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Abstract

We develop a geometric procedure to get all correlated equilibria in a 2×2 game. With this procedure we can actually 'see' all the correlated strategy profiles of a given game and compare it to the convex hull of the Nash equilibrium profiles. Games without dominant strategies fall into two different equivalence classes: (i) competitive games, that have a unique correlated equilibrium strategy, and (ii) coordination and anticoordination games, whose set of correlated equilibria is a polytope with five vertices for which we provide general closed-form expressions. In this latter case, there are either three or four vertices for the payoffs. In contrast, the convex hull of the Nash equilibrium strategies and payoffs always have three vertices.

1 Introduction

In his 1974's seminal paper, Aumann introduces the concept of correlated equilibrium that generalizes the non-cooperative equilibrium notion due to Nash by allowing players to communicate before their play. More precisely, a correlated equilibrium of a given game is a Nash equilibrium of the associated enlarged game where players receive private signals not affecting payoffs before the play, and make their choices based on the signals they have received. For finite games in normal form, a correlated equilibrium can be described by means of a distribution over strategy profiles, a correlated strategy.

Correlated equilibria have many appealing properties.[1] In particular, the set of correlated

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[1] See, e.g., Aumann (1987) for rationality foundations, Chwe (2000) for issues of identification, Hart and Mas-Colell (2000) for the link between correlated equilibrium and adaptive dynamics, Mailath *et al.* (1997) for an interpretation of correlated equilibria as Nash equilibria of local-interaction games and Myerson (1986) for a refinement of this equilibrium concept, etc.

equilibrium strategies is a non-empty (compact and convex) polytope which contains the convex hull of the Nash equilibria as a subset.[2] However, determining exactly how both sets, and the resulting sets of payoffs, differ is still an open question.[3]

Here, we focus on the particular case of 2×2 games. For such games, we provide a complete characterization of the set of correlated equilibrium strategies and payoffs, and relate those sets to the convex hull of the set of Nash equilibrium strategies and payoffs. More precisely, games without dominant strategies fall into three different equivalence classes for the set of correlated equilibrium strategies: competitive games, coordination games, and anticoordination games.[4] For competitive games, the correlated equilibria and Nash equilibria sets coincide and are reduced to one single point. For coordination and anticoordination games, the set of correlated equilibria is a polytope with five vertices for which we provide general closed-form expressions.[5] In contrast, the convex hull of the Nash equilibria has only three vertices. For such games, the set of correlated equilibrium payoffs is a polytope with either three or four vertices while, again, the convex hull of Nash equilibrium payoffs has always three vertices.

Our approach is geometric. First, we propose a planar geometric representation of correlated strategies of 2×2 games. Then, we develop a geometric procedure to get all correlated equilibria of any 2×2 game. With this procedure we can actually 'see' all the correlated equilibrium strategies of a given game and compare it to the convex hull of the Nash equilibrium strategies.

Section 2 presents all the results and all the proofs are gathered in Section 3.

2 The set of correlated equilibria of 2×2 games

Correlated equilibrium strategies of finite games Let $\gamma = \left(N, (S_i)_{i \in N}, (u_i)_{i \in N}\right)$ be a finite $n-$person game in strategic form. Let $S = \times_{i \in N} S_i$ denote the set of $n-$tuples strategies of γ and $\Delta(S)$ denote the set of probability distributions on S.

The original definition of a correlated equilibrium is in Aumann (1974). Here, we focus on the equivalent definition in terms of correlated strategies provided in Aumann (1987), where a

[2] A polytope is the convex hull of finitely many elements of $\mathbb{R}^k$. It is thus compact and convex.. Alternatively, a polytope is a bounded polyhedron, where a polyhedron is the non-empty intersection of finitely many halfspaces. A polyhedron is closed and convex.

[3] To quote Fudenberg and Tirole (1995): "One might also like to know when the set of correlated equilibria differs 'greatly' from the convex hull of the Nash equilibria, but his question has not yet been answered" (p. 58). Partial results relating both sets have been achieved so far. See, e.g., Evangelista and Raghavan (1996) for results relating the extreme points of both equilibrium sets, and Moulin and Vial (1978) for results regarding payoffs.

[4] Games that are dominant solvable have only one Nash equilibrium in pure strategies which coincides with the unique (degenerated) correlated equilibrium strategy of the game. We deal with games with weakly dominant strategies at the end of Section 2.

[5] In fact, the set of correlated equilibrium strategies for coordination and anticoordination games are isomorphic to each other, and both types of games may well be gathered into one single equivalence class. Yet, the distinction is useful for computational purposes.

correlated strategy is a point in $\Delta(S)$.

Definition 1 *A correlated strategy $\mu \in \Delta(S)$ is a correlated equilibrium of γ if and only if, for all $i \in N$ and $s = (s_i, s_{-i}) \in S$,*

$$\sum_{s_{-i} \in S_{-i}} \mu(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \mu(s_i, s_{-i}) u_i(s_i', s_{-i}), \text{ for all } s_i' \in S_i.$$

The set $CE(\gamma)$ of correlated equilibrium strategies of γ is defined by a finite number of linear inequalities (incentive constraints) on the set $\Delta(S)$ of correlated strategies. For finite games, $CE(\gamma)$ is a non-empty polytope which contains the convex hull of all the Nash equilibria $NE(\gamma)$.[6] Correlated strategies that belong to the intersection $CE(\gamma) \cap NE(\gamma)$ are product measures.

From now on, we focus on 2×2 games. Let $N = \{a, b\}$ be the set of players and $S_a = S_b = \{0, 1\}$ the set of strategies. Given two matrices $A = (a_{ij})_{(i,j) \in S_a \times S_b}$ and $B = (b_{ij})_{(i,j) \in S_a \times S_b}$, we denote by $\gamma(A, B)$ the 2×2 game with the following payoffs matrix:

	0	1
0	a_{00}, b_{00}	a_{01}, b_{01}
1	a_{10}, b_{10}	a_{11}, b_{11}

Let $\Delta_3 = \left\{(x_1, ..., x_4) \in \mathbb{R}_+^4 \mid x_1 + ... + x_4 = 1\right\}$ denote the 3−dimensional simplex of $\mathbb{R}^4$. A correlated strategy $\mu = (\mu_{00}, \mu_{11}, \mu_{10}, \mu_{01}) \in \Delta_3$ can be represented the following way:

μ_{00}	μ_{01}
μ_{10}	μ_{11}

where μ_{00} and μ_{11} are the probabilities assigned to the symmetric (diagonal) strategy profiles $(0, 0)$ and $(1, 1)$, respectively, while μ_{10} and μ_{01} are the probabilities corresponding to the asymmetric (out-of-diagonal) profiles $(1, 0)$ and $(0, 1)$, respectively.

