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**Equilibrium Limited Liability Contracts  
in a Landlord-Tenant Market**

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# Equilibrium Limited Liability Contracts in a Landlord-Tenant Market\*

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## Abstract

We propose a model based on competitive markets in order to analyse an economy with several homogeneous landlords and heterogeneous tenants. We model the landlord-tenant economy as a two-sided matching game and characterise the equilibrium of this market. In equilibrium, contracts are Pareto optimal, and the incremental surplus generated in a landlord-tenant relationship accrues to the tenant. We also suggest policy measures in relation to efficiency and income distribution.

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# 1 Introduction

Traditional literature contributing to the theory of incentives has sought to analyse optimal contracts in principal-agent relationships when there exist asymmetries of information. When this asymmetry concerns an action, or a decision to be made by the agent, a *moral hazard* problem emerges. Several works analyse optimal contracts when only one principal and one agent interact, including the seminal works by Pauly [15], Mirrlees [13], and Harris and Raviv [10]. The *principal-agent contracts* involve the provision of incentives and typically lead to inefficiency due to informational asymmetry.

The main goal of this paper is to analyse a landlord-tenant relationship not as an isolated entity but as part of an entire market where several landlords (principals) and tenants (agents) interact. In the framework that we propose, the utilities obtained by each tenant and each landlord are determined endogenously in the market. This allows us to improve upon the previous approach where the tenants' payoffs are exogenously determined and the landlords (principals) assume all the bargaining power.

We model the landlord-tenant economy as a two-sided matching game. An outcome of this economy is an *endogenous matching* and a set of contracts, one for each landlord-tenant pair formed under the matching. Roughly speaking, an outcome is said to constitute an equilibrium if there is no individual or no relevant pair objecting to the existing outcome. The paper studies equilibrium outcomes of this landlord-tenant market, using stability as the solution concept.

In particular, we consider an economy with several identical landlords and several tenants differentiated with respect to their initial wealth. A pair of individuals, one landlord and one tenant, can enter into a relationship by signing a contract. This contract specifies the contingent payments that are to be made by the tenant. Also it sets the level of investment, which together with a non-verifiable effort made by the tenant, determines the probability of having a high crop from the land the tenant cultivates. The initial wealth of the tenant may not cover the amount to be invested and hence, the wealth differences imply differences in liability.

We begin by providing a complete characterisation of the equilibrium outcomes of the landlord-tenant economy. The first simple property we prove is that all landlords earn the same payoff in equilibrium. The second feature is that the contracts offered in equilibrium are Pareto optimal, i.e., it is not possible to increase the utility level of the landlord (tenant) without making the tenant (landlord) strictly worse-off. More interestingly, in equilibrium, the matching itself is optimal, in the sense that it is the one that maximises productive

efficiency. If the tenants are in the long side of the market, only the wealthier ones, i.e., the more attractive ones are matched. Third, the productive efficiency of a contract signed in equilibrium increases with the wealth of a matched tenant. That is, the richer the tenant, closer his contract to the first-best. The additional surplus generated due to this increase in efficiency accrues to the tenant. Finally, the equilibrium contracts are more efficient than *principal-agent* contracts, i.e., the contracts signed when the landlords assume all the bargaining power in a relationship.

The previous characteristics of equilibrium outcomes have very relevant policy implications in this environment. Suppose that the public authority would like to improve the situations of the tenants by endowing them with some additional money. Our analysis suggests that the government will be interested in creating wealth asymmetries among tenants since otherwise, the landlords would appropriate all the incremental surplus intended to the tenants.

The use of matching games to model the landlord-tenant economy allows us to determine simultaneously the identity of the parties who meet (i.e., who are the tenants contracted) and the contracts they sign in an environment where each relationship is subject to moral hazard. Akerberg and Botticini [1] provide empirical evidence for endogenous matching in determining the contract forms in tenancy relationships in early Renaissance Tuscany. In analysing share tenancy relationships in Punjab (India), Bell, Raha and Srinivasan [5] show that an interesting matching problem emerges, with landlords looking for tenants with particularly low endowments of land relative to labour.

From the point of view of matching theory, one can see our model as a generalisation of the *assignment game* with several buyers and sellers described by Shapley and Shubik [20].<sup>1</sup> In the current model, a relationship is established through a contingent contract, rather than a price. The first distinguishing feature is that the surplus of each landlord-tenant pair, in our model, is determined endogenously. Next, the utility cannot be transferred between a landlord and a tenant on a one-to-one basis. In other words, unlike the assignment game, our model is a non-transferable utility game.

We further propose a simple non-cooperative game in which each of the tenants proposes a contract and each landlord chooses a tenant. We show that the equilibrium outcomes of this game coincide with the set of stable outcomes of the matching market.

Serfes [19] analyses an economy where the agents have different attitudes towards risk and the principals own assets which are subject to different exogenous variability. He also models the economy as a two-sided matching game and characterise the stable outcome where the

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<sup>1</sup>Roth and Sotomayor [17] provide an excellent review of the literature of matching models.

principals have all the bargaining power. In his model, a principal-agent pair cannot block an outcome with any contract, rather it is the principal who proposes a contract once a blocking pair is formed. The predictions of Serfes [19] are different from those of the standard risk model where an isolated principal-agent pair is studied. In particular, there can be a positive, negative, or non-monotonic relationship between risk and incentives.

In the context of an agricultural market, Shetty [21] explains the role of ex-post limited liability in determining the forms of tenancy contracts. Laffont and Matoussi [11] analyse the role of ex-ante financial constraint in the context of share-tenancy. Ray and Singh [16] propose a model where a set of principals compete for a continuum of agents in the presence of limited liability. Restricting themselves to linear contracts, they show that if the tenant's crop-share is unconstrained, wealthier tenants receive fixed-rent contracts, while poorer tenants receive sharecropping contracts. The role of limited liability in tenancy contracts are also analysed extensively by Basu [4] and Sengupta [18]. Assuming limited liability and moral hazard in choice of technique, Basu [4] proves that share contract is the optimal choice of the landlord.<sup>2</sup>

In an economy with a continuum of (heterogeneous) participants in both sides, Legros and Newman [12] present sufficient conditions for matchings to be monotone when utility is not fully transferable between partners. In contrast with the above paper, our framework can accommodate the analysis of economies with a few participants as well as those with a large number of participants. Dam [7], in the context of financial intermediation, analyses a principal-agent matching market under two-sided heterogeneity. Mookherjee and Ray [14] analyse the optimal short term contracts in an infinitely repeated interaction among principals and agents who are randomly matched at each period. Finally, the work of Barros and Macho-Stadler [3] looks into a situation where several principals compete for an agent. They also find that the competition among the principals make the incentive contracts more efficient.

The paper is organised as follows. In Section 2 we lay out the basic model. We describe the main results in Section 3. In the following section we discuss the characteristics of contracts in equilibrium. In Section 5 we propose a non-cooperative game that implements the set of equilibrium outcomes. In Section 6 we discuss the policy implications of the results found. In Section 7 we conclude.

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<sup>2</sup>A few other papers study agency problems with several principals and agents. See also Bhaskar [6], and Ghatak and Pandey [9] for further analyses of optimal contracts in presence of moral hazard and limited liability.

## 2 The Model

### 2.1 Landlords and Tenants

Consider an economy with a (finite) set of  $m$  risk neutral tenants. A tenant is identified by his level of initial wealth  $w^j$ . We arrange the tenants according to their wealth levels:  $w^1 \geq w^2 \geq \dots \geq w^m \geq 0$ . There are  $n$  risk neutral landlords, with a plot of land apiece. Lands are equally productive. A landlord  $l_i$  hires a tenant to cultivate her land. A plot of land yields *High output* ( $y > 0$ ) in case of success (which occurs with probability  $\pi$ ) and *Low output* (0) in the event of failure (with complementary probability). Land output depends on the level of effort ( $e$ ) that the tenant exerts and the monetary investment ( $K$ ) he makes. A tenant  $w^j$  can choose between *High effort* (equal to 1) and *Low effort* (equal to 0), both unobservable by the landlord. Investment  $K$  is financed entirely by the landlord.<sup>3</sup> Probability of success depends on effort and investment. Given effort  $e$  and investment  $K$ , the probability of success is given by  $\pi = \pi_e(K)$ . We assume:

- (a)  $\pi_1(K) > \pi_0(K)$ , for all  $K > 0$ ,
- (b)  $0 \leq \pi_e(K) \leq 1$ , for all  $K > 0$  and  $\pi_0(0) = 0$ ,
- (c)  $\pi'_e(K) > 0 > \pi''_e(K)$  for all  $K > 0$  and  $\lim_{K \rightarrow \infty} \pi'_e(K) = 0$ .

Part (c) guarantees that the solution in  $K$  is interior. We denote by  $\mathcal{M} \equiv \{\mathcal{L}, \mathcal{W}, \Pi\}$  the *landlord-tenant market*, where  $\mathcal{L}$  is the set of landlords,  $\mathcal{W}$  is the set of tenants (or, equivalently the vector of initial wealth), and  $\Pi$  represents the production technology.

