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Sequential choice and self-reinforcing rankings*

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Abstract

People's behavior is informed and influenced by other people's choices. In many online technologies, for instance, aggregate information about the choices of other individuals is encoded in the form of rankings. Such rankings, in turn, have a direct impact on people's future choices. What are the long-term dynamics of these rankings, and do the dynamics depend on specific assumptions about people's behavior? In this paper, we propose a general framework for modeling the dynamics in settings where information about peoples' past choices is recorded as a ranking and influences future choices. We find a general condition for convergence, show that it is satisfied by many important models in economics and beyond, and characterize the possible limits in terms of the choice probabilities.

JEL CLASSIFICATION: D01, D11, D83

Keywords— ranking, sequential choice, social learning, social influence, inattention, herding

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1 Introduction

In many economic settings people use the behavior of others as a source of information that can influence their own behavior. This problem, commonly referred to as *social learning*, has been extensively studied in economics and understandably so—social learning is crucial for understanding how information is aggregated and for predicting social outcomes and welfare (Banerjee, 1992; Ellison & Fudenberg, 1993; Smith & Sørensen, 2000; Golub & Jackson, 2012; Che & Hörner, 2018). A possible concern with this literature is the plethora of available models that many times reach very different conclusions about the possible limiting behavior. This concern becomes pressing when we think about the new economy and the dependence of many major websites on the sequential aggregation of information from a population of agents, especially in the form of rankings.

Online marketplaces, social media, and search engines rank content for their users and their algorithms rely on implicit or explicit user feedback, such as clicks, sales and likes to update and re-rank their content (i.e. products, webpages, etc.). There is plenty of evidence to suggest that people are more likely to choose or interact with items that appear higher on the list (Joachims, Granka, Pan, Hembrooke, & Gay, 2005; Salganik, Dodds, & Watts, 2006; Caplin, Dean, & Martin, 2011). Similar phenomena are encountered in offline settings where rankings convey socially aggregated information and influence people's choices, as in the cases of journal or department rankings (Laband, 2013; Espeland & Sauder, 2007). Ideally, we would like to reach a ranking that reflects the quality of the options available. It is not obvious though, whether aggregating feedback sequentially can lead to a ranking that is optimal, in the sense that options that are valuable to more people are ranked first.

In studying the dynamics of rankings that aggregate information from people's choices, one has to make assumptions about people's choice behavior, in particular how it is affected by the ranking. If different behavioral assumptions lead to very different limiting behaviors, as it seems to be the case in the social learning literature (Mobius & Rosenblat, 2014), any results would be relevant only in the limited contexts where these behavioural assumptions hold true (or approximately true). Since it is rarely possible to pin down entirely the way people behave when making choices, such dependence on specific assumptions can potentially be fatal for the usefulness of these results in predicting the long-term dynamics of rankings.

In this paper we show that, at least in some cases, it is possible to avoid this conclusion. We propose a general framework which addresses the complex dynamic nature of the problem discussed above and gives precise predictions for the limiting structure. Our main assumption is that the information available about other people's choices is in the form of a *popularity ranking*, i.e. which option is currently the most popular one, which one is second and so on. In this setting, we find quite general conditions under which the ranking converges and, moreover, we characterize the possible limit rankings in terms of the choice probabilities. The assumption we impose on the choice behavior of the agents is a weak form of preference for highly ranked options. Surprisingly, this assumption is satisfied by several prominent behavioural models like satisficing (Simon, 1955; Caplin et al., 2011), consideration set models (Manzini & Mariotti, 2014), drift-diffusion models (Fudenberg, Strack, & Strzalecki, 2018), models of position-dependent attention (Germano, Gómez, & Le Mens, 2019), but also by social learning models building on discrete choice theory (Brock & Durlauf, 2001) and even Banerjee's herding model (1992) in which agents are fully rational. This means that the framework we propose can be applied to get sharp predictions about the dynamics of popularity rankings under a variety of different behavioral assumptions.

Let us give some details about how this generality is achieved. Consider a set of items X, the collection $\mathcal{X}$ of all subsets of X and the set $\mathcal{R}$ of all possible rankings (orderings) of the items in X. A *ranking-based choice function* is a map $\pi : \mathcal{X} \times \mathcal{R} \to [0,1]$, such that $\sum_{A \in \mathcal{X}} \pi(A,r) = 1$ for each $r \in \mathcal{R}$. The quantity $\pi(A,r)$ is interpreted as the probability of choosing exactly the items in the set A when the items are ranked according to $r \in \mathcal{R}$. Thus, our choice model is similar to a choice from lists model (Rubinstein & Salant, 2006), except that a ranking must contain all elements of X, and the choice is stochastic. Also, the decision maker is allowed to choose a set of items from X, rather than a single item. The probability that $a \in X$ is included in the set of items chosen, when the ranking is r, is denoted by $q(a,r) = \sum_{A \ni a} \pi(A,r)$.

Our main assumption for π is a ranking-based reinforcement assumption: items that are ranked higher are generally more likely to be chosen. However, this condition may interact with the desirability of the different items. Specifically, we allow for an asymmetric relation P on X, with aPb roughly interpreted as a being preferred to b by the average agent. This relation doesn't have to be transitive nor complete[1], and it might as well be the empty relation. Our reinforcement assumption is roughly saying that if a is more highly ranked than b in the ranking r and $b\not{P}a$ (i.e. b is *not* more desirable than a), then $q(a, r) > q(b, r)$. No assumption is made if bPa. In terms of a search engine, for example, this would mean that if result a is both at least as relevant as result b (i.e. $b\not{P}a$) *and* it appears higher, then it should be more likely for a to be chosen. It turns out that this assumption is weak enough to be satisfied by a variety of models in the literature.

Next we consider a large number of agents that choose sequentially from the set X, according to the choice function π. Agent $n \in \mathbb{N}$ observes the popularity ranking R_{n-1}, that is the ranking of the items in X by number of choices among the first $n-1$ agents. This would be the case, for example, if an online marketplace displayed the various items in a category ranked by numbers of sales, since each user would see them ranked by popularity in terms of purchases by previous agents. We thus assume that the choice probabilities for the n-th agent are given by $\pi(\cdot, R_{n-1})$. With this sequential choice structure, and under the reinforcement assumption for π (see previous paragraph), we show that as $n \to \infty$, the popularity ranking R_n converges to some constant ranking with probability 1. However, there are in principle multiple possible limits. We characterize these possible limits in terms of the map π. If we know the limit ranking, we can also find the market shares of the various items, that is the proportion of agents that choose each of the items, again in terms of the map π.

Each item may also carry a utility for an agent that chooses it. We are interested in the average utility experienced by a large number of agents. We show that this average utility converges, with its limit being a function of the limit ranking. This implies in particular that we can compare the various rankings in terms of the average utility that they result in. The fact that there are generally multiple possible limit rankings implies that we may end up at a

[1] We adopt the economics convention and call *complete* a relation satisfying that for any $a \neq b$, either aPb or bPa. In mathematics such relations are called *connected*.

sub-optimal limit ranking in terms of average utility.

Importantly, the utility function may be intertwined with the ranking; on one hand, agents might gain utility by choosing popular items, so we allow the utility to depend on the ranking. On the other hand, the utility might affect the choice probabilities, as is the case for example for rational, utility-maximizing agents. We model the dependence of the choice probabilities on the utility by adding structure to the choice function, now written as $\pi(A, r, w)$, where w is the agent type and may carry information about the utility function of the agent. It turns out that all our results continue to hold as long as agent types are chosen i.i.d. from a given distribution.

The rest of the paper is organized as follows: Section 2 introduces our framework and assumptions. In Section 3 we treat the important case of two items, which is simpler and intuitive, but already contains many of the ideas involved in the general case. Section 4 presents the theoretical results for the general case, i.e. for any finite number of items. In Section 5 we show how several important behavioral models from the literature satisfy our assumptions, while in Section 6 we describe some alternative interpretations of our framework.

2 Ranking-based choice functions

2.1 Rankings

A ranking r of set X intuitively is a way to order the elements of X, such that some are placed first, second, etc., which allows for ties. More precisely:

Definition 2.1. *A ranking of a finite set X is a map $r : X \to \{1, \ldots, |X|\}$ with the property that for each $a \in X$,*

$$card\{x \in X : r(x) < r(a)\} = r(a) - 1. \tag{1}$$

The number $r(a)$ will be called the *rank* or *position* of a in r. We'll say that a is ranked higher than b in r and write $a \succ_r b$, if $r(a) < r(b)$, and we'll say that a and b are ranked equally in r and write $a \approx_r b$, if $r(a) = r(b)$. The relations $\succeq_r$, $\prec_r$ and $\preceq_r$ are defined analogously. Definition 2.1 requires that, for each a, exactly $r(a) - 1$ elements are ranked higher than a. Rankings are mathematically equivalent to weak orders.

Note that every bijection $r : X \to \{1, \ldots, |X|\}$ is a ranking, and it has the property that no two items in X are ranked equally. Such rankings will be called *strict* rankings.

2.2 The choice model

Let X be a finite set, $\mathcal{X}$ the collection of all subsets of X, and $\mathcal{R}$ the set of all possible rankings of the items in X. A *ranking-based choice function* is a map of the form $\pi : \mathcal{X} \times \mathcal{R} \to [0, 1]$, such that $\sum_{A \in \mathcal{X}} \pi(A, r) = 1$ for each r. The quantity $\pi(A, r)$ is interpreted as the probability that the agent chooses exactly the set of items A, given a ranking r of X. The probability that a particular item a is chosen (perhaps among other items) is

$$q(a, r) = \sum_{A \ni a} \pi(A, r). \tag{2}$$

We will use the notation $q(ab', r)$ to denote the probability that a is among the items chosen while b is not, that is

$$q(ab', r) = \sum_{\substack{A \in \mathcal{X}: \\ a \in A,\, b \notin A}} \pi(A, r). \tag{3}$$

We impose the following condition:

Assumption 2.2 (Ranking-based reinforcement)**.** *There exists some asymmetric relation P on X, such that for any $a, b \in X$ and any $r \in \mathcal{R}$ with $q(ba', r) > 0$:*

- *If $a \succ_r b$ and $b \not P a$, then $q(a, r) > q(b, r)$.*
- *If $a \approx_r b$ and aPb, then $q(ab', r) > 0$.*

The relation aPb can be roughly interpreted as a being preferable to b. The first part of the assumption says that if a is ranked higher than b and b is *not* preferable to a, then a is chosen with higher probability than b. The second condition says that whenever two items are equally ranked and one is preferable to the other, there is positive probability of choosing the preferable without choosing the less preferable one. Exceptions to both cases are allowed when $q(ba', r) = 0$, that is when a is chosen whenever b' is chosen[2]. A map $\pi : \mathcal{X} \times \mathcal{R} \to [0, 1]$ satisfying Assumption 2.2 will

[2]This takes care of the trivial case that a and b can only be chosen together, that is $q(ab', r) = q(ba', r) = 0$.

be called a *ranking-based choice function with reinforcement*. Several examples of choice models from the literature that take this form will be given in Section 5.