With a straightforward application of Definition 1, $\mu \in \Delta_3$ is a correlated equilibrium strategy of $\gamma(A, B)$ if and only if:

$$\begin{cases} \mu_{00}(a_{00} - a_{10}) \geq \mu_{01}(a_{11} - a_{01}) \\ \mu_{11}(a_{11} - a_{01}) \geq \mu_{10}(a_{00} - a_{10}) \\ \mu_{00}(b_{00} - b_{01}) \geq \mu_{10}(b_{11} - b_{10}) \\ \mu_{11}(b_{11} - b_{10}) \geq \mu_{01}(b_{00} - b_{01}) \end{cases} \tag{1}$$

[6] Non-emptiness of $CE(\gamma)$, thus, results from non-emptiness of $NE(\gamma)$. Hart and Schmeidler (1989) provide a direct existence proof for correlated equilibria, both for finite and infinite games, that relies on linear duality rather than on standard fixed-point arguments.

Equivalence classes for 2×2 games Denote by $\mathcal{G}$ the set of 2×2 games, and by $\mathcal{G}^{\circ}$ the set of 2×2 games without neither weakly nor strictly dominated strategies. We first focus on $\mathcal{G}^{\circ}$ and leave for the end of this section the case of games in $\mathcal{G} \backslash \mathcal{G}^{\circ}$.

For all $\gamma(A, B) \in \mathcal{G}^{\circ}$, let $\alpha = |a_{00} - a_{10}| / |a_{11} - a_{01}|$ and $\beta = |b_{00} - b_{01}| / |b_{11} - b_{10}|$. The requirement that $\gamma(A, B)$ does not possess any dominated strategy implies that $a_{00} \neq a_{10}$, $a_{11} \neq a_{01}$, $b_{00} \neq b_{01}$ and $b_{11} \neq b_{10}$. Therefore, both α and β are well-defined and $\alpha, \beta > 0$.

Define the three following games:

	0	1
0	α, β	$0, 0$
1	$0, 0$	$1, 1$

$\gamma_{\mathrm{I}}(\alpha, \beta)$: *coordination*

	0	1
0	$-\alpha, -\beta$	$0, 0$
1	$0, 0$	$-1, -1$

$\gamma_{\mathrm{II}}(\alpha, \beta)$: *anticoordination*

	0	1
0	$-\alpha, \beta$	$0, 0$
1	$0, 0$	$-1, 1$

$\gamma_{\mathrm{III}}(\alpha, \beta)$: *competitive*

Lemma 1 (equivalence classes) *Let $\gamma(A, B) \in \mathcal{G}^{\circ}$. Then, $CE(\gamma(A, B)) = CE(\gamma_{\ell}(\alpha, \beta))$, for some $\ell \in \{I, II, III\}$.*

The set of correlated equilibria of 2×2 games without dominated strategies can thus be partitioned into three equivalence classes for the set of correlated equilibrium strategies, corresponding to γ_{I}, γ_{II}, and γ_{III}–type games.[7] The criteria to assign some given game $\gamma(A, B) \in \mathcal{G}^{\circ}$ to its corresponding equivalence class are the following:

$$CE(\gamma(A, B)) = \begin{cases} CE(\gamma_{\mathrm{I}}(\alpha, \beta)) \text{ if and only if } a_{00} > a_{10}, a_{11} > a_{01}, b_{00} > b_{01}, \text{and } b_{11} > b_{10}; \\ CE(\gamma_{\mathrm{II}}(\alpha, \beta)) \text{ if and only if } a_{00} < a_{10}, a_{11} < a_{01}, b_{00} < b_{01}, \text{and } b_{11} < b_{10}; \\ CE(\gamma_{\mathrm{III}}(\alpha, \beta)) \text{ otherwise.} \end{cases}$$

We now characterize $CE(\gamma_{\ell}(\alpha, \beta))$, for all $\alpha, \beta > 0$ and $\ell \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$.

It is readily checked that $CE(\gamma_{\mathrm{III}}) = NE(\gamma_{\mathrm{III}})$, that is, the set of correlated equilibria and Nash equilibria of γ_{III} coincide and are reduced to one single point in Δ_3.[8] We thus focus on $CE(\gamma_{\mathrm{I}})$ and $CE(\gamma_{\mathrm{II}})$.

[7] Germano (2003) proposes a general procedure to obtain equivalence classes for general normal-form finite games that applies to different equilibrium concepts and based on the geometry of the correspondences of the equilibrium concept being considered. Germano's classification applied to the correlated equilibria of 2×2 games yields three different classes: (*i*) games that are dominance solvable (that we leave for the end of this section), (*ii*) γ_{III}–type games, and (*iii*) γ_{I} and γ_{II}–type games. For our purpose, although γ_{I} and γ_{II}–type games are equivalent up to a relabelling of the actions available to the players, the distinction we make is still useful as it allows to distinguish between diagonal and out-of-diagonal probability values for correlated strategies, which has some implications for the geometry of the equilibrium set and payoffs. Lemma 2 makes this point more precise.

[8] The set of Nash equilibria of $\gamma_{\mathrm{III}}(\alpha, \beta)$ is restricted to the mixed strategy: play 0 with probability $1/(1 + \alpha)$ for the column player and $1/(1 + \beta)$ for the row player. The corresponding correlated equilibrium distribution is the product measure of the mixed strategy play. Moulin and Vial (1978) refer to such games as *strategically zero-sum games.*

For all $z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4$, let $\tau(z) = (z_3, z_4, z_1, z_2)$, so that τ is a permutation swapping the first two coordinates with the last two coordinates while preserving the inner order for the pairs of coordinates being swapped.