### 2.2 Contracts and Payoffs

A landlord-tenant pair  $(l_i, w^j)$  signs a contract  $c$ , which specifies the payment made to the landlord in the event of success  $R_{ij}$ , the payment in case of failure  $r_{ij}$ , and the level of investment  $K_{ij}$ . We follow the convention that the tenant keeps the output. A tenant incurs an effort cost  $e$  if he chooses an effort level equal to  $e$ . Given a contract  $c = (R_{ij}, r_{ij}, K_{ij})$ , let  $e_c$  be defined as the effort that maximises the expected utility of tenant:<sup>4</sup>

$$e_c = \underset{e}{\operatorname{argmax}} \{ \pi_e(K_{ij})(y - R_{ij}) - (1 - \pi_e(K_{ij}))r_{ij} - e \}. \quad (\text{IC})$$

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<sup>3</sup>As wealth of the tenants are observable, all our findings remain (qualitatively) unaltered if a tenant with positive wealth finances part of the investment.

<sup>4</sup>Conventionally  $e_c = 1$  if both 1 and 0 maximise tenant's expected utility.

For a contract  $c$ , the effort chosen by the tenant will be  $e_c$  given that the effort is not contractible. This is the *incentive compatibility* constraint. Moreover, we normalise the per unit opportunity cost of lending (i.e., the cost of financing a project) to 1. Then the expected utilities of landlord  $l_i$  and the tenant  $w^j$  when they sign the contract  $c$  will be:

$$\begin{aligned} u_i(w^j, c) &= \pi_{e_c}(K_{ij})R_{ij} + (1 - \pi_{e_c}(K_{ij}))r_{ij} - K_{ij} \\ w^j(l_i, c) &= \pi_{e_c}(K_{ij})(y - R_{ij}) - (1 - \pi_{e_c}(K_{ij}))r_{ij} - e_c. \end{aligned}$$

Notice that we have defined the expected utility of tenant  $w^j$  *net* of his wealth. The gross expected utility of  $w^j$  would be  $u^j(l_i, c) + w^j$ . For further notational convenience, we denote by  $c^{null} = (0, 0, 0)$ , the *null contract*. Under  $c^{null}$ ,  $u_i(w^j, c^{null}) = u^j(l_i, c^{null}) = 0$ . We assume that, for a tenant, signing a contract  $c^{null}$  is equivalent to the situation where he is not contracted by any landlord, i.e., his *reservation utility* equals 0. Tenant's liability is limited to his current wealth. This imposes certain restrictions on the set of contracts. *Limited liability* implies:

$$R_{ij} \leq y + w^j, \quad (\text{LS})$$

$$r_{ij} \leq w^j. \quad (\text{LF})$$

The assumption of risk neutrality together with limited liability makes the incentive compatibility constraint costly and hence, it gives rise to *moral hazard* in tenant's effort choice. A contract signed by a landlord-tenant pair must satisfy the incentive compatibility and limited liability constraints. Furthermore, neither a tenant nor a landlord would accept a contract that yields negative expected utility. We say that a contract  $c$  is *acceptable* for a pair  $(l_i, w^j)$  if  $u_i(w^j, c) \geq 0$  and  $u^j(l_i, c) \geq 0$ . We club all these natural restrictions into the following definition.<sup>5</sup>

**DEFINITION 1** *A contract is **feasible for a tenant**  $w^j$  if it satisfies the restrictions of limited liability and acceptability.*

Denote by  $\mathcal{X}^j$  the set of contracts feasible for tenant  $w^j$ . From now on we will concentrate only on feasible contracts.

The incentive compatibility constraint implies that the tenant may choose any of the two effort levels (High or Low). In order to deal with interesting situations, we will assume, from now on, that the output  $y$  in case of success is high enough so that it is always optimal first, to establish a relationship and second, to set a contract that induces the tenant to exert high

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<sup>5</sup>Notice that the limited liability constraints are tenant specific.

effort. Hence, one can substitute the incentive compatibility constraint (IC) by the following:

$$(\pi_1(K_{ij}) - \pi_0(K_{ij}))(y - R_{ij} + r_{ij}) \geq 1. \quad (\text{IC}')$$

We denote by  $\mathcal{Z}^j \subset \mathcal{X}^j$  the set of feasible contracts that also satisfy the incentive compatibility constraint (IC'). One particular class of contracts are the *principal-agent* contracts, where the landlord (principal) assumes all the bargaining power. The *principal-agent* contract for the pair  $(l_i, w^j)$ , denoted by  $\bar{c}^j$ , solves the following programme:

$$\max_{c \in \mathcal{Z}^j} u_i(w^j, c). \quad (\text{P1})$$

Given the limited liability constraints, the moral hazard problem is typically costly for the landlord, i.e., she earns lower profits compared to the *first best* situation where she does not face any moral hazard problem. This happens if tenant's wealth is below the level that makes the limited liability constraints no longer binding. Denote by  $w^0$  this threshold level of wealth. Next, we show that, Under moral hazard, if the landlord has all the bargaining power, she strictly prefers hiring a tenant with higher wealth.

**PROPOSITION 1** *If  $w^k < w^j < w^0$ , then for any landlord  $l_i$ ,  $u_i(w^j, \bar{c}^j) > u_i(w^k, \bar{c}^k)$ .*

**PROOF** See Appendix B.  $\square$

## 2.3 Matching

Landlords and tenants are matched in pairs and when a pair is formed, they sign a contract. A (one-to-one) **matching** for the market  $\mathcal{M}$  is a *rule*  $\mu$  that assigns a landlord (tenant) to a tenant (landlord), or an individual stays alone. Formally, the partners of landlord  $l_i$  and tenant  $w^j$  in corresponding matches are denoted by  $\mu(l_i)$  and  $\mu(w^j)$ , respectively.<sup>6</sup> Matching refers to the bilateral nature of trades carried out in this market. If a tenant works in a land, then the owner of the land hires him. We have mentioned earlier that one of the main goals of this paper is to determine the equilibrium matching and contracts simultaneously. We refer to the couple  $(\mu, \mathcal{C})$  as an outcome for the market  $\mathcal{M}$  where  $\mathcal{C}$  is a list of feasible contracts one for each pair formed under the matching  $\mu$ .

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<sup>6</sup>If a landlord (tenant) stays alone, then we say that this individual is assigned to herself/himself. Also,  $\mu(\mu(l_i)) = l_i$  and  $\mu(\mu(w^j)) = w^j$ .



### 3 The Equilibrium

The outcomes of the market we describe here are endogenous. This endogeneity has two aspects. First, the contracts signed by the landlords and the tenants are endogenous. In the principal-agent theory, considerable attention has been paid to analyse the contracts that prevail in a given (isolated) principal-agent relationship. The second aspect is that the matching itself should be endogenous. We will approach this perspective in the same vein as matching theory. We require that an *equilibrium* outcome should be immune to be *blocked* by any landlord-tenant pair (as well as by any single individual). Consider an outcome  $(\mu, \mathcal{C})$ . If there is a landlord-tenant pair which can sign a different feasible contract such that both of them are strictly better-off under the new arrangement compared to their situation in the outcome  $(\mu, \mathcal{C})$ , then such an outcome cannot constitute an equilibrium. This idea corresponds to the notion of stability in matching games. Formally:

**DEFINITION 2** *An outcome  $(\mu, \mathcal{C})$  for the market  $\mathcal{M}$  is an **equilibrium** outcome if there does not exist any pair  $(l_i, w^j)$  and any feasible contract  $c'$  for tenant  $w^j$  such that  $u_i(w^j, c') > u_i(\mu(l_i), c_i)$  and  $u^j(l_i, c') > u^j(\mu(w^j), c^j)$ .*

The above definition makes sure that there does not exist any landlord-tenant pair that can *block* the current outcome, signing a feasible contract  $c'$ . Moreover, since all contracts in an equilibrium outcome are feasible (hence, acceptable), an equilibrium outcome is also *individually rational*. One can see immediately that an equilibrium outcome is **Pareto optimal** in the sense that, given a pair  $(l_i, w^j)$ , there is no possibility of improving the utility of one individual (by signing a different feasible contract) without making the other one worse-off.

In this section we characterise the equilibrium of the market  $\mathcal{M}$ . If an outcome constitutes an equilibrium then there is no landlord who can gain more than any of her counterpart does. The following lemma proves this assertion.<sup>7</sup>

**LEMMA 1** *In equilibrium all landlords consume the same utility.*

**PROOF** Suppose  $u_i(\mu(l_i), c_i) > u_k(\mu(l_k), c_k)$  in an equilibrium outcome  $(\mu, \mathcal{C})$ . We show that there exists a contract  $c' \in \mathcal{C}$  such that  $(l_k, \mu(l_i))$  blocks the outcome with  $c'$ . First, note that  $\mu(l_i) \in \mathcal{W}$ , otherwise  $u_i(\mu(l_i), c_i) = 0$ . Suppose  $c_i = (R_{ij}, r_{ij}, K_{ij})$  and consider  $c' \equiv c - \varepsilon = (R_{ij} - \varepsilon, r_{ij} - \varepsilon, K_{ij})$  with  $\varepsilon > 0$ .<sup>8</sup> It is easy to check that  $e_{c_i} = e_{c'}$ . Hence, for  $\varepsilon$  small enough,  $u_k(\mu(l_i), c') = u_i(\mu(l_i), c_i) - \varepsilon > u_k(\mu(l_k), c_k)$  and  $u_{\mu(l_i)}(l_k, c') =$

<sup>7</sup>This property is no longer valid if the lands are heterogeneous.

<sup>8</sup>In some proofs we will use the notation  $c - \varepsilon$  to refer to the contract  $(R_{ij} - \varepsilon, r_{ij} - \varepsilon, K_{ij})$ , when  $c = (R_{ij}, r_{ij}, K_{ij})$ .

