

Centre de Referència en Economia Analítica

Barcelona Economics Working Paper Series

Working Paper n° 42

**On Some Geometry and Equivalence
Classes of Normal Form Games**

Fabrizio Germano

September 2003

On Some Geometry and Equivalence Classes of Normal Form Games

Fabrizio Germano^{*}

Departament d'Economia i Empresa
Universitat Pompeu Fabra
Ramon Trias Fargas 25-27
08005 Barcelona, Spain
fabrizio.germano@upf.edu

Barcelona Economics WP n^o 42

September 2003

Abstract

Equivalence classes of normal form games are defined using the discontinuities of correspondences of standard equilibrium concepts like correlated, Nash, and robust equilibrium, or risk dominance and rationalizability. Resulting equivalence classes are fully characterized and compared across different equilibrium concepts for 2×2 games; larger games are also considered. It is argued that the procedure leads to broad and game-theoretically meaningful distinctions of games as well as to alternative ways of representing, comparing and testing equilibrium concepts. **Keywords** Non-cooperative games, classification and equivalence classes, experimental games, geometry of games. *JEL Classification* C70, C72, C78, C90, C91.

^{*}I thank Vince Crawford, Itzhak Gilboa, Jean-François Mertens, Rosemarie Nagel, Jack Robles, and Joel Sobel for insightful comments and conversations. Financial support from European Commission, TMR Network Grant ERBFMRXCT0055, "Cooperation and Information," as well as in the form of a Ramon y Cajal fellowship from the Spanish Ministry of Science and Technology are gratefully acknowledged. All errors are my own.

1 Introduction

The representation of a strategic situation by means of a normal form game is one of the most fundamental constructions of game theory. But, while much work has been done devising and studying a variety of equilibrium concepts for both normal and extensive form games, not much has been done to systematically differentiate the infinite number of games and strategic situations that naturally arise in theory and practice. In other words, questions like, What types of games are there? or, What are meaningful distinctions between different strategic situations? are hardly addressed in a systematic way in the game theory literature. (In Section 5, we discuss some exceptions and other related literature.)

The present paper attempts to address such questions by introducing a general geometric procedure of defining equivalence classes of normal form games from standard (non-cooperative) equilibrium concepts such as rationalizability, Nash, or correlated equilibrium. The idea is that the geometry of a given equilibrium correspondence reflects basic aspects of the logic or rationality underlying the equilibrium notion and can therefore be used to differentiate or identify games. The procedure associates to each equilibrium correspondence its equivalence classes of games. This is done by using, on one hand, the relabeling of strategies and/or of players to identify games, and on the other, and here is where the geometry comes in, the path components at which a given correspondence is continuous. Thus two games that belong to the same path component, possibly after relabeling strategies and/or players, are said to be equivalent relative to the given equilibrium correspondence.

Besides giving a sense of which aspects of normal form games different equilibrium concepts distinguish and which aspects they ignore, the obtained classifications each provide a compact organization of the underlying space of games into well-defined and typically finite equivalence classes of games. Not only does this help to better understand the underlying equilibrium concepts and the geometries of their equilibrium correspondences, it also provides a natural and transparent way of comparing equilibrium concepts with each other. At the same time, it forces one to think about what the fundamental characteristics describing a strategic situation may be and why.

An essential requirement of the procedure is that it should not distinguish “too many” games. In this sense, one reason for focusing on the geometry (or continuity) of equilibrium correspondences is that it allows to identify games that from many other points of view would be regarded as

different. Consider for example the game

$$\gamma_\epsilon = \begin{pmatrix} (1, 0) & (0, 1) \\ (0, 1) & (1, \epsilon) \end{pmatrix}.$$

For $\epsilon = 0$ this is a constant-sum game, while for any other value of ϵ it does not qualify as such. Yet, it seems clear that for values of ϵ close to zero relative to 1, the game retains the “character” of a constant-sum or at least of a matching pennies type game. Part of our task is to formally define notions of equivalence or of “types” of games in this sense.

More specifically, the procedure defining the equivalence classes (see Definition 1) is based on two ingredients: (i) discontinuities of the underlying (equilibrium) correspondence — these are the geometric features, and (ii) relabeling of the strategies and/or of the players. Take a correspondence $e : \mathbb{R}^m \rightarrow X$, where \mathbb{R}^m is a space of games, e.g., a space of payoff parameters, and where X is a set, e.g., the space of mixed strategies — e could for example be the Nash equilibrium correspondence. The discontinuities of e , or more precisely, the games at which e is discontinuous, divide the space of games \mathbb{R}^m into a certain number of path components. All games in the same path component are then said to be equivalent relative to the correspondence e . For example, it is easy to see that the Nash equilibrium correspondence is continuous on a neighborhood of the above game γ_ϵ with $\epsilon = 0$, implying that all games in such a neighborhood are what we call Nash equivalent. This is the first ingredient.

Moreover, following traditional game theory, two games that differ only by relabeling of the strategies and/or of the players are also necessarily equivalent — this is the second ingredient. Two games in two disconnected components will be equivalent if one can be obtained from the other through such relabeling of strategies and/or of players. For example, the game

$$\gamma'_\epsilon = \begin{pmatrix} (0, 1) & (1, 0) \\ (1, 0) & (\epsilon, 1) \end{pmatrix}.$$

is always equivalent to the game γ_ϵ above, since it is obtained from it by relabeling the two players. Therefore, from the above, we can infer that all games in a neighborhood of γ_ϵ with $\epsilon = 0$ are also Nash equivalent to γ'_ϵ with $\epsilon = 0$, but also to games in a neighborhood of the latter.

Given that for semi-algebraic correspondences, which include the rationalizable, Nash, and correlated equilibrium correspondences among others, the number of resulting path components is always finite, (see, e.g., Schanuel et al. (1991) and Blume and Zame (1994), and Lemmas 1 and 6 below), and since the number of path components is an upper bound for the

number of equivalence classes distinguished, this implies that the classifications obtained for semi-algebraic correspondences distinguish finitely many equivalence classes of games. While this already reduces the infinitely many normal form games to finite numbers of equivalence classes, depending on the correspondences, these numbers could in principle still be very large even for say the 2×2 games.

After formally defining the procedure, we consider the space of 2×2 two player games and show that reasonable distinctions are obtained in this case. In particular, we show that, applying the procedure to the rationalizability, Nash, and correlated equilibrium concepts, leads to the same three classes of generic 2×2 games being distinguished, namely, (a-i) games of the matching pennies type (or with zero pure Nash equilibria), (a-ii) games solvable by iterated strict dominance (or with unique pure Nash equilibrium), and (a-iii) coordination type games (or with two pure Nash equilibria); see Proposition 2.

On the other hand, when applied to the concept of robust equilibrium of Kajii and Morris (1997a, b), still within generic 2×2 games, the procedure yields a different classification,¹ namely, (b-i) games of the matching pennies type on one hand, and (b-ii) games solvable by iterated strict dominance *and* coordination games on the other; see Proposition 3. We view the fact the two classifications are not the same as significant. Essentially, it is related to the fact that games of the form

$$\hat{\gamma}_\epsilon = \begin{pmatrix} (1, 1) & (0, 0) \\ (0, 0) & (\epsilon, \epsilon) \end{pmatrix}$$

are all equivalent relative to the robust equilibrium concept, regardless of whether ϵ is positive or negative, as long as it is not equal to 1.² This is not true for the other three concepts, since the set of rationalizable strategies as well as the sets of correlated and Nash equilibria all change abruptly as ϵ goes from being negative to being positive. On the other hand, the correspondences of these three concepts, are all continuous at $\epsilon = 1$, unlike the robust equilibria.

We briefly argue in the paper that experimental evidence appears to be more supportive of the classification obtained with robust equilibria, at least in the 2×2 case; (see Section 4). However, we believe that experiments, testing equilibrium concepts relative to the equivalence classes

¹For generic 2×2 games, the robust equilibrium is unique and coincides with the unique risk dominant equilibrium in the sense of Harsanyi and Selten (1988); see Kajii and Morris (1997a, b).

²These games have a unique robust equilibrium, which is top-left if $\epsilon < 1$ and bottom-right if $\epsilon > 1$; relabeling the strategies makes them all equivalent relative to robust (or risk dominant) equilibrium.

implied, have only partially been performed, and think that, in principle, they could provide useful tests of equilibrium theories. In fact, the location of a game *within* an equivalence class may contain relevant information, and a measure of distance of a given game to the nearest discontinuity of the underlying equilibrium correspondence may prove a useful variable for explaining experimentally observed behavior.

The paper is organized as follows. Section 2 introduces preliminary notation and definitions, Section 3 defines and analyzes equivalence classes relative to the notions of rationalizability, Nash, and correlated equilibrium, in particular, showing that they coincide for 2×2 games, but are otherwise distinct and in a general sense unrelated for 2×3 and larger games. Section 4 considers robust and risk dominant equilibria, Section 5 discusses related literature, Section 6 concludes. Most proofs are relegated to the Appendix.

2 Preliminary Notions

Let $I = \{1, \dots, n\}$ denote the set of players, let S_i denote player i 's space of pure strategies, and $S = \times_{i \in I} S_i$ the space of pure strategy profiles. Let $\Sigma_i = \Delta(S_i)$ denote the set of probability distributions on S_i , $\Sigma = \times_{i \in I} \Sigma_i$ the space of mixed strategies, and $\Delta(S)$ the probability measures on S . Let also $S_{-i} = \times_{j \neq i} S_j$, and $\Sigma_{-i} = \times_{j \neq i} \Sigma_j$ and set $K_i = \#S_i$, $K = \sum_{i \in I} K_i$, $\kappa = \prod_{i \in I} K_i$. In what follows, we consider finite normal form games, i.e., where n and each K_i are finite, and fix both the set of players and the set of strategy profiles, so that we can identify a game with a point in Euclidean space $\gamma \in \mathbb{R}^{\kappa n}$ and, accordingly, the space of games with the whole Euclidean space $\mathbb{R}^{\kappa n}$. We also denote by $\gamma^i \in \mathbb{R}^{\kappa}$ the payoff array of player i .

