing on the available information of a survey in Nordic countries, Håkanson (1993) explores the firm decision regarding partner selection and the design of the agreements. In particular he conveys the attention to the reasons driving the failure of an agreement. Some agreements are established with a short-term objective and are dissolved on reaching that objective. Håkanson's survey suggests that the risk of failure lies in the technical and commercial uncertainty that may induce changes in the strategic priorities of the partners. This risk seems to be smoothed by a right matching between the 'organizational cultures' of the two partners.

Length is the second salient feature distinguishing firm agreements from joint ventures. Exploiting a sample of joint ventures with at least one US corporation, Reuer (2001) concludes that the average duration of a joint venture is about 7/8 years. Instead, the contractual R&D partnership has a short time-horizon, due to their project-based organization. Some empirical evidence helps to qualify this statement. In the US, NASA manages an important amount of cooperative agreements with large commercial firms whose length does not exceed 3 years (see http://www.hq.nasa.gov). Link and Scott (2001) find empirical support for such length in projects jointly funded by the Advanced Technology Program and the private partners. López-Bayón and González-Díaz (2004) evaluate the average duration of firm agreements in electronics in Spain. The most frequent values are 1 and 5 years.

1.2 Our contribution

As argued in the *open innovation* framework, in case of firm agreements, the partners are usually competitors (at different degrees) in the market of final goods, willing to cooperate at the production level. Hence, we focus on agreements where both parties benefit from the advantages of their collaboration in the research stage of the production process, while keeping their own identity and independence in the market. Also, we shall assume that a successful agreement will allow firms produce more efficiently, but will not allow to close the initial technological gap among them. From this viewpoint, the great challenge for the partners is to define as precisely as possible the object of the agreement and put the effort to get it, knowing that they will compete in the final good market. The importance of this practice should be directly related to the prospect that the agreement entails.

The empirical evidence quoted above helps to detect the basic features driving the creation of partnership agreements. Agreements may involve dissimilar partners and in such deals, usually the partner with better technology exchanges it against retail access in new markets. Uncertainty and asymmetric information about the real productive and research resources of the partners requires cautionary behavior in planning the activity in order to achieve successful results. One common strategy adopted by managers (and decision-makers in general) is to fix the length of the agreement with respect to the number and the technological background of the partners. Daily experience suggests that even successful agreements or ventures do not last forever. Hence, again, one of the basic features of an agreement is its length, namely the number of periods the partners decide to cooperate to achieve the objective of the contract. In particular, it could be useful to wonder the reasons why short agreements could be preferred to long ones. This question is not new. It can be strictly related to another important issue in contract theory (as long as an agreement can be considered as a contract between two or more firms). In that sense, short contracts are preferred to the long ones since one party needs to gather information on the other party, particularly about its trustworthiness and willingness to cooperate in the future (Aghion et al. 2002).

In such a spirit, we are especially interested in analyzing the time dimension of a R&D agreement and the related implication on (i) the decisions of firms to join it and (ii) its successfulness. There exists a wide range of contributions focusing on the elements supporting the creation of such a kind of agreements. This is particularly relevant in R&D settings where the problem of the appropriation of the issues of the R&D activities as well as their connected profits plays a crucial role (cf. Hinloopen (1997, Bureth et al. (1997) or Link and Scott (2001)).

Another strand of literature tackles the problem of defining an optimal contract supporting a stable agreement between equal or different firms. This topic is addressed in Pérez-Castrillo and Sandonís (1996). If projects are advantageous, it is always possible to find contracts acceptable to both firms giving incentives to disclose knowledge to the more advanced firm. In addition Veugelers et al. (1994) prove that the emergence of a stable joint venture is directly related to the importance of the synergies between the two partners. Indeed, the dominant strategy for the loyal partner is to comply with the agreement as far as it earns more from the venture than from the own development of a new technology.

Different from previous contributions, we address neither the stability problem of the agreements nor the design of an optimal contract. We propose an approach that joins the traditional duopoly framework with the temporal dimension embedded in the process of cumulation of knowledge involved by the joint action of two partners. Firms engaging in a partnership bring with them their technological backgrounds. By signing an agreement, they expect to improve their technologies (reducing their production costs), and thus, improve their competitiveness in the final good market. The main contribution of this paper is the modeling of the interaction between the partners to study how the initial technological conditions determine the success of an agreement. Our purpose is to examine whether firms' initial technological endowments, i.e. the technology they dispose at the moment they sign the contract, are relevant in the successful completion of an agreement in a dynamic framework where we introduce a learning process in time.

We propose a duopoly model of product differentiation where firms collaborate at the R&D stage and compete à la Bertrand in the final product market. They share the market demand according to the degree of substitutability of goods. The rationale of this choice relies on the purpose to focus on situations in which competitors, belonging to a same sector but with different technological endowments, compete in prices. An original feature of our model is the introduction of a learning process throughout the length of the agreement. Bureth et al. (1997) show that learning is a key factor in the evolution of firms' collaboration. Indeed, a continuous collaborative interaction may influence the decision to continue or not the agreement. In our paper, the length of an agreement turns out to be the crucial element in the cumulation of advantages stemming from the collaboration. It is the influence of an implicit learning process that eventually, allows for selecting the kind of initial technologies leading to successful collaborations.

The paper is organized as follows. Section 2 presents the main building blocks of the theoretical setting. Section 3 deals with the definition of the terms of the agreement, and section 4 presents the initial conditions suitable to ensure successful agreements. Section 5 discusses the implications of such results and section 6 concludes.

2 The model

Following Vives (1999) and Singh and Vives (1984), we consider a differentiated duopoly with two firms $i = 1, 2$. They use a constant, but different marginal cost technologies without fix costs. In the final product market, firms compete à la Bertrand. Market demand is linear and goods (respectively $1, 2$) produced by firms may be substitutes or complements.

