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Resource-Monotonicity for House Allocation Problems

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Abstract

We study a simple model of assigning indivisible objects (*e.g.*, houses, jobs, offices, *etc.*) to agents. Each agent receives at most one object and monetary compensations are not possible. We completely describe all rules satisfying *efficiency*, *independence of irrelevant objects*, and *resource-monotonicity*. The characterized rules assign the objects in a sequence of steps such that at each step there is either a dictator or two agents "trade" objects from their hierarchically specified "endowments."

JEL Classification: D63, D70

Keywords: indivisible objects, resource-monotonicity.

1   Introduction

We study the problem of allocating heterogeneous indivisible objects among a group of agents (for instance, houses, jobs, or offices) when monetary compensations are not possible. Agents have strict preferences over objects and

remaining unassigned. An assignment is an allocation of the objects to the agents such that every agent receives at most one object. A rule associates an assignment to each preference profile. This problem is known as "house allocation" and it is the subject of recent papers by Abdulkadiroğlu and Sönmez (1998,1999), Bogomolnaia and Moulin (2001), Ehlers (2002), Ehlers and Klaus (2002a,b), Ehlers, Klaus, and Pápai (2002), Ergin (2000,2002), Kesten (2003a,b), Pápai (2000,2001), and Svensson (1999).[1]

We consider situations where resources may change. When the change of the environment is exogenous, it would be unfair if the agents who were not responsible for this change were treated unequally. We apply this idea of solidarity and require that if additional resources become available, then either all agents (weakly) gain or they all (weakly) lose. This requirement is called *resource-monotonicity* (Chun and Thomson, 1988). Two recent studies of *resource-monotonicity* for economic environments with indivisibilities are Ehlers and Klaus (2003) for multiple assignment problems and Thomson (2003a) for the assignment of indivisible objects with money.[2] Ehlers and Klaus (2003) show that the only rules satisfying *resource-monotonicity* in combination with other desirable properties (*efficiency* and *coalitional strategy-proofness*) are serial dictatorships. Thomson (2003a) demonstrates that *resource-monotonicity* is not compatible with other desirable properties (*efficiency* and *weak identical preferences lower bound*).[3] Both results can be seen as part of a larger program devoted to the study of the possibility and the structure of solutions for resource allocation problems that satisfy appealing properties (for results and further references concerning other properties see Ehlers and Klaus (2003) and Thomson (2003a,b)).

We contribute to this line of research by applying *resource-monotonicity* to house allocation problems. Our main result is a characterization of a class of rules, called *mixed dictator-pairwise-exchange rules*, by *efficiency*, *independence of irrelevant objects*,[4] and *resource-monotonicity*. Mixed dictator-pairwise-exchange rules are essentially hierarchical since they allow "trading" of the objects by at most two agents at a time. Therefore, our result implies that *efficiency* and *resource-monotonicity* are essentially only feasi-

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[1]This list is not exhaustive.

[2]For the assignment of indivisible objects with money, Alkan, Demange, and Gale (1991) and Alkan (1994) consider *resource-monotonicity* properties in relation with *no-envy*.

[3]However, Moulin (1992) shows that an application of the Shapley value yields a rule satisfying the properties of Thomson's incompatibility result when preferences are quasi-linear, all objects are desirable (meaning each agent's monetary value is nonnegative for each object), and the monetary transfers sum up to zero.

[4]The rule only depends on preferences over the set of available objects.

ble either in the absence of initial individual ownership or in the absence of more than two owners at a time.

In Section 2 we introduce the house allocation problem with variable resources and define our main properties for rules. In Section 3 we first present the class of mixed dictator-pairwise-exchange rules. We state and discuss our characterization of this class of rules by *efficiency, independence of irrelevant objects*, and *resource-monotonicity* in the second part of Section 3. We prove our main result in Section 4.

2 House Allocation with Variable Resources

Let N denote a finite set of agents, $|N| \geq 2$. Let K denote a set of potential real objects. Not receiving any real object is called "receiving the null object." Let 0 represent the *null object*. Each agent $i \in N$ is equipped with a preference relation R_i over all objects $K \cup \{0\}$. Given $x, y \in K \cup \{0\}$, $x\, R_i\, y$ means that agent i weakly prefers x to y, and $x\, P_i\, y$ means that agent i strictly prefers x to y. We assume that R_i is strict, *i.e.*, R_i is a linear order over $K \cup \{0\}$. Let $\mathcal{R}$ denote the class of all linear orders over $K \cup \{0\}$, and $\mathcal{R}^N$ the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. Given $K' \subseteq K \cup \{0\}$, let $R_i|_{K'}$ denote the restriction of R_i to K' and $R|_{K'} = (R_i|_{K'})_{i \in N}$. Let $\mathcal{R}_0 \subsetneq \mathcal{R}$ denote the class of preference relations where the null object is the worst object. That is, if $R_i \in \mathcal{R}_0$, then all real objects are "goods": for all $x \in K$, $x\, P_i\, 0$.

An *allocation* is a list $a = (a_i)_{i \in N}$ such that for all $i \in N$, $a_i \in K \cup \{0\}$, and none of the real objects in K is assigned to more than one agent. Note that 0, the null object, can be assigned to any number of agents and that not all real objects have to be assigned. Let $\mathcal{A}$ denote the set of all allocations. Let $\mathcal{H}$ denote the set of all non-empty subsets H of K. A *(house allocation) problem* consists of a preference profile $R \in \mathcal{R}^N$ and a set of real objects $H \in \mathcal{H}$. Note that the associated set of *available objects* $H \cup \{0\}$ includes the null object which is available in any economy. An *(allocation) rule* is a function $\varphi : \mathcal{R}^N \times \mathcal{H} \to \mathcal{A}$ such that for all problems $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, $\varphi(R, H) \in \mathcal{A}$ is *feasible*, *i.e.*, for all $i \in N$, $\varphi_i(R, H) \in H \cup \{0\}$. By feasibility, each agent receives an available object. Given $i \in N$, we call $\varphi_i(R, H)$ the *allotment* of agent i at $\varphi(R, H)$.

A natural requirement for a rule is that the chosen allocation depends only on preferences over the set of available objects.

Independent of Irrelevant Objects: For all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$ and all $R' \in \mathcal{R}^N$ such that $R|_{H \cup \{0\}} = R'|_{H \cup \{0\}}$, $\varphi(R, H) = \varphi(R', H)$.

Next, a rule chooses only (Pareto) efficient allocations.

Efficiency: For all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, there is no feasible allocation $a \in \mathcal{A}$ such that for all $i \in N$, $a_i \, R_i \, \varphi_i(R, H)$, with strict preference holding for some $j \in N$.

When the set of objects varies, another natural requirement is *resource-monotonicity*. It describes the effect of a change in the available resource on the welfare of the agents. A rule satisfies *resource-monotonicity*, if after such a change, either all agents (weakly) lose or all (weakly) gain.