2.3 Sequential choice

We now consider a large number of agents who choose sequentially from X. Each agent chooses according to the popularity ranking at the time of their turn. Specifically, the agents choose according to a fixed ranking-based choice function $\pi : \mathcal{X} \times \mathcal{R} \to [0, 1]$, with the second argument determined by the current popularity ranking.

More precisely, let $A_n \subset X$ denote the set of items that agent n chooses. The popularity of item a at time n is

$$Z_n(a) = card\{k \leq n : a \in A_k\} + z(a), \tag{4}$$

where the additive constant $z(a)$, for some map $z : X \to \mathbb{R}$, allows for different starting points, possibly giving an initial advantage to some of the items. We denote by R_n the *popularity ranking* of the items at time n, that is the unique ranking $r \in \mathcal{R}$ that satisfies $r(a) < r(b)$ if and only if $Z_n(a) > Z_n(b)$, for each $a, b \in X$. It is easy to check that this ranking is given by

$$R_n(a) = card\{x \in X : Z_n(x) > Z_n(a)\} + 1. \tag{5}$$

Then, agent n is presented with the choice problem $\pi(\cdot, r)$, where $r = R_{n-1}$, and their choice is assumed independent of any previous agents' choices, given R_{n-1}. In other words,

$$\mathbb{P}\left(A_n = S \mid R_{n-1} = r, A_1, \dots, A_{n-1}\right) = \mathbb{P}\left(A_n = S \mid R_{n-1} = r\right) = \pi(S, r), \quad S \in \mathcal{X}. \tag{6}$$

The probability of any event involving the agents' choices can be determined by π and the initial condition z, through Eqs. (4) to (6). We will call the pair (π, z) a *ranking-based sequential choice system*. If π satisfies Assumption 2.2, then we will call (π, z) a *ranking-based sequential choice system with reinforcement*. In Section 4 we study the long-term behavior of R_n and Z_n for such systems.

2.4 Non-identical agents and utility

In the above setting, there is no assumption that there is a single type of agent. Although we have considered a unique ranking-based choice function π, agent heterogeneity could be built into it, as long as agents are drawn i.i.d. from a fixed distribution.

Agent heterogeneity can be made more explicit by considering a set $\mathcal{W}$ of agent types and a probability distribution Q over it. Each agent type has distinct choice probabilities, encoded by a generalized ranking-based choice function $\pi : \mathcal{X} \times \mathcal{R} \times \mathcal{W} \to [0, 1]$, with $\pi(A, r, w)$ denoting the probability that an agent of type $w \in \mathcal{W}$ chooses the set of items A, given a ranking r. Then, assuming that the agent is chosen according to the distribution Q, we may define the aggregate choice function

$$\pi(A, r) = \int_{w \in \mathcal{W}} \pi(A, r, w) \, dQ. \tag{7}$$

Further assuming that each agent's type is picked (from the distribution Q) independently of any previous agent types or choices, the sequential choice system described in the previous section remains the same. For later reference we also note that $q(a, r, w)$, the probability that an item $a \in X$ is in the choice set of an agent of type w, given the ranking r, may be defined analogously to Eq. (2), and by linearity of the integral it satisfies

$$q(A, r) = \int_{w \in \mathcal{W}} q(A, r, w) \, dQ. \tag{8}$$

Utility

From Eq. (6) it is clear that, given the aggregate choice function $\pi(A, r)$, specifying explicitly the agent types and agent type-specific choice probabilities $\pi(A, r, w)$ does not alter the statistics of the agent choices. However, other quantities might depend on the agent type and their statistics may vary even for the same aggregate choice function.[3] One quantity that we expect to depend on the agent type is the utility experienced from any given choice.

Specifically, we assume that each set of items $A \in \mathcal{X}$ carries some utility that will be expe-

[3] For example consider two agent types with the same choice probabilities, but different in other ways. If a function depends on the agent type, then its expected value will depend on the probability of each type. On the other hand, the choice function $\pi(A, r)$ will be unaffected by these probabilities.

rienced by an agent that chooses it, depending on the agent type. We also allow the utility to depend on the ranking, in order to allow for scenarios in which agents gain utility by making popular choices (Brock & Durlauf, 2001). More precisely, we denote by $u(A, r, w)$ the utility experienced by an agent of type $w \in W$ that chooses the set of items $A \in \mathcal{X}$, given the ranking $r \in \mathcal{R}$, where

$$u : \mathcal{X} \times \mathcal{R} \times \mathcal{W} \to \mathbb{R}. \tag{9}$$

The average utility experienced by the first n agents in a ranking-based sequential choice system is

$$\bar{u}_n = \frac{1}{n} \cdot \sum_{k=1}^{n} u(A_k, R_{k-1}, W_k), \tag{10}$$

where W_k denotes the k-th agent's type and A_k and R_{k-1} are as in the previous section.

Note that the fact that both the utility $u(A, r, w)$ and the choice function $\pi(A, r, w)$ take the agent type w as an argument allows to describe choice models that depend on the utilities of the various items in an arbitrary way. To see this, note that the agent type w gives full information about the utilities of the various choices under any ranking, through the map $u(\cdot, \cdot, w) : \mathcal{X} \times \mathcal{R} \to \mathbb{R}$. This information enters the choice function π as the third argument. As an example, in Section 5.1 and Appendix A.3 we show how one can describe in these terms the satisficing model (Simon, 1955), where an agent searches the items sequentially until they find one whose utility exceeds some threshold value.

3 Popularity dynamics of two items

Recall that we want to study the long-term behavior of a ranking-based sequential choice system (π, z), where $\pi : \mathcal{X} \times \mathcal{R} \to [0, 1]$ is a ranking-based choice function with reinforcement, and $z : X \to \mathbb{R}$ encodes the initial conditions. In particular, we'd like to find conditions for the ranking R_n to converge, and characterize the limit in terms of π and z. Ideally, we would also like to get some more refined results regarding the long-term behavior of the popularity, for instance about the convergence and limits of $\frac{Z_n(a)}{n}$, i.e. the proportion of agents that choose item a, for any $a \in X$.

In this section we deal with the important case of choice from a set of two items $X = \{a, b\}$, and with $z(a) = z(b) = 0$. Such binary choice problems have been extensively studied in economics, psychology, neuroscience and the management sciences (Brock & Durlauf, 2001; Busemeyer & Townsend, 1993; Fudenberg et al., 2018). This case has the benefit of straightforward arguments and we can obtain quantitative results for the probability of the possible long-term behaviors. Several of our results from this section will be generalized in the next section.

For simplicity we assume that each agent chooses exactly one of the items, although the arguments can easily be extended to allow for choosing both or none of the items. Note that this implies that the system can be characterized by the quantities $q(a, r) = \pi(\{a\}, r)$, $r \in \mathcal{R}$, since $\pi(\{a, b\}, r) = \pi(\emptyset, r) = 0$ and $\pi(\{b\}, r) = q(b, r) = 1 - q(a, r)$ for each $r \in \mathcal{R}$. To avoid trivial cases, we will also assume that $0 < q(a, r) < 1$ for all $r \in \mathcal{R}$, so that there is always positive probability of choosing any of the two items. We do not impose Assumption 2.2; rather we'll show below that it is equivalent to a simple statement regarding the $q(\cdot, r)$'s and that it naturally arises as a necessary and sufficient condition for certain convergence results.

The set $\mathcal{R}$ of the possible rankings of two items contains three elements, corresponding to the cases of a being ranked first, b being ranked first, and the two items being tied. We will denote these cases by "+", "−", and "0", respectively, so that $\mathcal{R} = \{+, -, 0\}$.

Recall that $Z_n(a)$ and $Z_n(b)$ are the numbers of agents, up to agent number n, that have chosen items a and b, respectively. We define their difference $Y_n = Z_n(a) - Z_n(b)$. Note that at each step exactly one of the $Z_n(a)$ or $Z_n(b)$ increases by 1, with probabilities depending on whether $Z_n(a) > Z_n(b)$, $Z_n(a) = Z_n(b)$, or $Z_n(a) < Z_n(b)$. Therefore, $\Delta Y_{n+1} = Y_{n+1} - Y_n$ takes values ± 1, with probabilities depending on whether $Y_n > 0$, $Y_n < 0$ or $Y_n = 0$[4]. More precisely, we have

$$
\begin{aligned}
\mathbb{P}\left(\Delta Y_{n+1} = 1 \mid Y_n > 0\right) &= q(a, +), \quad & \mathbb{P}\left(\Delta Y_{n+1} = -1 \mid Y_n > 0\right) &= q(b, +) = 1 - q(a, +), \\
\mathbb{P}\left(\Delta Y_{n+1} = 1 \mid Y_n < 0\right) &= q(a, -), \quad & \mathbb{P}\left(\Delta Y_{n+1} = -1 \mid Y_n < 0\right) &= q(b, -) = 1 - q(a, -), \\
\mathbb{P}\left(\Delta Y_{n+1} = 1 \mid Y_n = 0\right) &= q(a, 0), \quad & \mathbb{P}\left(\Delta Y_{n+1} = -1 \mid Y_n = 0\right) &= q(b, 0) = 1 - q(a, 0).
\end{aligned}
\tag{11}
$$

[4] Y_n performs a non-homogeneous random walk (Menshikov, Popov, & Wade, 2016).

Since we have assumed zero initial conditions, we have $Y_0 = 0$. After the first step, the random walk has positive probability of reaching either 1 or -1. Suppose that $Y_1 = 1$. Until the next time it returns to 0, Y_n performs a regular biased random walk, with probability of stepping to the right equal to $q(a, +)$. If $q(a, +) > 1/2$, the theory of random walks states that there is positive probability that Y_n will remain positive for all $n > 0$ and in fact, if it does so, then $Y_n \to \infty$. Since $Y_n = Z_n(a) - Z_n(b)$, this means that there is positive probability that item a will become increasingly more popular than item b in the long-run. In contrast, if $q(a, +) \leq 1/2$, then Y_n will eventually return to 0 with probability 1, i.e. item b will eventually catch up. Similar results hold if $Y_1 = -1$, for the cases $q(b, -) > 1/2$ and $q(b, -) \leq 1/2$.