Lemma 2 (duality) *Let $\alpha, \beta > 0$. Then, $\mu \in CE(\gamma_I(\alpha, \beta))$ if and only if $\tau(\mu) \in CE(\gamma_{II}(\alpha, 1/\beta))$.*

In words, given a correlated equilibrium strategy for some coordination game with parameters α and β, the strategy obtained by swapping the diagonal probabilities with the out-of-diagonal terms is a correlated equilibrium strategy for the anticoordination game with parameters α and $1/\beta$. And vice-versa. The class of coordination games and the class of anticoordination game are thus isomorphic to each other. We restrict our analysis to the former.

The set of correlated equilibrium strategies of 2×2 coordination games Let $\alpha, \beta > 0$ and consider $\gamma_{\mathrm{I}}(\alpha, \beta)$. This game has three Nash equilibria corresponding to the following correlated strategy profiles:

1	0
0	0

NE in pure strategies $(0,0)$

0	0
0	1

NE in pure strategies $(1,1)$

$\frac{1}{(1+\alpha)(1+\beta)}$	$\frac{\alpha}{(1+\alpha)(1+\beta)}$
$\frac{\beta}{(1+\alpha)(1+\beta)}$	$\frac{\alpha\beta}{(1+\alpha)(1+\beta)}$

NE in mixed strategies

The set of Nash equilibrium strategies of γ_{I}, denoted by $NE(\gamma_{\mathrm{I}})$, and the convex hull of this set, denoted by $cov(NE(\gamma_{\mathrm{I}}))$, are all correlated equilibrium strategies of γ_{I}, that is, $cov(NE(\gamma_{\mathrm{I}})) \subset CE(\gamma_{\mathrm{I}})$. But the inclusion may be strict. We first provide a complete characterization of $CE(\gamma_{\mathrm{I}})$ and then compare both sets.

For all $\alpha, \beta > 0$, define the following sets:

$$\mathcal{D}(\alpha, \beta) = \left\{(x, y) \in \mathbb{R}^2_+ \mid x + y + \min\{\beta x; y/\alpha\} + \min\{\alpha x; y/\beta\} \geq 1\right\}$$
$$\mathcal{O}(\alpha, \beta) = \left\{(x, y) \in \mathbb{R}^2_+ \mid x + y + \max\{x/\beta; y/\alpha\} + \max\{\alpha x; \beta y\} \leq 1\right\}$$

We have the following result.

Proposition 1 *Let $\alpha, \beta > 0$. Then, $CE(\gamma_I(\alpha, \beta)) = (\mathcal{D}(\alpha, \beta) \times \mathcal{O}(\alpha, \beta)) \cap \Delta_3$.*

Consider the following table.

μ	μ_{00}	μ_{11}	μ_{10}	μ_{01}
$\mu^*_C(\alpha, \beta)$	1	0	0	0
$\mu^*_D(\alpha, \beta)$	0	1	0	0
$\mu^*_E(\alpha, \beta)$	$\frac{1}{(1+\alpha)(1+\beta)}$	$\frac{\alpha\beta}{(1+\alpha)(1+\beta)}$	$\frac{\beta}{(1+\alpha)(1+\beta)}$	$\frac{\alpha}{(1+\alpha)(1+\beta)}$
$\mu^*_F(\alpha, \beta)$	$\frac{1}{1+\beta+\alpha\beta}$	$\frac{\alpha\beta}{1+\beta+\alpha\beta}$	$\frac{\beta}{1+\beta+\alpha\beta}$	0
$\mu^*_G(\alpha, \beta)$	$\frac{1}{1+\alpha+\alpha\beta}$	$\frac{\alpha\beta}{1+\alpha+\alpha\beta}$	0	$\frac{\alpha}{1+\alpha+\alpha\beta}$

(2)

A geometric characterization of the set of correlated equilibria that highlights the differences between $NE(\gamma_{\mathrm{I}})$ and $CE(\gamma_{\mathrm{I}})$ is the following.

Proposition 2 *$CE(\gamma_I(\alpha,\beta))$ is a polytope of Δ_3 with five vertices defined in (2).*[9]

Note that the correlated strategies $\mu^*_C(\alpha,\beta)$, $\mu^*_D(\alpha,\beta)$ and $\mu^*_E(\alpha,\beta)$, that correspond to the Nash equilibria of $\gamma_I(\alpha,\beta)$, are all vertices of $CE(\gamma_I(\alpha,\beta))$. This is consistent with Evangelista and Raghavan (1996) who establish that every extreme point of a maximal Nash set is an extreme point of the set of correlated equilibria. $\mu^*_F(\alpha,\beta)$ and $\mu^*_G(\alpha,\beta)$ are additional vertices of the correlated equilibrium set not belonging to the convex hull of the Nash equilibria.

Propositions 2 provides a simple tool to construct the correlated equilibrium polytope of any coordination game $\gamma(A,B) \in \mathcal{G}^\circ$. It suffices to compute α and β and get the five vertices of $CE(\gamma(A,B))$ with (2). If $\gamma(A,B) \in \mathcal{G}^\circ$ is an anticoordination game, the duality result in Lemma 2 implies that the vertices of $CE(\gamma(A,B))$ are the same that in (2) where β is replaced by $1/\beta$ and the out-of-diagonal probabilities are swapped with the diagonal terms.

Example 1. Consider the following game, $\gamma^{chicken}$, known as the 'chicken game':

	0	1
0	$6,6$	$2,7$
1	$7,2$	$0,0$

This game falls into the equivalence class of general anticoordination games, with $\alpha = 0.5 = \beta$. The five vertices of the correlated equilibrium polytope are the following.

μ	μ_{00}	μ_{11}	μ_{10}	μ_{01}
μ^*_C	0	0	1	0
μ^*_D	0	0	0	1
μ^*_E	$\frac{4}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$
μ^*_F	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
μ^*_G	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

[9] A point of a polyhedron is called an extreme point or a *vertex* if it is not a convex combination of two other elements of the polyhedron.

Figure 1 shows the set of correlated equilibrium strategies of $\gamma^{chicken}$ on Δ_3.