2.1 Consumers' program

According to Singh and Vives (1984) the system of inverse demands is given by,

$$\begin{aligned} p_1 &= \alpha_1 - \beta_1 q_1 - \gamma q_2, \\ p_2 &= \alpha_2 - \beta_2 q_2 - \gamma q_1, \end{aligned} \tag{1}$$

where goods are substitutes, independent or complements according to γ greater than, equal to or less than zero.

Let $\delta = \beta_1\beta_2 - \gamma^2, c = \gamma/\delta, a_i = (\alpha_i\beta_j - \gamma\alpha_j)/\delta, b_i = \beta_j/\delta$ for $i \neq j$ and $i = 1, 2$, we can write the direct demand functions as:

$$
\begin{aligned}
q_1 =& a_1 - b_1 p_1 + c p_2, \qquad (2) \\
q_2 =& a_2 - b_2 p_2 + c p_1.
\end{aligned}
$$

A quick inspection of this demand system reveals that the demand for a single good is downward sloping in its price and increasing in the price of the competitor if goods are substitutes.

2.2 Firms' program

We consider an asymmetric duopoly, as described in Vives (1999). Firms compete à la Bertrand. They use constant but different marginal cost technologies given by,

$$C_i(q_i) = \xi_i q_i, \quad i = 1, 2,$$

where $\xi_i \in [0, 1]$ is a known parameter linked to the efficiency in the reduction of costs (see below). For simplicity we normalize $\xi_1 = 1$ and assume that $\xi_2 = \xi \leq 1$. Firms use the same technology if $\xi = 1$, while the lower ξ the more efficient is firm 2 with respect to firm 1.

Solving firms' profit maximization problems, we obtain the system of reaction functions:

$$p_1 = \frac{a_1 + b_1 + c p_2}{2b_1}, \qquad (3)$$

$$p_2 = \frac{a_2 + \xi b_2 + c p_1}{2b_2}. \qquad (4)$$

Following Singh and Vives (1984) and Vives (1999), we consider prices net of marginal costs. Thus, we define

$$
\begin{aligned}
\hat{p}_1 &= p_1 - 1; \qquad \hat{a}_1 = a_1 - b_1 + c\xi, \\
\hat{p}_2 &= p_2 - \xi; \qquad \hat{a}_2 = a_2 - b_2\xi + c,
\end{aligned}
$$

so that firm i's profit function is $\Pi_i = \hat{p}_i(\hat{a}_i - b_i\hat{p}_i + c\hat{p}_j)$ with $i \neq j$ and $i = 1, 2$.

Equilibrium prices are,

$$\hat{p}_1^* = \frac{2b_2\hat{a}_1 + c\hat{a}_2}{4b_1b_2 - c^2}; \quad \hat{p}_2^* = \frac{2b_1\hat{a}_2 + c\hat{a}_1}{4b_1b_2 - c^2}, \tag{5}$$

In the case of independent goods (i.e. $c = 0$), markets are separated and we obtain monopoly prices:

$$\hat{p}_1^m = \frac{a_1 + b_1}{2b_1}; \quad \hat{p}_2^m = \frac{a_2 + b_2\xi}{2b_2}. \tag{6}$$

From the equilibrium prices (5), we compute the associated equilibrium quantities,

$$q_1^* = b_1\hat{p}_1^*; \quad q_2^* = b_2\hat{p}_2^*. \tag{7}$$

Finally, equilibrium profits are given by,

$$\Pi_1 = b_1\left(\frac{2b_2\hat{a}_1 + c\hat{a}_2}{4b_1b_2 - c^2}\right)^2; \quad \Pi_2 = b_2\left(\frac{2b_1\hat{a}_2 + c\hat{a}_1}{4b_1b_2 - c^2}\right)^2. \tag{8}$$

For future reference, monopoly profits are,

$$\Pi_1^m = \frac{1}{b_1}\left(\frac{a_1 - b_1}{2}\right)^2; \quad \Pi_2^m = \frac{1}{b_2}\left(\frac{a_2 - b_2\xi}{2}\right)^2. \tag{9}$$

2.3 The terms of the agreement

Generically, one can think of agreements between firms displaying similar or different technologies at the moment they create the agreement. This is the situation of R&D agreements between firms in industrialized countries that are technically similar and agreements where one firm is located in an industrialized area while the other belongs to a less developed country, like those between enterprises in Western and Eastern Europe. As we stated in the previous section this study aims at concentrating on the time dimension. In either case, we can envisage two scenarios. On the one hand, the agreement may be renewed period by period. On the other hand, firms agree in keeping the collaboration for more than one period and the issue of this lasting collaboration is a process of cumulation of knowledge (a sort of *learning*) that aims at improving the technology available for the two partners engaged in the agreement. We focus on the second kind of contracts.[1]

As we mentioned before, we assume that benefits stemming from the agreement do not allow the technological lagged firm to fill up the existing gap with respect to the other firm. We model the process of the cumulation of knowledge in the spirit of Chipman (1970): a learning process allows firms to improve the technology they dispose by reducing their cost of production. We define learning as the cumulative process of upgrading the existing technology firms dispose by increasing their stock of knowledge starting at the moment of the signature of the agreement (i.e. at $t = 0$) and lasting for t periods (with $t > 1$). We define the learning parameter as follows:

Assumption 1 Let λ_0 be the initial stock of knowledge shared by the two partners at the moment they sign the agreement ($t = 0$). It is the combination of the technologies the two partners are endowed with. Let us generically define:

$$\lambda_0 = \xi_1^{\alpha} \xi_2^{\beta} > 0, \text{ with } \xi_i, \alpha, \beta \in [0,1] \text{ for } i = 1, 2,$$

and by construction $\lambda_0 \in [0, 1]$.