For a given normal form game, $\gamma \in \mathbb{R}^{\kappa n}$, denote by $e_{\text{RAT}}(\gamma) \subset S$ the set of **rationalizable strategy profiles**, by $e_{\text{NE}}(\gamma) \subset \Sigma$ the set of **Nash equilibria**, and by $e_{\text{CE}}(\gamma) \subset \Delta(S)$ the set of **correlated equilibria**.³ Further, denote by e_{RAT} , e_{NE} , and e_{CE} the respective correspondences mapping games to corresponding sets of equilibria, and by $\eta_{\text{RAT}} \subset \mathbb{R}^{\kappa n} \times S$, $\eta_{\text{NE}} \subset \mathbb{R}^{\kappa n} \times \Sigma$, and $\eta_{\text{CE}} \subset \mathbb{R}^{\kappa n} \times \Delta(S)$ the respective graphs of the equilibrium correspondences.

The definition of equivalence class that is given below, builds on two ingredients; the first is based on continuity of equilibrium or related correspondences, and the second is based on the fact that relabeling players and their strategies does not really change the game. The following lemma

³See Fudenberg and Tirole (1991) or Myerson (1991) for definitions.

follows from the fact that e_{RAT} , e_{CE} , and e_{NE} are semi-algebraic.⁴

Lemma 1 *There exist closed, lower dimensional subsets of the space of games, D_{RAT} , D_{CE} , and $D_{NE} \subset \mathbb{R}^{\kappa n}$, such that e_{RAT} , e_{CE} , and e_{NE} are continuous respectively on the complements $\mathbb{R}^{\kappa n} \setminus D_{RAT}$, $\mathbb{R}^{\kappa n} \setminus D_{CE}$, and $\mathbb{R}^{\kappa n} \setminus D_{NE}$. Moreover, each of the complements consists of a finite number of connected components.*

The nature of the complements $\mathbb{R}^{\kappa n} \setminus D_{RAT}$ and $\mathbb{R}^{\kappa n} \setminus D_{NE}$ will play an important role in the classifications. Notice that, since rationalizability and iterated strict dominance are equivalent for two-player games, (see, e.g., Fudenberg and Tirole (1991), Ch. 2, p. 51), it follows that, at any game $\gamma \in \mathbb{R}^{2\kappa} \setminus D_{RAT}$, the number of strategies surviving iterated strict dominance is locally constant. Also, since, for two-player games, continuity of e_{NE} is equivalent to all Nash equilibria being locally unique, (see van Damme (1991), Ch. 3, p. 55, where strongly stable corresponds to what we call locally unique), implies that, at any game $\mathbb{R}^{\kappa n} \setminus D_{NE}$, the set of Nash equilibria consists of a finite (and odd) number of locally unique Nash equilibria.⁵

The other important ingredient in the definition of equivalence class, can be formalized by means of certain (linear) maps, which, following Nash (1951), we call symmetry operations. These maps identify games through relabeling of the players' strategies and/or of the players themselves.⁶ More precisely, let P_a be the set of all permutations $p_a : I \rightarrow I$ satisfying $K_{p_a(i)} = K_i$, $i \in I$, let P_s be the set of all permutations $p_s = (p_s^i : S_i \rightarrow S_i)_{i \in I}$, and let:

$$\begin{aligned} \Psi_a &= \{ \psi_a : \mathbb{R}^{\kappa n} \rightarrow \mathbb{R}^{\kappa n} \mid \psi_a(\gamma) = (\gamma_{k_{p_a^{-1}(1)}}, \dots, \gamma_{k_{p_a^{-1}(n)}})_{i \in I}, p_a \in P_a, \gamma \in \mathbb{R}^{\kappa n} \}, \\ \Psi_s &= \{ \psi_s : \mathbb{R}^{\kappa n} \rightarrow \mathbb{R}^{\kappa n} \mid \psi_s(\gamma) = (\gamma_{p_s^1(k_1)}, \dots, \gamma_{p_s^n(k_n)})_{i \in I}, p_s \in P_s, \gamma \in \mathbb{R}^{\kappa n} \}. \end{aligned}$$

We say a map $\psi : \mathbb{R}^{\kappa n} \rightarrow \mathbb{R}^{\kappa n}$ is a **symmetry operation** within the class of n -player $K_1 \times \dots \times K_n$ games, if $\psi = \psi_a^1 \psi_s^1 \dots \psi_a^p \psi_s^p$ for some $\psi_a^1, \dots, \psi_a^p \in \Psi_a$, $\psi_s^1, \dots, \psi_s^p \in \Psi_s$, $p \in \mathbb{N}$, and denote by Ψ the set of all such symmetry

⁴Schanuel et al. (1991) prove the case e_{NE} ; the cases e_{RAT} and e_{CE} (as well as of some of the other correspondences defined in this paper) are analogous; see Lemma 6 in the appendix for more details.

⁵Let $(\gamma, x) \in \eta$ be a point on the graph of the (equilibrium) correspondence $e : \mathbb{R}^{\kappa n} \rightarrow X$, we say the equilibrium $x \in X$ of $\gamma \in \mathbb{R}^{\kappa n}$ is **locally unique** if there exists a neighborhood U of γ and a neighborhood V of x such that (i) $\#(e(\gamma') \cap V) = 1$, for all $\gamma' \in U$, and (ii) the map $h : U \rightarrow V$, $h(\gamma') = e(\gamma') \cap V$ is continuous. Locally unique equilibria can be parameterized (locally) by games.

⁶For a thorough treatment of such symmetries within a more abstract space of games, the reader is referred to Mertens (1987) and the notions of ordinality defined there.

operations. This space depends on the number of players and the cardinality of the strategy spaces, i.e., on K_1, \dots, K_n , which, to save notation, we often leave out. The maps $p_a \in P_a, p_s \in P_s$ correspond respectively to relabeling of the players and of the players' strategies. The maps $\psi_a \in \Psi_a, \psi_s \in \Psi_s$ are the maps induced on the space of games by corresponding maps p_a and p_s . Hence, Ψ_a denotes the set of all possible relabelings of the players, while Ψ_s denotes the set of all possible relabelings of players' strategies. Notice that elements of Ψ, Ψ_a , and Ψ_s are always matrices with determinant $+1$ or -1 .

Next, we introduce the notion of equivalence, the main concept of this paper, which can be seen as a procedure associating to any given (equilibrium) correspondence, $e : \mathbb{R}^{\kappa n} \rightarrow X$, an equivalence relation on the underlying space of games $\mathbb{R}^{\kappa n}$.

Definition 1 *Let X be some set, and let $\mathbb{R}^{\kappa n}$ be the space of $K_1 \times \dots \times K_n$ games in normal form with n players and let $e : \mathbb{R}^{\kappa n} \rightarrow X$ be an (equilibrium) correspondence on this space of games. We say two games $\gamma_0, \gamma_1 \in \mathbb{R}^{\kappa n}$ are **equivalent** relative to e , which we denote by $\gamma_0 \sim \gamma_1$, if there exists a symmetry operation $\psi \in \Psi(K_1, \dots, K_n)$, and a continuous path $v : [0, 1] \rightarrow \mathbb{R}^{\kappa n}$, with $v(0) = \gamma_0$ and $v(1) = \psi(\gamma_1)$, such that, for all $\gamma \in v([0, 1])$, e is continuous at γ .*

Since we require that e be continuous at all games along the path v , including the games at the endpoints γ_0 and $\psi(\gamma_1)$, we strictly speaking define equivalence only for a subset of $\mathbb{R}^{\kappa n}$. As mentioned, if e is semi-algebraic, then the subset is open and dense in $\mathbb{R}^{\kappa n}$ and the set of games at which the e is discontinuous will be contained in a lower dimensional subset (see Lemmas 1 and 6). The following shows that Definition 1 indeed defines an equivalence relation.

Lemma 2 *The relation \sim defines an equivalence relation on the space of games in $\mathbb{R}^{\kappa n}$ at which e is continuous.*

3 Some Standard Equivalence Classes

Together with Definition 1, we say that two games $\gamma_0, \gamma_1 \in \mathbb{R}^{\kappa n}$ are respectively **rationalizable** and **Nash equivalent**, which we denote respectively by $\gamma_0 \sim_{RAT} \gamma_1$ and $\gamma_0 \sim_{NE} \gamma_1$, if they are equivalent relative to respectively e_{RAT} and e_{NE} . While we could define the equivalence classes analogously for correlated equilibria, to simplify the analysis, we say that two games $\gamma_0, \gamma_1 \in \mathbb{R}^{\kappa n}$ are **correlated equivalent**, which we denote by $\gamma_0 \sim_{CE} \gamma_1$, if they are equivalent relative to the correspondence

$$e_{dCE} : \mathbb{R}^{\kappa n} \rightarrow \mathbb{R}, e_{dCE}(\gamma) = \dim[e_{CE}(\gamma)],$$

which assigns to any game γ , the dimension of the corresponding set of correlated equilibria. Since the set of correlated equilibria of a given game is always a nonempty, compact, and convex polyhedron, defined by a finite number of linear inequalities, its dimension is always well-defined. It turns out that checking for changes in the dimension of the set of equilibria (continuity of e_{dCE}) is much easier than checking for continuity of e_{CE} . Although in principle, changes in the dimension could occur at points where e_{CE} is continuous, we view the changes in dimension themselves as significant geometric or topological events of the correspondence e_{CE} .

3.1 2×2 Games

In this section we consider the case where $n = 2$ and also $K_1 = K_2 = 2$. We will see how the equivalence classes defined above make game-theoretically meaningful distinctions, and lead to the same characterizations in this case. However, before giving the full characterizations of the rationalizable, correlated and Nash equivalence classes for this case, we first introduce the notion of a nondegenerate two-player game, (see von Stengel (2002)), which we use to compute and describe the standard equivalence classes.

Definition 2 *Let $\gamma \in \mathbb{R}^{2\kappa}$ be a two-player game in normal form. We say the game γ is **nondegenerate** if any mixed strategy $x \in \Sigma_i$ of player i has at most $|\text{supp}(x)|$ pure best responses, for $i = 1, 2$; we also say γ is **degenerate** if it is not nondegenerate.*

The following is a basic fact concerning nondegenerate games; see von Stengel (2002).