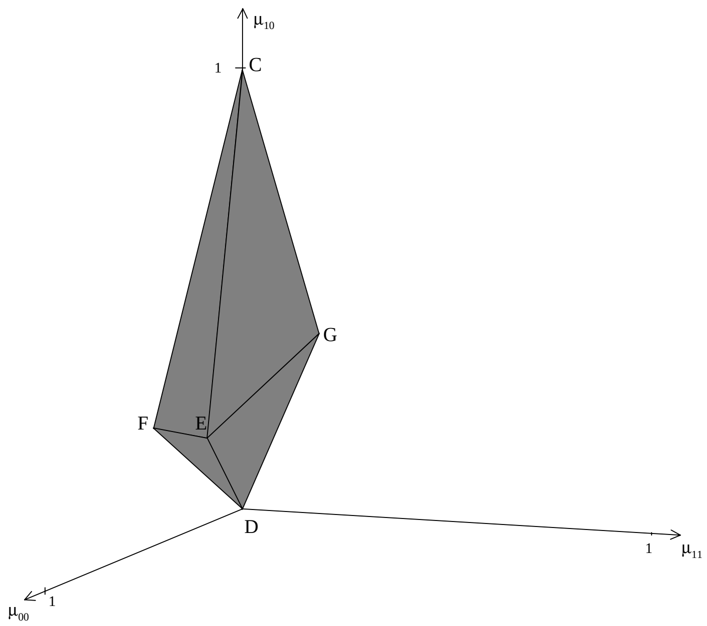

Figure 1. The correlated equilibrium set of $\gamma^{chicken}$.

In Figure 1, C and D correspond to the two Nash equilibria in pure strategies of $\gamma^{chicken}$, while E is the Nash equilibrium in (completely) mixed strategies. The convex hull of the Nash equilibria corresponds to the 'slice' CDE of $CE\left(\gamma^{chicken}\right)$. The vertices F and G are correlated equilibria not belonging to this convex hull.

The geometry of correlated equilibrium payoffs of 2×2 coordination games Let $\gamma(A,B) \in \mathcal{G}^{\circ}$ such that $CE(\gamma(A,B)) = CE(\gamma_{\mathrm{I}}(\alpha,\beta))$. For all $\mu \in CE(\gamma_{\mathrm{I}}(\alpha,\beta))$ denote by $u(\mu) = (u_a(\mu), u_b(\mu))$ the correlated equilibrium payoffs for players a and b. We have:

$$\begin{cases} u_a(\mu) = \mu_{00}a_{00} + \mu_{11}a_{11} + \mu_{10}a_{10} + \mu_{01}a_{01} \\ u_b(\mu) = \mu_{00}b_{00} + \mu_{11}b_{11} + \mu_{10}b_{10} + \mu_{01}b_{01} \end{cases} \tag{3}$$

Denote by $CEP(\gamma(A,B)) = \left\{(u_a(\mu), u_b(\mu)) \in \mathbb{R}^2 \mid \mu \in CE(\gamma(A,B))\right\}$ the set of correlated equilibrium payoffs of $\gamma(A,B)$. We have the following result.

Proposition 3 *Let $\gamma(A,B) \in \mathcal{G}^{\circ}$ be a coordination game. Then, $CEP(\gamma(A,B))$ is a polytope of $\mathbb{R}^2$ with either three or four different vertices. The pure Nash equilbrium payoffs $u(\mu^*_C(\alpha,\beta))$ and $u(\mu^*_D(\alpha,\beta))$ are always vertices, and either $u(\mu^*_F(\alpha,\beta))$, or $u(\mu^*_G(\alpha,\beta))$ or both are the other vertices.*

A sketch of the proof is as follows. First, any vertex of the polytope of payoffs is necessarily obtained with a vertex of the polytope of strategies. Therefore, $CEP(\gamma(A,B))$ has at most five different vertices. Second, by Proposition 2, the completely mixed strategy Nash equilibrium $\mu_E^*(\alpha,\beta)$ is a vertex of $CE(\gamma(A,B))$. Yet, it is a standard result that the mixed equilibrium payoffs of coordination games are (strictly) Pareto-dominated by some correlated equilibrium payoffs.[10] Observing that the same result applies to a suitable transformation of $\gamma(A,B)$, we deduce that mixed equilibrium payoffs is an interior point of the polytope of correlated equilibrium payoffs. Moreover, the mixed Nash equilibrium payoffs $u(\mu_E^*(\alpha,\beta))$ and the purely correlated payoffs $u(\mu_F^*(\alpha,\beta))$ and $u(\mu_G^*(\alpha,\beta))$ are Pareto-ranked, with the correlated payoffs being the extremal ones.

Propositions 2 and 3 altogether provide simple tools to construct the polytope of the correlated equilibrium strategies and payoffs of any 2×2 coordination game. Together with Lemma 2, they also cover the family of 2×2 coordination games.

Example 2. The polytope of correlated equilibrium payoffs of $\gamma^{chicken}$ is displayed in Figure 2 −grey-shaded area. The convex hull of the Nash equilibrium payoffs, delineated with a dashed line, is included in $CEP\left(\gamma^{chicken}\right)$.

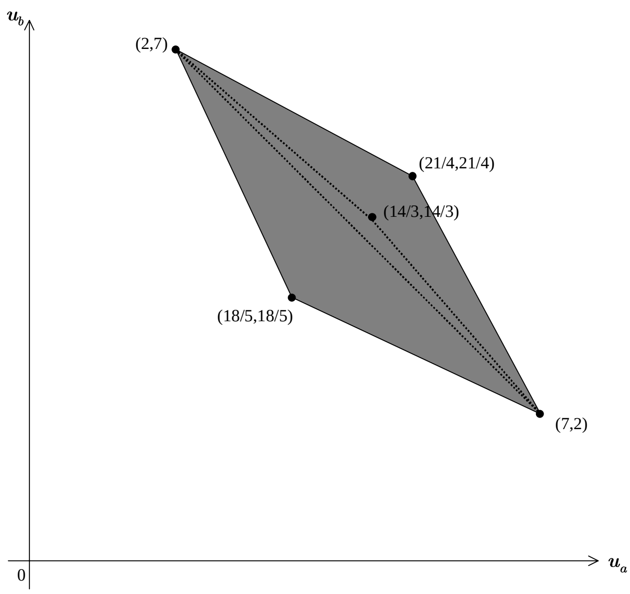

Figure 2. The correlated equilibrium payoffs of $\gamma^{chicken}$.