It is easy to see that in combination with *efficiency*, *resource-monotonicity* means that if for some fixed preference profile and some fixed set of objects, additional objects are available, then – this being good news – all agents (weakly) gain. Since we study *resource-monotonicity* together with *efficiency* we use the latter notion to formalize *resource-monotonicity*.

Resource-Monotonicity: For all $R \in \mathcal{R}^N$ and all $H, H' \in \mathcal{H}$, if $H \subseteq H'$, then for all $i \in N$, $\varphi_i(R, H') \, R_i \, \varphi_i(R, H)$.

3 Mixed Dictator-Pairwise-Exchange Rules

Our aim is to describe the class of rules that are *efficient*, *independent of irrelevant objects*, and *resource-monotonic*. We show that each such rule allocates the available objects in a sequence of steps as follows: At the first step there is either an agent who receives for all problems his most preferred object from the set of available objects—we call such an agent a dictator—or there are exactly two agents who partition the set of available real objects among themselves and for all problems their allotments result from a pairwise exchange using this partition as endowments. At the second step there is again either a dictator or a pairwise exchange (restricted to the remaining available objects); and so on. Here, we call such a rule a *mixed dictator-pairwise-exchange rule* (Ehlers, 2002). In Ehlers, Klaus, and Pápai (2002) we discuss essentially the same class of rules under the name "restricted endowment inheritance rules." After formally defining mixed dictator-pairwise-exchange rules, we briefly discuss another interpretation of these rules as so-called efficient priority rules.

For the formal description we use "(endowment) inheritance tables" (Pápai, 2000). For each real object $x \in K$, a one-to-one function $\pi_x : \{1, \ldots, |N|\} \to N$ specifies the inheritance of object x. Here agent $\pi_x(1)$ is initially endowed with x. If x is still available after $\pi_x(1)$ received an object, then agent $\pi_x(2)$ inherits x; and so on. Let Π^N denote the set of

all one-to-one functions from $\{1, \ldots, |N|\}$ to N. An *inheritance table* is a profile $\pi = (\pi_x)_{x \in K}$ specifying the inheritance of each real object. We call an inheritance table π a *mixed dictator-pairwise-exchange inheritance table with respect to* $S = (S_1, S_2, \ldots, S_m)$ if

(i) $(S_1, S_2, \ldots, S_m)$ is a partition of N into singletons and pairs, *i.e.*, for all $t \in \{1, \ldots, m\}$, $2 \geq |S_t| \geq 1$,

(ii) row 1 and row $|S_1|$ of the inheritance table contain exactly S_1, *i.e.*, $\{\pi_x(1) \,|\, x \in K\} = S_1$ and $\{\pi_x(|S_1|) \,|\, x \in K\} = S_1$, and

(iii) row $(1 + \sum_{l=1}^{t-1} |S_l|)$ and row $(\sum_{l=1}^{t} |S_l|)$ contain exactly S_t, *i.e.*, for all $t \in \{2, \ldots, m\}$, $\{\pi_x(1 + \sum_{l=1}^{t-1} |S_l|) \,|\, x \in K\} = S_t$ and $\{\pi_x(\sum_{l=1}^{t} |S_l|) \,|\, x \in K\} = S_t$.

Given $i \in N$ and $H \in \mathcal{H}$, let $\text{top}(R_i, H)$ denote agent i's most preferred object under R_i in $H \cup \{0\}$.

Mixed Dictator-Pairwise-Exchange Rules, $\varphi^{(\pi,S)}$: Given a mixed dictator-pairwise-exchange inheritance table π with respect to $S = (S_1, S_2, \ldots, S_m)$, for all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$ the allocation $\varphi^{(\pi,S)}(R, H)$ is inductively determined as follows:

Step 1:

(a) If $S_1 = \{i\}$, then $\varphi_i^{(\pi,S)}(R, H) = \text{top}(R_i, H)$.
(b) Let $S_1 = \{i, j\}$ $(i \neq j)$. If $\text{top}(R_i, H) = \text{top}(R_j, H) \equiv h_1 \in H$ and $\pi_{h_1}(1) = i$, then $\varphi_i^{(\pi,S)}(R, H) = h_1$ and $\varphi_j^{(\pi,S)}(R, H) = \text{top}(R_j, H \backslash \{h_1\})$. Otherwise, $\varphi_i^{(\pi,S)}(R, H) = \text{top}(R_i, H)$ and $\varphi_j^{(\pi,S)}(R, H) = \text{top}(R_j, H)$.

Step t: Let $H_{t-1} = \bigcup_{i \in \left(\cup_{l=1}^{t-1} S_l \right)} \{\varphi_i^{(\pi,S)}(R, H)\}$ denote the set of objects that are assigned up to Step t.

(a) If $S_t = \{i\}$, then $\varphi_i^{(\pi,S)}(R, H) = \text{top}(R_i, H \backslash H_{t-1})$.
(b) Let $S_t = \{i, j\}$ $(i \neq j)$. If $\text{top}(R_i, H \backslash H_{t-1}) = \text{top}(R_j, H \backslash H_{t-1}) \equiv h_t \in H \backslash H_{t-1}$ and $\pi_{h_t}(1 + \sum_{l=1}^{t-1} |S_l|) = i$, then $\varphi_i^{(\pi,S)}(R, H) = h_t$ and $\varphi_j^{(\pi,S)}(R, H) = \text{top}(R_j, H \backslash (H_{t-1} \cup \{h_t\}))$. Otherwise, $\varphi_i^{(\pi,S)}(R, H) = \text{top}(R_i, H \backslash H_{t-1})$ and $\varphi_j^{(\pi,S)}(R, H) = \text{top}(R_j, H \backslash H_{t-1})$.

Mixed dictator-pairwise-exchange rules are a subclass of endowment inheritance rules discussed in Ehlers, Klaus, and Pápai (2002).[5]

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[5]Endowment inheritance rules are based on *Gale's top trading cycle algorithm.* We

As already mentioned, instead of interpreting a mixed dictator-pairwise-exchange rule as an endowment inheritance rule, one can equivalently interpret it as a priority rule. Take the underlying inheritance table and use it as follows for each real object x: $\pi_x^{-1}(i) < \pi_x^{-1}(j)$ means "agent i has higher priority for object x than agent j." A rule violates the priority of agent i for object x if there is a preference profile under which i envies another agent j who obtains x even though i has a higher priority for x than j. A rule is a *priority rule* if it never violates the specified priorities. In an earlier version of his article, Ergin (2002) shows that efficient priority rules can be described through the so-called serial-bidictatorship algorithm, which for house allocation problems turns out to be equivalent to the algorithm underlying the corresponding mixed dictator-pairwise-exchange rules. For a further discussion of efficient priority rules we refer to Ehlers and Klaus (2002b) and Ergin (2002).[6]

Our main result also applies to the domain $\mathcal{R}_0^N$ where all real objects are "goods".