Lemma 3 *If $\gamma \in \mathbb{R}^{2\kappa}$ is nondegenerate two-player game in normal form, then no pure strategy of any player is weakly dominated by or payoff equivalent to a mixed strategy of the same player that is not itself strictly dominated.*

Accordingly, we say that two games $\gamma_0, \gamma_1 \in \mathbb{R}^{2\kappa}$ are **nondegenerate equivalent**, which we denote by $\gamma_0 \sim_{ND} \gamma_1$, if they are equivalent relative to the correspondence

$$e_{ND} : \mathbb{R}^{2\kappa} \rightarrow [0, 1], e_{ND}(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is degenerate} \\ 0 & \text{else .} \end{cases}$$

Furthermore, for any nondegenerate two-player game $\gamma \in \mathbb{R}^{2\kappa}$, let the set $[\gamma] = \{\gamma' \in \mathbb{R}^{2\kappa} \mid \gamma' \sim_{ND} \gamma\}$ denote the **nondegenerate equivalence class** associated to γ ; let also $[\gamma, \gamma'] = ([\gamma] \cup [\gamma'] \cup (\text{cl}[\gamma] \cap \text{cl}[\gamma']))^{\circ}$, for

$\gamma, \gamma' \in \mathbb{R}^{2\kappa}$ nondegenerate with $[\gamma] \cap [\gamma'] = \emptyset$, and where $\text{cl}(\cdot)$ denotes the closure and $(\cdot)^\circ$ denotes the interior.⁷

The importance of the nondegenerate games and nondegenerate equivalence classes for the equivalence classes resulting from the standard equilibrium concepts mentioned above, follows from the following result.

Proposition 1 *If two two-player games are nondegenerate equivalent, then they are also rationalizable, correlated, and Nash equivalent.*

This shows that the rationalizable, correlated, and Nash equivalence classes have the more basic nondegenerate equivalence classes in common, or, in other words, their intersections contain the nondegenerate equivalence classes. We view this as evidence that the players' best-reply correspondences are in a specific sense fundamental to the three equilibrium notions.

We are finally in a position to compute the standard equivalence classes for the 2×2 two-player games. It is easy to check that, in this case, the degenerate games are games with the property $\gamma_1^i = \gamma_2^i$ for $i = 1$ or $i = 2$ (or both). Moreover, the degenerate 2×2 games divide the space of all 2×2 games, (which we identify with \mathbb{R}^8), into 16 connected components of nondegenerate games. (Notice that by Proposition 1 all games within any one of the 16 connected components are rationalizable, correlated, and Nash equivalent.) Applying the symmetry operations of relabeling the players' strategies, further identifies some of the 16 components and leads to 5 classes of nondegenerate games, which can be unambiguously represented by the following games:

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} (1, 1) & (1, 0) \\ (0, 1) & (0, 0) \end{pmatrix}, \gamma_2 = \begin{pmatrix} (1, 1) & (1, 0) \\ (0, 0) & (0, 1) \end{pmatrix}, \gamma_3 = \begin{pmatrix} (1, 1) & (0, 0) \\ (0, 1) & (1, 0) \end{pmatrix}, \\ \gamma_4 &= \begin{pmatrix} (1, 0) & (0, 1) \\ (0, 1) & (1, 0) \end{pmatrix}, \gamma_5 = \begin{pmatrix} (1, 1) & (0, 0) \\ (0, 0) & (1, 1) \end{pmatrix}. \end{aligned}$$

Applying the symmetry operation of relabeling the players, allows to further identify γ_2 and γ_3 , (i.e., $\gamma_2 \sim_{ND} \gamma_3$), which means that there are exactly four nondegenerate equivalence classes: $[\gamma_1], [\gamma_2], [\gamma_4]$, and $[\gamma_5]$. We state this formally.

Lemma 4 *For the 2×2 two-player games, there are exactly four distinct nondegenerate equivalence classes, described by*

$$[\gamma_1], [\gamma_2], [\gamma_4], [\gamma_5].$$

⁷In the definition of $[\gamma, \gamma']$, besides the nondegenerate games in $[\gamma] \cup [\gamma']$, we also include all degenerate games that lie between two neighboring connected components of $[\gamma]$ and $[\gamma']$, which are precisely the games in $\text{cl}[\gamma] \cap \text{cl}[\gamma']$ without points on the boundary.

With Proposition 1, this implies that each of the correspondences e_{RAT} , e_{dCE} , and e_{NE} distinguishes at most the equivalence classes $[\gamma_1]$, $[\gamma_2]$, $[\gamma_4]$, $[\gamma_5]$. The next proposition provides a precise characterization.

Proposition 2 *For the 2×2 two-player games, the rationalizable, correlated, and Nash equivalence classes all coincide and distinguish exactly three classes of games, which are described by*

$$[\gamma_1, \gamma_2], [\gamma_4], [\gamma_5].$$

Proof. It follows directly from Proposition 1 that there are at most four rationalizable, correlated and Nash equivalence classes, since all games within the corresponding nondegenerate equivalence classes $[\gamma_1]$, $[\gamma_2]$, $[\gamma_4]$, and $[\gamma_5]$ are also equivalent with respect to these notions. Moreover, it is easy to check that the games γ_1, γ_2 can all be connected by paths of games along which e_{RAT} , e_{NE} , and e_{dCE} are all continuous, so that they are also rationalizable, correlated and Nash equivalent. This means that, at most, there are three classes $[\gamma_1, \gamma_2]$, $[\gamma_4]$, and $[\gamma_5]$ of rationalizable, correlated and Nash equivalent games.

Next we show that for each of the three equivalence concepts there are in fact exactly these three equivalence classes. We start by showing it for Nash equivalence and show $\gamma_1 \not\sim_{NE} \gamma_4$, $\gamma_1 \not\sim_{NE} \gamma_5$, and $\gamma_4 \not\sim_{NE} \gamma_5$. This immediately implies corresponding statements about correlated and rationalizable equivalence.

The proof is geometric. Take the game γ_4 . It has a unique Nash equilibrium in mixed strategies and is depicted in Figure 1a below together with the players' best-reply correspondences. Any game in the neighborhood has similar best-reply correspondences, which may be slightly shifted in direction of the arrows on the figure, parallel to the axes. Shifting the best-reply correspondences sufficiently will eventually lead to a game with a continuum of Nash equilibria such as the game

$$\gamma_{41} = \left(\begin{array}{cc} (1, \frac{1}{2}) & (\frac{1}{2}, \frac{1}{2}) \\ (0, 1) & (\frac{1}{2}, 0) \end{array} \right),$$

which is exactly halfway between γ_4 and γ_1 and which is depicted in Figure 1b. At such a game the correspondence e_{NE} is clearly discontinuous.

Similarly, it can be checked that games from the class $[\gamma_4]$ (and $[\gamma_1]$) have a unique correlated equilibrium and hence the dimension jumps from zero to one when reaching games like γ_{41} and it is clear from Figure 1a that such jumps are unavoidable. This shows $\gamma_1 \not\sim_{NE} \gamma_4$ and $\gamma_1 \not\sim_{CE} \gamma_4$, as well as $\gamma_4 \not\sim_{NE} \gamma_5$ and $\gamma_4 \not\sim_{CE} \gamma_5$. Obviously, $\gamma_1 \not\sim_{RAT} \gamma_4$ since the number

of strategies per player surviving iterated strict dominance is respectively one and two for the two games. To see $\gamma_4 \not\sim_{RAT} \gamma_5$ is somewhat more subtle. Both games γ_4 and γ_5 have the same number of rationalizable strategies, however it is not difficult to see that it is impossible to join the two games along a path where ϵ_{RAT} is everywhere continuous. Discontinuities at games between the two classes are unavoidable, since, in any arbitrarily small neighborhood around such games, ϵ_{RAT} goes from containing all strategies to just containing one strategy profile. (Notice, however, that this does not mean that a path between the two classes along which the number of rationalizable strategies is constant does not exist; as for example the path through the game where all entries are zero.)

Figure 1

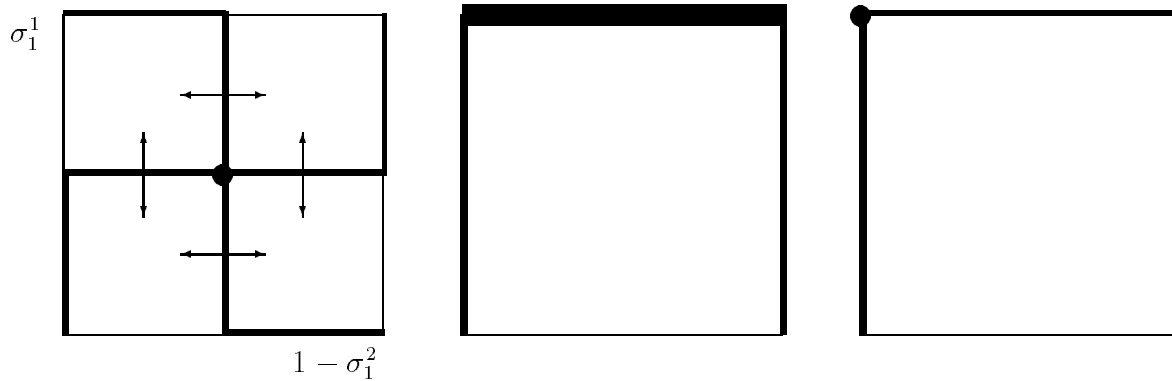


Figure 1a. Game γ_4

Figure 1b. Game γ_{41}

Figure 1c. Game γ_1

Next, take the game γ_5 . It has a unique mixed strategy equilibrium and two pure strategy equilibria and is also depicted in Figure 2a along with the best-reply correspondences.

Again, the arrows show possible (parallel) shifts of the best-reply correspondences for neighboring games. Also here one sees that whatever the direction of the shifts, it is impossible to leave the class of games $[\gamma_5]$ without encountering either a game with a continuum of Nash equilibria or games like the game

$$\gamma_{51} = \left(\begin{array}{cc} (1, 1) & (\frac{1}{2}, 0) \\ (0, \frac{1}{2}) & (\frac{1}{2}, \frac{1}{2}) \end{array} \right),$$

which is exactly halfway between γ_5 and γ_1 and which is depicted in Figure 2b. This game has exactly two Nash equilibria, one of which (the one on the lower-right corner) disappears as the payoffs further change towards the payoffs of γ_1 . This implies $\gamma_1 \not\sim_{NE} \gamma_5$ since, at such a game, the corre-

spendence e_{NE} is not continuous.