Note that the above argument holds not only at the beginning ($n = 0$), but whenever the random walk returns to 0: if either $q(a, +) > 1/2$ or $q(b, -) > 1/2$, then there is positive probability that Y_n escapes to $\pm\infty$ after any visit to 0. A simple argument then shows that $Y_n \to \pm\infty$ with probability 1. In contrast, if $q(a, +) \leq 1/2$ and $q(b, -) \leq 1/2$, then Y_n will keep returning to 0 indefinitely.

Finally, regarding the case that exactly one of the inequalities $q(a, +) > 1/2$ or $q(b, -) > 1/2$ or holds, we can be more specific about the limit: if $q(a, +) > 1/2$ and $q(b, -) \leq 1/2$, the above imply that $\mathbb{P}(Y_n \to \pm\infty) = 1$ and $\mathbb{P}(Y_n \to -\infty) = 0$, therefore $\mathbb{P}(Y_n \to \infty) = 1$. Similarly, if $q(a, +) \leq 1/2$ and $q(b, -) > 1/2$, then $\mathbb{P}(Y_n \to -\infty) = 1$. To summarize, we have shown the following (see also Table 1):

Proposition 3.1. *With notation as above,*

$$\mathbb{P}(Y_n \to \pm\infty) = \begin{cases} 1, & \text{if } \max\{q(a,+), q(b,-)\} > \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Moreover, $\mathbb{P}(Y_n \to \infty) > 0$ *if and only if* $q(a, +) > 1/2$*, and* $\mathbb{P}(Y_n \to -\infty) > 0$ *if and only if* $q(b, -) > 1/2$*.*

Since Y_n was defined as the difference of the popularities of items a and b ($Y_n = Z_n(a) - Z_n(b)$), the first part of Proposition 3.1 says that a necessary and sufficient condition for an item to dominate the market ($Z_n(a) - Z_n(b) \to \infty$) is that at least one of the $q(a, +)$ or $q(b, -)$ exceeds

		$q(a,+) > 1/2$: Yes	$q(a,+) > 1/2$: No
$q(b,-) > 1/2$	Yes	$\mathbb{P}\left(Y_n \to \pm\infty\right) = 1$	$\mathbb{P}\left(Y_n \to -\infty\right) = 1$
	No	$\mathbb{P}\left(Y_n \to +\infty\right) = 1$	$\mathbb{P}\left(Y_n \to \pm\infty\right) = 0$

Table 1: The different scenarios for convergence with two items. The left column corresponds to the case that item a is more likely than b to be chosen when it is more popular ($q(a,+) > 0.5$), while the right column to being equally or less likely than b to be chosen $q(a,+) \leq 0.5$. Similarly for the first and second row from the perspective of item b. If at least one item is more likely to be chosen when it is more popular, that is $q(a,+) > 0.5$ and/or $q(b,-) > 0.5$, then the popularity difference Y_n converges to $\pm\infty$ with probability 1, and if exactly one of the conditions holds, then we know whether $Y_n \to \infty$ or $Y_n \to -\infty$. If none of these conditions holds, then Y_n does not escape to infinity.

1/2. If this fails to hold, we get $q(b,+) \geq 1/2$ and $q(a,-) \geq 1/2$, meaning that the *less* popular item is always more (or equally) likely to be chosen than the more popular one, violating the principle of ranking-based reinforcement.

Indeed, Assumption 2.2, which we called ranking-based reinforcement assumption, is exactly equivalent to $\max\{q(a,+), q(b,-)\} > 1/2$. To see this, suppose first that Assumption 2.2 holds. Since the relation P is asymmetric, we may assume without loss of generality that $b \not P a$. By Assumption 2.2, we must have $q(a,+) > q(b,+)$, and since $q(b,+) = 1 - q(a,+)$, we get $q(a,+) > 1/2$. For the converse, assume without loss of generality that $q(a,+) > 1/2$. Then the (unique) asymmetric relation P on X that satisfies aPb can be easily seen to satisfy Assumption 2.2.

We therefore see that, in the case of two items, Assumption 2.2 is a necessary and sufficient condition for one item to dominate. As we will see later, for more than two items, Assumption 2.2 is still sufficient for convergence of the ranking, although not necessary in general.

Let us now return to the case that at least one of the $q(a,+)$ or $q(b,-)$ does exceed 1/2. The second part of Proposition 3.1 implies that if *exactly* one of them exceeds 1/2, then we know which item will end up winning the competition. But if both $q(a,+) > 1/2$ and $q(b,-) > 1/2$, then either outcome is possible. This leaves open two questions: (i) if both $q(a,+)$ and $q(b,-)$ exceed 1/2, how likely is it for either item to end up dominating, i.e. what are $\mathbb{P}\left(Y_n \to \infty\right)$ and $\mathbb{P}\left(Y_n \to -\infty\right)$? And (ii) if we know which item dominates, what proportion of agents choose each item?

These questions are answered by the following two propositions. The first one says that $q(a,+)$

and $q(b,+)$ are the proportions of agents that choose each of the two items in the case $Y_n \to \infty$, and similarly for the case $Y_n \to -\infty$.

Proposition 3.2 (Market shares - Two items)**.** *With notation as above, on the set $\{Y_n \to \infty\}$ we have*

$$\lim_{n\to\infty} \frac{Z_n(a)}{n} = q(a,+) \quad \textit{and} \quad \lim_{n\to\infty} \frac{Z_n(b)}{n} = q(b,+) \quad \textit{a.s.}, \tag{13}$$

while on $\{Y_n \to -\infty\}$ we have

$$\lim_{n\to\infty} \frac{Z_n(a)}{n} = q(a,-) \quad \textit{and} \quad \lim_{n\to\infty} \frac{Z_n(b)}{n} = q(b,-) \quad \textit{a.s.} \tag{14}$$

We skip the rigorous proof as this is a special case of Theorem 4.1 below, but the idea is simple: as long as $Y_n > 0$, the random variables $\Delta Z_{n+1}(a)$, $n \in \mathbb{N}$, behave like i.i.d Bernoulli variables with $\mathbb{P}\left(\Delta Z_{n+1}(a) = 1\right) = q(a,+)$. If $Y_n \to \infty$, then $Y_n > 0$ holds for all sufficiently large n, therefore by the Strong Law of Large Numbers, $Z_n(a)/n$ converges to $q(a,+)$. Similarly for item b, as well as for the case $Y_n \to -\infty$.

The next proposition answers the remaining question posed above, i.e. how likely each of $\{Y_n \to \infty\}$ and $\{Y_n \to -\infty\}$ is when both are possible. Its proof is given in Appendix A.2.

Proposition 3.3 (Probability of limits - Two items)**.** *Suppose $q(a,+)$ and $q(b,-)$ are both larger than $1/2$ and denote $s^+ = \frac{q(b,+)}{q(a,+)}$, $s^- = \frac{q(a,-)}{q(b,-)}$, and $s^0 = \frac{q(b,0)}{q(a,0)}$. Then,*

$$\mathbb{P}\left(Y_n \to \infty\right) = \frac{1 - s^+}{(1 - s^+) + s^0\left(1 - s^-\right)}. \tag{15}$$

Remark 3.4. Proposition 3.3 shows that in the case of choice from two items it is possible to calculate long term convergence probabilities in terms of the choice probabilities. This result can be useful in problems where the choice model can be retrieved from behavioral data and the probabilities of choosing each of the items can be reliably estimated. Although we consider this a valuable result, several of the results in our paper show that there is a lot to be said about sequential choice systems with ranking-based reinforcement even without knowing the exact choice probabilities, only by requiring that they satisfy certain inequalities (as in Proposition 3.1).

Example 3.5. *Suppose that there are two introductory textbooks of microeconomics, a and b, from which each agent chooses exactly one, with the following ranking-dependent probabilities: If textbook a is currently ranked higher (in terms of popularity), then it is chosen with probability $q(a,+) = 0.7$, while textbook b is chosen with probability $q(b,+) = 0.3$. If textbook b is ranked higher, then these probabilities are $q(a,-) = 0.4$ and $q(b,-) = 0.6$. For the case that the two textbooks are currently equally popular, we have $q(a,0) = 0.55$ and $q(b,0) = 0.45$. To summarize, we have the following table:*

	$q(\cdot,+)$	$q(\cdot,-)$	$q(\cdot,0)$
textbook a	0.7	0.4	0.55
textbook b	0.3	0.6	0.45

Note that these probabilities satisfy Assumption 2.2 with P being the empty relation.

Since $q(a,+)$ and $q(b,-)$ are both larger than $1/2$, Proposition 3.3 applies. Using the numbers in the above table, we calculate $s^+ = 3/7$, $s^- = 2/3$, and $s^0 = 9/11$. Therefore, Eq. (42) gives

$$\mathbb{P}\left(Y_n \to \infty\right) = \frac{\frac{4}{7}}{\frac{4}{7} + \frac{9}{11} \cdot \frac{1}{3}} = \frac{44}{65} \approx 0.68. \tag{16}$$

That is, with probability 68% (32%), textbook a (b) will eventually become and remain the more popular one.

Additionally, Proposition 3.2 implies that in the cases in which textbook a is more popular in the long run, the percentage of students who choose it converges to 70% ($q(a,+) = 0.7$) and that of textbook b to 30%. In the cases in which textbook b ends up being more popular, it is chosen by 60% of the agents ($q(b,-) = 0.6$), while textbook a is chosen by 40% of the agents. See Fig. 1.