Theorem 1. *Let $|K| > |N|$. On the domain $\mathcal{R}^N$ ($\mathcal{R}_0^N$), mixed dictator-pairwise-exchange rules are the only rules satisfying efficiency, independence of irrelevant objects, and resource-monotonicity.*

We present the proof of this characterization in Section 4. Note that Theorem 1 requires that there are more potential real objects than agents, *i.e.*, $|K| > |N|$. If $|K| = \infty$, then $|K| > |N|$ is trivially satisfied. We explain how to adjust the results if $|K| \leq |N|$ at the end of the section.

Theorem 1 is a characterization based on three relatively mild requirements. However, there is only a small class of rules satisfying all three axioms. Furthermore, Theorem 1 is one of the few characterizations in house allocation problems that does not require *strategy-proofness*:[7] no agent can ever benefit from misrepresenting his preferences. We use the notation $R_{-i} = R_{N \setminus \{i\}}$. For example, $(\bar{R}_i, R_{-i})$ denotes the profile obtained from R by replacing R_i by $\bar{R}_i$.

Strategy-Proofness: For all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, all $i \in N$, and all $\bar{R}_i \in \mathcal{R}$, $\varphi_i(R, H) \, R_i \, \varphi_i((\bar{R}_i, R_{-i}), H)$.

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omit their somewhat tedious definition and refer the interested reader to Pápai (2000). It is interesting to note that on the domain $\mathcal{R}_0^N$ some inheritance tables that do not satisfy the conditions of a mixed dictator-pairwise-exchange inheritance table may still generate an endowment inheritance rule that equals a mixed dictator-pairwise-exchange rule.

[6]For a more detailed discussion and comparison of endowment inheritance and priority rules see Kesten (2003a).

[7]Another characterization without *strategy-proofness* in house allocation problems can be found in Ergin (2000).

Next we demonstrate that in fact all mixed dictator-pairwise-exchange rules satisfy the stronger non-manipulability property of *coalitional strategy-proofness*: no group of agents can ever benefit by misrepresenting their preferences.

Given $R \in \mathcal{R}^N$ and $M \subseteq N$, let R_M denote the profile $(R_i)_{i \in M}$. It is the restriction of R to the subset M of agents.

Coalitional Strategy-Proofness: For all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$ and all $M \subseteq N$, there exists no $\bar{R}_M \in \mathcal{R}^M$ such that for all $i \in M$, $\varphi_i((\bar{R}_M, R_{-M}), H) \, R_i \, \varphi_i(R, H)$, with strict preference holding for some $j \in M$.

Note that *coalitional strategy-proofness* implies *strategy-proofness* (for $i \in N$ choose $M = \{i\}$).

Lemma 1. *Coalitional strategy-proofness implies independence of irrelevant objects.*

Proof. Let $R, R' \in \mathcal{R}^N$ be such that $R|_{H \cup \{0\}} = R'|_{H \cup \{0\}}$. Without loss of generality assume that $N = \{1, \ldots, n\}$. Then, let $R^0 \equiv R$ and for all $i \in N$, $R^i \equiv (R'_i, R^{i-1}_{-i})$. Thus, $R^n = R'$.

Assume that $\varphi(R, H) \neq \varphi(R^1, H)$. If $\varphi_1(R, H) \neq \varphi_1(R^1, H)$, then either (i) $\varphi_1(R^1, H) P_1 \varphi_1(R, H)$ or (ii) $\varphi_1(R, H) P_1 \varphi_1(R^1, H)$. (i) immediately contradicts *strategy-proofness*. Since $R_1|_{H \cup \{0\}} = R'_1|_{H \cup \{0\}}$, (ii) and feasibility imply $\varphi_1(R, H) P'_1 \varphi_1(R^1, H)$ contradicting *strategy-proofness*. Thus assume that $\varphi_1(R, H) = \varphi_1(R^1, H)$. Since $\varphi(R, H) \neq \varphi(R^1, H)$, there exists $j \neq 1$ such that $\varphi_j(R, H) \neq \varphi_j(R^1, H)$. Then either (i) $\varphi_j(R^1, H) \, P_j \, \varphi_j(R, H)$ or (ii) $\varphi_j(R, H) \, P_j \, \varphi_j(R^1, H)$. (i) immediately contradicts *coalitional strategy-proofness* for $M = \{1, j\}$. Since $R_j|_{H \cup \{0\}} = R'_j|_{H \cup \{0\}}$, (ii) and feasibility imply $\varphi_j(R, H) P'_j \varphi_j(R^1, H)$ contradicting *coalitional strategy-proofness* for $M = \{1, j\}$. Thus, $\varphi(R, H) = \varphi(R^1, H)$.

Similarly, we conclude that for all $i \in N$, $\varphi(R^{i-1}, H) = \varphi(R^i, H)$ and finally, $\varphi(R, H) = \varphi(R', H)$. $\square$

Note that no direct logical relation exists between *strategy-proofness* and *independence of irrelevant objects*. In order to show this, we use two "dictatorial rules." For simplicity, assume that $N = \{1, \ldots, n\}$.

The following dictatorial rule satisfies *strategy-proofness*, but not *independence of irrelevant objects*: Fix a real object $x \in K$. Given $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, let agent 1 choose his most preferred available object. If $x \, P_1 \, 0$, then the second dictator who chooses his most preferred object among the remaining available objects is agent 2. If $0 \, P_1 \, x$, then the second dictator

is agent n. All further dictators are determined by choosing the agent with the lowest index.[8]

Next we describe a dictatorial rule satisfying *independence of irrelevant objects*, but not *strategy-proofness*. Given $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, let agent 1 be the first dictator if he finds exactly one of the available objects in K acceptable. Otherwise, let agent n be the first dictator. All further dictators are determined by choosing the agent with the lowest index.

In an earlier version of this article we characterized the class of mixed dictator-pairwise-exchange rules by *efficiency*, *resource-monotonicity*, and *coalitional strategy-proofness*. Using Lemma 1, this result is now a corollary of Theorem 1.[9]

Corollary 1. *Let $|K| > |N|$. On the domain $\mathcal{R}^N$ ($\mathcal{R}_0^N$), mixed dictator-pairwise-exchange rules are the only rules satisfying efficiency, coalitional strategy-proofness, and resource-monotonicity.*

The following corollary of Theorem 1 is implied by the fact that any mixed dictator-pairwise-exchange rule is an endowment inheritance rule and such rules are *coalitionally strategy-proof* (Pápai, 2000). Alternatively, Corollary 2 follows from Theorem 1 and the fact that any efficient priority rule is *coalitionally strategy-proof* (Ergin, 2002).