Remark 1. *The way we model the combination of the technology (i.e. by a Cobb Douglas form) allows to fully capture the interaction (collaboration) among two partners by taking into account both the individual participation of each of them (to the realization of the project) and the externalities that can emerge by the joint action.*

It is a quite general, but complete form to model such kind of phenomena. The parameter λ_0 is the stock of knowledge that Firm 1 and Firm 2 share at the beginning of the agreement. We assume that once a firm subscribes an agreement she discloses the technological information she disposes (embedded in the marginal cost) to the partner. Parameters α and β stand for the relative weight that each firm

[1]We are assuming that the agreement states that no party can use the outcome of the project before its completion. This is what supports the fact that partners must wait for completing the project before using its outcome, so that none of them obtains a competitive advantage in advance in the final goods market.

has in the agreement. In particular, according to our previous hypothesis, given the normalization $\xi_1 = 1$, it follows that $\lambda_0 = 1$ when the two firms display the same technological endowment at $t = 0$. Once the two partners start collaborating in the common project, they acquire new knowledge so that their initial common stock of knowledge evolves. Therefore, we model this evolution as a diffusion process in discrete time.We are following some well known models in industrial organization literature, such as Mansfield (1961) or De Palma et al. (1991). In that sense we recover a very common feature in literature of the development and spreading of a new technology (see Mansfield, 1961). The rational of this choice is the following : the adoption of a new technology as well as its development does move along a path at a constant rate. There are differences in time of adoption simply because potential adopters are heterogeneous and react differently to the new technology. As stated in Mansfield (1961) and the following papers, a logistic process is the most suitable process to model such a development, since it bears the difference in the speeding of a adoption of a new technologic along the development path. Such a process entails that the returns from the learning are higher at the beginning of the collaboration (because of the novelty effect) than it slows down and finally it keeps a quite constant motion.At the moment the two firms sign the agreement they can enjoy the most of the benefits while the returns proportionally reduces as far as time passes. The main criticism addressed to this framework is that we implicitly assume that technology does not change over time (Baptista, 1999). In our framework this is not a crucial point: firms agree just for a particular project running for a short-period, while the criticism addressed above deserve most attention when considering an adoption process in the long run. Here, firms joint their effort in developing a unique technique and this project follows its own path separated from the remaining part of the activity of both the partners, even if, at the end the entire production of each firm can enjoy the results of the partnership. [2]

Definition 1. *Let us consider an agreement lasting for t periods (with $t > 1$). We*

[2]Empirical evidence mentioned above does not cite any case of important technological losses in charge of one or more partner in case of agreement failure.

consider that the accumulation of knowledge across time affects the technological parameter of the two partners engaged in the agreement and it follows a recursive diffusion process[3] *as:*

$$F^{t+1}(\lambda_0) = \lambda_{t+1} = \mu\lambda_t(1-\lambda_t),\ \textit{for}\ t = 0, 1, ..., n,\ \mu > 0,\ \lambda_0 > 0. \tag{10}$$

The process we have just described is a quadratic function, that for particular values will lead to a chaotic behavior. In the next section we will precisely define its domain of existence and we will define its structure.

Equation (10) states that λ_t increases a lot from one period to the next when it is small, while decreases when it is large. The parameter μ is a multiplier of this dynamics. It affects the steepness of the hump in the curve. This process captures a cumulation process that appears when the agreement lasts for several periods. In terms of our model, this process can be interpreted as follows. By construction, $\lambda_0 \in [0, 1]$ and equation (10) is built around λ_0. Hence, there exists a continuum of possible agreements that span from the case in which firms participating to the agreement display different technologies (λ_0 small) to the case in which firms are very similar in technology (λ_0 large). The expected benefits of the two extreme types of agreement are different. The maximum is reached at a point where although technologies are not identical they match in an optimal way. This is so because the law of motion of λ_t given by (10) is quadratic and concave in λ_t.

Taking for granted that the optimal contracts supporting such agreements exist (see Pérez-Castrillo and Sandonís (1996) and Veugelers and Kesteloot (1994)), our concern is to find the initial technological conditions allowing two firms to join the agreement leading to an optimal and successful result.

We split our analysis into two parts. First, we study agreements that do not span in time, and thus contain no learning process. We want to study the constellation of parameter values allowing firms to benefit from the agreement. Next, we will introduce the time dimension. Also, we can easily imagine that the degree of differentiation of the products supplied by the firms may range from independent

[3]see May(1976) and Li-Yorke (1975)

goods (so that firms serve separate markets) to some level of substitutability, so that markets will be interrelated. We will consider both cases as well.

3 Static agreements

To get familiar with the model, let us consider agreements not involving any learning process, that is, agreements signed in a static environment. First, we will present the case of independent goods. Next, we will consider the case of substitute goods.

3.1 Separate markets

We consider an environment where firms' markets are separated and firms sign an agreement lasting just one period. Firms produce products so differentiated that they hold monopoly status in their respective markets (i.e. $c = 0$). We are looking for the conditions under which firms both with similar and very different technologies are willing to engage in an agreement.

Firms participating to an agreement benefit from a better cost saving technology once the objective of the agreement is achieved.

Definition 2. *Consider two firms signing a R&D agreement. The cost function for each firm is,*

$$C_1 = F(\lambda_0)q_1 = \mu\lambda_0(1-\lambda_0)q_1,$$
$$C_2 = F(\lambda)\xi q_2 = \mu\lambda_0(1-\lambda_0)\xi q_2.$$

Proposition 1. *When firms are local monopolies, they are willing to engage in an agreement for $\mu > 4$ when $\lambda_0 \in \left[0, \frac{1}{2} - \frac{\overline{\mu}}{2}\right] \cup \left[\frac{1}{2} + \frac{\overline{\mu}}{2}, 1\right]$, where $\overline{\mu} = \left(\frac{\mu-4}{\mu}\right)^{1/2} \in (0,1)$.*

Proof. We start by computing the corresponding equilibrium prices, quantities and profits for both firms.