Computing the vertices of the correlated equilibrium polytope with (2) is a crucial step to identifying the set of payoffs that can be achieved with a correlated equilibrium in 2×2 games. Identifying this set of payoffs, in turn, is crucial to distinguish between the payoffs belonging to the convex hull of the Nash equilibrium payoffs and that can be achieved through jointly controlled lotteries,[11] from the 'purely' correlated equilibrium payoffs that require other types of mechanisms

[10] See, e.g., Moulin and Vial (1978).
[11] See Aumann and Maschler (1995) for details on jointly controlled lotteries.

to be implemented.[12]

Games with dominated strategies It is straightforward to check that strictly dominated strategies cannot be used with positive probability in any correlated equilibrium of a finite game. The set of correlated equilibria of a dominance solvable 2×2 game thus coincides with the set of Nash equilibria of that game.

Consider now the case of weakly dominated strategies. Games with weakly dominated strategies are such that either $\alpha = 0$, or $\beta = 0$ or both. Denote by $\overline{\mathcal{D}}(\alpha, \beta)$ and $\overline{\mathcal{O}}(\alpha, \beta)$ the extensions of $\mathcal{D}(\alpha, \beta)$ and $\mathcal{O}(\alpha, \beta)$, respectively, to $\mathbb{R}_+^2$ defined as follows:

$$\begin{cases} \overline{\mathcal{D}}(\alpha, \beta) = \mathcal{D}(\alpha, \beta), \text{ for all } \alpha, \beta > 0 \\ \overline{\mathcal{D}}(0, \beta) = \left\{(x, y) \in \mathbb{R}_+^2 \mid (1 + \beta) x + y \geq 1\right\} \end{cases}$$

and the natural extension to $\beta = 0$,

$$\begin{cases} \overline{\mathcal{O}}(\alpha, \beta) = \mathcal{O}(\alpha, \beta), \text{ for all } \alpha, \beta > 0 \\ \overline{\mathcal{O}}(0, \beta) = \left\{(x, y) \in \mathbb{R}_+^2 \mid (1 + 1/\beta) x + (1 + \beta) y \leq 1\right\} \\ \overline{\mathcal{O}}(\alpha, 0) = \left\{(x, y) \in \mathbb{R}_+^2 \mid (1 + \alpha) x + (1 + 1/\alpha) y \leq 1\right\} \\ \overline{\mathcal{O}}(0, 0) = \left\{(x, y) \in \mathbb{R}_+^2 \mid x + y \leq 1\right\} \end{cases}$$

We have the following result.

Corollary 1 *Let $\alpha, \beta \geq 0$. Then,* $CE\left(\gamma_I(\alpha, \beta)\right) = \left(\overline{\mathcal{D}}(\alpha, \beta) \times \overline{\mathcal{O}}(\alpha, \beta)\right) \cap \Delta_3$.

Let $\gamma(A, B) \in \mathcal{G} \backslash \mathcal{G}^\circ$ without strictly dominant strategies. Therefore, $\gamma(A, B)$ has weakly dominant strategies. Small perturbations of payoffs yield a game in $\mathcal{G}^\circ$, without neither weakly nor strictly dominant strategies. Proposition 1, together with Lemma 2, and the criteria for assigning an equivalence class to any game in $\mathcal{G}^\circ$ provide a full characterization of the set of correlated equilibrium strategies for the perturbed game. $CE(\gamma(A, B))$ is then obtained by a limit argument as the payoffs' perturbations vanish.[13]

[12] Such as, for instance, the long mediated talk of Lehrer (1994), or the one-shot public mediated talk of Lehrer and Sorin (1997).

[13] Suppose, for instance, that $\gamma(A, B)$ is a general coordination game except that $a_{00} = a_{10}$, that is, 0 is a weakly dominated strategy for player a. The set of correlated strategies is then deduced from Figure 4 in Section 3 by letting $\alpha = 0$. In particular, $\mu_{01} = 0$ for any $\mu \in CE(\gamma(A, B))$. In the language of Myerson (1986), the strategy profile $(0, 1)$, that includes the weakly dominated strategy for a, is an *unacceptable* action.

3 Proofs

Proof of Lemma 1: Suppose first that $a_{00} > a_{10}$, $a_{11} > a_{01}$, $b_{00} > b_{01}$, and $b_{11} > b_{10}$. Then, (1) is equivalent to

$$\begin{cases} \mu_{00}\alpha \geq \mu_{01} \\ \mu_{11} \geq \mu_{10}\alpha \\ \mu_{00}\beta \geq \mu_{10} \\ \mu_{11} \geq \mu_{01}\beta \end{cases}, \tag{4}$$

implying that $CE(\gamma(A,B)) = CE(\gamma_{\mathrm{I}}(\alpha,\beta))$. The cases where $a_{00} < a_{10}$, $a_{11} < a_{01}$, $b_{00} < b_{01}$, and $b_{11} < b_{10}$, or $a_{00} < a_{10}$, $a_{11} < a_{01}$, $b_{00} > b_{01}$, and $b_{11} > b_{10}$, are similar. ∎

Proof of Lemma 2: Let $\mu = (\mu_{00}, \mu_{11}, \mu_{10}, \mu_{01}) \in CE(\gamma_{\mathrm{I}}(\alpha,\beta))$. Specializing (1), this is equivalent to

$$\begin{cases} \mu_{00}\alpha \geq \mu_{01} \\ \mu_{11} \geq \mu_{10}\alpha \\ \mu_{00}\beta \geq \mu_{10} \\ \mu_{11} \geq \mu_{01}\beta \end{cases} \Leftrightarrow \begin{cases} -\mu_{10}\alpha \geq -\mu_{11} \\ -\mu_{01} \geq -\mu_{00}\alpha \\ -\mu_{10}/\beta \geq -\mu_{00} \\ -\mu_{01} \geq -\mu_{11}/\beta \end{cases}$$

equivalent to $\tau(\mu) = (\mu_{10}, \mu_{01}, \mu_{00}, \mu_{11}) \in CE(\gamma_{\mathrm{II}}(\alpha, 1/\beta))$. ∎