Corollary 2. *Let $|K| > |N|$. On the domain $\mathcal{R}^N$ ($\mathcal{R}_0^N$), if a rule satisfies efficiency, independence of irrelevant objects, and resource-monotonicity, then it is coalitionally strategy-proof.*

In Ehlers, Klaus, and Pápai (2002) we consider house allocation with variable sets of agents instead of variable resources. Mixed dictator-pairwise exchange rules can be naturally extended to accommodate changes in the sets of agents and the resources (see Ehlers and Klaus, 2002a). When keeping resources fixed, but allowing for changes in the sets of agents, Ehlers, Klaus, and Pápai (2002) characterize mixed dictator-pairwise exchange rules by *efficiency*, *strategy-proofness*, and the solidarity property *population-monotonicity*[10]. Similarly as in Ehlers, Klaus, and Pápai (2002) we conclude

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[8]Note that this rule satisfies *efficiency* and *resource-monotonicity*.

[9]The use of *coalitional strategy-proofness* greatly simplifies the proof, which then is very similar to the proof of the characterization presented in Ehlers, Klaus, and Pápai (2002). Kesten (2003b), using a result from Ergin (2002), presents a short proof for the general preference domain $\mathcal{R}$.

[10]If some agents leave the economy, then as a result either all remaining agents (weakly) gain or they all (weakly) lose.

that guaranteeing solidarity comes with a price. Whereas without the solidarity property agents can "trade" objects arbitrarily, *resource-monotonicity* restricts the assignment of individual property rights, and therefore, "trading" to two agents at a time.

For multiple assignment problems where agents can consume sets of objects, Ehlers and Klaus (2003) show that the only rules satisfying *efficiency*, *resource-monotonicity*, and *coalitional strategy-proofness* are serial dictatorship rules. In light of this result, our characterization is a bit more positive since some of the rules we identify are non-dictatorial. In spite of the hierarchical nature of mixed dictator-pairwise exchange rules, they are not unappealing, and they offer flexibility in selecting the hierarchy itself and choosing the splitting of the endowments in the case of "twin-dictators."

It is easy to see that *efficiency*, *independence of irrelevant objects*, and *resource-monotonicity* are logically independent. Below we discuss how our results change if $|K| \leq |N|$.

Adjustments for $|K| \leq |N|$

For our main result (Theorem 1) we require $|K| > |N|$ for technical convenience. Now suppose $|K| \leq |N|$ and denote $k \equiv |K|$ and $n \equiv |N|$. For the larger domain $\mathcal{R}$, all our results remain true. However, if $n \geq k$, then on the domain $\mathcal{R}_0$, where the null object is always ranked last, we derive a slightly larger set of rules. They are essentially mixed dictator-pairwise-exchange rules except that, loosely speaking, the last two objects may be arbitrarily inherited.

These allocation rules still require that the inheritance table reflects that at most two agents trade. However, since every agent who leaves the market receives a real object, given the preference domain $\mathcal{R}_0$, only the first k rows of the inheritance table are relevant if $n > k$. In fact, inheritance of an object by an agent in row $(k-1)$ implies that there are no more than two objects left, given that $(k-2)$ agents have already received their allotments. Thus, independently of the structure of the inheritance table after row k, these rules still do not allow trading by more than two agents. This is illustrated by the following example.

Example 1. Let $N \equiv \{1, 2, 3, 4\}$ and $K \equiv \{x, y, z\}$. Consider the inheri-

tance table π specified below.

π_x	π_y	π_z
1	1	1
2	3	4
3	4	2
4	2	3

Note that when preferences are restricted to $\mathcal{R}_0$, φ^π satisfies *efficiency*, *resource-monotonicity*, and *coalitional strategy-proofness*. Furthermore, even though the second and third rows contain three different agents each, an agent in the second row inherits an object only if agent 1 has already received a real object. Thus, exactly two agents inherit objects in the second row, and trading by three agents is excluded. $\diamondsuit$

There are two difficulties regarding the specification of mixed dictator-pairwise-exchange inheritance tables when $n \geq k$. The first one has to do with the uniqueness of the inheritance tables. When agents may rank the null object first (for preferences in $\mathcal{R}$), each mixed dictator-pairwise-exchange inheritance table π uniquely defines a mixed dictator-pairwise-exchange rule, in the sense that for two different mixed dictator-pairwise-exchange inheritance tables there always exists at least one preference profile at which the resulting allocations differ. This follows since, in situations where agents want to consume the null object and thus can leave the market without any real object, each entry in a mixed dictator-pairwise-exchange inheritance table π matters, given that the potential inheritance indicated by each entry is realized in certain cases. By contrast, if agents always rank the null object last (the case we examined here), a mixed dictator-pairwise-exchange inheritance table may not uniquely define a rule.[11] Note that in Example 1, the entire last two rows are redundant: since 2, 3, and 4 (in fact, only two of these agents) inherit one object each from 1, given the second row of the table, further inheritances will not occur at any preference profile.

The second difficulty is that when inheritances indicated by entries in the last (k-th) row of the table are not redundant, they may depend on the allocation of the objects to agents who have already received their assignments. For example, if agents 1 and 2 share exclusively the first two rows of an inheritance table, given the same setup as in Example 1, then we may specify the inheritance of object z (which is not redundant in this case) as follows: let 3 inherit z if 1 receives x and 2 receives y, and let 4 inherit z

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[11]This possibility of non-uniqueness is reflected in the definition of the more general class of endowment inheritance rules. For a further discussion of uniqueness see Pápai (2000).

if 1 receives y and 2 receives x. This "history-dependent" specification of inheritance, unlike in any other case, does not violate *resource-monotonicity* in the current case, since on the preference domain $\mathcal{R}_0$ agents 3 or 4 never inherit from 1 or 2 if there is less than the full set of three objects to allocate. Note that this type of conditional inheritance cannot be described by an inheritance table, and that in fact these rules do not form a subclass of the endowment inheritance rules.[12]

Both of the above difficulties are avoided if $n < k$, that is, if the full set of real objects contains more objects than there are agents. The reason for this is that in this case, *resource-monotonicity* has implications for the last two rows of the inheritance table as well. This is even true on the smaller preference domain $\mathcal{R}_0$, which precludes agents who have endowments from leaving the market without a real object. Note, furthermore, that the rules satisfying the required properties when $n \geq k$ are not significantly different from the mixed dictator-pairwise-exchange rules: the differences only concern the allocation of the last two objects, and these rules only offer more flexibility in choosing the last two recipients, while trade is still restricted to at most two agents. It follows from the proof of Theorem 1 that the first $(k-1)$ rows of the inheritance table are still uniquely defined[13] and up to row $(k-2)$, there is always a dictator or a pairwise exchange. In order to avoid tedious details, we state and prove our theorem on the preference domain $\mathcal{R}_0$ for the case $n < k$. Finally, note that we do not compromise much with this assumption, since it is a very reasonable assumption in the context of variable resources: it simply says that we have more *potentially* available real objects than the fixed number of agents, but, since resources may vary, it may be the case that the set of *actually* available real objects contains fewer objects than there are agents.