Figure 2

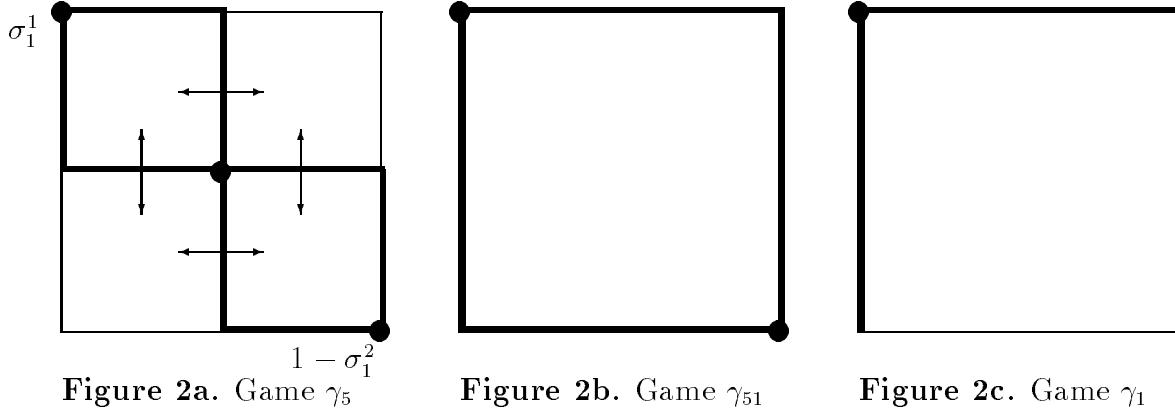


Figure 2a. Game γ_5

Figure 2b. Game γ_{51}

Figure 2c. Game γ_1

On the other hand, the games γ_5 and γ_1 cannot be correlated equivalent, since the dimensions of the corresponding sets of correlated equilibria are three and zero respectively. Similarly, they cannot be rationalizable equivalent, since again the number of strategies per player surviving iterated strict dominance is respectively two and one. This completes the proof. \square

Somewhat surprisingly, the equivalence classes for the three standard concepts of rationalizability, Nash, and correlated equilibrium all coincide and consist of (i) games with only one pure Nash equilibrium, $[\gamma_1, \gamma_2]$, (ii) games with only one mixed Nash equilibrium, $[\gamma_4]$, and (iii) games with one mixed and two pure Nash equilibria, $[\gamma_5]$. Examples of games in the different classes are (i) the prisoners' dilemma games and other strict dominance solvable games, (ii) matching pennies type games, and (iii) chicken, battle of the sexes, and other coordination games.

Notice that the games in (ii) and (iii) have all strategies rationalizable, but are nonetheless not rationalizable equivalent. Similarly, all games in (i) and (ii) have a unique correlated and a unique Nash equilibrium, yet are neither correlated nor Nash equivalent. Showing these non-equivalences constitutes the main part of the proof; a basic step consists in showing that discontinuities of all three correspondences e_{RAT} , e_{NE} , and e_{CE} are inevitable when leaving the class of games $[\gamma_4]$. This also shows that simply counting the number of rationalizable strategies or of Nash (or correlated) equilibria is not sufficient to obtain the corresponding classification.

The above classification for 2×2 games has already been obtained for example by Rapoport and Guyer (1966), typically based on purely game-theoretic considerations, and is also often used when discussing or describing 2×2 games. We take this as evidence that the present procedure can yield game-theoretically meaningful distinctions. However, we will come back to 2×2 games further below in the context of robust and risk dominant equilibria, where we also discuss some experimental literature.

3.2 Larger Games

We next consider games larger than 2×2 games and show that the classifications of the rationalizable, correlated, and Nash equivalence classes distinguish different games in general.

2×3 games. It can be shown that the space of 2×3 is divided by the degenerate games into 336 connected components of nondegenerate games, which after applying the symmetry operations reduce to 28 nondegenerate equivalence classes. The equivalence classes for the 2×3 games are all different across the three mentioned standard equilibrium concepts. It can be shown that there are four distinct rationalizable equivalence classes consisting of a class represented by games with one strategy per player surviving iterated strict dominance, two classes of games with two strategies per player surviving iterated strict dominance, and games with all strategy profiles surviving iterated strict dominance. Examples of the latter games are

$$\gamma_6 = \begin{pmatrix} (1,3) & (1,2) & (0,0) \\ (0,0) & (0,2) & (1,3) \end{pmatrix}, \gamma_7 = \begin{pmatrix} (1,3) & (0,2) & (1,0) \\ (0,0) & (1,2) & (0,3) \end{pmatrix},$$

which satisfy $\gamma_6 \sim_{RAT} \gamma_7$. It can further be shown that there are also four distinct correlated equivalence classes represented by the games γ'_1 , γ'_4 , and γ'_5 , obtained from γ_1 , γ_4 , γ_5 by adding a strictly dominated strategy for player 2 to make them 2×3 games, and games like γ_6 or γ_7 which can be shown to also satisfy $\gamma_6 \sim_{CE} \gamma_7$, and where the dimension of the set of correlated equilibria is five for both games. Finally, it can be shown that there are five distinct Nash equivalence classes represented by the games γ'_1 , γ'_4 , γ'_5 , γ_6 , and γ_7 , which can all be shown not to be Nash equivalent. The latter two are representatives of games with two pure strategy and one mixed strategy equilibria, γ_6 , where all pure strategy profiles are in the support of some Nash equilibrium, and of games with with one pure strategy and two mixed strategy equilibria, γ_7 .

Hence, unlike the case of 2×2 games, the classifications for the three standard equivalence concepts are all different in the 2×3 games and can

be summarized as follows. There are four distinct rationalizable equivalence classes described by

$$[\gamma'_1, \gamma'_2, \dots], [\gamma'_4, \dots], [\gamma'_5, \dots], [\gamma''_4, \gamma_6, \gamma_7],$$

there are also four distinct correlated equivalence classes, which slightly differ from the rationalizable equivalence classes, and are described by

$$[\gamma'_1, \gamma'_2, \dots], [\gamma'_4, \gamma''_4, \dots], [\gamma'_5, \dots], [\gamma_6, \gamma_7],$$

and there are five distinct Nash equivalence classes described by

$$[\gamma'_1, \gamma'_2, \dots], [\gamma'_4, \gamma''_4, \dots], [\gamma'_5, \dots], [\gamma_6], [\gamma_7].$$

For reasons of space and since they can be unambiguously determined from the mentioned set of 28 distinct nondegenerate equivalence classes, we have not listed all the representatives of the nondegenerate equivalence classes and have instead referred to them by the dots. The nondegenerate equivalence class $[\gamma''_4]$ is what distinguishes the rationalizable and the correlated equivalence classes, while the nondegenerate equivalence classes $[\gamma_6]$ and $[\gamma_7]$ are the ones distinguishing the Nash and the correlated equivalence classes. The games γ'_4 and γ''_4 are discussed below.

Deciding the equivalence class directly from the game. The above shows that within the nondegenerate 2×2 and 2×3 games, knowing the number of Nash equilibria $\#e_{NE}(\gamma)$ is not sufficient to decide the equivalence class of a game, (see γ_1 and γ_4). However, knowing the Nash equilibrium distributions, i.e., knowing $e_{NE}(\gamma)$, is sufficient to place the game γ unambiguously within its Nash equivalence class. This fails with the 3×3 games as is illustrated by the following two games

$$\gamma_8 = \begin{pmatrix} (5, 3) & (0, 0) & (5, 2) \\ (4, 3) & (2, 2) & (2, 0) \\ (0, 0) & (5, 2) & (0, 3) \end{pmatrix}, \gamma_9 = \begin{pmatrix} (5, 3) & (0, 0) & (5, 2) \\ (0, 3) & (5, 2) & (0, 0) \\ (4, 0) & (2, 2) & (2, 3) \end{pmatrix}.$$

Both games have the same Nash equilibria, namely the pure equilibrium, where both play strategy 1, the mixed equilibrium where players 1 and 2 mix between respectively strategies 2 and 3, and 1 and 2, and, finally, the completely mixed equilibrium, yet it can be shown that the two games are not Nash equivalent. To understand why, one may notice that the indices of the two mixed equilibria do not coincide. However, it can also be shown that having the same Nash equilibrium distributions with same indices is still not sufficient to imply Nash equivalence of the games, thus

implying that besides the index, further (topological) invariants also matter in determining the Nash equivalence class; see Germano (1998) for further discussion and definitions.

Further, we already saw that within the 2×2 games, knowing $e_{RAT}(\gamma)$ or $e_{dCE}(\gamma)$ was not sufficient to place γ in its respective equivalence class, as the games γ_4 and γ_5 showed for rationalizable equivalence and γ_1 and γ_4 showed for correlated equivalence.⁸

An important step for characterizing equivalence classes of given equilibrium concepts will be to find (necessary and) sufficient conditions which need to be satisfied by two games in order to be equivalent. The conditions above provide some necessary conditions.

Relations between (standard) equivalence classes. We here show that the coincidence of the rationalizable, Nash, and correlated equivalence classes for 2×2 games is indeed very special for this class of games. As the following 2×3 matching pennies type games show two games that are Nash or correlated equivalent (or both) need not be rationalizable equivalent

$$\gamma'_4 = \begin{pmatrix} (1, 0) & (0, 1) & (2, -2) \\ (0, 1) & (1, 0) & (-2, -2) \end{pmatrix}, \gamma''_4 = \begin{pmatrix} (1, 0) & (0, 1) & (2, -2) \\ (0, 1) & (1, 0) & (-2, 2) \end{pmatrix}.$$

Both γ'_4 and γ''_4 have a the same unique Nash and correlated equilibrium (involving only the first two strategies) and are clearly Nash and correlated equivalent. However, the set of rationalizable strategies changes, as player 2's third strategy is no longer strictly dominated in γ''_4 .

Similarly, two games that are rationalizable or correlated equivalent (or both) need not be Nash equivalent, as the games γ_6 and γ_7 above show.

Furthermore, the following rock-scissors-paper type games, γ_{10} and γ_{11} , show that two games that are rationalizable or Nash equivalent (or both) need not be correlated equivalent

$$\gamma_{10} = \begin{pmatrix} (1, 1) & (3, 0) & (0, 3) \\ (0, 3) & (1, 1) & (3, 0) \\ (3, 0) & (0, 3) & (1, 1) \end{pmatrix}, \gamma_{11} = \begin{pmatrix} (2, 2) & (3, 0) & (0, 3) \\ (0, 3) & (2, 2) & (3, 0) \\ (3, 0) & (0, 3) & (2, 2) \end{pmatrix}.$$

Both games have all three strategies rationalizable and they have the same unique, completely mixed Nash equilibrium. However, the dimension of the correlated equilibrium correspondence, is zero at γ_{10} (the unique Nash equilibrium is also the unique correlated equilibrium) and eight at γ_{11} .