$u_{\mu(l_i)}(l_i, c_i) + \varepsilon > u_{\mu(l_i)}(l_i, c_i)$ . Therefore,  $(l_k, \mu(l_i))$  blocks  $(\mu, \mathcal{C})$  with  $c'$  and hence the lemma.  $\square$

Since an equilibrium contract is Pareto optimal, a contract signed by a matched pair  $(l_i, w^j)$  must maximise the expected utility of one party taking into account that the other gets at least a certain utility level. One particular class of optimal contracts are the *principal-agent* contracts, which we have discussed in Section 2.2.

The utility possibility frontier for any landlord-tenant pair is the set of utilities generated by the contracts that solve a programme similar to (P1) where the reservation utility of the tenant can take value not only equal to zero as in (P1), but any number. The same set of optimal contracts results in if one maximises tenant's utility subject to a participation constraint of the landlord (PCL). We denote by  $c^j(\hat{u})$  the optimal contract that solves the following programme (as before we take tenant's utility net of his wealth  $w^j$ ):

$$\begin{cases} \max_{c \in \mathcal{Z}^j} u^j(l_i, c) \\ \text{s.t. } u_i(w^j, c) \geq \hat{u} \end{cases} \quad (\text{PCL}) \quad (\text{P2})$$

Notice that the contract that solves (P2) is acceptable for  $w^j$  only if  $\hat{u}$  is not too high. More precisely,  $u^j(l_i, c^j(\hat{u})) \geq 0$  if and only if  $\hat{u} \leq u_i(w^j, \bar{c}^j)$ . In the following theorems we characterise the equilibrium of the landlord-tenant market. The properties that the contracts in equilibrium are optimal and that all landlords earn same payoffs provide a partial characterisation. In the following theorem, we consider the situation where there are more tenants than landlords ( $m > n$ ) in the economy.<sup>9</sup>

**THEOREM 1** *If  $m > n$ , then an outcome  $(\mu, \mathcal{C})$  constitutes an equilibrium for the market  $\mathcal{M}$  if and only if the following three conditions hold:*

- (a)  $\mu(l_i) \in \mathcal{W}$  for all  $l_i \in \mathcal{L}$ ,  $\mu(w^j) \in \mathcal{L}$  if  $w^j > w^{n+1}$  and  $\mu(w^j) = w^j$  if  $w^j < w^n$ ,
- (b)  $u_i(\mu(l_i), c_i) = \hat{u} \in [u_i(w^{n+1}, \bar{c}^{n+1}), u_i(w^n, \bar{c}^n)]$  for all  $l_i \in \mathcal{L}$ , and
- (c)  $c^j = c^j(\hat{u})$  if  $\mu(A^j) \in \mathcal{L}$  and  $c^j = c^{null}$  if  $\mu(w^j) = w^j$ .

**PROOF** We first prove that (a)-(c) are necessary conditions for an equilibrium.

(a) Let  $c_i$  and  $c^j$  denote the contracts signed by a landlord  $l_i$  and a tenant  $w^j$ , respectively. Suppose first, that in an equilibrium outcome  $(\mu, \mathcal{C})$  a landlord  $l_i$  is not matched. Then  $u_i(\mu(l_i), c_i) = 0$ . Now consider a tenant  $w^j$  who is initially unmatched under  $\mu$ . Then the

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<sup>9</sup>This case seems the most reasonable in reality. For completeness (and because the framework may be useful in other applications), see Theorem 2 of Dam and Pérez-Castrillo [8] for an analysis of economies where the number of tenants (agents) is smaller than or equal to the number of landlords (principals).

contract  $\bar{c}^j - \varepsilon \in \mathcal{Z}^j$  yields strictly higher payoffs to both  $l_i$  and  $w^j$ . Hence,  $(l_i, w^j)$  with  $\bar{c}^j - \varepsilon$  blocks  $(\mu, \mathcal{C})$ . Second, we show that  $w^j$  is matched if  $w^j > w^{n+1}$ . Suppose, on the contrary, that  $w^j$  is unmatched under  $\mu$  and hence,  $u^j(w^j, c^j) = 0$ . Because of the previous part, under  $\mu$  there are  $n$  tenants matched. Suppose,  $w^k$  is a matched tenant such that  $w^k \leq w^{n+1}$ . Following Proposition 1,  $u_{\mu(w^k)}(w^j, \bar{c}^j) > u_{\mu(w^k)}(w^k, \bar{c}^k)$ . Given that  $u_{w^k}(\mu(w^k), c^k) \geq 0$  (since, the contract is feasible),  $u_{\mu(w^k)}(w^k, c^k) \leq u_{\mu(w^k)}(w^k, \bar{c}^k) < u_{\mu(w^k)}(w^j, \bar{c}^j)$ . Take  $c' = \bar{c}^j - \varepsilon$ , with  $\varepsilon$  small enough. It is easy to see that  $(\mu(w^k), w^j)$  with the contract  $c'$  will block the outcome, which is a contradiction. For the last part of (a), suppose on the contrary that  $w^j$  is matched under  $\mu$  and  $w^j < w^n$ . Since  $n$  tenants are matched, take  $w^k$  such that this tenant is not matched in equilibrium and  $w^k > w^n$ . Applying the same argument as before, it is easy to show that  $(\mu(w^j), w^k)$  with the contract  $\bar{c}^k - \varepsilon$  will block the current outcome.

**(b)** We know that in equilibrium the utilities of the landlords must be equal. Denote by  $\hat{u}$  the common utility of the landlords. First we show that in an equilibrium outcome  $(\mu, \mathcal{C})$ ,  $\hat{u} \geq u_i(w^{n+1}, \bar{c}^{n+1})$ . Suppose on the contrary,  $\hat{u} < u_i(w^{n+1}, \bar{c}^{n+1})$ . From part (a) we know that any tenant with wealth lower than  $w^n$  cannot be matched in an equilibrium. Suppose this tenant is  $w^{n+1}$  and consider any landlord  $l_i$ . Then there is a contract  $c' = \bar{c}^{n+1} - \varepsilon$ , with  $\varepsilon$  small enough, such that (1)  $u_i(w^{n+1}, c') = u_i(w^{n+1}, \bar{c}^{n+1}) - \varepsilon > \hat{u}$  and (2)  $u^{n+1}(l_i, c') \geq \varepsilon > 0 = u^{n+1}(\mu(w^{n+1}), \bar{c}^{n+1})$ . Hence,  $(l_i, w^{n+1})$  blocks the outcome. Second, from Proposition 1 we know that  $u_i(w^j, \bar{c}^j) > u_i(w^k, \bar{c}^k)$  if and only if  $w^j > w^k$ . In equilibrium  $(\mu, \mathcal{C})$ , a tenant with wealth higher than  $w^{n+1}$ , say  $w^n$  is matched with some landlord,  $l_i$ . Then  $u_i(w^n, c_i) = \hat{u} > u_i(w^n, \bar{c}^n)$  which implies that  $u^n(l_i, c_i) < u^n(l_i, \bar{c}^n)$ . This is not possible in equilibrium.

**(c)** Let  $(\mu, \mathcal{C})$  be an equilibrium outcome. Any contract  $c \in \mathcal{C}$  is optimal and  $c^j$  is such a contract. So, given that  $(\mu, \mathcal{C})$  constitutes an equilibrium,  $c^j = c^j(\hat{u})$  if  $\mu(w^j) \in \mathcal{L}$ .

We now prove that any outcome  $(\mu, \mathcal{C})$  satisfying **(a)**-**(c)** indeed constitutes an equilibrium. Suppose  $\mu(w^j) \in \mathcal{L}$  and consider any landlord  $l_i$  who, because of part (a), is matched. Clearly,  $(l_i, w^j)$  cannot block the outcome with any contract. Indeed, there does not exist a contract such that  $l_i$  gets more than  $\hat{u}$  and  $w^j$  gets more than  $u^j(\mu(w^j), c^j(\hat{u}))$  since  $c^j(\hat{u})$  is optimal by (c). Now suppose  $\mu(w^j) = w^j$  and choose any arbitrary  $l_i$  (we can do so, since all landlords have the same utility). By (a), we know that  $w^j \leq w^{n+1}$ . Then the maximum utility  $l_i$  can get by contracting  $w^j$  such that  $u^j(\dots) \geq 0$  is  $u_i(w^j, \bar{c}^j) \leq u_i(w^{n+1}, \bar{c}^{n+1})$ . Given that  $\hat{u} \geq u_i(w^{n+1}, \bar{c}^{n+1})$  (because of (b)), there is no room for the pair  $(l_i, w^j)$  to block  $(\mu, \mathcal{C})$ .  $\square$

We have already established that in equilibrium all landlords get the same utility. When there are too many tenants, this uniform utility cannot be less than the surplus that can be created by the *richest* unmatched tenant and it cannot be more than the surplus that can be created by the *poorest* matched tenant.

The above theorem characterises the equilibrium of this landlord-tenant economy. First important thing to note is the optimality property of the outcomes in equilibrium. Optimality in this market has in fact two aspects. The contracts signed are optimal for the parties involved. On the other hand, part (a) of the above theorem makes sure that the matching itself is optimal. This is the case because, in equilibrium, when there are more tenants than landlords, only the *best* (wealthier) tenants are the ones who get contracted.

The second important property is that the payoffs of the landlords are equal, because in equilibrium there emerges competition among the landlords for the wealthier tenants.