4 Proof of Theorem 1

It is easy to verify that mixed dictator-pairwise-exchange rules satisfy *efficiency*, *independence of irrelevant objects*, and *resource-monotonicity* since no more than two agents "trade" at any step. In proving the converse, let $|K| > |N|$ and let φ be a rule satisfying *efficiency*, *independence of irrelevant*

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[12]The *hierarchical exchange rules* defined in Pápai (2000) include "history-dependent" inheritances and thus further explanations may be found there.

[13]The $(k-1)$th row is defined by taking $(k-1)$ objects and assigning $(k-2)$ objects to the first $(k-2)$ agents. Since there are k objects, there is one real object which is not available in the economy. This object can then be used to show that the definition of the $(k-1)$th row is independent of the $(k-2)$ objects assigned to the first $(k-2)$ agents.

objects, and *resource-monotonicity*. We give all proofs for the domain $\mathcal{R}_0$, since the proofs for the larger domain $\mathcal{R}$ are completely analogous.

Since all agents always rank the null object last, for all $H \in \mathcal{H}$ and all $R, R' \in \mathcal{R}_0^N$ such that $R|_H = R'|_H$ we have

$$\varphi(R, H) = \varphi(R', H). \tag{1}$$

Equation (1) reflects *independence of irrelevant objects* on $\mathcal{R}_0$, denoted by *IIO* in the sequel.

We prove that we can calculate the allocations assigned by φ in a sequence of steps that correspond to the algorithm for a unique mixed dictator-pairwise-exchange rule.

1. At most two agents trade in Step 1

Let $R \in \mathcal{R}_0^N$. For all $h \in K$, we define $f^h(1) \equiv i$ if and only if $\varphi_i(R, \{h\}) = h$. By *efficiency*, $f^h(1)$ is well-defined. Note that (1) and the fact that the null object is ranked last imply that the definition of $f^h(1)$ is independent of the choice of R. We call agent $f^h(1)$ the *dictator over object h* and define $\pi_h(1) \equiv f^h(1)$.

The first lemma proves that the first row of the inheritance table contains at most two agents.

Lemma 2. $\left|\{f^h(1) \mid h \in K\}\right| \leq 2$.

Proof. Suppose that $\left|\{f^h(1) \mid h \in K\}\right| \geq 3$. Let $1, 2, 3 \in N$, $a, b, c \in K$, and

$$\begin{aligned}
\varphi_1(R, \{a\}) &= a, \\
\varphi_2(R, \{b\}) &= b, \\
\varphi_3(R, \{c\}) &= c.
\end{aligned}$$

Let $R' \in \mathcal{R}_0^N$ be such that

$$\begin{aligned}
b \, P_1' \, c \, P_1' \, a, \\
a \, P_2' \, b \, P_2' \, c, \\
a \, P_3' \, c \, P_3' \, b.
\end{aligned}$$

Since $R|_{\{a\}} = R'|_{\{a\}}$, $\varphi_1(R', \{a\}) = a$. Similarly, $\varphi_2(R', \{b\}) = b$ and $\varphi_3(R', \{c\}) = c$. Hence, by *efficiency* and *resource-monotonicity*,

$$\begin{aligned}
\varphi_1(R', \{a, b\}) &= b, \\
\varphi_2(R', \{a, b\}) &= a, \\
\varphi_1(R', \{a, c\}) &= c, \\
\varphi_3(R', \{a, c\}) &= a.
\end{aligned}$$

Now, *resource-monotonicity* yields the desired contradiction $\varphi_2(R', \{a, b, c\}) = a$ and $\varphi_3(R', \{a, b, c\}) = a$. $\square$

Let $a, b \in K$ be such that $a \neq b$. Let $\bar{R} \in \mathcal{R}_0^N$ be such that $a \, \bar{P}_{f^a(1)} \, b$ and for all $i \in N \backslash \{f^a(1)\}$, $b \, \bar{P}_i \, a$. We define $f_b^a(2) \equiv i$ if and only if $\varphi_i(\bar{R}, \{a, b\}) = b$.

The following argument shows that $f_b^a(2) \in N \backslash \{f^a(1)\}$. By *IIO*, $\varphi_{f^a(1)}(\bar{R}, \{a\}) = a$. Thus, by $a \, \bar{P}_{f^a(1)} \, b$ and *resource-monotonicity*, $\varphi_{f^a(1)}(\bar{R}, \{a, b\}) = a$. Then *efficiency* implies that $f_b^a(2) \in N \backslash \{f^a(1)\}$. Similarly as in Lemma 2 the following holds.

Lemma 3. $|\{f_b^a(2) \mid b \in K \backslash \{a\}\}| \leq 2$.

Assume that agent i is the dictator over objects a and b, i.e., $i = f^a(1) = f^b(1)$. Then, a second dictator over a third object does not depend on which of the first two objects agent i picks.

Lemma 4. *Let $a, b \in K$ be such that $f^a(1) = f^b(1)$. Then for all $c \in K \backslash \{a, b\}$ we have $f_c^a(2) = f_c^b(2)$.*

Proof. Without loss of generality, let $f^a(1) = f^b(1) = 1$ and $f_c^a(2) = 2$. Let $R \in \mathcal{R}_0^N$ be such that $b \, P_1 \, a \, P_1 \, c$ and for all $i \in N \backslash \{1\}$, $\text{top}(R_i, K) = c$. By $f^a(1) = 1$, *IIO*, and *resource-monotonicity*, $\varphi_1(R, \{a, c\}) = a$. By $f_c^a(2) = 2$ and $R|_{\{a,c\}} = \bar{R}|_{\{a,c\}}$, $\varphi_2(R, \{a, c\}) = c$. By *resource-monotonicity*, $\varphi_2(R, \{a, b, c\}) = c$ (otherwise 2 would lose). Since $f^b(1) = 1$, $\varphi_1(R, \{b, c\}) = b$. Thus, by *resource-monotonicity*, $\varphi_2(R, \{b, c\}) = c$ (otherwise, some agent $j \notin \{1, 2\}$ gains when object a is removed from the economy $(R, \{a, b, c\})$). Thus, by *IIO*, $f_c^b(2) = 2 = f_c^a(2)$. $\square$

By Lemma 2, there are two cases. In Case 1 we can derive the allotment for one agent by a "dictator step" in the first step of the definition of a mixed dictator-pairwise-exchange rule. Case 2 deals with a pairwise exchange step.

Case 1: Step 1 is a dictator step, *i.e.*, without loss of generality, for all $h \in K$, $f^h(1) = 1$. Thus, for all $h \in K$, $\pi_h(1) = 1$ and $S_1 = \{1\}$.