How would the results in the above example change if textbook a was more likely to be chosen even when it was ranked lower? This would be the case if we had for example $q(a,-) = 0.6$ and $q(b,-) = 0.4$. In this case Assumption 2.2 would be satisfied with P containing only aPb. Proposition 3.3 wouldn't apply (because $q(b,-) < 1/2$)), but Proposition 3.1 would imply that $\mathbb{P}\left(Y_n \to \infty\right) = 1$. That is, with certainty, textbook a would eventually become the more popular

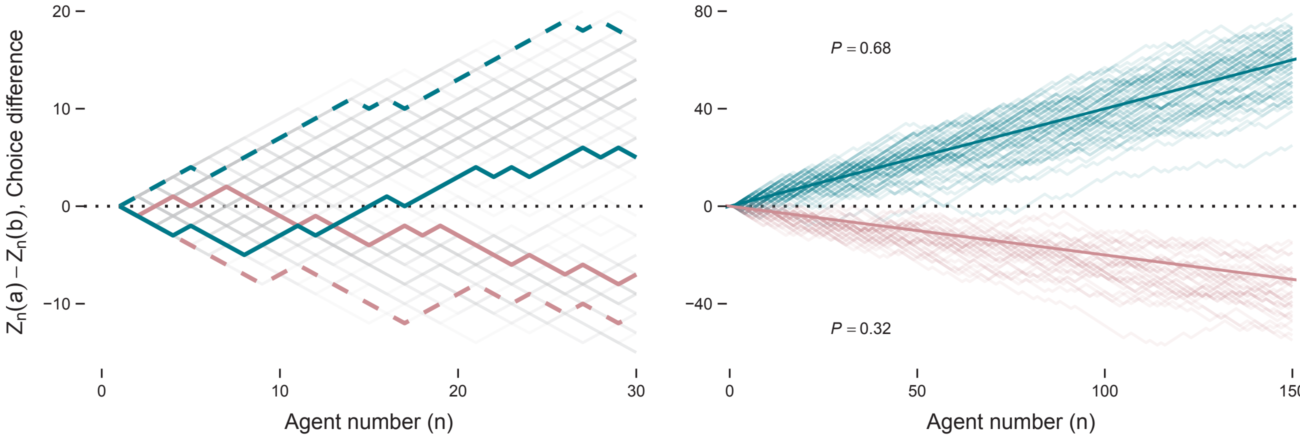

Figure 1: Simulation of a large number of realizations of the ranking-based sequential choice system of Example 3.5. **Left:** $Z_n(a) - Z_n(b)$ for $n \leq 30$. A large number of trajectories are overlaid in gray color. The solid blue line is one for which textbook b has an initial advantage, but eventually textbook a overtakes. The dashed blue line shows a case where textbook a is chosen by the first agent and stays ahead for all $n \geq 1$. Similarly for solid and dashed pink lines. **Right:** Zoomed-out version of the image on the left, where the trajectories can be seen for $n \leq 150$. The trajectories now have been color-coded according to whether $Z_n(a) - Z_n(b) \to \infty$ or $Z_n(a) - Z_n(b) \to -\infty$. About 68% of the trajectories are blue. The solid lines are $y = 0.4x$ and $y = -0.2x$, demonstrating that $\frac{Z_n(a)-Z_n(b)}{n} \to q(a,+) - q(b,+) = 0.4$ for blue trajectories and $\frac{Z_n(b)-Z_n(a)}{n} \to q(b,-) - q(a,-) = 0.2$ for pink trajectories.

one and stay like this forever after. By Proposition 3.2 we can be more precise and say with certainty that, in the long-run, 70% of the students will choose textbook a.

Discussion The results of this section are agnostic with regards to the underlying choice model. Choice probabilities like the ones in Table 1 can be produced by an array of models in economics and beyond. For example, in discrete choice or random utility models (Brock & Durlauf, 2001) such ranking-dependent choice probabilities could occur if the popularity rank of a product is considered as an attribute (e.g. encoding status) taken into account in the decision making process. This idea is developed further in section 5.5. Most other models presented as applications in section 5 can also be applied in binary choice scenarios.

A family of models that deal specifically with binary choices are drift-diffusion models (Busemeyer & Townsend, 1993; Fudenberg et al., 2018; Krajbich, Armel, & Rangel, 2010). In these models evidence is accumulated dynamically until an evidence threshold is reached, whereby the final decision is made. Evidence might be accumulated faster (higher drift rate) for salient products due to the increased attention they receive, or people might reach a decision with less evidence in favor of popular products (the starting point of the drift is closer to the threshold, also see

Tump, Pleskac, and Kurvers, 2020). In both cases, the choice probabilities are affected. As long as the probabilities depend solely on the ranking of the two options, the results of this section apply.[5]

4 The general case

We now turn to the general case, that is $|X| = N \in \mathbb{N}$ and the initial condition z being arbitrary. As in Section 3, our goal is to find conditions under which the ranking and the market shares converge, and to characterize the possible limits.

Our first result generalizes the first part of Proposition 3.1.

Theorem 4.1. *For any ranking-based sequential choice system with reinforcement, the popularity ranking R_n converges (i.e. stops changing) with probability* 1.

This theorem guarantees convergence of the ranking, with the only restriction on π being that Assumption 2.2 is satisfied (that's the *with reinforcement* part). To see the connection with Proposition 3.1, observe that for $N = 2$ there are only two strict rankings, to which in Section 3 we referred using the superscripts "+" and "−". Convergence to "+" is equivalent to $Y_n \to \infty$ and convergence to "−" is equivalent to $Y_n \to -\infty$. Proposition 3.1 stated that R_n converges if and only if $\max\{q(a,+), q(b,-)\} > 1/2$, which as shown in Section 3, is equivalent to Assumption 2.2. Note however that Theorem 4.1 only states sufficiency of this condition for convergence.

Although Theorem 4.1 guarantees that the ranking converges, it doesn't say what the possible limits are, that is for which rankings r we have $\mathbb{P}(R_n \to r) > 0$. We won't answer this question for all pairs of ranking-based choice functions π and initial conditions z; instead for a given π, we will be able to say whether there is *some* initial condition z such that $\mathbb{P}_z(R_n \to r) > 0$ (Proposition 4.3), where the subscript z is now needed since z is not fixed. Then we will give a sufficient condition such that $\mathbb{P}_z(R_n \to r) > 0$ for *all* z (Proposition 4.6). This motivates the following definition.

[5] We note that for the simplest models in this family, such as the linear ballistic accumulator model (Brown & Heathcote, 2008), the choice probabilities can be analytically calculated, while in other cases they can be approximated numerically.

Definition 4.2. *Let $\pi : \mathcal{X} \times \mathcal{R} \to [0,1]$ be a ranking-based choice function. A ranking $r \in \mathcal{R}$ is (weakly) terminal for π, if there exists some initial condition z such that $\mathbb{P}_z(R_n \to r) > 0$. A ranking r is strongly terminal for π, if $\mathbb{P}_z(R_n \to r) > 0$ for all initial conditions z.*

According to the above definition, a ranking r is weakly terminal, if one can find initial conditions such that the popularity ranking R_n may converge to r. At first sight this might not seem a very useful property, since we would normally be interested in a specific initial condition z, but note that the negative form does provide useful information: if r is *not* terminal, then $\mathbb{P}_z(R_n \to r) = 0$ for *any* initial condition z.

The following proposition gives an easy way to check whether a ranking is terminal, based on the choice function π. Recall that $q(ab', r)$ denotes the probability that item $a \in X$ is chosen and $b \in X$ is not, given the ranking r (Eq. (3)).

Proposition 4.3. *A ranking $r \in \mathcal{R}$ is weakly terminal for a ranking-based choice function π if and only if, for any $a, b \in X$:*

- *If $a \succ_r b$ and $q(ba', r) > 0$, then $q(a, r) > q(b, r)$.*
- *If $a \approx_r b$, then $q(ab', r) = q(ba', r) = 0$.*

Note that Proposition 4.3 does not require π to satisfy Assumption 2.2.

If the cases $q(ab', r) = 0$ can be excluded, i.e. if there is always positive probability that an item a is chosen when b is not, for any $a, b \in X$, then the above conditions can be simplified, as follows:

Corollary 4.4. *If $q(ab', r) > 0$ for all $a, b \in X$ and all $r \in \mathcal{R}$, then a ranking $r \in \mathcal{R}$ is weakly terminal if and only if it is a strict ranking (i.e. it contains no ties) and $q(a, r) > q(b, r)$ whenever $a \succ_r b$. Equivalently, r is weakly terminal if and only if it is a strict ranking and*

$$q(r^{-1}(1), r) > \ldots > q\left(r^{-1}(N), r\right), \tag{17}$$

where r^{-1} is the inverse of r and $N = |X|$.

Proof. The first part of the corollary follows immediately from Proposition 4.3. For Eq. (17), note that by definition we have $r(r^{-1}(i)) = i$ for all $i = 1, \ldots, N$, hence $r(r^{-1}(1)) < \ldots < r(r^{-1}(N))$,

$(r(a), r(b), r(c))$	$(r^{-1}(1), r^{-1}(2), r^{-1}(3))$	$q(r^{-1}(1), r)$	$q(r^{-1}(2), r)$	$q(r^{-1}(3), r)$
$(1, 2, 3)$	(a, b, c)	0.8	0.12	0.032
$(1, 3, 2)$	(a, c, b)	0.8	0.08	0.072
$(2, 1, 3)$	(b, a, c)	0.6	0.32	0.032
$(2, 3, 1)$	(c, a, b)	0.4	0.48	0.072
$(3, 1, 2)$	(b, c, a)	0.6	0.16	0.192
$(3, 2, 1)$	(c, b, a)	0.4	0.36	0.192

Table 2: Probabilities of choosing items in Example 4.5. The first column shows all possible strict rankings of the items a, b, c. In the second column the inverse function is shown; $r^{-1}(i)$ is the item ranked at the i-th position. The items are searched in this order. Columns 3-5 show the probabilities of choosing the item ranked first, second, and third, respectively, calculated using Eq. (20). Note that their sum is less than 1, because there is positive probability that no item is chosen. By Corollary 4.4, a strict ranking r is terminal if and only if $q(r^{-1}(1), r) > q(r^{-1}(2), r) > q(r^{-1}(3), r)$. Here the first, second, third, and sixth rankings are terminal.

which is equivalent to $r^{-1}(1) \succ_r \ldots \succ_r r^{-1}(N)$. Therefore, by the first part of this corollary, $r \in \mathcal{R}$ is terminal if and only if $q(r^{-1}(1), r) > \ldots > q(r^{-1}(N), r)$. $\square$

Example 4.5. *Let $X = \{a, b, c\}$ and assume that the choice probabilities $q(\cdot, \cdot)$ for the six possible strict rankings of X are as in Table 2. These probabilities were calculated assuming that the agents choose based on the satisficing model of Section 5.1, with $p(a) = 0.8$, $p(b) = 0.6$, and $p(c) = 0.4$ (see Eq. (20)), but other models could also produce the same choice probabilities. If a different model is assumed, we also require that $q(xy', r) > 0$ for all $x, y \in X$ and $r \in \mathcal{R}$, which for the satisficing model is true by Eq. (23).*

Observe that there are four rankings (first, second, third, and sixth line of the main body of Table 2) for which $q(r^{-1}(1), r) > q(r^{-1}(2), r) > q(r^{-1}(3), r)$. By Corollary 4.4, these are the terminal rankings for the underlying choice function.