Proof of Proposition 1: Let $(\mu_{00}, \mu_{11}) \in \mathbb{R}^2_+$. Then, (μ_{00}, μ_{11}) are the diagonal probabilities of some correlated equilibrium strategy of $\gamma_{\mathrm{I}}(\alpha,\beta)$ if and only if there exists some $(\mu_{10}, \mu_{01}) \in \mathbb{R}^2_+$ such that $\mu = (\mu_{00}, \mu_{11}, \mu_{10}, \mu_{01})$ satisfies (4), and $\mu_{00} + \mu_{11} + \mu_{10} + \mu_{01} = 1$. Note that (4) is equivalent to

$$\begin{cases} \min\{\mu_{00}\beta; \mu_{11}/\alpha\} \geq \mu_{10} \\ \min\{\mu_{00}\alpha; \mu_{11}/\beta\} \geq \mu_{01} \end{cases}$$

Therefore, it is possible to find two out-of-diagonal probabilities (μ_{10}, μ_{01}) satisfying the above requirements if and only if $(\mu_{00}, \mu_{11}) \in \mathcal{D}(\alpha,\beta)$.

Let now $(\mu_{10}, \mu_{01}) \in \mathbb{R}^2_+$. By Lemma 2, (μ_{10}, μ_{01}) are the out-of-diagonal probabilities of some correlated equilibrium strategy of $\gamma_{\mathrm{I}}(\alpha,\beta)$ if and only if there are the diagonal probabilities of some correlated equilibrium strategy of $\gamma_{\mathrm{II}}(\alpha, 1/\beta)$. Specializing (1) for $\gamma_{\mathrm{II}}(\alpha, 1/\beta)$ we have:

$$\begin{cases} \mu_{00} \geq \max\{\mu_{10}/\beta; \mu_{01}/\alpha\} \\ \mu_{11} \geq \max\{\mu_{10}\alpha; \mu_{01}\beta\} \end{cases}$$

for $\mu = (\mu_{10}, \mu_{01}, \mu_{00}, \mu_{11}) \in CE(\gamma_{\mathrm{II}}(\alpha, 1/\beta))$. With a similar reasoning than before, we deduce that (μ_{10}, μ_{01}) are the out-of-diagonal probabilities of some correlated equilibrium strategy of $\gamma_{\mathrm{I}}(\alpha,\beta)$ if and only if $(\mu_{10}, \mu_{01}) \in \mathcal{O}(\alpha,\beta)$. ∎

Proof of Proposition 2: We first describe a planar geometric representation of Δ_3.

Consider a squared Edgeworth box of size unity. Any $z = (z_1, z_2, z_3, z_4) \in \Delta_3$ can be represented on the Edgeworth box as a couple of points (Z_d, Z_o), where $Z_d = (z_1, z_2)$ (resp. $Z_o = (z_3, z_4)$)

corresponds to the first two (resp. last two) coordinates of z. Let $\delta = z_1 + z_2$. Then, $z_3 + z_4 = 1 - (z_1 + z_2) = 1 - \delta$. Therefore, for a given Z_d, the set of possible loci for Z_o is given by the segment $\left\{(x_3, x_4) \in \mathbb{R}^2 \mid x_3 + x_4 = 1 - \delta\right\}$.[14] See Figure 3.

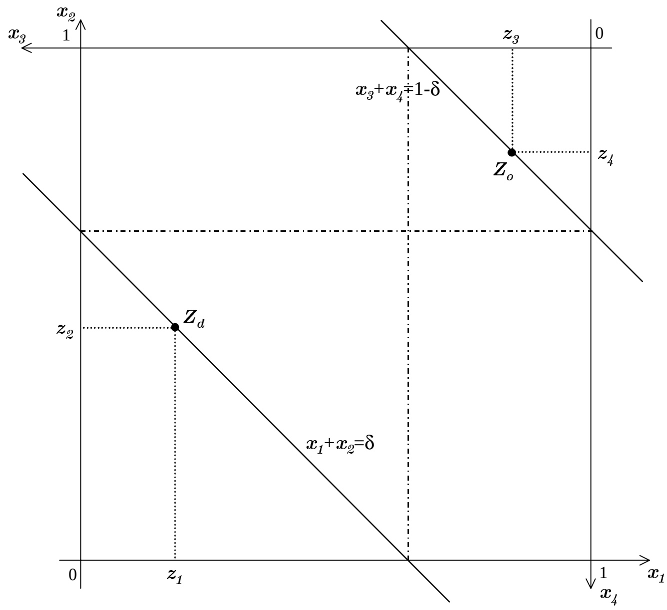

Figure 3. A planar geometric representation of Δ_3.

Using the previous planar representation for elements of Δ_3, a correlated equilibrium strategy $\mu = (\mu_{00}, \mu_{11}, \mu_{10}, \mu_{01})$ is identified by a couple of points (M_d, M_o) on the Edgeworth box, where $M_d = (\mu_{00}, \mu_{11})$ are the diagonal probabilities and $M_o = (\mu_{10}, \mu_{01})$ the out-of-diagonal probabilities.

Let $\alpha, \beta > 0$. The grey-shaded are in Figure 4 represents $CE\left(\gamma_{\mathrm{I}}\left(\alpha, \beta\right)\right)$ as characterized by

[14] $Z_d = (z_1, z_2)$ is simply the projection of z on the subset $\left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\right\}$ and $Z_o = (z_3, z_4)$ the projection of z on the subset $\left\{(x_3, x_4) \in \mathbb{R}^2 \mid x_3 + x_4 \leq 1, x_3 \geq 0, x_4 \geq 0\right\}$.

Proposition 1.