Note that *resource-monotonicity* implies that agent 1 receives for all economies his most preferred object. First, we show that the definition of $f_c^a(2)$ is independent of $R_{N \backslash \{1\}}$. Let $f_c^a(2) = 2$.

Lemma 5. *For all $R \in \mathcal{R}_0^N$, if $a \, P_1 \, c$, then $\varphi_2(R, \{a, c\}) = c$.*

Proof. Let $b \in K\backslash\{a,c\}$. Let $\bar{R} \in \mathcal{R}_0^N$ be such that $a\,\bar{P}_1\,b\,\bar{P}_1\,c$ and for all $i \in N\backslash\{1\}$, $\bar{R}_i|_{\{a,c\}} = R_i|_{\{a,c\}}$ and $c\bar{P}_i b$. By Lemma 4 and $f^a(1) = f^b(1) = 1$, we have $f_c^a(2) = f_c^b(2) = 2$. By *IIO*,

$$\varphi(\bar{R}, \{a,c\}) = \varphi(R, \{a,c\}). \tag{2}$$

By *IIO* and the construction of $\bar{R}$, $\varphi_1(\bar{R}, \{b,c\}) = b$ and $\varphi_2(\bar{R}, \{b,c\}) = c$. By *resource-monotonicity* and $f^a(1) = 1$, $\varphi_1(\bar{R}, \{a,b,c\}) = a$, and therefore, $\varphi_2(\bar{R}, \{a,b,c\}) = c$. Since $f^a(1) = 1$, $\varphi_1(\bar{R}, \{a,c\}) = a$. Thus, by *resource-monotonicity*, $\varphi_2(\bar{R}, \{a,c\}) = c$ (otherwise, some agent in $N\backslash\{1,2\}$ gains when object b is removed from $(\bar{R}, \{a,b,c\})$). Hence, by (2), $\varphi_2(R, \{a,c\}) = 2$. $\qquad\square$

The next lemma is the important step for Case 1 in proving that the next row of the inheritance table contains at most two agents.

Lemma 6. $|\{f_c^a(2) \mid a,c \in K \text{ and } a \neq c\}| \leq 2$.

Proof. Suppose that $|\{f_c^a(2) \mid a,c \in K \text{ and } a \neq c\}| \geq 3$. Hence, $|N| \geq 4$. By Lemmas 3 and 4, for some $a \in K$ we have $|\{f_x^a(2) \mid x \in K\backslash\{a\}\}| = 2$ and for some $h \in K\backslash\{a\}$, $f_a^h(2) \notin \{f_x^a(2) \mid x \in K\backslash\{a\}\}$. Let

$$\left|\{f_a^h(2), f_b^a(2), f_c^a(2)\}\right| = 3. \tag{3}$$

Without loss of generality, suppose $h \neq b$. By Lemma 4, $f_a^b(2) = f_a^h(2)$. Since $|N| \geq 4$ and $|K| > |N|$, there exists $d \in K\backslash\{a,b,c\}$. By Lemma 4, we have $f_a^d(2) = f_a^b(2)$, $f_b^d(2) = f_b^a(2)$, and $f_c^d(2) = f_c^a(2)$. Thus, by (3), $\left|\{f_h^d(2) \mid h \in K\backslash\{d\}\}\right| \geq 3$, which contradicts Lemma 4. $\qquad\square$

Case 2: Step 1 is a pairwise exchange step, *i.e.*, without loss of generality, $\{f^h(1) \mid h \in K\} = \{1,2\}$. Thus, for all $h \in K$, $\pi_h(1) \in \{1,2\}$ and $S_1 = \{1,2\}$.[14]

First note the following feature of mixed dictator-pairwise-exchange rules. Suppose that $f^a(1) = 1$ and for all $h \in K\backslash\{a\}$, $f^h(1) = 2$, *i.e.*, 1 owns object a only. Then, 1 either receives a, or he inherits another object from 2, or he exchanges a for another object. In particular, 2 never "physically" inherits object a.

The following lemma shows that if agent 1 owns at least two objects, then agent 2 inherits agent 1's objects.

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[14]If $\pi_h(1) = 1$, define $\pi_h(2) = 2$ and if $\pi_h(1) = 2$, define $\pi_h(2) = 1$.

Lemma 7. If $f^a(1) = f^b(1) = 1$, then $f_b^a(2) = 2$.

Proof. By Case 2, there is $c \in K\backslash\{a,b\}$ such that $f^c(1) = 2$. Let $R \in \mathcal{R}_0^N$ be such that $a\,P_1\,c\,P_1\,b$ and for all $i \in N\backslash\{1\}$, $b\,P_i\,c\,P_i\,a$. By *efficiency*, *resource-monotonicity*, $f^b(1) = 1$, and $f^c(1) = 2$, we have $\varphi_1(R, \{b,c\}) = c$ and $\varphi_2(R, \{b,c\}) = b$. By *resource-monotonicity* and $f^a(1) = 1$, $\varphi_1(R, \{a,b,c\}) = a$ and $\varphi_2(R, \{a,b,c\}) = b$. Since $f^a(1) = 1$, $\varphi_1(R, \{a,b\}) = a$. Thus, by *resource-monotonicity*, $\varphi_2(R, \{a,b\}) = b$ (otherwise, some agent in $N\backslash\{1,2\}$ gains when object c is removed from $(R, \{a,b,c\})$). Thus, by *IIO*, $f_b^a(2) = 2$, the desired conclusion. □

Lemma 7 implies that whenever there are only two objects, then agents 1 and 2 receive them and the other agents receive null objects. By *efficiency* and *resource-monotonicity* in Case 2 the first step is a pairwise exchange.

Next we define $f_c^{ab}(3)$. Let $\bar{R} \in \mathcal{R}_0^N$ be such that $a\,\bar{P}_1\,b\,\bar{P}_1\,c$, $b\,\bar{P}_2\,a\,\bar{P}_2\,c$, and for all $i \in N\backslash\{1,2\}$, $c\,\bar{P}_i\,a\,\bar{P}_i\,b$. Notice that at $\bar{R}$, since the first step is a pairwise exchange, whenever a and b are present, agent 1 receives object a and agent 2 object b. Thus, $\varphi_1(\bar{R}, \{a,b,c\}) = a$ and $\varphi_2(\bar{R}, \{a,b,c\}) = b$. We define $f_c^{ab}(3) \equiv j$ if and only if $\varphi_j(\bar{R}, \{a,b,c\}) = c$. By *efficiency*, $f_c^{ab}(3)$ is well-defined.

Next, we show that the definition of $f_c^{ab}(3)$ is independent of object b that agent 2 picks. Let $d \in K\backslash\{a,b,c\}$.

Lemma 8. $f_c^{ab}(3) = f_c^{ad}(3)$.