As noted above, for a ranking that is not terminal, we know that $\mathbb{P}_z(R_n \to r) = 0$ for any initial condition z, but for a terminal ranking r we can't immediately tell for which initial conditions z we have $\mathbb{P}_z(R_n \to r) > 0$. This changes if r is *strongly* terminal, in which case $\mathbb{P}_z(R_n \to r) > 0$ holds for all z. The following proposition gives a sufficient condition for all weakly terminal rankings to be strongly terminal.

Proposition 4.6. *Let π be a ranking-based choice function satisfying that $\pi(\{a\}, r) > 0$ for all $r \in \mathcal{R}, a \in X$. Then, all terminal rankings are strongly terminal. In particular, for any initial* *condition z, $\mathbb{P}_z(R_n \to r) > 0$ if and only if r is terminal.*

The condition $\pi(\{a\}, r) > 0$ says that a has positive probability of being the *only* item chosen, and this has to be true no matter the ranking r. Again, this condition can be easily checked given the choice function π.

Finally, we have the following generalization of Proposition 3.2. It gives the proportion of agents that choose each item, if we know the limit ranking r.

Proposition 4.7. *Let (π, z) be a ranking-based sequential choice system. For any $r \in \mathcal{R}$, on the set $\{R_n \to r\}$ we have $Z_n(a)/n \to q(a, r)$ a.s.*

4.1 Utility

We now consider the long-term behavior of the average utility experienced by the agents. Recall from Section 2.4 that we denote by $u(A, r, w)$ the utility experienced by an agent of type $w \in \mathcal{W}$, who chooses the set of items A when presented with a ranking r, and that each agent's type is picked independently according to a probability distribution Q on $\mathcal{W}$. To make sure that certain expected values below exist, we assume that u is bounded, that is there exists some $M > 0$ such that $|u(A, r, w)| \leq M$ for all A, r, w.[6]

Let A_n denote the choice of the n-th agent and W_n their type. Then, conditioned on the ranking at time $n-1$ being r, the expected utility experienced by this agent is given by

$$\begin{aligned} \mathbb{E}[u(A_n, R_{n-1}, W_n) \mid R_{n-1} = r] &= \mathbb{E}[u(A_n, r, W_n)] \\ &= \sum_{A \in \mathcal{X}} \int_{w \in W} \pi(A, r, w) \cdot u(A, r, w) \, dQ, \end{aligned} \tag{18}$$

whenever the first expression makes sense, i.e. $\mathbb{P}(R_{n-1} = r) > 0$. Notice that the last expression in Eq. (18) does not depend on n, and it can be interpreted as the expected utility of a randomly picked agent, given that the ranking at the time of their choice is r. We'll denote this by $V(r)$, that is

[6] In fact it would be enough that for any $A \in \mathcal{X}$ and $r \in \mathcal{R}$, $\int_{w \in \mathcal{W}} |u(A, r, w)| \, dQ < \infty$. Since $\mathcal{X}$ and $\mathcal{R}$ are finite sets, this implies that the expectation of $u(A, R, W)$ exists and is finite for arbitrary random elements A, R.

$$V(r) = \sum_{A \in \mathcal{X}} \int_{w \in W} \pi(A, r, w) \cdot u(A, r, w)\, dQ, \tag{19}$$

Our main result in this section says that the average utility $\bar{u}_n$ experienced by the first n agents in a ranking-based sequential choice system (see Eq. (10)) converges to $V(r)$ as $n \to \infty$, if r is the limit of the popularity ranking.

Theorem 4.8. *For any ranking-based sequential choice system with bounded utility function, on the set $\{R_n \to r\}$, we have $\overline{u}_n \to V(r)$ a.s.*

Theorem 4.8 does not require Assumption 2.2, but if that assumption is satisfied, then we know from Theorem 4.1 that the ranking converges a.s. In that case, Theorem 4.8 implies that the average utility also converges a.s., but with its limit depending on the limit ranking. Since there can be multiple possible limit rankings, the long-term average utility may also have multiple possible limits, implying unpredictability in terms of the final outcome. The fact that $\bar{u}_n \to V(r)$ where r is the limit ranking implies that the average utility is maximized if and only if the system converges to the ranking r that maximizes $V(r)$, which we may call the *optimal* ranking.

5 Behavioral models

In this section we show how various well-known models from the literature, or slight variations of them, can be described in terms of a ranking-based choice function with reinforcement.

5.1 Search and satisficing

The reinforcement assumption (2.2) naturally applies to models in which people go through a number of items in a prespecified order and choose the first item with utility exceeding a certain threshold. Simon's (1955) satisficing model was the first model of this type, although at the time the order in which items were searched was not emphasized[7]. Many recent models in economics share the same sequential search structure and also stress the importance of the order in which the items are examined (Caplin et al., 2011; Reutskaja, Nagel, Camerer, & Rangel,

[7]Optimal stopping models (DeGroot, 2005) also share the same sequential structure, but assume that people search at random and the stopping threshold is optimally derived.

2011; Athey & Ellison, 2011). Very similar assumptions, but in a simpler utility framework, have been used by modelers in computer science. When the utility is binary (i.e. satisfying/not satisfying), the same assumptions correspond to the well-studied cascade model in information retrieval (Craswell, Zoeter, Taylor, & Ramsey, 2008). According to this model, agents search an online interface (e.g. search engine, recommender system) until they find the first item relevant to them. More generally, the utility may be drawn from an arbitrary, item-specific distribution, but independently for different items. In order to determine the choice probabilities the only thing that matters is the probability $p(x)$ that the utility of item $x \in X$ exceeds the threshold T.

Below we calculate the choice probabilities for the satisficing model in terms of the $p(x)$'s, and show that they give rise to a ranking-based choice function with reinforcement. To avoid corner cases, we assume that $p(x) \in (0, 1)$ for all $x \in X$. Also, if no item's utility exceeds T, the agent chooses nothing.

Consider first a strict ranking $r \in \mathcal{R}$, i.e. one that doesn't contain any ties. For an item $a \in X$ to be chosen, its utility must exceed the threshold, and the utilities of all items that are ranked higher than a must fail to exceed it. That is

$$q(a, r) = p(a) \cdot \prod_{x \succ_r a} (1 - p(x)). \tag{20}$$

Now consider an arbitrary ranking $r \in \mathcal{R}$, i.e. one that may also contain ties. Let S_r denote the set of all bijections $\sigma : X \to \{1, \ldots, |X|\}$ that are consistent with the ranking r, in the sense that $\sigma(a) < \sigma(b)$ whenever $r(a) < r(b)$[8]. These bijections are the different possible search orders for the given ranking, i.e. the different ways that the agent may go through the items while respecting the ranking.

We assume that one of the possible search orders is chosen uniformly randomly, and that this choice is independent of the agent type. Then, since each σ can be regarded as a strict ranking, we get

$$q(a, r) = \frac{1}{|S_r|} \cdot \sum_{\sigma \in S_r} q(a, \sigma). \tag{21}$$

[8]The point here is that both cases $\sigma(a) > \sigma(b)$ and $\sigma(a) < \sigma(b)$ are allowed when $r(a) = r(b)$.

That is, $q(a, r)$ is the average of $q(a, \sigma)$, where σ runs through all strict rankings that satisfy $\sigma(a) < \sigma(b)$ whenever $r(a) < r(b)$.

Note that since the agent is allowed to choose at most one item from X, the quantities $q(a, r)$, $a \in X, r \in \mathcal{R}$, are enough to determine the ranking-based choice function π. Specifically,

$$\pi(A, r) = \begin{cases} q(a, r), & \text{if } A = \{a\} \\ 0, & \text{if } |A| \geq 2, \\ 1 - \sum_{x \in X} q(x, r), & \text{if } A = \emptyset. \end{cases} \tag{22}$$

We also note for later reference that for the probability of choosing a and *not* b we have $q(ab', r) = q(a, r)$, for any $a, b \in X, r \in \mathcal{R}$, and that since $p(x) \in (0, 1)$ for all $x \in X$, we get by Eqs. (20) and (21) that

$$q(ab', r) = q(a, r) > 0, \quad a, b \in X. \tag{23}$$

It remains to specify the relation P on X for which π satisfies Assumption 2.2. We define it as

$$aPb \text{ if and only if } p(a) > p(b). \tag{24}$$

Clearly, this is an asymmetric relation. To check Assumption 2.2, note that the second part is satisfied trivially by Eq. (23). For the first part, observe that $b\not P a$ implies $p(b) \leq p(a)$, and $a \succ_r b$ implies that the product $\prod_{x \succ_r b} (1 - p(x))$ includes all factors in $\prod_{x \succ_r a} (1 - p(x))$ and at least one additional factor, namely $(1 - p(a))$. Therefore, if r doesn't contain any ties, we get from Eq. (20) that $q(b, r) < q(a, r)$. For a ranking r that contains ties, the above argument can be applied to each term $q(i, \sigma)$ in Eq. (21) (recall that $a \succ_r b$ implies $a \succ_\sigma b$), so we again get $q(b, r) < q(a, r)$.

We therefore see that the satisficing model of Simon satisfies Assumption 2.2, meaning that it can be thought of as ranking-based choice with reinforcement. The reinforcement in this model is conveyed by the order that the items are searched. Since this is the first behavioral model we examine, we discuss the implications of our results in some detail. In subsequent models we will largely skip this step and will rather focus on just showing that our assumptions are satisfied.

To begin with, Theorem 4.1 implies that for the ranking-based sequential choice system (π, z), where π is given by Eqs. (20) to (22) and z is any initial condition, R_n converges with probability 1, i.e. the ranks of the various items will eventually stabilize. If $R_n \to r$, then the market share of each item will converge to its choice probability in that ranking, that is $\frac{Z_n(a)}{n} \to q(a,r)$, by Proposition 4.7.