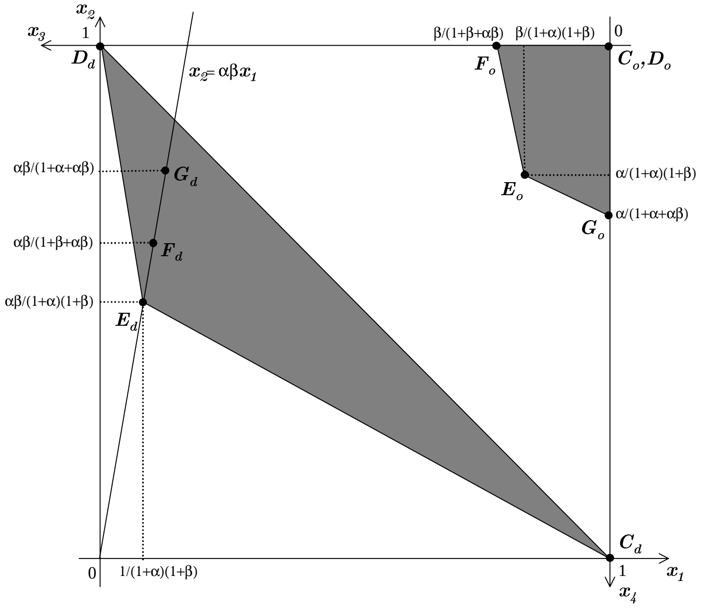

Figure 4. The set of correlated equilibrium strategies of $\gamma_{\mathrm{I}}(\alpha, \beta)$.

In Figure 4, (C_d, C_o) and (D_d, D_o) correspond to the two Nash equilibria in pure strategies of γ_{I}, and (E_d, E_o) is the Nash equilibrium in (completely) mixed strategies of γ_{I}. Finally, (F_d, F_o) and (G_d, G_o) correspond to the vertices μ_F^* and μ_G^* in (2), respectively.

Let $\mathcal{V} = \{C, D, E, F, G\}$. For all $i \in \mathcal{V}$, let $\mu_i^* = \left(\mu_{d,i}^*, \mu_{o,i}^*\right)$, where $\mu_{d,i}^*$ (resp. $\mu_{o,i}^*$) denotes the diagonal (resp. out-of-diagonal) probabilities of μ_i^*. We show that $\mathcal{V}$ is the set of vertices of the polytope $CE(\gamma_{\mathrm{I}}(\alpha, \beta))$.

Given some $A \subset \mathbb{R}^4$, let $\mathrm{Pr}_d(A) = \{(x_1, x_2) \mid \exists (x_3, x_4) : (x_1, x_2, x_3, x_4) \in A\}$. This is the projection of A on the first two coordinates of $\mathbb{R}^4$. Similarly, define the projection of A on the two second coordinates as $\mathrm{Pr}_o(A) = \{(x_3, x_4) \mid \exists (x_1, x_2) : (x_1, x_2, x_3, x_4) \in A\}$.

We proceed in two steps.

Step 1: We first prove that any $\mu \in CE(\gamma_{\mathrm{I}}(\alpha, \beta))$ is a convex combination of $\mu_i^*, i \in \mathcal{V}$. Let $\mu = (\mu_d, \mu_o) \in CE(\gamma_{\mathrm{I}}(\alpha, \beta))$. Noting that $\mathrm{Pr}_o\, CE(\gamma_{\mathrm{I}}(\alpha, \beta)) = cov(C_o, E_o, F_o) \cup cov(C_o, E_o, G_o)$, we can assume, without loss of generality, that $\mu_o \in cov(C_o, E_o, F_o)$. Let thus λ_i, $i \in \{C, E, F\}$ such that $\lambda_i \geq 0$, $\lambda_C + \lambda_E + \lambda_F = 1$, and $\mu_o = \lambda_C \mu_{o,C}^* + \lambda_E \mu_{o,E}^* + \lambda_F \mu_{o,F}^*$. For all $\rho \in [0, 1]$, define:

$$\mu_d(\rho, \lambda) = \rho \lambda_C \mu_{d,C}^* + (1 - \rho) \lambda_C \mu_{d,D}^* + \lambda_E \mu_{d,E}^* + \lambda_F \mu_{d,F}^*$$

The set $\{\mu_d(\rho, \lambda) \mid \rho \in [0, 1]\}$ is a segment that belongs to the set of diagonal probabilities that, together with μ_o, form a correlated strategy (that is, the sum of the corresponding probabilities is equal to one). We denote by $\mu_d(0,\lambda)$ and $\mu_d(1,\lambda)$ its endpoints. See Figure 5.

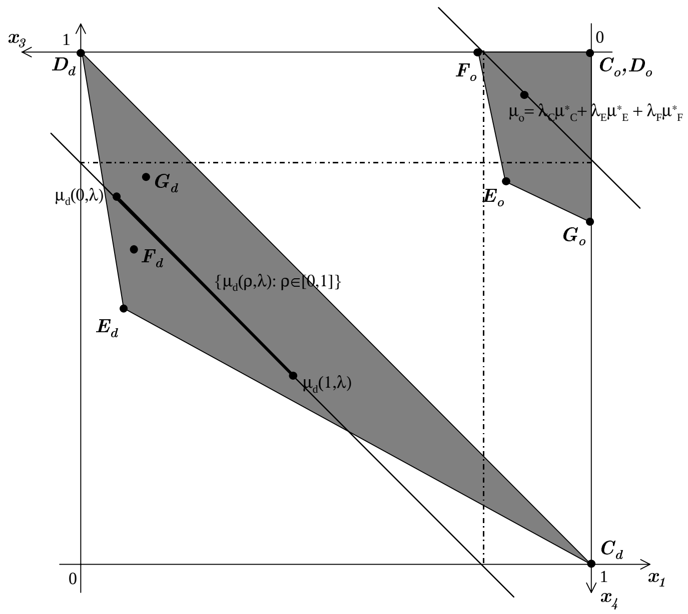

Figure 5. The segment $\{\mu_d(\rho,\lambda) : \rho \in [0,1]\}$.