Third, in equilibrium, all the tenants whose wealth level is above the wealth of the poorest tenant contracted obtain a strictly higher utility than that under a *principal-agent* contract. In fact, there are equilibria where the same is true even for the poorest tenant contracted. To understand this, notice that had the tenants been symmetric, i.e., if they had equal initial wealth, and they were large in number, the landlords would assume all the bargaining power. In this case, the equilibrium would involve a *principal-agent* contract for each tenant hired. The asymmetry among the tenant does not let the landlords appropriate all the incremental surplus generated in a relationship, even when there are more tenants than landlords. Rather, the competition among landlords makes the incremental surplus accrue to the tenants.

Finally, as is usual in classical matching models, the set of equilibrium outcomes in our economy has a nice structure. First, if  $(\mu, \mathcal{C})$  constitutes an equilibrium and  $\mu'$  is an optimal matching, then  $(\mu', \mathcal{C})$  also constitutes an equilibrium. That is, the set of equilibrium outcomes is the Cartesian product of the set of optimal matchings and a set of menus of optimal contracts. Second, if one equilibrium outcome  $(\mu, \mathcal{C})$  is better for a tenant than another equilibrium outcome  $(\mu', \mathcal{C}')$ , then  $(\mu, \mathcal{C})$  is better than  $(\mu', \mathcal{C}')$  for all tenants hired and worse for all landlords matched. In particular, out of all equilibrium outcomes there exists an outcome which is the best for the landlords, and another that is best for all the tenants. In this economy, these two extreme points in the set of equilibrium outcomes correspond to the outcomes in which the utilities of the landlords are  $\hat{u} = u_i(w^n, \bar{c}^n)$  and  $\hat{u} = u_i(w^{n+1}, \bar{c}^{n+1})$ , respectively.<sup>10</sup> The first point is the landlords' optimal outcome (we refer to this as *L-optimium*), while the second is the tenants' optimal outcome (call this *W-optimium*).

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<sup>10</sup>If  $w^n = w^{n+1}$ , then the common utility consumed by all landlords,  $\hat{u}$ , is equal to  $u_i(w^n, \bar{c}^n) = u_i(w^{n+1}, \bar{c}^{n+1})$ . Moreover, any tenant obtains the same utility in all the equilibrium outcomes.

In our framework, transactions occur via contracts. The major difference between this economy and a market where transactions go through prices (as in the assignment game analysed by Shapley and Shubik [20]) is that the total surplus produced in a particular relationship does depend on the way in which the surplus is shared between the landlord and the tenant and on the design of the contract. The size of the surplus that accrues to the tenant influences the extent to which the limited liability constraints are binding and hence the total surplus.

## 4 Equilibrium Contracts

In this section, we provide the characteristics of the contracts signed in equilibrium. We have already shown that any such contract solves the maximisation programme (P2). To better illustrate the main qualitative features, we develop the analysis under the following assumption. In the appendix we comment on the qualitative changes if the opposite assumption holds.<sup>11</sup>

*Assumption 1*  $\pi_1(K)\pi'_0(K) - \pi'_1(K)\pi_0(K) > 0$  for all  $K > 0$ .

Assumption 1 implies that the derivative of  $\frac{\pi_0}{\pi_1}$  with respect to  $K$  is positive. That is, the higher the level of investment, the lower is the difference between  $\pi_0$  and  $\pi_1$ , and hence, the influence of making a high effort.

The *first-best* level of investment,  $K^0$  is given by the following equation:

$$\pi'_1(K^0)y = 1. \tag{1}$$

In the first-best contract,  $K^0$  is the level of investment that would be chosen if there was no moral hazard problem, or equivalently, if the limited liability (in case of failure) would not have any bite. In order to analyse the programme (P2), one can identify two disjoint ranges of values of  $w^j$  where the optimal solutions are different. First, for a very high level of tenant's wealth both the incentive compatibility constraint and limited liability constraint (in the event of failure) are not binding.<sup>12</sup> This is tantamount to saying that there is no moral hazard problem. The threshold level of initial wealth,  $w(\hat{u})$ , beyond which the optimal investment reaches its first-best level  $K^0$  depends on the utility of the landlord,  $\hat{u}$ , and is

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<sup>11</sup>The appendix provides a complete analysis of the solution to (P2).

<sup>12</sup>One can easily check that the limited liability constraint in the event of success is automatically satisfied for the problem and that  $R_{ij}$  can be calculated from (PCL).

given by:

$$w(\hat{u}) \equiv -\pi_1(K^0)y + K^0 + \hat{u} + \frac{\pi_1(K^0)}{\pi_1(K^0) - \pi_0(K^0)}.$$

For low levels of initial wealth,  $w^j \leq w(\hat{u})$ , both the incentive and the limited liability constraints bind. In this region the moral hazard problem becomes important and hence, the optimal investment is lower than its first-best level. The optimal investment  $\hat{K}(w^j; \hat{u})$  is implicitly defined by the following equation:

$$-\pi_1(\hat{K})y + \hat{K} + \hat{u} + \frac{\pi_1(\hat{K})}{\pi_1(\hat{K}) - \pi_0(\hat{K})} = w^j.$$

Given the above assumption, the optimal investment increases with tenants' wealth. The optimal investment is summarised in the following equation:

$$K_{ij} = \begin{cases} \hat{K}(w^j; \hat{u}) & \text{if } w^j < w(\hat{u}) \\ K^0 & \text{if } w^j \geq w(\hat{u}). \end{cases}$$

We also describe in brief the characteristics of the state contingent transfers. Notice that, for  $w^j \geq w(\hat{u})$ , any combination of  $(R_{ij}, r_{ij})$  that satisfies the constraints can be candidate for the optimum. One possible optimum corresponds to  $r_{ij} = w^j$ . In case where the constraints (IC') and (LF) are binding (for  $w^j \leq w(\hat{u})$ ),  $r_{ij} = w^j$  is also an optimum. Using the participation constraint of the landlord, one can then easily calculate the optimal transfer in case of success which is given by the following:

$$R_{ij} = \begin{cases} \frac{\hat{u} + \hat{K}(w^j; \hat{u}) - (1 - \pi_1(\hat{K}(w^j; \hat{u})))w^j}{\pi_1(\hat{K}(w^j; \hat{u}))} & \text{if } w^j < w(\hat{u}) \\ \frac{\hat{u} + K^0 - (1 - \pi_1(K^0))w^j}{\pi_1(K^0)} & \text{if } w^j \geq w(\hat{u}) \end{cases}$$

From the above it is not difficult to see that if one approximates the two-point payment schedule by a linear contract (which involves a fixed rent component and a share component), our results suggest that, in equilibrium, richer tenants tend to get more fixed rent contracts and the poorer ones, more share contracts.<sup>13</sup>

Once we know the characteristics of the solutions to program (P2), we use Theorem 1 to provide a description of the contracts in equilibrium. Consider first a situation with many tenants where the wealth of most of them is zero, i.e.,  $m > n$  and  $w^n = w^{n+1} = 0$ . In this economy, the contracts signed in all the equilibrium outcomes are unique. The contract signed

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<sup>13</sup>It is clear that the fixed rent component increases with tenant's wealth. The payment in case of success,  $R$ , is determined by the endogenous level of investment which also increases with  $w^j$ . Hence it is not so obvious to see that tenant's share of crop decreases with his wealth.

by the hired tenants with zero wealth will be the corresponding *principal-agent* contract, while the contract signed by the richer tenants will correspond to the solution of program (P2), for  $\hat{u} = u_i(w^n, \bar{c}^n)$ . Figure 1 depicts the level of investments in equilibrium.<sup>14</sup> For comparison, the diagram also includes the level of investments  $K(w^j)$  that would be made if all tenants would sign a *principal-agent* contract. In this figure,  $\bar{K}$  is the minimum level that would be invested by the tenants with very low level of wealth (say, less than  $\bar{w}$ ). The investment level is closer to the first-best level  $K^0$  as the wealth of a tenant is higher. That is, the productive efficiency of the relationship increases with the tenant's wealth. The investment level coincides with the first-best level if the tenant, say  $w^1$ , is rich enough, i.e.,  $w^1 \geq w(u_i(w^n, \bar{c}^n))$ . It is also worth noting that these investments are always higher than those under *principal-agent* contracts, unless the tenant's wealth is very high, more than  $w \geq w^0$ .

[Insert Figure 1 about here]

For the same economy, Figure 2 depicts tenant' net and gross utility levels (the common landlords' utility is  $u_i(w^n, \bar{c}^n)$ ). Tenants' net utility increases with the wealth level (unless the level of wealth is already above  $w(u_i(w^n, \bar{c}^n))$ ). The utility of wealthier tenants is not only higher because of the initial wealth levels, they also profit from the increase in surplus due to a more efficient (i.e., closer to the first-best) contracts.

[Insert Figure 2 about here]

For completeness, Figure 3 depicts the set of investment levels in equilibrium  $w^n > w^{n+1}$  and  $w^{n+1}$  is high. The line corresponding to the level of investments in a particular equilibrium outcome, say  $K^s(w^j)$  is quite similar to that in Figure 1 (although it starts from a level higher than  $\bar{K}$ ). This line will be placed at a higher (or a lower) position depending if we are in an equilibrium closer to (or farther from) the *W-optimum*. In particular, the lowest line (that starts from  $K(w^n)$ ) corresponds to the investment levels in the *L-optimum*.<sup>15</sup>

[Insert Figure 3 about here]

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<sup>14</sup>For sake of tangibility, all the figures are drawn for  $\pi_1(K) = \frac{K}{1+K}$  and  $\pi_0(K) = \frac{K}{2+K}$ . Our results, although, hold good for a very general class of probability functions satisfying our assumptions.