⁸Using the discontinuities of e_{CE} to define correlated equivalence would have allowed to distinguish $e_{CE}(\gamma_1)$ and $e_{CE}(\gamma_4)$, but we do not pursue this here.

To summarize then, unlike with the 2×2 case, where the three equivalence notions, \sim_{RAT} , \sim_{CE} , and \sim_{NE} , coincide, somewhat surprisingly, it is possible to find examples, already within the 3×3 games, showing that Nash or correlated equivalence (or both) need not imply rationalizable equivalence; that rationalizable or correlated equivalence (or both) need not imply Nash equivalence; and that Nash or rationalizable equivalence (or both) need not imply correlated equivalence. This means that equivalence with respect to one concept does not imply equivalence for any of the other two and, moreover, also intersecting the equivalence classes of two of the concepts does not imply equivalence with respect to the other.⁹ However, we found one notion, nondegenerate equivalence, whose equivalence classes, at least for two player games, are always contained in the equivalence classes of these three concepts, hence also in their intersection; (see Proposition 1). We turn to a further equilibrium notion.

4 Robust Equilibria and Risk Dominance

We consider a fourth equilibrium concept, namely that of robust equilibrium of Kajii and Morris (1997a, b).¹⁰ The equilibrium concept involves consideration of neighboring incomplete information games. Let T_i^s denote a countable set of **standard** types of player i , and let $T_i^c = S_i$ denote the set of **committed** types of player i ; the set of all possible types of player i is $T_i = T_i^s \cup T_i^c$, let also $T = \times_{i \in I} T_i$ and $T_{-i} = \times_{j \neq i} T_j$. Notice that a committed (or “crazy”) type of player i of type t_i will have a strictly dominated strategy to play pure strategy t_i . From a complete information game $\gamma \in \mathbb{R}^{\kappa^n}$ and a probability distribution over types $P \in \Delta(T)$, we can derive an incomplete information game with payoffs

$$g^i(s; t) = \begin{cases} \gamma^i(s) & \text{if } t^i \in T_i^s \\ 1 & \text{if } s^i = t^i \in T_i^c \\ 0 & \text{if } s^i \neq t^i, t^i \in T_i^c, \end{cases}$$

for $s \in S$, $t \in T$, and $i \in I$. The following defines incomplete information games that are ϵ -close to the complete information game γ .

Definition 3 *A canonical ϵ -elaboration consists of a complete information game γ and a probability distribution $P \in \Delta(T)$ such that $P[T^s] = 1 - \epsilon$*

⁹Looking at Hannan sets (see Hart and Mas-Colell (2003)) and the equivalence classes implied by their correspondence (or dimension) may provide more common ground between these three equivalence notions. We leave this for further research.

¹⁰Carlsson and van Damme (1993), Morris et al. (1995) are related approaches for two player games, see also Morris and Shin (2003) for further related literature; Cabrales et al. (2002) provide experimental evidence.

for $\epsilon \in [0, 1]$. A strategy profile $\sigma \in \Sigma$ is **robust to all canonical elaborations** if, $\forall \delta > 0, \exists \bar{\epsilon} > 0$ such that, $\forall \epsilon \leq \bar{\epsilon}$, every ϵ -elaboration of γ has a Bayesian Nash equilibrium μ with $\mu(\sigma) \geq 1 - \delta$. Finally, we say a strategy profile is a **robust equilibrium** if it is robust to all canonical elaborations.

Again with Definition 1, we say that two games $\gamma_0, \gamma_1 \in \mathbb{R}^{kn}$ are **robust equivalent**, which we denote by $\gamma_0 \sim_{RE} \gamma_1$, if they are equivalent with respect to e_{RE} , where e_{RE} is the robust equilibrium correspondence.

To get some intuition about robust equivalence classes, consider the 2×2 games. The analog of Proposition 1 does not hold for robust equivalence. In fact, it can be shown that the correspondence e_{RE} is not always continuous at nondegenerate games. To see this, notice that it follows from Kajii and Morris (1997b) that, for 2×2 games with two strict equilibria, the robust equilibria coincide with the risk dominant equilibria (see Harsanyi and Selten (1988) or van Damme (2002)). One implication of this is that there is a set Υ of nondegenerate games in $[\gamma_5]$ at which the robust equilibrium correspondence is *not* continuous, namely,

$$\Upsilon = \left\{ \gamma \in [\gamma_5] \mid \begin{array}{l} (\gamma_{11}^1 - \gamma_{21}^1)(\gamma_{11}^2 - \gamma_{12}^2) = (\gamma_{22}^1 - \gamma_{12}^1)(\gamma_{22}^2 - \gamma_{21}^2) \text{ if } \gamma_{11}^1 > \gamma_{21}^1 \\ (\gamma_{12}^1 - \gamma_{22}^1)(\gamma_{12}^2 - \gamma_{11}^2) = (\gamma_{21}^1 - \gamma_{11}^1)(\gamma_{21}^2 - \gamma_{22}^2) \text{ if } \gamma_{11}^1 < \gamma_{21}^1 \end{array} \right\}.$$

These are precisely games in $[\gamma_5]$ where the deviation losses for the two (strict) equilibria are equal. An example of such a game is the game γ_5 itself; it is easy to see that with arbitrarily small variations of the payoffs, the robust equilibrium moves from the top-left to the bottom-right strategy profile. Furthermore, Kajii and Morris (1997b) have also shown that nondegenerate games in $[\gamma_1]$, $[\gamma_2]$, and $[\gamma_4]$ all have a unique robust equilibrium which is also the unique Nash equilibrium. We summarize this as follows.

Lemma 5 *Every nondegenerate 2×2 two-player game not in Υ has a unique robust equilibrium, which coincides with the unique risk dominant Nash equilibrium of the game.*

Now, we can state the characterization of the robust equivalence classes for 2×2 games.

Proposition 3 *For the 2×2 two-player games, there are two distinct robust equivalence classes described by*

$$[\gamma_1, \gamma_2, \gamma_5] \setminus \Upsilon \text{ and } [\gamma_4].$$

Proof. The proof is in two steps. First, we show that any two games in $[\gamma_1, \gamma_2, \gamma_5] \setminus \Upsilon$ as well as in $[\gamma_4]$ are robust equivalent. Second, we show that games from the two different classes are not robust equivalent.

Clearly, all games in $[\gamma_1, \gamma_2]$ are robust equivalent, and the same holds for games in $[\gamma_4]$. Similarly, despite the discontinuity at Υ , all games in $[\gamma_5] \setminus \Upsilon$ are robust equivalent, due to symmetry operations identifying games with strict equilibria at the top-left and bottom-right profiles with corresponding games with strict equilibria at the bottom-left and top-right profiles. Finally, to see equivalence of a game from $[\gamma_1, \gamma_2]$ with a game from $[\gamma_5] \setminus \Upsilon$, consider the following game

$$\hat{\gamma}_\epsilon = \begin{pmatrix} (1, 1) & (0, 0) \\ (0, 0) & (\epsilon, \epsilon) \end{pmatrix}, \quad (1)$$

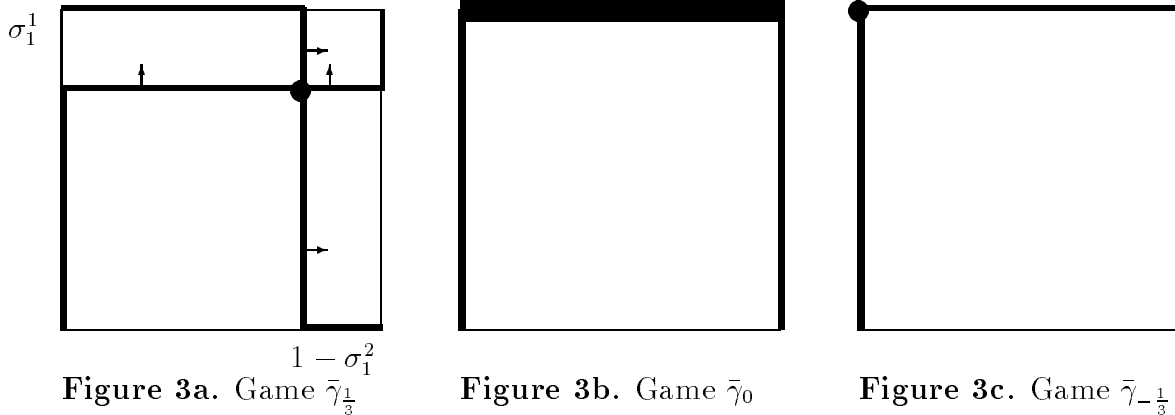
where $\epsilon < 1$. For $\epsilon \in (0, 1)$, we have $\hat{\gamma}_\epsilon \in [\gamma_5] \setminus \Upsilon$, while for $\epsilon < 0$, we have $\hat{\gamma}_\epsilon \in [\gamma_1, \gamma_2]$. In particular, we have that $\hat{\gamma}_{\frac{1}{2}} \in [\gamma_5] \setminus \Upsilon$ and $\hat{\gamma}_{-\frac{1}{2}} \in [\gamma_1, \gamma_2]$. On the other hand, for all $\epsilon < 1$, the game $\hat{\gamma}_\epsilon$ has a unique robust equilibrium at the top-left strategy profile, and, moreover, the correspondence e_{RE} is continuous at any such game. This shows in particular $\hat{\gamma}_{\frac{1}{2}} \sim_{RE} \hat{\gamma}_{-\frac{1}{2}}$ and hence the first step.

To see the second step, consider the game

$$\bar{\gamma}_\epsilon = \begin{pmatrix} (1, 0) & (0, \epsilon) \\ (0, 1) & (\epsilon, 0) \end{pmatrix},$$

where again $\epsilon < 1$. For $\epsilon \in (0, 1)$, we have $\bar{\gamma}_\epsilon \in [\gamma_4]$, while for $\epsilon < 0$, we have $\bar{\gamma}_\epsilon \in [\gamma_1, \gamma_2, \gamma_5] \setminus \Upsilon$. However, as can be seen in Figure 3, and analogous to the reasoning in the proof of Proposition 2, discontinuities of e_{RE} are unavoidable when going from games in $[\gamma_4]$ to games in $[\gamma_1, \gamma_2, \gamma_5] \setminus \Upsilon$.