Proof. Let $R \in \mathcal{R}_0^N$ be such that $a\,P_1\,b\,P_1\,d\,P_1\,c$, $d\,P_2\,b\,P_2\,a\,P_2\,c$, and for all $i \in N\backslash\{1,2\}$, $c\,P_i\,a\,P_i\,b\,P_i\,d$. By definition and *IIO*, $\varphi_1(R, \{a,c,d\}) = a$, $\varphi_2(R, \{a,c,d\}) = d$, and $\varphi_{f_c^{ad}(3)}(R, \{a,c,d\}) = c$. By *resource-monotonicity*, $\varphi_1(R, \{a,b,c,d\}) = a$, $\varphi_2(R, \{a,b,c,d\}) = d$, and $\varphi_{f_c^{ad}(3)}(R, \{a,b,c,d\}) = c$. Since the first step is a pairwise exchange, $\varphi_1(R, \{a,b,c\}) = a$ and $\varphi_2(R, \{a,b,c\}) = b$. Thus, by *resource-monotonicity*, $\varphi_{f_c^{ad}(3)}(R, \{a,b,c\}) = c$ (otherwise, some agent in $N\backslash\{1,2,f_c^{ad}(3)\}$ gains when object d is removed from $(R, \{a,b,c,d\})$). Hence, by definition of $f_c^{ab}(3)$ and *IIO*, $f_c^{ab}(3) = f_c^{ad}(3)$, the desired conclusion. □

The proofs of the following lemma and of Lemma 8 are similar.

Lemma 9. $f_c^{ab}(3) = f_c^{db}(3)$.

Lemmas 8 and 9 together with *IIO* imply that $f_c^{ab}(3) = f_c^{ba}(3)$,[15] *i.e.*, the definition of the third agent is independent of whether agent 1 picks

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[15]We have $f_c^{ab}(3) = f_c^{ad}(3) = f_c^{bd}(3) = f_c^{ba}(3)$.

object a and agent 2 object b or vice versa. Furthermore, Lemmas 8 and 9 imply

$$\left|\{f_a^{hh'}(3) \mid h, h' \in K\backslash\{a\} \text{ and } h \neq h'\}\right| = 1. \tag{4}$$

Next we show that if agent 1 owns object a, then the definition of $f_c^{ab}(3)$ is independent of whether agent 2 prefers object a to b or object b to a.

Lemma 10. *Let* $f^a(1) = 1$ *and* $R \in \mathcal{R}_0^N$ *be such that* $a\, P_1\, b\, P_1\, c$, $a\, P_2\, b\, P_2\, c$, *and for all* $i \in N\backslash\{1,2\}$, $c\, P_i\, a\, P_i\, b$. *Then* $\varphi_{f_c^{ab}(3)}(R, \{a, b, c\}) = c$.

Proof. Let $d \in K\backslash\{a, b, c\}$. Let $R' \in \mathcal{R}$ be such that $a\, P_1'\, b\, P_1'\, d\, P_1'\, c$, $d\, P_2'\, a\, P_2'\, b\, P_2'\, c$, and for all $i \in N\backslash\{1,2\}$, $c\, P_i'\, a\, P_i'\, d\, P_i'\, b$.

By Lemma 8, $f_c^{ad}(3) = f_c^{ab}(3)$. Thus, by definition of R' and *IIO*, we have $\varphi_{f_c^{ad}(3)}(R', \{a, c, d\}) = \varphi_{f_c^{ab}(3)}(R', \{a, c, d\}) = c$. By *resource-monotonicity*, $\varphi_{f_c^{ab}(3)}(R', \{a, b, c, d\}) = c$. Since the first step is a pairwise exchange, $\varphi_1(R', \{a, b, c\}) = a$ and $\varphi_2(R', \{a, b, c\}) = b$. Thus, by *resource-monotonicity*, $\varphi_{f_c^{ab}(3)}(R', \{a, b, c\}) = c$ (otherwise, some agent in $N\backslash\{1, 2, f_c^{ab}(3)\}$ gains when object d is removed from $(R', \{a, b, c, d\})$). By construction, $R'|_{\{a,b,c\}} = R|_{\{a,b,c\}}$, and by *IIO*, $\varphi(R', \{a, b, c\}) = \varphi(R, \{a, b, c\})$, the desired conclusion. $\qquad\square$

Similarly to the proof of Lemma 10, we can show that whenever agents 1 and 2 receive objects a and b, always the same agent receives object c if $H = \{a, b, c\}$, even if agent 1 or agent 2 rank object c as their second best object among the objects in H (and not as their worst, as assumed in Lemma 10). This fact and Lemma 10 imply that the definition of $f_c^{ab}(3)$ is independent of the preferences of agents 1 and 2 (given that they strictly prefer both objects to c).

The following lemma is similar to Lemma 3.

Lemma 11. $\left|\{f_h^{ab}(3) \mid h \in K\backslash\{a, b\}\}\right| \leq 2$.

The next lemma is similar to Lemma 6. We prove that the third row of the inheritance table contains at most two agents.

Lemma 12. $\left|\{f_{\bar{h}}^{hh'}(3) \mid h, h', \bar{h} \in K \text{ and } |\{h, h', \bar{h}\}| = 3\}\right| \leq 2$.

Proof. Suppose that $\left|\{f_{\bar{h}}^{hh'}(3) \mid h, h', \bar{h} \in K \text{ and } |\{h, h', \bar{h}\}| = 3\}\right| \geq 3$. Hence, $|N| \geq 5$ and $|K| > 5$. By (4) there exist $a, b, c \in K$ such that $\left|\{f_a^{bc}(3), f_b^{ac}(3), f_c^{ab}(3)\}\right| = 3$ and $|\{a, b, c\}| = 3$. Let $d, e \in K\backslash\{a, b, c\}$. By (4), we have $f_a^{de}(3) = f_a^{bc}(3)$, $f_b^{de}(3) = f_b^{ac}(3)$, and $f_c^{de}(3) = f_c^{ab}(3)$. But then, $\left|\{f_h^{de}(3) \mid h \in K\backslash\{d, e\}\}\right| \geq 3$, which contradicts Lemma 11. $\qquad\square$

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We already proved that whenever agents 1 and 2 receive objects a and b, always the same agent receives object c if $H = \{a, b, c\}$ and for all agents $i \in N\backslash\{1, 2\}$, $c \, P_i \, a \, P_i \, b$. Finally, we show that the agent who receives object c does not depend on the preferences of agents in $N\backslash\{1, 2\}$.

Lemma 13. *Let* $R \in \mathcal{R}_0^N$ *be such that* $\varphi_1(R, \{a, b, c\}) = a$ *and* $\varphi_2(R, \{a, b, c\}) = b$. *Then* $\varphi_{f_c^{ab}(3)}(R, \{a, b, c\}) = c$.