<sup>15</sup>The graphical representation of an economy with more landlords than tenants is very similar to figures 1 and 2. The levels of investment and of net and gross utilities are as in figures 1 and 2, with the only difference that they all start at a higher level than  $\bar{K}$  and  $\bar{w}$ .

## 5 Equilibrium Outcomes through a Non-cooperative Game

In this section we show that the set of equilibrium outcomes that we have characterised in Theorem 1 are also the equilibrium outcomes of a very simple and natural non-cooperative interaction between landlords and tenants. We propose a simple two-stage game form, called  $\Gamma^w$ . This is a two-stage game where in the first stage each tenant proposes a contract. In the second stage of the game, each landlord contracts a tenant.<sup>16</sup> Formally, in the first stage, each tenant simultaneously announces a (feasible) contract,  $s^j$ . In the second stage, knowing these announcements, each landlord  $l_i$  names a tenant,  $s_i$ . The outcome function  $g(\cdot)$  associates to each vector  $s = (s_1, \dots, s_n, s^1, \dots, s^m)$  a matching,  $\mu^s$ , and a menu of contracts,  $\mathcal{C}(s)$ , such that  $\mu^s(w^j)$  is the smallest indexed landlord of the set  $\mathcal{L}^j = \{l_i \in \mathcal{L} \mid s_i = w^j\}$  if  $\mathcal{L}^j \neq \emptyset$  and  $\mu^s(w^j) = w^j$ , otherwise. Moreover,

$$c^j(s) = \begin{cases} s^j & \text{if } \mu^s(w^j) \in \mathcal{L} \\ c^{null} & \text{, otherwise.} \end{cases}$$

The natural solution concept used here is *Subgame Perfect Equilibrium*. We analyse the Subgame Perfect Equilibria in pure strategies (SPE).

**THEOREM 2** *The set of SPE outcomes of the game  $\Gamma^w$  coincides with the set of equilibrium outcomes for the market  $\mathcal{M}$ .*

PROOF See Appendix E.  $\square$

The above theorem shows that one can propose a very simple non-cooperative game to implement the set of equilibrium outcomes of the economy.

## 6 Efficiency and Wealth Distribution

In a seminal work, Shetty [21] shows that wealth differences among tenants play a key role in determining the credit contracts when there exists a possibility of default on tenant's rental commitments. Difference in initial wealth implies difference in liability of the tenants. Hence, in the case where there is significant moral hazard problem due to limited liability, wealthier

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<sup>16</sup>The proposed game form adapts to our framework the mechanisms suggested by Alcalde, Pérez-Castrillo and Romero-Medina [2]. The two main differences are that the participants now sign contracts, more complex than a salary as in the previous paper, and it is a one-to-one matching model which imposes some additional rigidities on the working of the mechanism.



tenants are always preferred for a better contractual structure, since possibility of default is less with wealthier tenants.

The results in the previous sections can be used to analyse situations when a set of landlords interact with a set of tenants through tenancy relations. It is very common that the same person acts as landlord-cum-moneylender in villages by leasing land and lending money to the same person (here, the tenant). The contracts described for the market  $\mathcal{M}$  also capture these components. The state contingent transfers,  $(R, r)$  are the payments made to the landlord and  $K$  is the amount borrowed from the landlord that is invested eventually in land. In this economy, the tenants cannot seek loans in the formal credit market due to lack of sufficient collateral, while the landlords can. Consequently, the landlords become the only sources of credit to the hapless borrowers.

The properties highlighted in Theorem 1 have important implications with respect to distributive (in)equality and efficiency. It suggests that for a very low level of aggregate wealth, more is the inequality in the distribution of tenant's wealth, higher is the total investment and more efficient is the relationship. Indeed, as the wealth level of the poorest tenant hired decreases, the bargaining power of the other tenants increases. Consequently, these other tenants take more profit from a relationship and the contract terms are more efficient (i.e., the investment level is closer to the first-best).

From a normative point of view, the analyses suggest that if the public authority has a small amount of money to distribute which could serve as collateral in tenancy relations, it may need to induce inequality among the tenants in order to increase both the efficiency of the contracts and the utility of (some of) the tenants. Suppose all the tenants have no initial wealth. If the public authority distributes to every tenant a small (but same) amount (less than  $\bar{w}$  in Figure 1). Then in equilibrium, all the tenants will sign the *principal-agent* contracts investing a level  $\bar{K}$  which is the same they would do with zero wealth. Hence, the efficiency of the relationship will remain the same as that prior to the distribution. Moreover, the gross utility of all the tenants hired will be the same as before. That is, the landlords will appropriate the additional amount distributed, which was intended to improve the welfare of the tenants. On the other hand, if the public authority distributes the money among a few tenants (a number smaller than the number of landlords), then the contracts signed by these tenants will be more efficient than before, and their gross utility will increase by more than the additional money they receive. Hence, targeting a small group rather than all tenants improves the welfare of this group and overall efficiency.

## 7 Concluding Remarks

In this paper we model a landlord-tenant economy as a two-sided matching market and completely characterise the set of equilibrium outcomes of this economy. As we have mentioned earlier, our model can be seen as a generalisation of the assignment game described by Shapley and Shubik [20]. The main task of this paper lies in suggesting a general (competitive) equilibrium model of a landlord-tenant economy. Using the restriction of limited liability should be taken as a very simple way to tackle incentive problems. We also show that our results are not only the outcome of a cooperative game, but can be reached through very simple non-cooperative interactions between the landlords and the tenants.

Our paper leaves several avenues open to further research. First, we have assumed that the landlords are identical. Although some of the conclusions of our analyses can immediately be extended to apply to economies with heterogeneous landlords, the characteristics of the contracts signed in equilibrium can be different from those identified in the current work. On the one hand, the results that the contracts signed in equilibrium are optimal and the matching itself is efficient (in the sense that it maximises the total surplus) hold also in a framework with heterogeneous landlords. On the other, there is no unique way to model the differences among the landlords and the contracts will be different depending on the type of heterogeneity one would like to introduce. Second, ours is a one-to-one matching model. If we consider the situation where several independent tenants are matched with each landlord, then the conclusions will remain unchanged. But these will be different in a more interesting situation where the action of a tenant is dependent on that of others. This kind bears similarity with the agency problem in a multi-agent situation. A natural way to analyse this would be to make use of a many-to-one matching model.

## Appendix

### A. The Principal-Agent Contracts

We solve for the optimal contract for a pair  $(l_i, w^j)$ :

$$\left\{ \begin{array}{l} \text{maximise}_{\{R_{ij}, r_{ij}, K_{ij}\}} \quad u_i = \pi_1(K_{ij})R_{ij} + (1 - \pi_1(K_{ij}))r_{ij} - K_{ij} \\ \text{subject to} \quad (PC) \quad \pi_1(K_{ij})(y - R_{ij} + r_{ij}) - r_{ij} \geq 1 \\ \quad (IC') \quad [\pi_1(K_{ij}) - \pi_0(K_{ij})](y - R_{ij} + r_{ij}) \geq 1 \\ \quad (LS) \quad R_{ij} \leq y + w^j \\ \quad (LF) \quad r_{ij} \leq w^j. \end{array} \right. \quad (P1)$$

(i) Consider the region where, at the optimum, (IC') binds. In this region we can write the constraint with equality. Using this, one can replace  $R_{ij}$  in the objective function and in the other three constraints. Moreover, if (PC) and (LF) are satisfied, (LS) also holds good. Hence, the above programme reduces to the following:

$$\left\{ \begin{array}{l} \text{maximise}_{\{r_{ij}, K_{ij}\}} \quad \pi_1(K_{ij})y - \frac{\pi_1(K_{ij})}{\pi_1(K_{ij}) - \pi_0(K_{ij})} + r_{ij} - K_{ij} \\ \text{subject to} \quad (PC') \quad \frac{\pi_1(K_{ij})}{\pi_1(K_{ij}) - \pi_0(K_{ij})} - r_{ij} - 1 \geq 0 \\ \quad (LF) \quad w^j - r_{ij} \geq 0. \end{array} \right. \quad (P1')$$

We denote  $\mu_1$  and  $\mu_2$  the Lagrangean multipliers of (P1'). Then, the Kuhn-Tucker (first-order) conditions are given by:<sup>17</sup>

$$y\pi_1' - 1 + (1 - \mu_1) \frac{\pi_1'\pi_0 - \pi_1\pi_0'}{(\pi_1 - \pi_0)^2} = 0 \quad (2)$$

$$1 - \mu_1 - \mu_2 = 0 \quad (3)$$

$$\mu_1 \left( \frac{\pi_1}{\pi_1 - \pi_0} - r_{ij} - 1 \right) = 0 \quad (4)$$

$$\mu_2 (w^j - r_{ij}) = 0 \quad (5)$$

$$\frac{\pi_1}{\pi_1 - \pi_0} - r_{ij} - 1 \geq 0 \quad (6)$$

$$w^j - r_{ij} \geq 0 \quad (7)$$

$$\mu_1, \mu_2 \geq 0 \quad (8)$$

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<sup>17</sup>The hypotheses on  $\pi_1(K_{ij})$  and  $y$  make sure the optimal  $K_{ij}$  must be interior and it satisfies the first-order conditions. The corner solution for  $r_{ij}$  is explicitly taken into account.