Furthermore, since $\pi(\{a\}, r) = q(a,r) > 0$ for all $a \in X$ and $r \in \mathcal{R}$, we get by Proposition 4.6 that $r \in \mathcal{R}$ is a possible limit of R_n (i.e. $\mathbb{P}(R_n \to r) > 0$) if and only if it is terminal. By Corollary 4.4 and Eq. (23), a ranking r is terminal if and only if Eq. (17) holds, which can easily be checked once we have the $q(a,r)$'s. In Example 4.5 we find the terminal rankings for a case with three items.

Regarding the average utility experienced by the agents, Theorem 4.8 says that if $R_n \to r$, then $\bar{u}_n \to V(r)$, where $\bar{u}_n$ is the average utility for the first n agents and $V(r)$ the expected utility for an agent given the ranking r. In order to apply this proposition, however, we need to describe the utility as a function of the form of Eq. (9) for a suitable set $\mathcal{W}$ of agent types and a distribution Q over it, calculate the per agent-type choice probabilities $q(a,r,w)$, and show that the choice probabilities $q(a,r)$ derived from this distribution through Eq. (7) are the same as Eq. (20). This is done in Appendix A.3.

If there are multiple rankings r for which $\mathbb{P}(R_n \to r) > 0$, then $\bar{u}_n$ has positive probability of converging to $V(r)$ for different r's. Unless all these rankings happen to have the same expected utility $V(r)$ and maximum among all $r \in \mathcal{R}$, it follows that $\bar{u}_n$ has positive probability of converging to a suboptimal value.

5.2 Multiplicative attention models

A second family of models for which our results apply are multiplicative attention models. In contrast to the satisficing model, there is no assumption about any sequential consideration of the options, but instead each position is associated with a fixed coefficient that modulates the choice probability, capturing the fact that decision-makers are more likely to pay attention to items that are higher on a list, (Chuklin, Markov, & Rijke, 2015; Joachims, Swaminathan, & Schnabel, 2017). Here we describe the model appearing in a recent publication of Germano et al. (2019), where the authors explicitly consider a scenario in which people choose one after the other, from a set of options presented in order of descending popularity, exactly like the sequential choice model that we have studied (see Section 2.3).

Their behavioral model, similarly to the satisficing model, assumes that each agent chooses exactly one option, but now according to the following rule: each option $a \in X$ has some propensity $v(a) > 0$ of being chosen and this propensity is multiplied by a factor $\beta(r(a)) > 0$, where $r(a)$ is the ranking of item a according to popularity, and $\beta : \{1, \ldots, |X|\} \to (0, \infty)$ is a strictly decreasing map. The product is then normalized, so that the probability of choosing item a given the ranking r is[9]

$$q(a, r) = \frac{\beta(r(a)) \cdot v(a)}{\sum_{x \in X} \beta(r(x)) \cdot v(x)}. \tag{25}$$

The factor $\beta(r(a))$ can be thought of as an attention modulation factor, and note that it depends only on the item's rank.

To see that this model fits into our framework, define P to be the strict (partial) order induced by $v(\cdot)$, i.e. aPb whenever $v(a) > v(b)$. Then, it is easy to see from Eq. (25) and the fact that $a \succ_r b \Leftrightarrow \beta(r(a)) > \beta(r(b))$ that Assumption 2.2 is satisfied. As a result, Theorem 4.1 implies that, in a sequential choice system based on the resulting choice function, the ranking will eventually converge, and by Corollary 4.4 and Proposition 4.6 the possible limits are those for which $q(r^{-1}(1), r) > \ldots > q(r^{-1}(|X|), r)$. This is an intuitive result that was used in (Germano et al., 2019) to find the limit ranking. Our results provide theoretical justification for this intuition.

Randomized rankings

In both the satisficing and the multiplicative attention model we were assuming that the items are always presented in the order of popularity. However, this assumption can be relaxed; our theory continues to apply even if we introduce some randomness in the presentation order, as long as the more popular items are more likely to be presented higher. Randomization strategies are

[9]The actual model in Germano et al. (2019) is slightly more complex, with three types of agents instead of one, but the adaptation is straightforward. Also, in that reference, β has the specific form $\beta(i) = \alpha^i$, for some $\alpha \in (0, 1)$.

commonly used in online interfaces to debias algorithms and get better estimates of the quality of content (Pandey, Roy, Olston, Cho, & Chakrabarti, 2005). Randomization of the ranking could also occur because different individuals may perceive or search a ranked list in a stochastic manner (Aguiar, Boccardi, & Dean, 2016).

To demonstrate how randomization can be taken into account with an example, suppose that with probability $1 - \epsilon$ the order of presentation is the same as the popularity ranking and with probability ϵ the order of presentation is completely random, where $\epsilon \in (0, 1)$. Then, in the multiplicative attention model described above, the probability of choosing item $a \in X$ when the *popularity* ranking is r would be

$$q(a, r) = (1 - \epsilon) \cdot \frac{\beta(r(a)) \cdot v(a)}{\sum\limits_{x \in X} \beta(r(x)) \cdot v(x)} + \epsilon \cdot \frac{1}{N!} \cdot \sum_{\sigma} \frac{\beta(\sigma(a)) \cdot v(a)}{\sum\limits_{x \in X} \beta(\sigma(x)) \cdot v(x)}, \tag{26}$$

where σ runs through all $N!$ bijections from X to $\{1, \ldots, N\}$. It can be shown that Assumption 2.2 is satisfied for the same relation P as the original model, for any $\epsilon \in (0, 1)$. If ϵ is sufficiently small, then the terminal rankings will be the same as in the case $\epsilon = 0$, but for larger ϵ we may have different terminal rankings, which can again be found applying Corollary 4.4. Similar ranking randomization techniques can be applied to other models.

5.3 Consideration sets

When an agent has a large number of options, they may restrict their attention to a subset of those, ignoring the rest (Stigler, 1961; De los Santos, Hortaçsu, & Wildenbeest, 2012). The set of options that the agent pays attention to is commonly called a *consideration set* (Shocker, Ben-Akiva, Boccara, & Nedungadi, 1991; Masatlioglu, Nakajima, & Ozbay, 2012; Manzini & Mariotti, 2014; Hauser, 2014). Choice models based on consideration set formation thus assume a two-step decision process: first restrict attention to a subset of the items, and then choose one among this smaller set of items.

Here we will assume that the agent considers only the top-K ranked options, with K a random positive integer. Consequently, whether an item is in the consideration set $C \subset X$ depends on

K and the item's rank. We assume that $\mathbb{P}(K=k)>0$ for all k, and that all items that are tied in the K-th position (or above) are included in the consideration set, so that $a \in C$ if and only if $r(a) \leq K$. Note that this implies that the consideration set includes at least K items, but possibly more.

Given the consideration set C, we will assume that the agent chooses an item from C according to a Luce rule (1959), independently of K. That is, there are Luce parameters $\beta(a)$, $a \in X$, such that for any $a \in X, r \in \mathcal{R}$,

$$q(a,r)=\sum_k \mathbb{P}(K=k)\cdot \frac{\beta(a)}{\sum\limits_{x:r(x)\leq k}\beta(x)}\cdot \mathbf{1}_{r(a)\leq k}. \tag{27}$$

The second factor in the above product encodes the choice according to the Luce rule from among the items that are ranked k or higher. The last factor says that we only include the term for a if its rank is k or higher (so that a is in the consideration set).

We define the relation P to mean "larger Luce parameter", that is aPb if and only if $\beta(a) > \beta(b)$. Let us check that Assumption 2.2 is satisfied. For any $a,b \in X$ and any $r \in \mathcal{R}$ we have

$$q(a,r)-q(b,r) \quad =\sum_k \frac{\mathbb{P}(K=k)}{\sum\limits_{x:r(x)\leq k}\beta(x)}\cdot\left[\beta(a)\cdot \mathbf{1}_{r(a)\leq k}-\beta(b)\cdot \mathbf{1}_{r(b)\leq k}\right]. \tag{28}$$

Assume $b\not{P}a$ and $a \succ_r b$. The former implies that $\beta(a)\geq\beta(b)$, while the latter implies $\mathbf{1}_{r(a)\leq k}\geq \mathbf{1}_{r(b)\leq k}$ for all k, and $\mathbf{1}_{r(a)\leq k} > \mathbf{1}_{r(b)\leq k}$ for $k=r(a)$, therefore from Eq. (28) we get $q(a,r)-q(b,r)>0$, which shows the first part of Assumption 2.2. Now assume that aPb and $a \approx_r b$. The former implies that $\beta(a)>\beta(b)$ and the latter implies $r(a)=r(b)$, hence from Eq. (28) we get again that $q(a,r)-q(b,r)>0$, from where the second part of Assumption 2.2 follows.

5.4 Random utility models

Here we consider random utility models (Manski, 1977) in which the utility contains an additive deterministic component that is a function of the ranking. The ranking term could express utility derived from the higher status of options higher in the hierarchy (Podolny, 1993) or a preference for consuming more popular items. Specifically, for each item $a \in X$, its utility is the sum of a fixed term $v(a)$, a random term $\epsilon(a)$, and a position-based bonus $b(r(a))$, for some decreasing map $b : \{1, \ldots, |X|\} \to \mathbb{R}$, giving the total utility

$$u(a, r) = v(a) + b(r(a)) + \epsilon(a). \tag{29}$$

This is very similar to the social interactions model of Brock and Durlauf (2001) and Blume and Durlauf (2003), except that in these references the middle term is a linear or quadratic function of the (perceived) expected proportion of agents that choose each item, rather than a function of the ranking of these proportions.

Equation (29) can also be written in the form of Eq. (9), in a similar manner as is done for the satisficing model in Appendix A.3, but we don't do it here to keep notation simple. We assume $\epsilon(a)$ to be i.i.d. and Gaussian. Assuming that, for a given ranking r, the agent chooses the item that maximizes $u(\cdot, r)$, the probability that item a is chosen is

$$q(a, r) = \mathbb{P}\left(u(a, r) > \max_{x \neq a} \{u(x, r)\} \right). \tag{30}$$

Given that the $\epsilon(a)$'s are i.i.d. and Gaussian, this probability is positive and strictly increasing in $v(a) + \beta(r(a))$, that is

$$q(a, r) > q(b, r) \Leftrightarrow v(a) + \beta(r(a)) > v(b) + \beta(r(b)) \quad a, b \in X, r \in \mathcal{R}. \tag{31}$$

We define the relation P as follows: aPb if and only if $v(a) > v(b)$. We show that this model satisfies Assumption 2.2 under either of two conditions:

- β is *strictly* decreasing: Assuming that $a \succ_r b$ and $b \not{P} a$, we get $v(a) \geq v(b)$ and $\beta(r(a)) > \beta(r(b))$, respectively, so $q(a, r) > q(b, r)$ by Eq. (31).