$$\tilde{p}_1^m = \frac{a_1 + \mu\lambda_0(1-\lambda_0)b_1}{2b_1}; \quad \tilde{q}_1^m = \frac{a_1 - \mu\lambda_0(1-\lambda_0)b_1}{2},$$
$$\tilde{\Pi}_1^m = \frac{1}{b_1}\left(\frac{a_1 - \mu\lambda_0(1-\lambda_0)b_1}{2}\right)^2, \tag{11}$$
$$\tilde{p}_2^m = \frac{a_2 + \mu\lambda_0(1-\lambda_0)b\xi}{2b_1}; \quad \tilde{q}_2^m = \frac{a_2 - \mu\lambda_0(1-\lambda_0)b_2\xi}{2},$$
$$\tilde{\Pi}_2^m = \frac{1}{b_2}\left(\frac{a_2 - \mu\lambda_0(1-\lambda_0)b_2\xi}{2}\right)^2. \tag{12}$$

Not surprisingly, equilibrium values are symmetric. Hence, we can concentrate on firm 1 and extend the conclusions to firm 2. Comparing profits firm 1 gets in (9) and in (11), it is easy to see that firm 1 will participate in the agreement if and only if,

$$\frac{1}{b_1}\left(\frac{a_1 - \mu\lambda_0(1-\lambda_0)b_1}{2}\right)^2 > \frac{1}{b_1}\left(\frac{a_1 - b_1}{2}\right)^2,$$

that reduces to a quadratic function of λ_0,

$$b_1[1 - \mu\lambda_0(1-\lambda_0)] > 0. \tag{13}$$

Given that $b_1 > 0$ by assumption, we need to verify that $[1-\mu\lambda_0(1-\lambda_0)] > 0$. This inequality admits real roots for $\mu > 4$. These are $\lambda_{1,2} = \frac{1}{2} \pm \frac{\overline{\mu}}{2}$ with $\overline{\mu} = \left(\frac{\mu-4}{\mu}\right)^{1/2}$. Note that $0 < \frac{1-\overline{\mu}}{2} < \frac{1+\overline{\mu}}{2} < 1$. Therefore, inequality (13) is fulfilled for $\lambda_0 \in \left[0, \frac{1}{2} - \frac{\overline{\mu}}{2}\right] \cup \left[\frac{1}{2} + \frac{\overline{\mu}}{2}, 1\right]$. □

Figure 1 summarizes the discussion.

3.2 Interrelated markets

Next, we extend the previous setting to the situation in which final goods may be substitutes or complements so that the two firms interact in the market. We have now two degrees of freedom to characterize the conditions under which firms may engage in an agreement. On the one hand the degree of substitutability or complementarity given by c; on the other hand, the degree of technical similarity between firms given by λ_0.

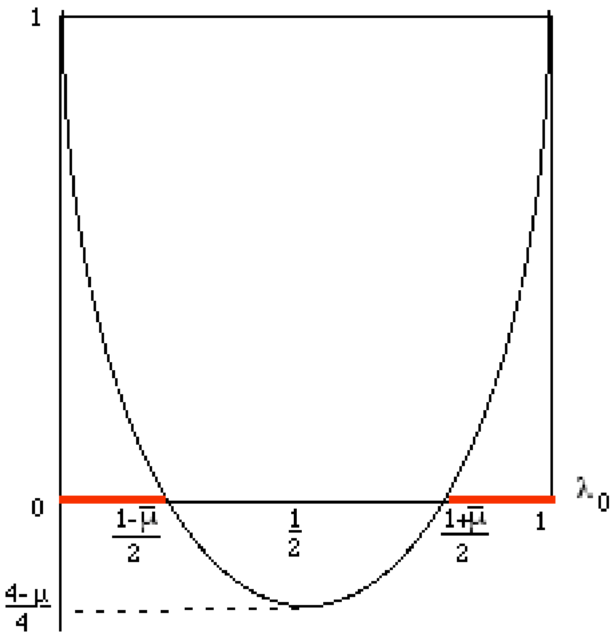

Figure 1:

Proposition 2. *When markets interact, firms are willing to engage in an agreement if $\mu > 4$ and goods are either poor substitutes or close substitutes. Namely, (i) for $c \to 0$ technological conditions making the agreement sustainable are described by $\lambda_0 \in [0, \frac{1-\overline{\mu}}{2}] \cup [\frac{1+\overline{\mu}}{2}, 1]$, and (ii) for values of c large enough the agreement is sustainable for $\lambda_0 \in (\frac{1-\overline{\mu}}{2}, \frac{1+\overline{\mu}}{2})$, where $\overline{\mu} = \left(\frac{\mu-4}{\mu}\right)^{1/2}$.*

Proof. Now firms compete à la Bertrand in the market. The equilibrium prices, quantities, and profits are,

$$\overline{p}_1 = \frac{2b_2\overline{a}_1 + c\overline{a}_2}{4b_1b_2 - c^2}; \quad \overline{q}_1 = b_1\overline{p}_1; \quad \overline{\Pi}_1 = b_1\left(\frac{2b_2\overline{a}_1 + c\overline{a}_2}{4b_1b_2 - c^2}\right)^2, \tag{14}$$

$$\overline{p}_2 = \frac{2b_1\overline{a}_2 + c\overline{a}_1}{4b_1b_2 - c^2}; \quad \overline{q}_2 = b_2\overline{p}_2; \quad \overline{\Pi}_2 = b_2\left(\frac{2b_1\overline{a}_2 + c\overline{a}_1}{4b_1b_2 - c^2}\right)^2, \tag{15}$$

where $\overline{a}_1 = a_1 - \mu\lambda_0(1-\lambda_0)[b_1 - c\xi]$ and $\overline{a}_2 = a_2 - \mu\lambda_0(1-\lambda_0)[b_2\xi - c]$.