Figure 3



(Notice that the bold dots in Figure 3 indicate the (unique) robust equilibria.) This shows that games in $[\gamma_4]$ are not robust equivalent to games in $[\gamma_1, \gamma_2, \gamma_5] \setminus \Upsilon$ and so completes the proof. \square

Robust equivalence identifies games in $[\gamma_5] \setminus \Upsilon$ with games in $[\gamma_1, \gamma_2]$, i.e., games with two pure Nash equilibria with games with unique pure Nash equilibrium. We view this as a fundamental departure from the equivalence classes obtained in the previous section, also since most casual discussions or other classifications in the literature typically distinguish the two classes of games, see, e.g., Rapoport et al. (1976), Carlsson and van Damme (1993), Roth (1995), or Friedman (1996).

Moreover, we view experimental evidence as providing support for the robust equivalence classes in the following sense. Take the game $\hat{\gamma}_\epsilon$ of Eq. (1) and suppose that ϵ is arbitrarily close to zero (units could be millions of euros). In this case, while it matters for rationalizable, correlated, and Nash equivalence, whether ϵ is positive or negative, (since a discontinuity occurs at $\epsilon = 0$), it does not matter for robust equivalence, as long as ϵ is sufficiently small; further, when ϵ is equal to 1, then a discontinuity of the set of robust equilibria occurs, (since $\hat{\gamma}_1$ is in Υ), while each of the rationalizable, correlated, and Nash correspondences are all continuous at such a game. In both cases, experimental evidence seems to be better explained by the predictions of the robust equilibria; because of the usual risk dominance argument for ϵ close to 0, and because of the issue of which strategy profile to coordinate on for ϵ close to 1; see Ochs (1995), Straub (1995), Friedman (1996), Crawford (1997), for experimental evidence, and also Cabrales et al. (2002), who test further connections with global games.

At the same time, all four equilibrium concepts identify all games in $[\gamma_1, \gamma_2]$ as well as all games in $[\gamma_5]$ or $[\gamma_5] \setminus \Upsilon$. In particular, this means that, for games in $[\gamma_1, \gamma_2]$, whether the (unique) profile surviving iterated dominance, which is also the unique Nash, correlated and robust equilibrium is also a Pareto efficient outcome or not, is irrelevant to the equilibrium concepts in the sense of the induced equivalence classes; on the other hand, there is experimental evidence suggesting that at least some extra variance is observed when the unique equilibrium point does not coincide with the Pareto efficient outcome, as in the prisoners' dilemma; see, e.g., Rapoport et al. (1976) and Roth (1995), and Ahn et al. (2002) for some experiments on such games. Similarly, for games in $[\gamma_5]$ or $[\gamma_5] \setminus \Upsilon$, whether or not the risk dominant equilibrium is also Pareto efficient or whether and what type of a battle of the sexes game it is, although not relevant to the studied equivalence classes, seems to matter experimentally; see, e.g., Ochs (1995), Crawford (1997), Battalio et al. (2001), and van Huyck and Battalio (2002).

A variable that may be worth considering when testing experimental behavior is a measure of the distance of the given game to the next closest discontinuity of the relevant equilibrium correspondence under study. Finally, the class $[\gamma_4]$, which is also distinguished by all four concepts, appears to exhibit relatively more homogenous behavior; see Erev and Roth (1998) for experimental evidence on such games as well as further references.

5 Related Literature

As mentioned in the Introduction, the question of classifying or identifying games has not received much attention in the game theory literature. There are some exceptions, however, mainly from the literature on experiments and games, combinatorics and games, psychology, and bounded rationality. We discuss these in turn.

Rapoport et al. (1976) consider ordinal 2×2 two player games which, after distinguishing 78 strategically different games, they classify into 24 different classes. Their distinctions are based on several notions, including whether the “natural outcome” (this is the intersection of the maxmin profiles) is a Nash equilibrium, whether it is Pareto efficient, or what kinds of threats the players can exercise against each other. In particular, the following three (coordination type) games all belong to different classes,

$$\gamma_{61} = \begin{pmatrix} (4, 4) & (1, 3) \\ (3, 1) & (2, 2) \end{pmatrix}, \gamma_{64} = \begin{pmatrix} (3, 4) & (2, 1) \\ (1, 2) & (4, 3) \end{pmatrix}, \gamma_{68} = \begin{pmatrix} (3, 4) & (2, 2) \\ (1, 1) & (4, 3) \end{pmatrix},$$

where the indices here correspond to the labels of Rapoport et al. (1976). As Bárány et al. (1992) point out, it is not clear how their procedure extends to larger games, since certain parts of their classification are defined specifically for the 2×2 games. Their analysis has a very strong experimental motivation, which may explain the fact that their distinctions are not necessarily related to the standard non-cooperative equilibrium concepts. Rapoport and Guyer (1966) and Harris (1972) study related distinctions; see also Rapoport et al. (1976) for further related references.

Bárány et al. (1992) take a combinatorial approach and classify $m \times n$ two player ordinal games. They also start from the set of strategically different (ordinal) games and look at the numbers of possible outcome sets generated by the payoff matrices, as well as the numbers of shapes of the corresponding convex hulls. They then derive asymptotic estimates for the number of strategically different games ($\approx (mn)^{2mn}$), for the number of different outcome sets ($\approx (mn)^{mn}$), and for the number of different convex hulls of the outcome sets ($\approx e^{(mn)^{2/3}}$). In the case of 2×2 two player games,

they start from the 78 strategically different games and obtain 24 different outcome sets and 7 different shapes of the convex hulls. In particular, the games $\gamma_{61}, \gamma_{64}, \gamma_{68}$ also belong to different outcome sets which also have different shapes; while on the other hand, the following four games,

$$\left(\begin{array}{cc} (4, 4) & (3, 3) \\ (2, 2) & (1, 1) \end{array} \right), \left(\begin{array}{cc} (3, 4) & (2, 1) \\ (1, 2) & (4, 3) \end{array} \right), \left(\begin{array}{cc} (2, 3) & (4, 1) \\ (1, 4) & (3, 2) \end{array} \right), \left(\begin{array}{cc} (2, 3) & (4, 1) \\ (3, 2) & (1, 4) \end{array} \right),$$

all have the same shape of the convex hull, and the first two have the same outcome set, which is different from the outcome set of the last two. (Notice that, while the outcomes for all four games lie on a straight line, the first two are coordination games, the third is a dominance solvable game and the last a matching pennies type game; they correspond to games under Fig. 1 and Fig. 24 in the appendix of Bárány et al. (1992).) Although Bárány et al. also mention the possibility of using properties of the best reply or other non-cooperative equilibrium correspondences to derive alternative classifications, it is clear from the games above that their classification is not really related to the ones implied by the (non-cooperative) equilibrium concepts of this paper. It should also be pointed out that, already for the 3×3 games, the number of strategically different ordinal games, which are the basis for the equivalence classes of Bárány et al. (1992) (and also Rapoport et al. (1976)), amounts to over 65 billion games (see Bárány et al. (1992), p. 280). In contrast, there are just over 110,000 different path components separated by the degenerate games, which after applying the symmetry operations reduce to just above 530 different nondegenerate equivalence classes, which further reduce to 32 different Nash equivalence classes for the 3×3 case (see Germano (1998)).

Within the mathematics literature, Conway (2001) shows how equivalence classes of games can have a nice mathematical structure. He considers the class of so-called nim-type games, (these are two player zero sum games where agents make sequential moves, one after the other, choosing from a given set of strategies that decreases as they make moves), and shows that the set of equivalence classes of these nim-type games can be identified with the space of nonstandard or surreal numbers. This shows that equivalence classes of games can correspond to interesting mathematical objects. (For further literature on nim-type and related games see also Guy (1991) and Berlekamp et al. (2003).)

Partly motivated by literature aiming to extend certain results from potential games to larger classes of games, Morris and Ui (2002) define and characterize notions of best-reply, better-reply, and von Neumann and Morgenstern equivalence. These are increasingly restrictive equivalence notions, and the weakest one, best-reply equivalence, which obtains when two

games have the same best-reply correspondence, is already a substantially stronger notion than our notion of nondegenerate equivalence. Already within 2×2 games, best-reply equivalence distinguishes uncountably many matching pennies and coordination games. (See Morris and Ui (2002) for related literature and applications.)

Also related are the notions of analogy or similarity, typically studied within the context of decision situations. Besides an extensive literature in psychology on analogical and case-based reasoning, see for example Holyoak and Thagard (1995) and references there, related work has also been done in the economics literature. Rubinstein (1988) defines a notion of similarity, which he uses to define rules of thumb via a procedural rationality approach. Gilboa and Schmeidler (1995, 2001) formulate a case-based decision theory which, in a general way, allows decision makers to make use of similarities when assessing the performance of different strategies. Similarity functionals are also derived from agents' preferences and memories. Samuelson (2001) models analogies as part of a model which studies the consequences of individuals economizing on their reasoning resources; van Huyck and Battalio (2002) test whether experimental subjects use certain rules to play similar, but different, coordination games; Jehiel (2002) develops an equilibrium concept, analogy-based expectation equilibrium, which is also based on agents using analogies to obtain simplified representations of games and of reactions of their opponents.

6 Conclusion

We study a procedure for identifying games into broad and typically finitely many well-defined equivalence classes. These constitute a classification of the games relative to the given equilibrium concept, thus reflecting basic aspects of the logic and rationality of the underlying equilibrium concept. The overall procedure gives an alternative way of comparing and testing equilibrium concepts as well as of thinking about strategic situations through the geometries of equilibrium correspondences.

At the same time, many questions are left unanswered. First, understanding the structure of the set of equivalence classes for rationalizable, Nash, correlated, or robust equilibria for any given space of normal form games, and to see how these change when strategies and/or players are added is not an easy task. Our analysis focused on small two player games and it seems worthwhile studying larger games in more detail. It would also be interesting to obtain a clearer understanding of general relationships between the rationalizable, correlated, Nash, and robust equivalence classes.

The stronger notion of nondegenerate equivalence was seen to provide a first basic link at least for the first three concepts; the Hannan sets, which are compact convex polyhedra containing the correlated equilibrium sets (Hart and Mas-Colell (2003)) may also be worth studying in this context.