Now we study different regions where the Kuhn-Tucker conditions can be satisfied. For simplicity, we develop the analysis when  $\pi_1' \pi_0 - \pi_1 \pi_0' < 0$ .

CASE 1:  $\mu_1 = \mu_2 = 0$  (Both the constraints are non binding)

From (3), we can see that this case is not possible.

CASE 2:  $\mu_1 > 0, \mu_2 = 0$  ((LF) is non-binding and (PC') is binding)

From (3),  $\mu_1 = 1$ . Then from (2), we have  $y\pi_1'(K^0) = 1$ , where  $K^0$  is the first-best level of investment. Using (PC') and (LF), one has

$$w^j \geq \frac{\pi_1(K^0)}{\pi_1(K^0) - \pi_0(K^0)} - 1 \equiv w^0.$$

Hence, if  $w^j \geq w^0$  a candidate for optimal solution exists involving  $K_{ij} = K^0$ . In particular, an optimal payment vector is  $(R_{ij} = y + w^j - \frac{1+w^j}{\pi_1(K^0)}, r_{ij} = w^j)$ .

CASE 3:  $\mu_1 = 0, \mu_2 > 0$  ((LF) is binding and (PC') is non-binding)

From (3),  $\mu_2 = 1$ . Then (2) implicitly defines the level of optimum investment  $\bar{K}$ ,

$$y\pi_1'(\bar{K}) - 1 + \frac{\pi_1'(\bar{K})\pi_0(\bar{K}) - \pi_1(\bar{K})\pi_0'(\bar{K})}{(\pi_1(\bar{K}) - \pi_0(\bar{K}))^2} = 0.$$

From (LF), we also have  $r_{ij} = w^j$ . Moreover,  $R_{ij}$  is determined by (IC') as  $R_{ij} = y + w^j - \frac{1}{\pi_1(\bar{K}) - \pi_0(\bar{K})}$ . And from the non-binding (PC') we have

$$w^j \leq \frac{\pi_1(\bar{K})}{\pi_1(\bar{K}) - \pi_0(\bar{K})} - 1 \equiv \bar{w}.$$

That is, the previous contract can only be a candidate if  $w^j \leq \bar{w}$ .

CASE 4:  $\mu_1 > 0, \mu_2 > 0$  (Both the constraints are binding)

From (LF),  $r_{ij} = w^j$ . Then (PC') defines the optimal  $K_{ij}$  as an implicit function of  $w^j$ . Denote this by  $K(w^j)$ , which must satisfy the following condition

$$\frac{\pi_1(K(w^j))}{\pi_1(K(w^j)) - \pi_0(K(w^j))} = w^j + 1. \quad (9)$$

Finally,  $R_{ij}$  is determined by (IC'). Previously found  $R_{ij}$ ,  $r_{ij}$  and  $K(w^j)$  are indeed the candidates for optimum if the Lagrange multiplier,  $\mu_1$ , implicitly defined by (2) lies in the

interval  $[0, 1]$  (so that constraints (3) and (8) are satisfied). Given that  $\pi'_1\pi_0 - \pi_1\pi'_0 < 0$ ,  $\mu_1 < 1$  if and only if

$$y\pi'_1(K(w^j)) - 1 > 0 \Rightarrow K(w^j) < K^0.$$

Again using  $\pi'_1\pi_0 - \pi_1\pi'_0 < 0$ ,  $K(w^j) < K^0$  is optimal if

$$\frac{\pi_1(K^0)}{\pi_1(K^0) - \pi_0(K^0)} < w^j + 1 \Rightarrow w^j < w^0.$$

Similarly,  $\mu_1 > 0$  if and only if

$$y\pi'_1(K(w^j)) - 1 + \frac{\pi'_1(K(w^j))\pi_0(K(w^j)) - \pi_1(K(w^j))\pi'_0(K(w^j))}{(\pi_1(K(w^j)) - \pi_0(K(w^j)))^2} < 0.$$

The above inequality implies  $K(w^j) > \bar{K} \Rightarrow \frac{\pi_1(\bar{K})}{\pi_1(\bar{K}) - \pi_0(\bar{K})} < 1 + w^j \Rightarrow w^j > \bar{w}$ . Hence, the optimal contract corresponds to the solution found in Case 3 when  $w^j < \bar{w}$ , is the candidate found in Case 4 when  $\bar{w} < w^j < w^0$ , and it is the first-best contract of Case 2 when  $w^0 \leq w^j$ .

(ii) The region where, at the optimum, (IC') does not bind is easier to analyse. In this case, it is not difficult to check that the constraint (PC) must be binding, while (LS) must not. Also, the optimal investment must be  $K^0$ , and because of (PC), the candidate does satisfy (IC') if and only if

$$r_{ij} \geq \frac{\pi_1(K^0)}{\pi_1(K^0) - \pi_0(K^0)} - 1 = w^0.$$

However, this is compatible with constraint (LF) if and only if  $w^j \geq w^0$ . Therefore, this region gives the same candidate as Case 2 in Region (i).

## B. Proof of Proposition 1

We are to show that if  $w^j > w^k$  in the region  $w^j < w^0$ , then  $u_i(w^j, \bar{c}^j) > u_i(w^k, \bar{c}^k)$ . From the previous section one can write the value function  $v(w^j) = u_i(w^j, \bar{c}^j)$ . Using the Envelope theorem, we get  $v'(w^j) = \mu_2 > 0$  and hence the proposition.

### C. Equilibrium Contracts

Let us rewrite (P2) as follows:

$$\left\{ \begin{array}{l} \text{maximise}_{\{R_{ij}, r_{ij}, K_{ij}\}} \quad w^j = \pi_1(K_{ij})(y - R_{ij}) - (1 - \pi_1(K_{ij}))r_{ij} - 1 \\ \text{subject to} \quad (PCL) \quad \pi_1(K_{ij})R_{ij} + (1 - \pi_1(K_{ij}))r_{ij} - K_{ij} \geq \hat{u} \\ \quad (IC') \quad [\pi_1(K_{ij}) - \pi_0(K_{ij})](y - R_{ij} + r_{ij}) \geq 1 \\ \quad (LS) \quad R_{ij} \leq y + w^j \\ \quad (LF) \quad r_{ij} \leq w^j. \end{array} \right. \quad (10)$$

As we have pointed out in the paper, this programme is individually rational for the tenant only if  $\hat{u} \leq u_i(w^j, \bar{c}^j)$ . Denote by  $w^{min}(\hat{u})$  the level of wealth such that  $\hat{u}$  is the utility of a landlord who hires a tenant with this wealth under a *principal-agent* contract. Programme (P2) is only well defined for  $w^j \geq w^{min}(\hat{u})$ . At the optimum, (PCL) binds. Hence, one can substitute for  $R_{ij}$  in the objective function and the rest of the constraints. Also, if both (IC') and (LF) hold, then (LS) becomes redundant. Then one has the above programme reduced as the following:

$$\left\{ \begin{array}{l} \text{maximise}_{\{r_{ij}, K_{ij}\}} \quad \pi_1(K_{ij})y - \hat{u} - K_{ij} - 1 \\ \text{subject to} \quad (IC'') \quad \pi_1(K_{ij})y - \frac{\pi_1(K_{ij})}{\pi_1(K_{ij}) - \pi_0(K_{ij})} + r_{ij} - K_{ij} - \hat{u} \geq 0 \\ \quad (LF) \quad r_{ij} \leq w^j. \end{array} \right. \quad (P2')$$

Let  $\nu_1$  and  $\nu_2$  be the Lagrange multipliers for (IC'') and (LF), respectively. The Kuhn-Tucker (first-order) conditions are

$$y\pi_1' - 1 + \nu_1 \left( y\pi_1' - 1 + \frac{\pi_1'\pi_0 - \pi_1\pi_0'}{(\pi_1 - \pi_0)^2} \right) = 0 \quad (11)$$

$$\nu_1 \left( \frac{\pi_1 - \pi_0}{\pi_1} \right) - \nu_2 = 0 \quad (12)$$

$$\nu_1 \left( (\pi_1 - \pi_0) \left( y - \frac{\hat{u} - r_{ij} + K_{ij}}{\pi_1} \right) - 1 \right) = 0 \quad (13)$$

$$\nu_2(w^j - r_{ij}) = 0 \quad (14)$$

$$\left( (\pi_1 - \pi_0) \left( y - \frac{\hat{u} - r_{ij} + K}{\pi_1} \right) - 1 \right) \geq 0 \quad (15)$$

$$w^j - r_{ij} \geq 0 \quad (16)$$

$$\nu_1, \nu_2 \geq 0 \quad (17)$$

Now we study different regions for the Kuhn-Tucker conditions to be satisfied.

CASE 1:  $\nu_1 = 0, \nu_2 > 0$  ((LF) is binding and (IC''), non-binding)

Using (12), one can see that this case is not possible.

CASE 2:  $\nu_1 > 0, \nu_2 = 0$  ((LF) is non-binding and (IC''), binding)

From (12), it is clear that this case is not possible either.