As before, given the symmetry of the problem we concentrate on the behavior of firm 1. Firm 1 evaluates the benefits she can get from the agreement comparing the level of profits with and without the agreement. That is, it compares profits in

(8) and (14). Participating in an agreement will be profitable if and only if,

$$b_1 \left(\frac{2b_2 \overline{a}_1 + c \overline{a}_2}{4b_1 b_2 - c^2} \right)^2 > b_1 \left(\frac{2b_2 \hat{a}_1 + c \hat{a}_2}{4b_1 b_2 - c^2} \right)^2 .$$

After some algebraic computations, the previous inequality reduces to,

$$b_1[1 - \mu\lambda_0(1 - \lambda_0)](2b_1 b_2 - b_2 c\xi - c^2) = b_1[1 - F(\lambda_0)](2b_1 b_2 - b_2 c\xi - c^2) > 0. \tag{16}$$

Note that (16) differs from (13) in the term in brackets. This term is quadratic in c, has a positive root and a negative one, and is concave in c. Therefore, for values of c around zero in between the two roots, the term $(2b_1 b_2 - b_2 c\xi - c^2)$ is positive, and inequality (16) behaves as (13). Thus, we obtain the same result as in the monopoly case. In contrast, for large enough values of c (beyond the respective roots), the term $(2b_1 b_2 - b_2 c\xi - c^2)$ is negative, so that the inequality is fulfilled when $[1 - \mu\lambda_0(1 - \lambda_0)] < 0$ that is, for $\lambda_0 \in (\frac{1-\overline{\mu}}{2}, \frac{1+\overline{\mu}}{2})$. $\square$

In the remaining part of this study, we implicitly consider the case of substitute goods because this is the sensible case in our context. Results are robust to the substitute or complementary nature of the final goods. The relevant feature is the degree of competition among firms. In other words, R&D agreements may arise between firms supplying either the same or different kinds of products. It is their degree of competition that qualify the results we can achieve.

4 Dynamic successful agreements

Now we extend the results obtained in the previous section by introducing the time dimension and, as a consequence, the process of cumulation of knowledge. In other words, we assume that when a firm takes her decision, she is aware that the advantages she can get from the agreement follow an iterating process given by (10).

We will proceed in two steps. First, we will identify the conditions guaranteeing that subscribing an agreement lasting for more than one period is profitable for each firm. That is, we will examine whether there are combinations of technologies, embodied in the variable λ_0, giving firms the incentive to maintain their collaboration for $t > 1$ periods (lemma 6). Narula and Hagedoorn (1999) point out that firms signing agreements look for profits in the short run. We transpose this evidence in our setting by imposing the (strict) condition that we only admit agreements that guarantee positive profits period by period (and not allowing for intertemporal monetary compensation). In particular, we concentrate on a situation in which two monopolies may decide to extend the length of an existing agreement and we evaluate under which conditions such a decision may be a successful. Next, we will illustrate, by means of an example, how the set of solutions depends on the time horizon.

Lemma 1. *Consider two local monopolists and assume $\mu > 4$. For an existing $(t-1)$ period agreement, there is a range of values of λ_0 that in a t-period iterative learning process among firms allows them to improve their level of profits. It is given by $\lambda_0 \in (\frac{3}{4}, 1]$.*

Proof. Given the structure of the iterative learning function, $\lambda_1 = \mu\lambda_0(1-\lambda_0), \ldots, \lambda_t = \mu\lambda_{t-1}(1-\lambda_{t-1}), \lambda_{t+1} = \mu\lambda_t(1-\lambda_t)$.

As a consequence, the sequence of profits for, say, firm 1 in every iteration t are,

$$\widetilde{\Pi}_{1t}^m = \frac{1}{b_1}\left(\frac{a_1 - \lambda_t b_1}{2}\right)^2, \; t = 1, 2, \ldots \tag{17}$$

Our local monopolist will be willing to extend the agreement from period $t-1$ to period t if and only if,

$$\widetilde{\Pi}_{1t}^m > \widetilde{\Pi}_{1t-1}^m. \tag{18}$$

Note that from the expressions of profits it follows that $sign[\widetilde{\Pi}_{1t}^m - \widetilde{\Pi}_{1t-1}^m] = sign[\lambda_{t-1} - \lambda_t]$. Accordingly, inequality (18) reduces to studying the values of λ satisfying $\lambda_{t-1} - \lambda_t > 0$.

Given that $\lambda_t = \mu\lambda_{t-1}(1-\lambda_{t-1})$, the previous expression holds for $\lambda_{t-1} > 1 - \frac{1}{\mu}$. Given that $\mu > 4$, firms will be willing to extend the agreement from period $t-1$ to t if $\lambda_{t-1} > \frac{3}{4}$. $\square$

Lemma 6 gives the consistency conditions ensuring that given an agreement of length t, there are no incentives to break it at an earlier period. These conditions involve firms' technologies being sufficiently similar. Note that equation (10), describing the diffusion of the technological change, considers λ_0 as the initial (exogenous) condition. That is the description, before the agreement, of the technological differences between firms. Thus, the lemma proves that, given some initial conditions, firms will maintain their collaboration period after period as long as the diffusion process maintains their technologies similar enough. Note also that the degree of feasible similarity is increasing in time although the less efficient firm never ends up catching up with its partner. Moreover, according the expected length of the agreement, the magnitude of the benefits over the costs of production varies.

We illustrate the dynamics just described thinking of a local monopolist forecasting the impact on its profits period by period when planning to sign an agreement lasting for t periods.[4]

4.1 The two-period agreement

Consider an agreement lasting for two periods. Firm 1 evaluates the profits it will get at the end of period two, according to the technology available at that time.[5] Then, it compares these profits with the ones in absence of agreement. Namely firm 1 compares profits in (9) with profits given by (17). [6] It turns out that $\widetilde{\Pi}_{12}^m > \Pi_1^m$ if $b_1(1-\lambda_2) > 0$, that is,

[4]In general, this is the kind of cost-benefit analysis that firms carry out when they evaluate the convenience of joining an agreement. Firms look at the evolution of profits over a finite horizon from the actual situation by computing the present (discounted) value of the flow of future profits. In addition, we are comparing stock variables at different moments in time and we implicitly discount them at the same discount rate. It is important to remind that we are considering the extreme case where the agreement must be profitable every single period. Midler assumptions would consider comparing aggregate discounted profits over a certain number of periods. Then, opportunities for successful collaboration should appear more easily.

[5]We would like to remind that firms can exploit the benefits they get from the agreement just at the end of period two.