On the other hand, a somewhat unsatisfactory aspect of the present analysis is that, while it leads to finitely many equivalence classes (e.g., when the correspondence is semi-algebraic), the number of classes distinguished may increase very rapidly with the number of players and/or strategies. For example, McKelvey and McLennan (1997) show that games with ten players with two strategies each, can have up to 1.3 million completely mixed (regular) equilibria; (see Keiding (1997), McLennan (1997), and von Stengel (1999) for more literature on the maximal numbers of Nash equilibria.) Although it needs to be understood exactly to what extent such numbers are related to say the number of Nash equivalence classes, it nonetheless suggests that such numbers may also be very large.

Something that has not been considered here, is whether the classifications can give insights about degenerate or extensive form games and refinements of equilibria. One could look at (standard) equivalence classes around given degenerate games. A further possibility would be to consider refinements concepts and define equivalence classes directly on spaces of degenerate or extensive form games, i.e., apply the procedure to concepts like perfect or sequential equilibria on these (sub)spaces of games.

The general procedure of defining equivalence classes from discontinuities of (equilibrium) correspondences could also be applied to cooperative games or games in characteristic function form and respective solution concepts. In fact, one could use transformations, mapping normal form games (assuming transferable payoffs) to characteristic function games (i.e., cooperative games with transferable payoffs; see e.g., Myerson (1991)), and apply the procedure to cooperative solution concepts such as the core to obtain classifications of both the characteristic function and the normal form games.

But while the procedure applies to almost arbitrary spaces of games and equilibrium concepts, it may be useful to find ways of defining equivalence classes that are as unrestricted as possible by the specifics of the actual game form or of the underlying space of games; as e.g., the notion of a coordination or competitive game does not depend on whether the game is a 2×2 or a $3 \times 3 \times 3$ game. Despite some of its rigidity, we view the present approach as a first step. More in this direction, but within a different context, Samuelson (2001) uses automata to characterize agents and thus distinguishes fairly different types of games such as tournaments, ultimatum and bargaining games based on complexity measures of the automata; Gilboa

and Schmeidler (2001) and Jehiel (2002) contain yet other approaches of implicitly identifying games into potentially broad and unrestricted classes.

Appendix

Proof of Lemma 1. For the sake of completeness, we prove the following more comprehensive statement.

Lemma 6 *The correspondences $e_{RAT}, e_{CE}, e_{dCE}, e_{NE}$, and e_{ND} are all semi-algebraic and hence there exist closed lower dimensional subsets of the space of games, $D_{RAT}, D_{CE}, D_{dCE}, D_{NE}, D_{ND} \subset \mathbb{R}^{\kappa n}$, such that the correspondences are continuous respectively on the complements $\mathbb{R}^{\kappa n} \setminus D_{RAT}, \mathbb{R}^{\kappa n} \setminus D_{CE}, \mathbb{R}^{\kappa n} \setminus D_{dCE}, \mathbb{R}^{\kappa n} \setminus D_{NE}$, and $\mathbb{R}^{\kappa n} \setminus D_{ND}$. Moreover, the complements all consist of a finite number of connected components.*

We only need to check that the correspondences are semi-algebraic; the remaining statements are standard, see, e.g., Schanuel et al. (1991) and Blume and Zame (1994).

(i) e_{RAT} : For all $i \in I$, let $\Sigma_i^0 = \Sigma_i$ and recursively define

$$\Sigma_i^{t+1} = \{\sigma_i \in \Sigma_i^t \mid \exists \sigma_{-i} \in \times_{j \neq i} \tilde{\Sigma}_j^t, \forall \sigma'_i \in \Sigma_i^t, \text{ with } \gamma_i(\sigma_i, \sigma_{-i}) \geq \gamma_i(\sigma'_i, \sigma_{-i})\},$$

where $\tilde{\Sigma}_j^t$ is the convex hull of Σ_j^t . Then the set of rationalizable strategies of player i is the set $R_i = \bigcap_{t=0}^{\infty} \Sigma_i^t$. However, as Pearce (1984) shows, the elimination procedure ends after a finite number of iterations for finite games, which implies that each R_i is a semi-algebraic set and so also e_{RAT} .

(ii) e_{CE} and e_{dCE} : The case of e_{CE} follows directly from Eq. (2) below, which shows that the set of correlated equilibria is defined by a finite number of linear inequalities. This also implies that e_{dCE} must be semi-algebraic, since computing the dimension of a finite polyhedron is a semi-algebraic operation.

(iii) e_{NE} : This is shown in Schanuel et al. (1991).

(iv) e_{ND} : This follows from the fact that checking whether a game is nondegenerate or not can be reduced to computing determinants of a finite number of matrices, see von Stengel (2002), p. 1740, which are semi-algebraic operations. \square

Proof of Lemma 2. It needs to be shown that the equivalence relation is reflexive, symmetric, and transitive. This is straightforward, see Germano (1998). \square

Proof of Proposition 1. Let $\gamma_0, \gamma_1 \in \mathbb{R}^{2\kappa}$ be two two-player games with $\gamma_0 \sim_{ND} \gamma_1$. By definition of \sim_{ND} both games are nondegenerate. We need to show that (i) γ_0, γ_1 have no pure strategies that are weakly dominated or payoff equivalent to a mixed strategy that is not strictly dominated; (ii) $\gamma_0 \sim_{RAT} \gamma_1$; (iii) $\gamma_0 \sim_{CE} \gamma_1$; and (iv) $\gamma_0 \sim_{NE} \gamma_1$.

(i) This is shown in von Stengel (2002), p. 1740.

(ii) Since a nondegenerate game has neither redundant nor weakly dominated strategies that are not strictly dominated strategies, and since all games along the path between γ_0 and $\psi(\gamma_1)$ are nondegenerate, the number of strategies surviving iterated strict dominance must be constant along the path and hence $\gamma_0 \sim_{RAT} \gamma_1$.

(iii) To see that two nondegenerate equivalent games are correlated equivalent, we first recall the following well-known lemma, (see for example Fudenberg and Tirole (1991), Ch. 2, p. 57).

Lemma 7 *Let $\gamma = (A, B) \in \mathbb{R}^{2mn}$ be a two player game in normal form. Then $p \in \mathbb{R}^{mn}$ is a correlated equilibrium of γ if and only if it is a solution to the linear inequalities:*

$$Cp \geq 0, \quad 1_{mn}p = 1, \quad \text{and } p \geq 0, \quad (2)$$

where C is the $(m(m-1) + n(n-1)) \times mn$ matrix defined by:

$$\left(\begin{array}{cccccccc} a_{11} - a_{21} & \cdots & a_{1n} - a_{2n} & & & & & \\ \vdots & & \vdots & & & & & 0 \\ a_{11} - a_{m1} & \cdots & a_{1n} - a_{mn} & & & & & \\ & & & \ddots & & & & \\ & & & & a_{m1} - a_{11} & \cdots & a_{mn} - a_{1n} & \\ & & & & \vdots & & \vdots & \\ & & & & a_{m1} - a_{m-1,1} & \cdots & a_{mn} - a_{m-1,n} & \\ b_{11} - b_{12} & \cdots & 0 & & b_{m1} - b_{m2} & \cdots & 0 & \\ \vdots & & \vdots & \cdots & \vdots & & \vdots & \\ b_{11} - b_{1n} & \cdots & 0 & & b_{m1} - b_{mn} & \cdots & 0 & \\ & & \ddots & & & & \ddots & \\ 0 & \cdots & b_{1n} - b_{11} & & 0 & \cdots & b_{mn} - b_{m1} & \\ \vdots & & \vdots & \cdots & \vdots & & \vdots & \\ 0 & \cdots & b_{1n} - b_{1,n-1} & & 0 & \cdots & b_{mn} - b_{m,n-1} & \end{array} \right).$$

Let E be the matrix consisting of all rows of $(C \ 1_{mn} \ I_{mn})^T$, that hold with equality for all vectors $p \in \mathbb{R}^{mn}$ satisfying (2) at a given game γ .¹¹ Then, as a direct application of a well-known result from polyhedral theory, (see, e.g., Nemhauser and Wolsey (1988), Ch. 2, p. 87), it follows that the dimension of the set of correlated equilibria of γ is equal to $mn - \text{rank}(E)$. Therefore, it suffices to show that the rank of E is locally constant at γ . We will show that it cannot increase. To avoid confusion, we now fix E to consist of all rows holding with equality at the given, nondegenerate game γ , so that as we change γ , we keep the set of rows of E fixed while allowing the entries to vary. This means that the only way the dimension of the correlated equilibria can increase as γ changes is (i) that the rank of E increases or (ii) that there exists a row of $(C \ 1_{mn} \ I_{mn})^T$ not belonging to E at γ that holds with equality for all correlated equilibria of a close by game γ' and moreover is linearly independent from the rows of E at γ' . We show that neither case can arise if γ is nondegenerate. First, notice that if a row of $(C \ 1_{mn} \ I_{mn})^T$ does not belong to E , then it must be linearly independent from the rows of E at γ . And if it is linearly independent from E , and the span of the rows of E does not increase with small perturbations of γ , it will continue to be linearly independent from the rows of E , and moreover, there will be correlated equilibria of the perturbed games at which such a row will continue to hold with strict inequality. Therefore, it remains to show that the span of the rows of E cannot increase with slight perturbations of γ when γ is nondegenerate.

Suppose that γ is nondegenerate. Then, by Proposition 3, it has neither redundant nor weakly dominated strategies that are not strictly dominated. To see that the number of linearly independent rows of E cannot increase locally, we make use of the following.

Lemma 8 *Let $\gamma = (A, B) \in \mathbb{R}^{2mn}$ be nondegenerate two player game in normal form and C as above, let D be any submatrix obtained from $(C \ 1_{mn})^T$ by deleting all rows corresponding to strictly dominated strategies and possibly more,¹² then D has full row rank, i.e., $\text{rank}(D) = \min \{mn, \text{number of rows of } D\}$.*

Before proving this lemma, we finish the proof of the theorem. Notice that any strictly dominated strategy has probability zero at any correlated equilibrium, which implies that any row of C corresponding to a strictly dominated strategy either holds trivially with equality for all correlated

¹¹We say a row of C or I_{mn} holds with equality or strict inequality at γ , if the corresponding inequality of (2) holds with equality or with strict inequality.