CASE 3:  $\nu_1 = \nu_2 = 0$  (Both the constraints are non-binding)

From (11),  $K_{ij} = K^0$ , the first best level of investment. The payment made to the landlord in case of failure,  $r_{ij}$  is calculated from (PCL). For example,  $r_{ij} = w^j$  and  $R_{ij} = \frac{\hat{u} + K^0 - (1 - \pi_1(K^0))w^j}{\pi_1(K^0)}$  are optimal. From (IC'') and (LF), the above is only possible if

$$w^j \geq -\pi_1(K^0)y + K^0 + \hat{u} + \frac{\pi_1(K^0)}{\pi_1(K^0) - \pi_0(K^0)} \equiv w(\hat{u}).$$

CASE 4:  $\nu_1 > 0, \nu_2 > 0$  (Both the constraints are binding)

In this case,  $r_{ij} = w^j$  and optimal investment is a function of  $w^j, \hat{K}(w^j; \hat{u})$ , that is implicitly defined by the condition

$$-\pi_1(\hat{K}(w^j; \hat{u}))y + \hat{K}(w^j; \hat{u}) + \hat{u} + \frac{\pi_1(\hat{K}(w^j; \hat{u}))}{\pi_1(\hat{K}(w^j; \hat{u})) - \pi_0(\hat{K}(w^j; \hat{u}))} = w^j. \quad (18)$$

Notice that, from (11), for  $\hat{K}(w^j; \hat{u}) \leq K^0$ ,  $y\pi_1' - 1 + \frac{\pi_1'\pi_0 - \pi_1\pi_0'}{(\pi_1 - \pi_0)^2} \geq 0$ . From the above expression, this immediately implies that  $\hat{K}(\cdot)$  is increasing in  $w^j$ . The previous values of  $r_{ij}$ ,  $R_{ij}$  and  $\hat{K}(w^j; \hat{u})$  are optimal solutions to the above programme if the multipliers  $\nu_1$  and  $\nu_2$  defined in equations (11) and (12) satisfy (17), i.e., they are non-negative. Notice that (11) implies  $\nu_2 > 0$  if and only if  $\nu_1 > 0$ . To check when  $\nu_1 > 0$ , notice that if  $w^j > w(\hat{u})$ , then it is necessary that

$$\begin{aligned} w^j &= -\pi_1(\hat{K}(w^j; \hat{u}))y + \hat{K}(w^j; \hat{u}) + \hat{u} + \frac{\pi_1(\hat{K}(w^j; \hat{u}))}{\pi_1(\hat{K}(w^j; \hat{u})) - \pi_0(\hat{K}(w^j; \hat{u}))} \\ &< -\pi_1(K^0)y + K^0 + \hat{u} + \frac{\pi_1(K^0)}{\pi_1(K^0) - \pi_0(K^0)} \equiv w(\hat{u}). \end{aligned}$$

Now we can characterise the optimal contract as follows.

$$K_{ij} = \begin{cases} \widehat{K}(w^j; \widehat{u}) & \text{if } w^j < w(\widehat{u}) \\ K^0 & \text{if } w^j \geq w(\widehat{u}). \end{cases}$$

$$R_{ij} = \begin{cases} \frac{\widehat{u} + \widehat{K}(w^j; \widehat{u}) - (1 - \pi_1(\widehat{K}(w^j; \widehat{u})))w^j}{\pi_1(\widehat{K}(w^j; \widehat{u}))} & \text{if } w^j < w(\widehat{u}) \\ \frac{\widehat{u} + K^0 - (1 - \pi_1(K^0))w^j}{\pi_1(K^0)} & \text{if } w^j \geq w(\widehat{u}) \end{cases}$$

and  $r_{ij} = w^j$

Here we also want to prove that for any level of  $w^j \geq w^{\min}(\widehat{u})$ ,  $\widehat{K}(w^j) \geq K(w^j)$ . First of all we know that,  $\widehat{K}(w^j) > \overline{K}$ . Comparing (9) and (17), it is clear that proving  $\widehat{K}(w^j) \geq K(w^j)$  is equivalent to showing that  $\pi_1(\widehat{K})y - \widehat{K} - \widehat{u} \geq 1$ . Suppose that  $w^{\min}(\widehat{u}) \leq \overline{w}$ . Then  $\widehat{u}$  is given by

$$\widehat{u} = \pi_1(\overline{K})y - \frac{\pi_1(\overline{K})}{\pi_1(\overline{K}) - \pi_0(\overline{K})} + w^{\min}(\widehat{u}) - \overline{K}.$$

Using the above together with (6), it is easy to see that  $\pi_1(\widehat{K})y - \widehat{K} - \widehat{u} > 1$ . This also proves that  $w(\widehat{u}) \leq w^0$ . We now do the same considering  $w^{\min}(\widehat{u}) > \overline{w}$ . Notice that, in this case  $\widehat{u} = \pi_1(K(w^{\min}(\widehat{u})))y - K(w^{\min}(\widehat{u}))$ . Also,  $[\pi_1(\widehat{K})y - \widehat{K}] - [\pi_1(K(w^{\min}(\widehat{u})))y - K(w^{\min}(\widehat{u}))] > 0$ , since investment is increasing in wealth. These previous two facts imply the above assertion that  $\widehat{K}(w^j) \geq K(w^j)$  for all  $w^j \geq w^{\min}(\widehat{u})$ .

#### D. The Case when $\pi_1(K)\pi'_0(K) < \pi'_1(K)\pi_0(K)$

In the paper we have analysed our model under the assumption that  $\pi_1\pi'_0 > \pi'_1\pi_0$ . We also asserted that, all the qualitative results of our model would hold good under the assumption that  $\pi_1\pi'_0 < \pi'_1\pi_0$ . Under this assumption, the findings in Appendix A imply  $\overline{K} > K(w^j) > K^0$  and  $K(w^j)$  is decreasing for  $w^j \in (\overline{w}, w^0)$ . The reason behind this is the following. When  $\pi_1(K)$  is increasing relative to  $\pi_0(K)$ , for a high level of initial investment, giving incentives is much easier. Because of this, for low level of wealth, the landlord gives over incentives to the tenant by lending more money (equivalently, the optimal investment is higher). Similarly, under this assumption, the findings of Appendix C imply that  $\widehat{K}(w^j; \widehat{u}) > K^0$  for  $w^j > w(\widehat{u})$ .

#### E. Proof of Theorem 2

Consider  $m > n$ . First we prove that each SPE outcome constitutes an equilibrium. We



do that through several claims. **(a)** At any SPE, the contracts accepted are (among) the ones yielding the highest utility to the landlords. Otherwise, a landlord accepting a contract that yields lower utility would have incentives to switch to a better contract that has not already been taken. **(b)** At any SPE, all the contracts that are accepted provide the same utility to all the landlords. Otherwise, on the contrary, consider one of the (at most  $n - 1$ ) contracts that gives the maximum utility to the landlords. If one of the tenants slightly decreases the payments offered at the first stage, his contract will still be accepted at any Nash equilibrium (NE) of the second-stage game for the new set of offers (because of **(a)**). **(c)** At any SPE, precisely  $n$  contracts are accepted. To see this, suppose on the contrary that at most  $n - 1$  contracts are accepted. Then there is a (unmatched) landlord with zero utility. This is not possible since **(b)** holds. **(d)** The contracts that are finally accepted are those offered by the wealthiest tenants. Suppose  $w^k > w^j$  and the contract offered by  $w^j$  is accepted, but not the one by  $w^k$ . Then  $w^k$  can offer a slightly better (for the landlords) contract than  $s^j$ . Given **(a)**-**(c)**, this new contract will be accepted at any NE of the second-stage game. This is a contradiction. **(e)** All the contracts signed are optimal. Otherwise, a tenant offering a non-optimal contract could improve it for both (any landlord and himself). This new contract will certainly be among the  $n$  best contracts for the landlords (since the previous contract was) and hence, will be accepted at any SPE outcome. **(f)** Finally, any SPE outcome constitutes an equilibrium. It only remains to prove that the common utility level of the landlords at an SPE, denoted by  $\hat{u}$ , lies in  $[u_i(w^{n+1}, \bar{c}^{n+1}), u_i(w^n, \bar{c}^n)]$ . First,  $\hat{u} \leq u_i(w^n, \bar{c}^n)$ , because otherwise, some tenants would be better-off by not offering any contract. Secondly,  $\hat{u} \geq u_i(w^{n+1}, \bar{c}^{n+1})$  for tenant  $w^{n+1}$  not to have incentives to propose a contract that would have been accepted.

We now prove that any equilibrium outcome can be supported by an SPE strategy. Let  $(\mu, \mathcal{C})$  constitutes an equilibrium where each landlord gets utility  $\hat{u}$ . Consider the following strategies of each tenant  $w^j$  for all  $j$  and of each landlord  $l_i$  for all  $i$ :

$$\hat{s}^j = \begin{cases} c_{\mu(w^j)} & \text{if } \mu(w^j) \in \mathcal{L} \\ \hat{c} \text{ s.t. } u_i(w^j, \hat{c}) = \hat{u} & \text{; for any } l_i \in \mathcal{L}, \text{ otherwise.} \end{cases}$$

And  $\hat{s}_i = \mu(l_i)$  if  $\hat{s}$  is played in the first stage. Otherwise, landlords select any tenant compatible with an NE in pure strategies given any other announcement  $s$  in the first period. These strategies constitute an SPE yielding the equilibrium outcome  $(\mu, \mathcal{C})$ . To see this, notice that given any message  $s^j \neq \hat{s}^j$ , landlords play their NE strategies. Given that  $\hat{s}$  is played in the first stage, by deviating any landlord  $l_i$  cannot gain more than  $\hat{u}$ . This is true because any contract offered in the first stage yields the same utility  $\hat{u}$  to any landlord. Now

consider deviations by the tenants. Given that  $\hat{u} \geq u_i(w^{n+1}, \bar{c}^{n+1})$ , by the equilibrium conditions, there does not exist any contract that would be offered by an unmatched tenant that guarantees him a positive utility while yielding at least  $\hat{u}$  to a landlord. Hence, unmatched tenants do not have incentives to deviate. Also, given the efficiency of the contracts in a stable outcome, there does not exist a different contract that a matched tenant could offer at which he could have strictly improved while still guaranteeing at least  $\hat{u}$  to the landlords. If there is a plethora of contracts that yields utility  $\hat{u}$  to the landlords, it is easy to check that there is no NE of the game at which a contract providing utility lower than  $\hat{u}$  is accepted by a landlord. Hence, the matched tenants do not have any incentive to deviate from  $\hat{s}$ .