[6]This is so because we are assuming to be in the case of optimal long-term non renegotiable contracts.

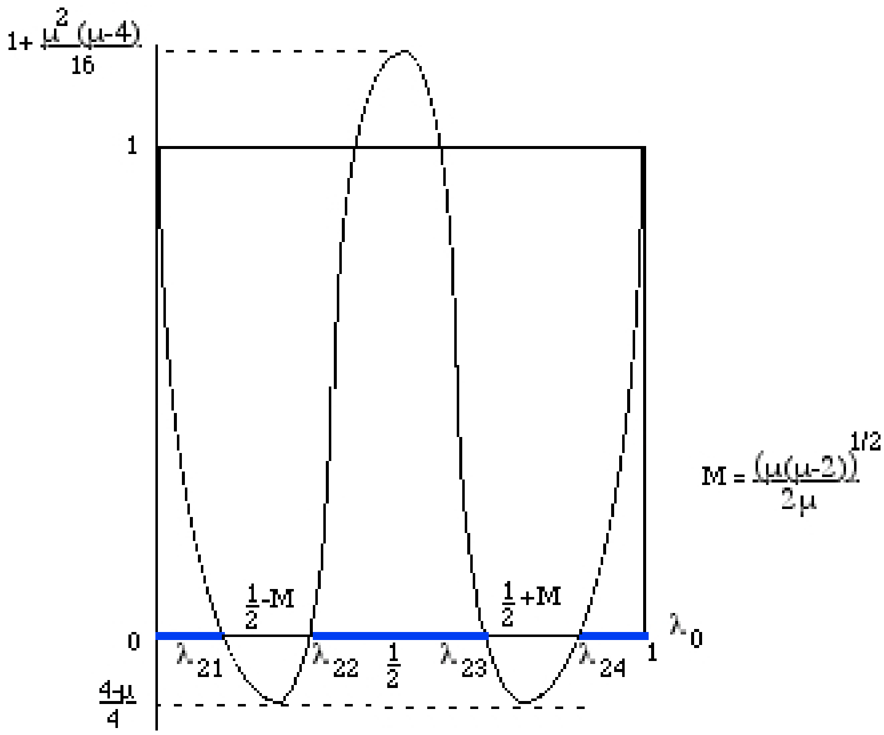

Figure 2:

$$b_1\{[1-\mu^2\lambda_0(1-\lambda_0)][1-\mu\lambda_0(1-\lambda_0)]\} = b_1[1-F^2(\lambda_0)] > 0. \qquad (19)$$

As displayed in Figure 2 for $\mu > 4$, inequality (19) admits four strictly positive critical points ($0 < \lambda_{21} < \lambda_{22} < \lambda_{23} < \lambda_{24} < 1$), where

$$\lambda_{2i} = \frac{1}{2} \pm \frac{\sqrt{\mu^2 - 2\mu(1 \pm \overline{\mu})}}{2\mu}.$$

As before $\overline{\mu} = \left(\frac{\mu-4}{\mu}\right)^{1/2} \in (0,1)$, and $i = 1,2,3,4$ according to the combination of positive or negative signs of the square roots chosen. Therefore, (19) is satisfied for $\lambda_0 \in [0, \lambda_{21}] \cup [\lambda_{22}, \lambda_{23}] \cup [\lambda_{24}, 1]$.

Finally, combining the range of admissible values of λ_0 just obtained for period 2 with the corresponding ones in period 1 (see Proposition 1) we get the range of values of λ_0 for which the two-period agreement is profitable:

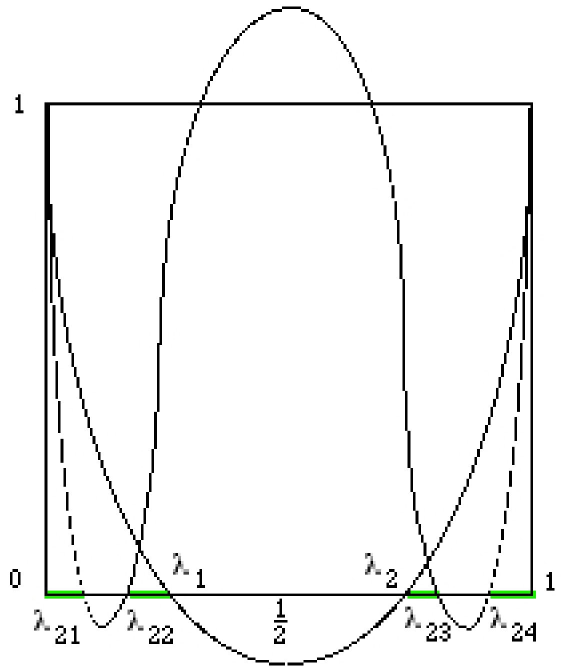

Figure 3:

$$\lambda_0 \in [0, \lambda_{21}] \cup \left[\lambda_{22}, \frac{1}{2} - \frac{\overline{\mu}}{2}\right] \cup \left[\frac{1}{2} + \frac{\overline{\mu}}{2},\ \lambda_{23}\right] \cup [\lambda_{24}, 1]\,.$$

4.2 N-period agreement

As it is well displayed by this example, and Figure 3 illustrates, the different intervals of solutions shrink as far as the number of iterations increases, i.e. the length of the agreement expands.

Hence, the question to tackle is to determine for which value of λ_0 an agreement can be successful given its length, knowing that the set of admissible values of λ_0 shrinks when the time dimension increases.[7]

We may sum-up the evolution of the process in the following way. The set of $\lambda_0 - values$ we are interested in are those for which the conditions (13), (16) and

[7] The progressive shrinking of the set of values is a consequence of the dynamics involved by the specification of the function for the cumulation of knowledge. Such quadratic function follows a complex dynamics for $\mu > 4$.

the corresponding ones for agreements lasting more than two periods, are satisfied. Let us rewrite those conditions in the following way:

- For agreements lasting one period ($t = 1$), the possible values of initial technological endowments entailing a successful result of the agreement are the values of $\lambda_0 \in \Lambda_1 \subset [0, 1]$ such that $G(\lambda_0) \equiv 1 - F(\lambda_0) \geq 0$,
- For agreements lasting two periods ($t = 2$), the possible values of initial technological endowments entailing a successful result of the agreement are the values of $\lambda_0 \in \Lambda_2 \subset \Lambda_1$ such that $G^2(\lambda_0) \equiv 1 - F^2(\lambda_0) \geq 0$.