- $v(a) \neq v(b)$ for any $a \neq b$: In this case $b \not{P} a$ implies $v(a) > v(b)$, while $a \succ_r b$ implies $\beta(r(a)) \geq \beta(r(b))$, hence we get again $q(a, r) > q(b, r)$ by Eq. (31).

Note that since $q(r, a) > 0$ for all $a \in X$ and $r \in \mathcal{R}$, the second part of Assumption 2.2 is trivially satisfied.

As a boundary case we get the classical random utility model without any ranking bonus. This corresponds to the case where β is a constant function. As shown above, this model still satisfies Assumption 2.2, as long as the $v(a)$'s are all distinct (i.e. $v(a) \neq v(b)$ for any $a \neq b$).

5.5 Non-strictly-ranking dynamics

Our formulation can model popularity dynamics even if the probability of choosing an option does not depend merely on how the options rank relative to each other, but also on whether the difference in terms of popularity is larger or smaller than a fixed threshold. We demonstrate this by showing how our framework applies to Banerjee's (1992) seminal herding model, which turns out to be similar to a lexicographic semi-order model (Manzini & Mariotti, 2012; Tversky, 1969) where popularity is the first considered piece of information and the threshold is two.

In this model there are d options, one of which is considered correct, in principle unknown to the agents. Each agent might have a private signal, with probability $p > 0$, that indicates which the correct option is, but the signal itself might be wrong when it exists, with probability $p' < 0.5$, same for all agents. Each agent has information about the choices of previous agents, but not the other agents' private signals or whether they have such a signal. Agents are assumed rational, i.e. they always choose the option that is most likely to be correct based on the information they have. In case that two or more options are tied as the most likely to be correct, one is chosen at random, with one exception: if one of the options tied as most likely to be correct is the one that the agent's private signal has indicated, then they trust their private signal and choose this option over others[10].

In Banerjee (1992) it is shown that the above assumptions imply the following behavior for the agents: if the agent's private signal indicates an option i that is either the most popular one (perhaps tied with other options) or just a single vote behind the most popular item(s), then the agent chooses i. Otherwise, the agent chooses one of the items tied in the first place, uniformly randomly.

[10] In (Banerjee, 1992) there is one more exception: if the agent has no signal and all items have popularity 0, then no item is chosen. We can avoid this situation by assuming that the first agent always has a private signal (equivalently, by ignoring all initial agents with no signal).

At first sight it might seem that our framework cannot model this choice rule, because the probability of choosing an option depends on the actual differences of the various options (whether they are larger than 1 or not), and not merely on their ranks. We can however bypass this problem as we will show below. The idea is to augment our initial list of items, numbered 1 through d, with the virtual items $d+1,\dots,2d$. The popularity component $Z_n(d+i)$ will record the value of $Z_n(i)$ minus 1. For example, if there are three options with 4, 11, and 0 votes respectively, then $Z_{15} = (4, 11, 0, 3, 10, -1)$. The motivation for this definition is the following: if an option j is just one vote behind $Z_n(i)$, then it will be tied with $Z_n(i+d)$, which is a piece of information that can be extracted from the ranking. In order for the vector Z_n to record the above information, our agents will be required to choose not single items, but pairs of items and their corresponding virtual items, that is pairs of the form $\{i, i+d\}$.

To make the above precise, let $X = \{1, \dots, 2d\}$, where d is the number of items among which an agent has to find the correct one. We define $\pi(\{i, d+i\}, r)$ to be the probability that the agent chooses item i in the initial problem, given the popularity ranking r of the real and virtual items. We also define $\pi(A, r) = 0$ for any A not of the form $\{i, d+i\}$. Note that this implies that for the item-specific probabilities of choice we have $q(i, r) = q(i+d, r) = \pi(\{i, d+i\}, r)$, for any $i \in \{1, \dots d\}$.

Let i^* denote the correct choice (among the initial d items). In Appendix A.4 we show that Assumption 2.2 is satisfied with the following relation P:

- iPj for any virtual item i and real item $j \neq i^*$, and
- i^*Pj and $(i^*+d)P(j+d)$ for any real item $j \neq i^*$.

Therefore, our results apply and in particular the ranking R_n converges to some $r \in \mathcal{R}$ as $n \to \infty$ with probability 1. We will show that this implies the herding behavior that was also shown in Banerjee (1992), with all but finitely many agents choosing the same item.

Let i_* be such that $r(i_*) = 1$ for the limit ranking r. Then clearly $r(i_* + d) = 2$ and $q(i_*, r) = q(i_* + d, r) > 0$, therefore by Proposition 4.3 no real item $j \neq i$ can be equally ranked with i or $i+d$, implying that any real item $j \neq i$ is at least two votes behind i_*, hence it can never be chosen. Consequently, $q(i_*, r) = 1$, i.e. once this ranking is established, all agents will be choosing item i_*.

Note that the most popular option does not have to be the correct one, that is we may have $i_* \neq i^*$, which would lead to an incorrect herding.

The above results can of course be deduced more directly, as is done in Banerjee (1992). However, our method puts the system in context and demonstrates that the results are a consequence of a certain reinforcement condition based on a generalized form of ranking. The idea of augmenting the item list with virtual items and casting the choice in terms of the ranking of the augmented list can be applied more generally, namely whenever the agent's choice depends on comparisons of the type $Z_n(i) > Z_n(j) + c$ or $Z_n(i) = Z_n(j) + c$ for a finite number of different constants c.

6 Extensions and other interpretations

So far we have considered problems in which the choices are made by different individuals, and where $Z_n(a)$ encodes the popularity of item a at time n. The same concepts, however, can apply to problems where choices are made by a single agent, as long as there is a sequential choice structure with ranking-based reinforcement. To illustrate that, we describe two problems that fit into our framework, where decision are made by a single agent and the vector Z_n has a different interpretation.

Exposure effect

Consider the following scenario: a person dines out every Saturday, choosing from a fixed list of N restaurants they know. Some restaurants might have no availability on a particular Saturday, so their available choices are somewhat restricted. From among the available restaurants, the person chooses the one that maximizes a measure of desirability $Z_n(i)$, that varies with the week n. Specifically, $Z_n(i)$ is the sum of a fixed term $z(i)$, representing the quality of the restaurant, and a term that increases every time the person visits the restaurant, reflecting the appeal of the restaurant due to the person's familiarity with it. More precisely, if $A_n \in \{0, 1, \ldots, N\}$ is the person's choice on week n, then

$$Z_n(i) = card\{k \leq n : A_n = i\} + z(i), \tag{32}$$

where $A_n = 0$ implies that the person didn't dine out because all restaurants were fully booked. We thus recover essentially Eq. (4), but this time with a different interpretation.

Equation (32) is used in Cerigioni (2017) to define the Exposure-Biased Luce Model, where the probability of choosing restaurant i is proportional to $Z_n(i)$. Here, instead, we will assume that the agent always prefers the restaurant that maximizes $Z_n(i)$ among those that have availability. In other words, the person goes through the available restaurants in order of desirability and chooses the first one that has availability. Assuming that the probability of a restaurant having availability is constant over days, and independent over days and restaurants, it is clear that this problem is equivalent to the satisficing model presented in Section 2.2.

Research and development

Last, let us consider a scenario of technology choice with a government or a funding body as the decision maker. Such problems have been studied in sequential choice settings with many agents (Arthur, 1989). Suppose that every time the funding body wants to develop a new technology or product in a certain field (e.g. Pharma), it invests at the same time in several companies, or research teams that could potentially develop such a product. At the end the funding body chooses to adopt the product that it considers the best solution. As a rule, the funding body holds a record of how many times in the past each company has achieved the best solution and it invests a larger amount to the most successful company so far, a smaller amount to the second most successful and so on. Of course, larger investments make a company more likely to succeed in developing the best solution. The dynamics of this system can be described within our framework, if we define $Z_n(a)$ as the number of times, among the first n projects, that company a's solution was chosen by the funding body, and A_n as the company chosen for the n-th project. Clearly, the probability that a company wins a project depends on the amount of money invested by the government, which depends on their ranking based on $Z_n(a)$, hence it can be expressed in the form of a ranking-based choice function. Moreover, since a company with a better research team and a higher amount invested in it is more likely to be successful, Assumption 2.2 is satisfied if we take the relation P to reflect the companies' inherent ability to do research in the area. We omit the details.

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A Appendix

A.1 Proofs of Theorem 4.1 and Propositions 4.3, 4.6 and 4.7

Theorem 4.1 and Propositions 4.3, 4.6 and 4.7 are direct applications of the theory of ranking-based processes developed in Analytis, Gelastopoulos, and Stojic (2021), henceforth abbreviated AGS. We begin by explaining how a ranking-based sequential choice system fits into the framework of that reference.

We index the items of X by the numbers $1, \dots, N$. The popularity map Z_n can then be thought of as a vector in $\mathbb{R}^N$, that is $Z_n = (Z_n^1, \dots, Z_n^N)$, where Z_n^i is the popularity of the i-th item. The difference $\Delta Z_{n+1} = Z_{n+1} - Z_n$ is then a vector containing zeros and ones, with $\Delta Z_{n+1}^i = 1$ if item i is chosen by the n-th agent and 0 otherwise. Equivalently, for any $A \subset X$ denote by $\delta_A = (\delta_A^1, \dots, \delta_A^N) \in \mathbb{R}^N$ the vector of zeros and ones with $\delta_A^i = 1$ if and only if $i \in A$. Then $\Delta Z_n = \delta_{A_n}$ where A_n is the set of items chosen by agent n (see Section 2.3).

Now recall that given the popularity ranking $R_n = r$, the next agent's choice is independent of the past (see Eq. (6)) and the choice probabilities are given by $\pi(\cdot, r)$. That is, for any $A \subset X$,

$$\mathbb{P}\left(\Delta Z_{n+1} = \delta_A \mid R_n = r, Z_1, \dots, Z_n\right) = \mathbb{P}\left(\Delta Z_{n+1}^i = \delta_A \mid R_n = r\right) = \pi(A, r). \tag{33}$$

This makes the process Z_n a *ranking-based process* as defined in Section 2.2 of AGS; the σ-algebra $\mathcal{F}_n$ in the definition can be taken to be $\mathcal{F}_n = \sigma(Z_1, \dots, Z_n)$, while μ^r is the singular measure that puts weight $\pi(A, r)$ on δ_A, for each $A \subset X$.