Proof. Since $|K| > |N|$ we assume that $|K| \geq 5$ (otherwise $|N| = 3$ and the conclusion of Lemma 13 is trivial). Let $d, e \in K\backslash\{a, b, c\}$. By *IIO*, we may suppose that c is ranked worst in $\{c, d, e\}$ for agents 1 and 2, 1 prefers a to both d and e, and 2 prefers b to both d and e. By (4),

$$f_c^{de}(3) = f_c^{ab}(3). \tag{5}$$

Let $R' \in \mathcal{R}_0^N$ be such that $R'_{\{1,2\}} = R_{\{1,2\}}$ and for all $i \in N\backslash\{1, 2\}$, $R'_i|_{\{a,b,c\}} = R_i|_{\{a,b,c\}}$ and $c \, P'_i \, d \, P'_i \, e$. By *IIO*,

$$\varphi(R', \{a, b, c\}) = \varphi(R, \{a, b, c\}). \tag{6}$$

By definition and (5), $\varphi_{f_c^{ab}(3)}(R', \{c, d, e\}) = c$. By *resource-monotonicity*, $\varphi_1(R', \{a, b, c, d, e\}) = a$ and $\varphi_2(R', \{a, b, c, d, e\}) = b$. Thus, by *resource-monotonicity*, $\varphi_{f_c^{ab}(3)}(R', \{a, b, c, d, e\}) = c$. Again, by *resource-monotonicity*, $\varphi_{f_c^{ab}(3)}(R', \{a, b, c\}) = c$ (otherwise, some agent in $N\backslash\{1, 2, f_c^{ab}(3)\}$ gains when objects d and e are removed from $(R', \{a, b, c, d, e\})$). Hence, by (6), $\varphi_{f_c^{ab}(3)}(R, \{a, b, c\}) = c$. $\qquad\square$

Cases 1 and 2 imply that **at Step 2 at most two agents trade** (Lemmas 6 and 12) independently of the preference profile and the allotments of the dictator(s) at Step 1 (Lemmas 5, 10, and 13). Similarly as for Step 1, we can now prove that Step 2 is either a dictator step or a pairwise exchange step and that again **at Step 3**, if there is one, **at most two agents trade** independently of the preference profile and the allotments of the dictator(s) at Steps 1 and 2, *etc.*

2. General Induction Step: In the first part of the proof, we have shown that φ allocates the objects through a dictatorship or a pairwise exchange in both Steps 1 and 2. As already indicated, in proving the general induction step we use the arguments of the first part of the proof (they give much insight into how the axioms work) and the following definitions.

Suppose that we have defined the mixed dictator-pairwise exchange rule up to Step t. Let $S_1, S_2, \ldots, S_t$ be the members of the ordered partition up

to Step t, where each member is a singleton or a pair. Let $s \equiv \sum_{l=1}^{t} |S_l|$ (so, the first s rows of the mixed dictator-pairwise-exchange inheritance table are defined). Without loss of generality, let $\bigcup_{l=1}^{t} S_l = \{1, 2, \ldots, s\}$.

We define the $(s+1)$st row as follows. For all $H^s \subseteq K$ such that $|H^s| = s$ and all $c \in K \backslash H^s$, order H^s in an arbitrary manner, say $H^s = \{h_1, h_2, \ldots, h_s\}$, and let $R \in \mathcal{R}_0^N$ be such that

$$h_1 \, P_1 \, h_2 \, P_1 \, h_3 \, P_1 \cdots P_1 \, h_s \, P_1 \, c,$$
$$h_2 \, P_2 \, h_1 \, P_2 \, h_3 \, P_2 \cdots P_2 \, h_s \, P_2 \, c,$$
$$h_3 \, P_3 \, h_1 \, P_3 \, h_2 \, P_3 \cdots P_3 \, h_s \, P_3 \, c,$$
$$h_4 \, P_4 \, h_1 \, P_4 \, h_2 \, P_4 \cdots P_4 \, h_s \, P_4 \, c,$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$h_s \, P_s \, h_1 \, P_s \, h_2 \, P_s \cdots P_s \, h_{s-1} \, P_s \, c,$$

and for all $i \in N \backslash \{1, \ldots, s\}$,

$$c \, P_i \, h_1 \, P_i \, h_2 \, P_i \ldots P_i \, h_s.$$

Note that for all $i_s \in \{1, \ldots, s\}$, $\varphi_{i_s}(R, H^s \cup \{c\}) = h_s$. We define[16]

$$f_c^{(h_1, \ldots, h_s)}(s+1) \equiv j \Leftrightarrow \varphi_j(R, H^s \cup \{c\}) = c.$$

Let $d \in K \backslash (H^s \cup \{c\})$. Similarly to Lemma 8 we can show the following.

Lemma 14. $f_c^{(d, h_2, \ldots, h_s)}(s+1) = f_c^{(h_1, h_2, \ldots, h_s)}(s+1)$.

Using Lemma 14, it follows that $f_c^{(h_1, h_2, \ldots, h_s)}(s+1)$ does not depend on the order of H^s. For example, $f_c^{(h_2, h_1, h_3, \ldots, h_s)}(s+1) = f_c^{(h_1, h_2, \ldots, h_s)}(s+1)$. Then $f_c^{H^s}(s+1) = f_c^{(h_1, h_2, \ldots, h_s)}(s+1)$ is well-defined. Similarly to (4) we have then

$$\left| \{ f_c^{H^s}(s+1) \mid H^s \subseteq K \backslash \{c\} \text{ and } |H^s| = s \} \right| = 1.$$

Then, similarly to Lemmas 10 and 12 in Case 2 of the first part of the proof, we can show that the $(s+1)$st row of the inheritance table contains at most two agents (Lemma 15), independently of the order of H^s and of the set of s objects H^s.

Lemma 15. $\left| \{ f_h^{H^s}(s+1) \mid H^s \cup \{h\} \subseteq K \text{ and } |H^s \cup \{h\}| = s+1 \} \right| \leq 2.$

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[16]Note that before we wrote $f_c^{ab}(3)$ instead of $f_c^{(a,b)}(3)$.

Then, similarly to Lemma 13 it follows that $f_c^{H^s}(s+1)$ receives c at any profile where agents $\{1, \ldots, s\}$ receive H^s in the first t steps.

Lemma 16. *Let $H^s \subseteq K$ be such that $|H^s| = s$, $c \in K\backslash H^s$, and $R \in \mathcal{R}_0^N$ be such that $\{\varphi_1(R, H^s \cup \{c\}), \ldots, \varphi_s(R, H^s \cup \{c\})\} = H^s$. Then $\varphi_{f_c^{H^s}(s+1)}(R, H^s \cup \{c\}) = c$.*

Finally, from *resource-monotonicity* it follows that Step $s+1$ is a dictator step or a pairwise exchange step.

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