  .....

- For agreements lasting N periods ($t = N$), the possible values of initial technological endowments entailing a successful result of the agreement are the values of $\lambda_0 \in \Lambda_N \subset \Lambda_{N-1}$ such that $G^N(\lambda_0) \equiv 1 - F^N(\lambda_0) \geq 0$.

Such behavior is induced by the iterative structure of function $F^N(\lambda_0)$. At the limit, when $t \to \infty$ we obtain a infinite collection of points as the set of solutions. These points are precisely the (infinite) roots of the polynomial (of infinite degree) resulting from the comparison of profits between signing an infinite horizon agreement and no agreement at all. To clarify this argument, define A_t as the set of λ_0-points that escape from the interval $I = [0, 1]$ at iteration $t + 1$. That is, those points that were admissible at iteration t but are no longer solutions after iteration $t + 1$. Formally,

$$A_t = \{\lambda_0 \in I \,|\, G^t(\lambda_0) < 0 \text{ and } G^\tau(\lambda_0) \in I,\ \tau < t\}.$$

This set of the solutions (Λ), in the case of an infinite number of iterations, reduces to:

$$\Lambda = I \setminus \bigcup_{t=0}^{\infty} A_t.$$

We will prove that Λ is a Cantor set, namely that it is a closed, perfect and totally disconnected subset of I.

Proposition 3. Λ *is a Cantor set for* $\mu > 4$.

Proof. See Appendix. □

Intuitively, note that A_t are open sets. Thus, Λ is formed by (sequentially) suppressing from the interval I a collection of open sets that are disjoint intervals. In other words, Λ is the union of closed and disjoint intervals, and thus closed. Incidentally, note that Λ is not empty because at least contains the extreme points of the suppressed intervals,

Next, by definition, a set is perfect if it does not contain isolated points, that is, all its points are limit points. Let us assume, on the contrary, that $x \in \Lambda$ is an isolated point. Then x must be an extreme point common to two adjacent intervals. But as we have argued before, Λ is a collection of disjoint intervals. Hence, those adjacent intervals do not have points in common. Accordingly, x cannot be an isolated point.

Finally, a set is totally disconnected if it does not contain any open interval. Again let us proceed by contradiction. Assume that there exists an open interval $\delta \in \Lambda$. Then δ has to be contained in one of the open intervals obtained in an iteration τ. But this is not possible since as $\tau \to \infty$, the length of the intervals tend to zero. Thus, at the limit Λ has infinitely many points.

5 Discussion

As we have seen, every iteration eliminates an open set of λ_0-values that were solutions in the previous iteration. The extreme points of those intervals remain in Λ though. It is important to bear in mind that a value of λ_0 that has been eliminated as a solution after an iteration, it remains out of Λ forever, i.e. it cannot be considered as solution again as the number of iterations increase.

Given the learning process we consider, as firms envisage longer and longer agreements, an increasing number of smaller intervals are excluded as solutions. Indeed, in the limit as $t \to \infty$, we obtain a (countable) set of solutions with infinitely many points. Formally, at every iteration t, the admissible values of λ_0

supporting an agreement of length t is characterized by a polynomial of degree 2^t. The roots of the successive polynomials associated to every iteration always remain in Λ. As t increases the length of admissible intervals shrinks, so that at the limit we have a polynomial of degree infinite characterizing intervals of measure zero. That is, only the points corresponding to the infinite solutions remain in Λ as solutions of an agreement of infinite length.

To help to visualize the evolution of the set of solutions, think of a firm willing to sign a short-term agreement. It can find a compatible partner almost effortlessly. As the commitment the firm is willing to engage in becomes deeper and deeper, the difficulty to find a suitable partner is also increasing. The reason behind this difficulty is *not* that there are less partners available (there are always infinite), but that getting to know about them and matching with the good one is increasingly hard.

In addition, conditions encountered for parameter λ_0 in Lemma 1 imply that lasting agreements are those signed by firms displaying similar technological endowments (i.e. high values of λ_0). Yet, we need to keep in mind the meaning of this result. Knowing the length of the agreement, a firm evaluates the advantages it can get before signing it. According to the initial conditions (λ_0) it will be able or not to fulfill its expectations. Moreover, the iteration process imposes that firms need to be very careful when choosing the agreement (a partner and a time horizon), given their initial technologies. In other words, if a firm wants to get the expected benefits from the agreement, needs to be extremely precise in choosing the right counterpart) allowing to fulfill its expectations. Put differently, with an infinite number of iterations, there is just a number of *discrete* points ensuring the success of the agreement. These correspond to the optimal combinations of initial technologies available at the firms level.

So far, we have only considered firms operating in separate markets. Recall that in the previous section studying agreements that do not span in time, we obtained the same qualitative results for both the case of local monopolies and of firm interaction. The introduction of time in the analysis involves a learning process but it does not change the dynamics of the decision process of firms. Hence, we should not expect to obtain qualitatively different results either. That is, if firms operate in the same market, we should expect to obtain also a Cantor set of solutions as the number of iterations increase.