To prove that the equilibrium outcomes can be supported by an SPE strategy, let  $(\mu, \mathcal{C})$  be an equilibrium outcome where each landlord gets utility  $\hat{u}$ . Consider the following strategies of each tenant  $w^j$  for all  $j$  and of each landlord  $l_i$  for all  $i$ :

$$\hat{s}^j = c^j(0) \text{ for any } w^j$$

And  $\hat{s}_i = \mu(l_i)$  if  $\hat{s}$  is played in the first stage. Otherwise, landlords select any tenant compatible with an NE in pure strategies given any other message  $s$  sent in the first period.

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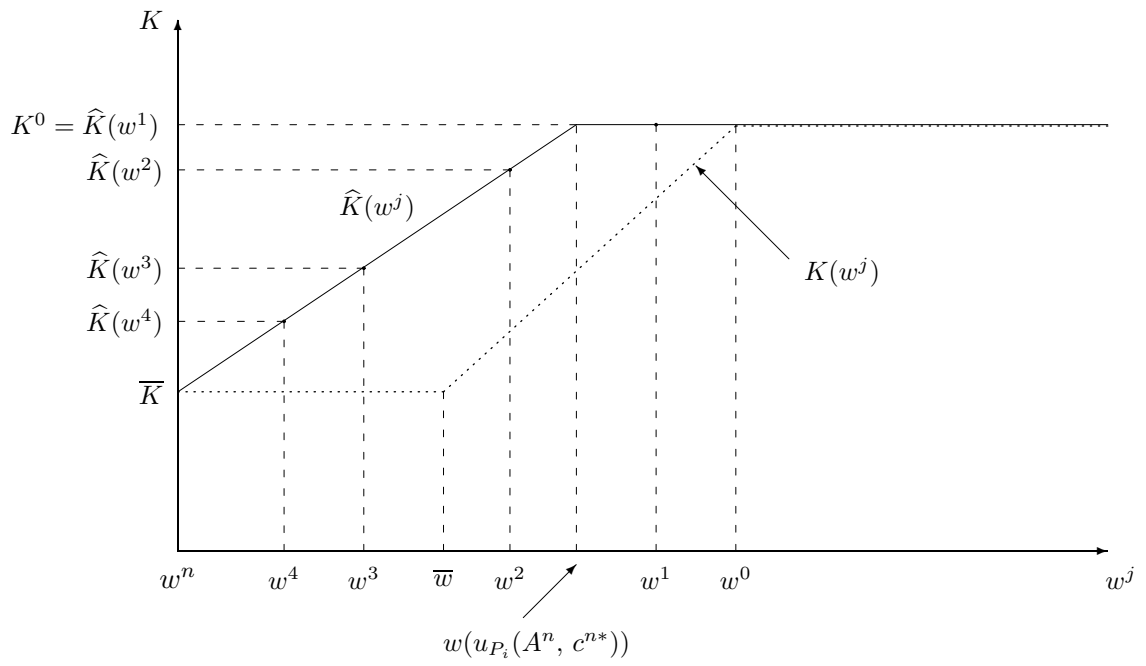


Figure 1: THE ENDOGENOUS INVESTMENT LEVELS WHEN  $w^n = w^{n+1} = 0$

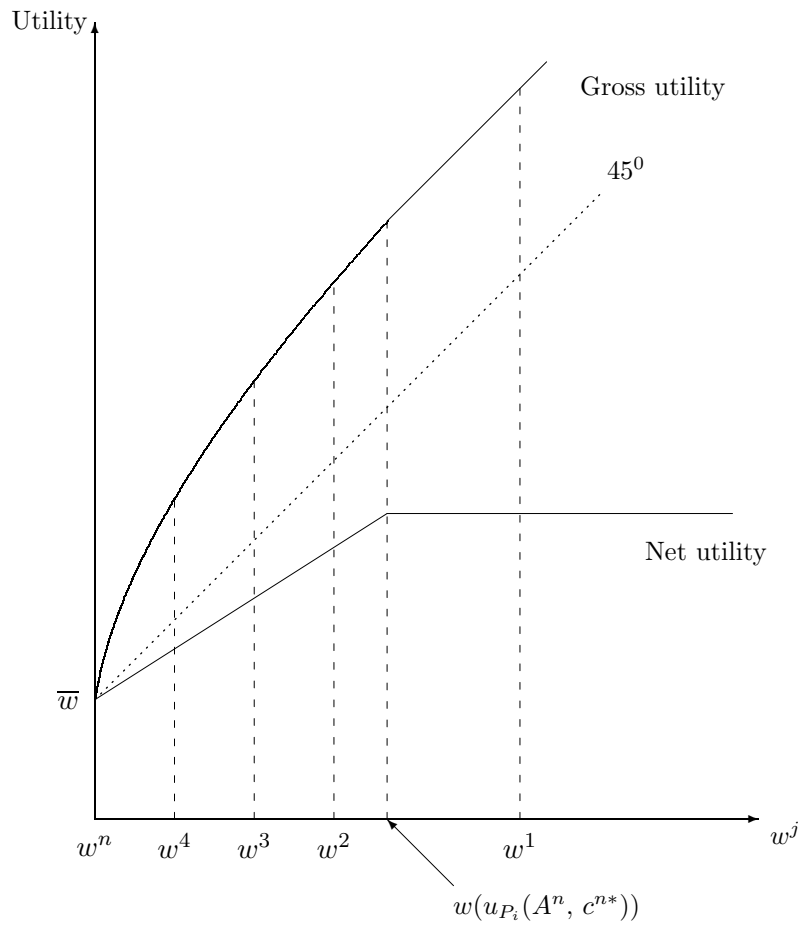


Figure 2: GROSS AND NET UTILITIES OF A TENANT

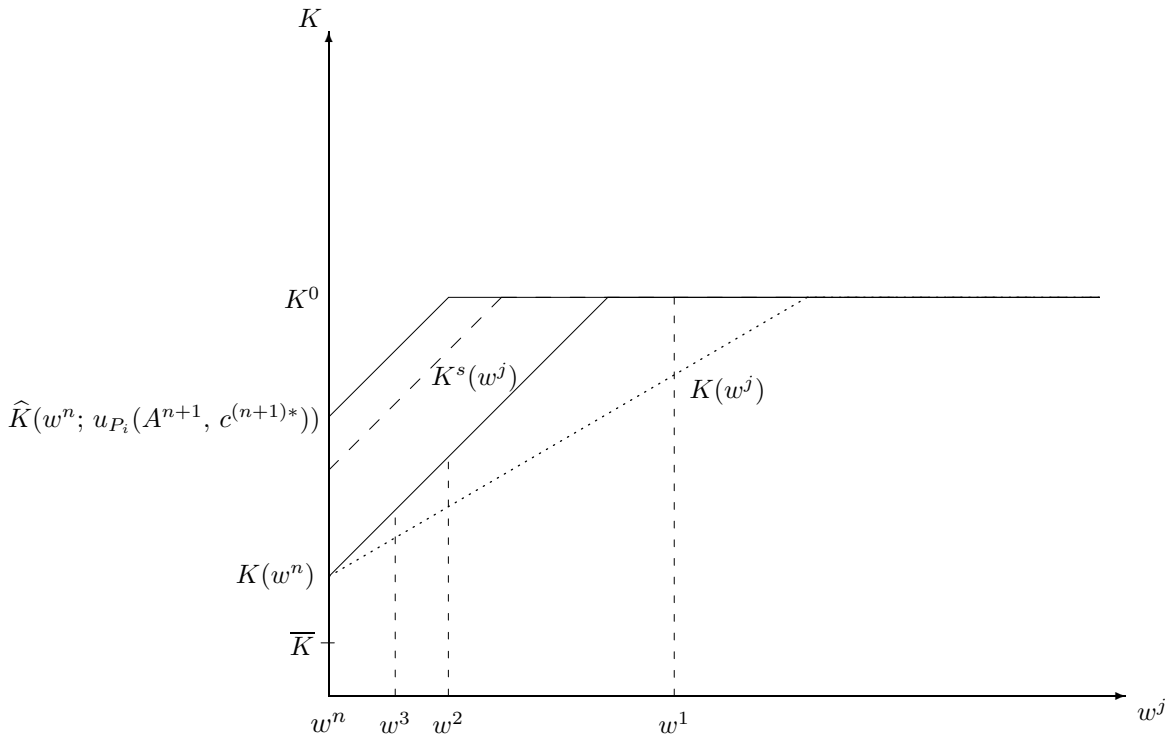


Figure 3: OPTIMAL INVESTMENT WHEN  $w^n > w^{n+1}$