¹²We say a row of C corresponds to a strategy $s_k^i \in S_i$ if it contains a_{ks} , for some $s = 1, \dots, n$, when $i = 1$ or if it contains b_{tk} for some $t = 1, \dots, m$, when $i = 2$.

equilibria (i.e., because all corresponding entries of the correlated equilibria are zero), or it holds with strict inequality for some correlated equilibrium. (This can be seen by inspection of the matrix C above.) This implies that all the rows of E that correspond to strictly dominated strategies are ones where all corresponding entries of all correlated equilibria are zero, which in turn implies that the corresponding rows of I_{mn} must also be contained in E . In other words the rows of E that correspond to strictly dominated strategies are spanned by corresponding rows of I_{mn} . Hence the span of these rows together with the corresponding rows of I_{mn} cannot be increased by varying γ locally. It remains to see whether it is possible to increase the span of the remaining rows, i.e., whether it is possible to increase the rank of the matrix D obtained from E after deleting all rows corresponding to strictly dominated strategies as well as the corresponding rows of I_{mn} . By the lemma above, such a matrix has full row rank. This implies that it is not possible, by perturbing its entries to increase its rank. As a consequence, perturbing γ will not increase the rank of such a submatrix D . But then, we have shown that perturbing γ will not affect the rank of E , and hence the dimension of the set of correlated equilibria must remain locally constant.

(iv) Finally, to see that two nondegenerate equivalent games are Nash equivalent, it follows from van Damme (1991), Ch. 3, pp. 52-55, that at a nondegenerate game all Nash equilibria are locally unique and hence ϵ_{NE} is continuous at such games. \square

Proof of Lemma 4. It is shown in the text that at most there can be the equivalence classes $[\gamma_1], [\gamma_2], [\gamma_4], [\gamma_5]$. Direct inspection of the payoffs, shows that games in different classes cannot be nondegenerate equivalent since ties must occur in the payoffs when going across the different classes. \square

Proof of Lemma 5. This follows from Kajii and Morris (1997a, b). \square

Proof of Lemma 8. It suffices to consider the statement of the lemma for the case where D is the full matrix C , assuming that the original game has no strictly dominated strategies. For, if a game is nondegenerate, then, the reduced game obtained after iteratedly eliminating all strictly dominated strategies will continue to be nondegenerate. Moreover, it will have neither redundant nor weakly dominated strategies. But then, inspection of the matrix C shows that it must have full row rank since the columns of any set of at least mn rows of C are linearly independent. To see this it may be useful to rewrite the matrix C as the difference of two matrices C_1 and C_2 in the obvious way, and recall that regularity of a game with no strictly dominated strategies implies that the submatrices of A and B^T appearing

in C_2 will have full row rank (see von Stengel (2002), p. 1740). This implies that any set of at most mn rows of $C = C_1 - C_2$ are linearly independent, and the statement of the lemma follows. \square

References

- [1] Ahn, T.K., Ostrom, E., Schmidt, D., Shupp, R., and J. Walker (2002) “Cooperation in PD Games: Fear, Greed, and History of Play,” *Public Choice*, **106**: 137-155.
- [2] Bárány, I., J. Lee, and M. Shubik (1992) “Classification of Two-Person Ordinal Bimatrix Games,” *International Journal of Game Theory*, **21**: 267-290.
- [3] Battalio, R., L. Samuelson, and J. van Huyck (2001) “Optimization Incentives and Coordination Failure in Laboratory Stag Hunt Games,” *Econometrica*, **69**: 749-764.
- [4] Berlekamp, E.R., J.H. Conway, and R.K. Guy (2003) *Winning Ways for Your Mathematical Plays, I–IV*, 2nd ed., A.K. Peters, San Jose, CA.
- [5] Blume, L.E. and W.R. Zame (1994) “The Algebraic Geometry of Perfect and Sequential Equilibrium,” *Econometrica*, **62**: 783-794.
- [6] Cabrales, A., R. Nagel, and R. Armenter (2002) “Equilibrium Selection through Incomplete Information in Coordination Games: An Experimental Study,” UPF Working Paper No. 601.
- [7] Carlsson, H. and E. van Damme (1993) “Global Games and Equilibrium Selection,” *Econometrica*, **61**: 989-1018.
- [8] Conway, J.H. (2001) *On Numbers and Games*, 2nd ed., A.K. Peters, San Jose, CA.
- [9] Crawford, V.P. (1997) “Theory and Experiment in the Analysis of Strategic Interaction,” in D.M. Kreps and K.F. Wallis (eds.), *Advances in Economics and Econometrics: Theory and Applications, I, Seventh World Congress of the Econometric Society*, Cambridge University Press, Cambridge.
- [10] Erev, I., and A.E. Roth (1998) “Modeling how people play games: Reinforcement Learning in Experimental Games with Unique Mixed Strategy Equilibria,” *American Economic Review*, **88**: 848-881.

- [11] Friedman, D. (1996) "Equilibrium in Evolutionary Games: Some Experimental Results," *Economic Journal*, **106**: 1-25.
- [12] Fudenberg, D. and J. Tirole (1991) *Game Theory*, MIT Press, Cambridge, MA.
- [13] Germano, F. (1998) "On Nash Equivalence Classes of Generic Normal Form Games," CORE Discussion Paper No. 9833.
- [14] Gilboa, I., and D. Schmeidler (2001) "Case-Based Decision Theory," *Quarterly Journal of Economics*, **110**: 605-639.
- [15] Gilboa, I., and D. Schmeidler (2001) *A Theory of Case-Based Decisions*, Cambridge University Press, Cambridge.
- [16] Guy, R.K. (1991) "What is a Game?," *Proceedings of Symposia in Applied Mathematics*, **43**: 1-22.
- [17] Harris, R.J. (1972) "An Interval Scale Classification for all 2×2 Games," *Behavioral Science*, **17**: 371-383.
- [18] Harsanyi, J., and R. Selten (1988) *A General Theory of Equilibrium Selection in Games*, MIT Press, Cambridge, MA.
- [19] Hart, S., and A. Mas-Colell (2003) "Regret-Based Continuous-Time Dynamics," *Games and Economic Behavior*, forthcoming.
- [20] Holyoak, K.J., and P. Thagard (1995) *Mental Leaps: Analogy in Creative Thought*, MIT Press, Cambridge, MA.
- [21] Jehiel, P. (2002) "Analogy-Based Expectation Equilibrium," Mimeo, CERAS, Paris.
- [22] Kajii, A. and S. Morris (1997a) "The Robustness of Equilibria to Incomplete Information," *Econometrica*, **65**: 1283-1309.
- [23] Kajii, A. and S. Morris (1997b) "Refinements and Higher Order Beliefs: A Unified Survey," Mimeo, Yale University.
- [24] Keiding, H. (1997) "On the Maximal Number of Nash Equilibria in an $n \times n$ Bimatrix Games," *Games and Economic Behavior*, **21**: 148-160.
- [25] McKelvey, R.D., and A. McLennan (1997) "The Maximal Number of Regular Totally Mixed Nash Equilibria," *Journal of Economic Theory*, **72**: 411-425.

- [26] McLennan, A. (1997) “The Maximal Generic Number of Pure Nash Equilibria,” *Journal of Economic Theory*, **72**: 408-410.
- [27] Mertens, J.-F. (1987) “Ordinality in Non-Cooperative Games,” CORE Discussion Paper No. 8782.
- [28] Morris, S., R. Rob, and H.S. Shin (1995) “ p -Dominance and Belief Potential,” *Econometrica*, **63**: 145-157.
- [29] Morris, S., and H.S. Shin (2003) “Global Games: Theory and Applications,” in M. Dewatripont, L.P. Hansen, S.J. Turnovsky (eds.), *Advances in Economics and Econometrics: Theory and Applications, Eighth World Congress of the Econometric Society*, Cambridge University Press, Cambridge.
- [30] Morris, S., and T. Ui (2002) “Best Response Equivalence,” Mimeo, Yale University and Yokohama National University.
- [31] Myerson, R.B. (1991) *Game Theory*, Harvard University Press, Cambridge, MA.
- [32] Nash, J. (1951) “Non-Cooperative Games,” *Annals of Mathematics*, **54**: 286-295.
- [33] Nemhauser, G.L. and L.A. Wolsey (1988) *Integer and Combinatorial Optimization*, Wiley and Sons, New York.
- [34] Ochs, J. (1995) “Coordination Problems,” in J.H. Kagel and A.E. Roth (eds.), *Handbook of Experimental Economics*, Princeton University Press, Princeton, NJ.
- [35] Pearce, D.G. (1984) “Rationalizable Strategic Behavior and the Problem of Perfection,” *Econometrica*, **52**: 1029-1050.
- [36] Rapoport, A., and M. Guyer (1966) “A Taxonomy of 2×2 Games,” *General Systems*, **11**: 203-214.
- [37] Rapoport, A., M.J. Guyer, and D.G. Gordon (1976) *The 2×2 Game*, University of Michigan Press, Ann Arbor.
- [38] Roth, A.E. (1995) “Introduction to Experimental Economics,” in J.H. Kagel and A.E. Roth (eds.), *Handbook of Experimental Economics*, Princeton University Press, Princeton, NJ.
- [39] Rubinstein, A. (1988) “Similarity and Decision-Making Under Risk,” *Journal of Economic Theory*, **46**: 143-153.

- [40] Samuelson, L. (2001) “Analogies, Adaptation, and Anomalies,” *Journal of Economic Theory*, **97**: 320-366.
- [41] Schanuel, S.H., L.K. Simon, and W.R. Zame (1991) “The Algebraic Geometry of Games and the Tracing Procedure,” in R. Selten (ed.), *Game Equilibrium Models, II: Methods, Morals, and Markets*, Springer Verlag, Berlin.
- [42] Straub, P.G. (1995) “Risk Dominance and Coordination Failures in Static Games,” *The Quarterly Review of Economics and Finance*, **35**: 339-363.
- [43] Van Damme, E. (1991) *Stability and Perfection of Nash Equilibria*, 2nd ed., Springer Verlag, Berlin.
- [44] Van Damme, E. (2002) “Strategic Equilibrium,” in R. Aumann and S. Hart (eds.), *Handbook of Game Theory, III*, Elsevier, Amsterdam.
- [45] Van Huyck, J., and R. Battalio (2002) “Prudence, Justice, Benevolence, and Sex: Evidence from Similar Bargaining Games,” *Journal of Economic Theory*, **104**: 227-246.
- [46] Von Stengel (1999) “New Maximal Numbers of Equilibria in Bimatrix Games,” *Discrete and Computational Geometry*, **21**: 557-568.
- [47] Von Stengel, B. (2002) “Computing Equilibria for Two-Person Games,” in R. Aumann and S. Hart (eds.), *Handbook of Game Theory, III*, Elsevier, Amsterdam.