6 Conclusion

In this paper, we study the consequences that a given level of technological endowment may exert on the successfulness of the results of a firm agreement. Based on a duopoly setting in which firms compete à la Bertrand, we prove that not all initial technologies are suitable for getting advantages from such an agreement. Indeed, according to the expected length of the agreement, there exists just a particular and precise set of initial conditions (evaluated as the technology available at firm level at the moment they create the agreement) ensuring firms to benefit from all the advantages that agreement can carry out. The central issue of this analysis is related to the existence of a learning process throughout the length of the contract. As the number of iterations increase, an increasing number of smaller intervals of values of λ_0 are excluded as solutions. In the limit, when considering agreements lasting forever, we obtain a countable set of infinitely many points characterized as a Cantor set. According to the structure of our framework, this last outcome means that in the case of agreements lasting for long periods, firms can benefit as much as possible from the advantages issued by the agreement just in the case they succeed in finding the proper combination of technological initial conditions. Put differently, not all the agreements are suitable for all the firms. Of course, to get this result we assume that, *a priori*, firms have perfect foresight of the status of the agreement from the initial period on. Indeed, it is this assumption that allow them to deal properly with the cost-benefit analysis of the agreement to detect the optimal combination of initial technological conditions. In other words, our model provides a rationalization of the prevalence of short-run agreements. Indeed, our main conclusion can be described in terms of the probability that a firm finds a suitable partner to engage in an agreement. Such probability is decreasing with the length of the contract.

Some extensions deserve attention. This framework could fit the analysis in other topics where the matching condition is fundamental, such as the labor market or the marriage matching problems. Accounting for uncertainty and technical development should complete the picture of present results. Also, an effort to give structure to λ_0 is in order be able to model a full dynamic learning process. Moreover, it could be also interesting to think of the possibility that a firm can leave the agreement before its completion and observing the way results can vary.

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Appendix: Proof of proposition 3

We will structure the proof in three steps following Devaney (1985) as guideline.

1. Λ is a closed set.

Let us define $G(\lambda_0) = 1 - F(\lambda_0)$ and re-write it as $G \equiv 1 - F$. By construction A_i is an open interval centered around 1/2 (see Figure 1 or 2). Let us focus on Figure 1 (one iteration), namely concentrating on A_0. In that case, the function G maps both the intervals $I_0 = [0, \lambda_1]$ and $I_1 = [\lambda_2, 1]$ monotonically onto I. Moreover, G is decreasing on the first interval and increasing on the second. Since $G(I_0) = G(I_1) = I$ there is a pair of intervals (one in I_0 and the other in I_1) which are mapped into A_0 by G. These intervals define the set A_1.Next, let us consider $\Lambda_1 = I - (A_0 \cup A_1)$. This set consists of four closed intervals (see Figure 2) and G maps them monotonically onto either I_0 or I_1, but, as before, each of the four intervals contains an open subinterval which is mapped by G_2 onto A_0, i.e. the points of this interval escape from I after the third iteration of G. By applying this iterative process, we note that A_t consists of 2^t disjoint open intervals and $\Lambda_t = I - (A_0 \cup .. \cup A_t)$ consists of 2^{t+1} closed intervals. Hence, Λ is a nested intersection of closed intervals, and thus, a closed set.

2. Λ is a perfect set.

Note that all endpoints of $A_t,\ (t = 1, ...)$ are contained in Λ. Such points are eventually mapped to the fixed point of G at 1, and they stay in I under iteration. If a point $x \in \Lambda$, were isolated, each nearby point must leave I under iteration and, hence, these points must belong to some A_t. Two possibilities arise. We can think of a sequence of endpoints of A_t converging to x. In this case the endpoints of A_t map to 1 and so, they are in Λ. Alternatively, all points in a deleted area nearby x are mapped out of I by some iteration of G. In this case, we may assume that G_τ maps x to 1 and all the other nearby points are mapped in the positive axis above 1. Then, G_τ has a minimum at x, i.e. $G'_\tau(x) = 0$. This iterative process ensures that it must be so for some $t < \tau$. Hence, $G_t(x) = 1/2$, but then $G_{t+1}(x) \notin I$ and $G_\tau(x) \to -\infty$, contradicting the fact that $G_\tau(x) = 1$.

3. Λ is a totally disconnected set.

Let us focus in the first iteration and assume μ is large enough so that $|G'(x)| > 1$ for all $x \in I_0 \cup I_1$. For those values of μ, there exists $\gamma > 1$ such that $|G'(x)| > \gamma$ for all $x \in \Lambda$. Our iterative process yields $|G'_\tau(x)| > \gamma_\tau$. We want to prove that Λ does not contain any interval. Let us proceed by contradiction and assume that there is a closed interval $[x, y] \in \Lambda$, $x, y \in I_0 \cup I_1$, $x \neq y$. In this case, $|G'_\tau(z)| > \gamma_\tau$, for all $z \in [x, y]$. Choose τ so that $\lambda_\tau |y - x| > 1$. Applying the Mean Value Theorem, it follows that $|G_\tau(y) - G_\tau(x)| \geq \gamma_\tau \mid y - x \mid > 1$ implying that either $G_\tau(y)$ or $G_\tau(x)$ lies outside of I. But this contradicts our main hypothesis, hence Λ does not contain intervals. It remains to determine the μ-values for which the previous argument holds. Finding the values of μ allowing $|G'(x)| > 1$ means to identify μ values for which $[-\mu (1 - 2x)]^2 > 1$. When $G = 0$, this inequality holds for $\mu > 2 + \sqrt{5}$. Thus we have proved that Λ is totally disconnected for $\mu > 2 + \sqrt{5}$. Recall that we have already imposed a condition on μ, namely $\mu > 4$. Hence, we need to verify whether Λ is also totally disconnected for $\mu \in (4, 2 + \sqrt{5}]$. We appeal to Kraft (1999) who establishes that Λ is a Cantor set for $\mu > 4$. The idea behind the proof is that for $\mu \in (4, 2 + \sqrt{5}]$ it turns out that $|G'(x)| \lesseqqgtr 1$. Kraft argues that the iteration process shrinks some components of I, and stretches some others. His proof thus, consists in showing that in the interval $(4, 2 + \sqrt{5})$ the stretching is dominated by the shrinking. To this end, he proves that Λ is an hyperbolic set, namely that $|G'_\tau(x)| > k\delta_\tau > 1$ for $x \in \Lambda$, $k > 0$, $\delta > 1$.