We prove that, in fact, $\{\mu_d(\rho,\lambda) \mid \rho \in [0,1]\} = \{\widetilde{\mu}_d \mid (\widetilde{\mu}_d, \mu_o) \in CE(\gamma_{\text{I}}(\alpha,\beta))\}$. Let $\rho \in [0,1]$. The correlated strategy $(\mu_d(\rho,\lambda), \mu_o)$ can be written as:

$$(\mu_d(\rho,\lambda), \mu_o) = \rho\lambda_C\mu_C^* + (1-\rho)\lambda_C\mu_D^* + \lambda_E\mu_E^* + \lambda_F\mu_F^*,$$

It is thus a convex combination of correlated equilibrium strategies. As such, it is also an equilibrium strategy. Therefore, $\{\mu_d(\rho,\lambda) \mid \rho \in [0,1]\} \subset \{\widetilde{\mu}_d \mid (\widetilde{\mu}_d, \mu_o) \in CE(\gamma_{\text{I}}(\alpha,\beta))\}$. We now establish equality. The endpoint $\mu_d(0,\lambda)$ is a convex combination of $\mu_{d,C}^*, \mu_{d,E}^*$ and $\mu_{d,F}^*$ with weights equal to, respectively, λ_C, λ_E and λ_F. Therefore,

$$\begin{cases} \mu_{00}(0,\lambda) = \lambda_E\mu_{00,E}^* + \lambda_F\mu_{00,F}^* \\ \mu_{10}(0,\lambda) = \lambda_E\mu_{10,E}^* + \lambda_F\mu_{10,F}^* \end{cases}.$$

Note that (see Figure 4) E and F are such that

$$\begin{cases} \mu_{00,E}^*\beta = \mu_{00,E}^*/\alpha = \mu_{10,E}^* \\ \mu_{00,F}^*\beta = \mu_{00,F}^*/\alpha = \mu_{10,F}^* \end{cases}.$$

implying $(\mu_d(0,\lambda), \mu_o)$ satisfies the equilibrium condition $\min\{\mu_{00}\beta; \mu_{11}/\alpha\} \geq \mu_{10}$ with equality. Any correlated strategy $(\widetilde{\mu}_d, \mu_o)$ such that $\widetilde{\mu}_{00} < \mu_{00}(0,\lambda)$ violates this condition. With a similar reasoning for $\mu_d(1,\lambda)$ we deduce that:

$$\{\mu_d(\rho,\lambda) \mid \rho \in [0,1]\} = \{\widetilde{\mu}_d \mid (\widetilde{\mu}_d, \mu_o) \in CE(\gamma_{\text{I}}(\alpha,\beta))\}$$

Therefore, there exists a unique $\widehat{\rho} \in [0,1]$ such that $\mu_d = \mu_d(\widehat{\rho}, \lambda)$. With such $\widehat{\rho}$, we have:

$$\mu = \widehat{\rho}\lambda_C \mu_C^* + (1-\widehat{\rho})\lambda_C \mu_D^* + \lambda_E \mu_E^* + \lambda_F \mu_F^*.$$

Step 2: It is clear from Figure 5 that, for all $i \in \mathcal{V}$, μ_i^* can not be obtained as a convex combination of the $\mu_j^*, j \in \mathcal{V} \backslash \{i\}$. ▮

Proof of Proposition 3: We first show that the set of vertices of $CEP(\gamma_{\mathrm{I}}(\alpha, \beta))$ is a subset of $\{u(\mu_i^*) \mid i \in \mathcal{V}\}$. Suppose not. Let $\mu \in CE(\gamma_{\mathrm{I}}(\alpha, \beta))$ such that $u(\mu)$ is a vertex of $CEP(\gamma_{\mathrm{I}}(\alpha, \beta))$, and $\mu \notin \{\mu_i^* \mid i \in \mathcal{V}\}$. Let λ_i, $i \in \mathcal{V}$ such that $\lambda_i \geq 0$, $\sum_{i \in \mathcal{V}} \lambda_i = 1$ and $\mu = \sum_{i \in \mathcal{V}} \lambda_i \mu_i^*$. Given that $\mu \notin \{\mu_i^* \mid i \in \mathcal{V}\}$, at least two λ_i are non zero. By (3), we have $u(\mu) = \sum_{i \in \mathcal{V}} \lambda_i u(\mu_i^*)$ with at least two λ_i non zero, which is a contradiction.

Second, we show that the completely mixed Nash equilibrium payoffs are not a vertex of $CEP(\gamma(A, B))$ whenever $\gamma(A, B)$ is either a coordination or an anticoordination game. Both general coordination and anticoordination games are such that $(a_{00} + a_{11} - a_{01} - a_{10})(b_{00} + b_{11} - b_{01} - b_{10}) > 0$. Corollary 4, p. 210, and Table 1, p. 211 of Moulin and Vial (1978) imply that, for such games, the completely mixed equilibrium payoffs are (strictly) Pareto-dominated by some correlated equilibrium payoffs. Let $\gamma(A, B)$ be a general coordination game and $u(\mu)$ the payoffs corresponding to some $\mu \in \Delta_3$. Let $\overline{\mu}$ be the completely mixed Nash equilibrium of $\gamma(A, B)$ yielding payoffs $u(\overline{\mu})$. By the previous observation, there exists some $\mu' \in CE(\gamma(A, B))$ such that $u(\mu') \geq u(\overline{\mu})$, where $\geq$ is the component-wise ordering, and with at least one strict inequality.

Let now

$$A^* = \begin{bmatrix} a_{10} & a_{11} \\ a_{00} & a_{01} \end{bmatrix}$$

and B^* defined similarly. Then, $\gamma(-A^*, -B^*)$ is a general coordination game with (completely) mixed Nash equilibrium payoffs $-u(\overline{\mu})$. Using again the previous observation, there exists some μ'' $\in CE(\gamma(-A^*, -B^*))$ whose payoffs Pareto-dominate $-u(\overline{\mu})$. Specializing (1) for $\gamma(-A^*, -B^*)$, one can check that $\tau(\mu'') \in CE(\gamma(A, B))$ and $-u(\tau(\mu'')) \geq -u(\overline{\mu})$, that is, the mixed Nash equilibrium payoffs of $\gamma(A, B)$ Pareto-dominate those stemming from $\tau(\mu'')$. ▮

Proof of Corollary 1: the result derives from a simple limit argument on perturbed payoffs, that we leave to the reader. ▮

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