In the proofs below we refer to statements in AGS in which appear expressions of the form $\mu^r(x_i \geq x_j)$ or similar. In our case this translates as $\mathbb{P}\left(\Delta Z_{n+1}^i \geq \Delta Z_{n+1}^j \mid R_n = r\right)$ or analogously. Simplifications of such expressions are often possible, since ΔZ_{n+1}^i may only take values 0 or 1. Specifically,

$$\mathbb{P}\left(\Delta Z_{n+1}^i > \Delta Z_{n+1}^j \mid R_n = r\right) = \mathbb{P}\left(\Delta Z_{n+1}^i = 1, \Delta Z_{n+1}^j = 0 \mid R_n = r\right) = q(ij', r), \tag{34}$$

$$\mathbb{P}\left(\Delta Z_{n+1}^i \neq \Delta Z_{n+1}^j \mid R_n = r\right) = q(ij', r) + q(ji', r), \tag{35}$$

and

$$
\begin{aligned}
&\mathbb{P}\left(\Delta Z_{n+1}^1 > \Delta Z_{n+1}^2 \geq \ldots \geq \Delta Z_{n+1}^N \mid R_n = r\right) \\
&= \mathbb{P}\left(\Delta Z_{n+1}^1 = 1, \Delta Z_{n+1}^2 = \ldots = \Delta Z_{n+1}^N = 0 \mid R_n = r\right) \\
&= \pi(\{i\}, r).
\end{aligned} \tag{36}
$$

Additionally, in AGS q_i^r denotes the conditional expectation of ΔZ_{n+1}^i, conditioned on $R_n = r$. This coincides with $q(i,r)$ in the current manuscript, since

$$
\mathbb{E}\left[\Delta Z_{n+1}^i \mid R_n = r\right] = \mathbb{P}\left(\Delta Z_{n+1}^i = 1 \mid R_n = r\right) = q(i,r). \tag{37}
$$

For the proofs of our main theorems, we will need a couple of lemmas.

Lemma A.1. *Let $i, j \in \{1, \ldots, N\}$ and $r \in \mathcal{R}$. The following two statements are equivalent:*

- *Either $q(i,r) > q(j,r)$ or $q(ji', r) = 0$.*
- *Either $q(i,r) > q(j,r)$ or $q(ij', r) = q(ji', r) = 0$.*

Proof. Clearly the second statement implies the first. For the converse, suppose that the first statement is true. If $q(i,r) > q(j,r)$, the second statement follows immediately, so assume instead that $q(i,r) \leq q(j,r)$ and $q(ji', r) = 0$. The latter means that j may only be chosen when i is chosen. If $q(ij', r) > 0$, i.e. i has positive probability of being chosen without j being chosen, then necessarily $q(i,r) > q(j,r)$, which contradicts our assumption. Therefore, it must be the case that $q(ij', r) = 0$ as well. □

Lemma A.2. *Assumption 2.2 in the current manuscript implies assumption 2.5 of AGS.*

Proof. Assumption 2.5 in AGS requires that for any pair of i, j, either one of them dominates the other, or they quasi-dominate each other, according to Definition 2.3 in that reference (see below).

Let P be the asymmetric relation of Assumption 2.2, and let $i, j \in \{1, \ldots, N\}$. We have two cases:

- If iPj, then necessarily $j \not P i$, hence Assumption 2.2 implies that for any $r \in \mathcal{R}$:
  - If $i \succ_r j$, then $q(i,r) > q(j,r)$ or $q(ji', r) = 0$.

– If $i \approx_r j$, then $q(ij', r) > 0$ or $q(ji', r) = 0$.

By Lemma A.1 and Eq. (35), the above can be written as

– If $i \succ_r j$, then $q(i, r) > q(j, r)$ or $\mathbb{P}\left(\Delta Z^i_{n+1} \neq \Delta Z^j_{n+1} \mid R_n = r\right) = 0$.

– If $i \approx_r j$, then $q(ij', r) > 0$ or $\mathbb{P}\left(\Delta Z^i_{n+1} \neq \Delta Z^j_{n+1} \mid R_n = r\right) = 0$.

This is the definition of dominance (i dominates j), according to Definition 2.3 in AGS.

- If jPi, we get as above that j dominates i.
- If $i\not{P}j$ and $j\not{P}i$, then Assumption 2.2 and Lemma A.1 imply that for any $r \in \mathcal{R}$:

– If $i \succ_r j$, then $q(i, r) > q(j, r)$ or $q(ij', r) = q(ji', r) = 0$.

– If $j \succ_r i$, then $q(j, r) > q(i, r)$ or $q(ij', r) = q(ji', r) = 0$.

This means that i and j quasi-dominate each other, according to Definition 2.3 in AGS.

We have shown that, if Assumption 2.2 holds, then for each pair i, j, either one dominates the other or they quasi-dominate each other, which is exactly Assumption 2.5 in AGS. □

We are now ready to prove Theorem 4.1 and Propositions 4.3, 4.6 and 4.7.

Proof of Theorem 4.1. By Lemma A.2, Z_n satisfies Assumption 2.5 of AGS. Therefore, Theorem 2.7 in AGS implies that R_n converges with probability 1. □

Proof of Proposition 4.3. Theorem 2.9 of AGS states that a ranking r is (weakly) terminal if and only if for each $i, j \in \{1, \ldots, N\}$:

- If $i \succ_r j$, then either $q(i, r) > q(j, r)$ or $\mathbb{P}\left(Z^i_{n+1} \neq Z^i_{n+1} \mid R_n = r\right) = 0$.
- If $i \approx j$, then $\mathbb{P}\left(Z^i_{n+1} \neq Z^j_{n+1} \mid R_n = r\right) = 0$.

In view of Eq. (35) and Lemma A.1, the above can be rewritten as:

- If $i \succ_r j$, then either $q(i, r) > q(j, r)$ or $q(ji', r) = 0$.
- If $i \approx j$, then $q(ij', r) + q(ji', r) = 0$.

These conditions are identical to those of Proposition 4.3. □

Proof of Proposition 4.6. Proposition 2.17 in AGS states that all weakly terminal rankings are strongly terminal, if for any $r \in \mathcal{R}$ and any permutation σ of $\{1, \ldots N\}$,

$$\mathbb{P}\left(\Delta Z_{n+1}^{\sigma_1} > \Delta Z_{n+1}^{\sigma_2} \geq \ldots \geq \Delta Z_{n+1}^{\sigma_N} \mid R_n = r\right) > 0. \tag{38}$$

By Eq. (36) this is equivalent to $\pi(\{\sigma_1\}, r) > 0$ for any $r \in \mathcal{R}$ and any permutation σ. This can also be stated as $\pi(\{i\}, r) > 0$ for all $i \in \{1, \ldots, N\}$ and $r \in \mathcal{R}$, which is exactly the assumption of Proposition 4.6.

For the last statement in Proposition 4.6, note that if r is terminal, then we have that it is strongly terminal (by the first part of the same proposition), hence $\mathbb{P}_z(R_n \to r) > 0$ for all z by definition of strong terminality. If it is not terminal, then $\mathbb{P}_z(R_n \to r) = 0$ for all z, by the definition of weak terminality. □

Proof of Proposition 4.7. This is the first part of Proposition 2.13 in AGS. □

A.2 Other proofs

Here we prove Theorem 4.8 and Proposition 3.3. We begin with Theorem 4.8, which we repeat here for easy reference.

Theorem (Theorem 4.8)**.** *For any ranking-based sequential choice system with bounded utility function, if $R_n \to r$, then $\overline{u}_n \to V(r)$ with probability* 1.

Here $\bar{u}_n$ denotes the average utility experienced by the first n agents (Eq. (10)), and $V(r)$ denotes the expected utility experienced by any agent, given that the ranking at the time of their choice is r (Eqs. (18) and (19)).

Proof. For any $r \in \mathcal{R}$, by Eq. (18) we have that $\mathbb{E}[u(A_k, r, W_k)] = V(r)$ for all k, hence the Strong Law of Large Numbers gives that

$$\frac{1}{n} \cdot \sum_{k=1}^{n} u(A_k, r, W_k) \to V(r), \quad a.s. \tag{39}$$

Now observe that the fact that R_n converges to r means that there exists some $n_0 \in \mathbb{N}$, such that $R_n = r$ for all $n > n_0$. In other words,

$$\{R_n \to r\} = \bigcup_{n_0=1}^{\infty} \{R_n = r \ \text{ for all } \ n > n_0\}. \tag{40}$$

Fix some $n_0 \in \mathbb{N}$. On the set $\{R_k = r \ \text{ for all } \ k > n_0\}$ we have that for any $n > n_0$,

$$\bar{u}_n = \frac{1}{n} \cdot \sum_{k=1}^{n} u(A_k, R_{k-1}, W_k) = \frac{1}{n} \cdot \sum_{k=1}^{n_0} u(A_k, R_{k-1}, W_k) + \frac{1}{n} \cdot \sum_{k=n_0+1}^{n} u(A_k, r, W_k). \tag{41}$$

As $n \to \infty$, the first term in the last expression converges to 0 a.s., while the second term converges to $V(r)$, by Eq. (39). Therefore, on the set $\{R_k = r \text{ for all } k > n_0\}$, $\bar{u}_n \to V(r)$ a.s. By Eq. (40), this completes the proof. $\square$

Next we prove Proposition 3.3, which we repeat below. The process Y_n and the quantities $q(\cdot, \cdot)$ are defined in Section 3.

Proposition (Proposition 3.3)**.** *Suppose $q(a, +)$ and $q(b, -)$ are both larger than $1/2$ and denote $s^+ = \frac{q(b,+)}{q(a,+)}$, $s^- = \frac{q(a,-)}{q(b,-)}$, and $s^0 = \frac{q(b,0)}{q(a,0)}$. Then,*

$$\mathbb{P}\left(Y_n \to \infty\right) = \frac{1 - s^+}{(1 - s^+) + s^0\left(1 - s^-\right)}. \tag{42}$$

Proof. Let s_1 (s_2) be the probability that the first step is to the right (left) and Y_n never returns to 0, that is (recall that we assume $Y_0 = 0$)

$$s_1 = \mathbb{P}\left(\bigcap_{k=1}^{\infty} \{Y_k > 0\}\right), \quad \text{and} \quad s_2 = \mathbb{P}\left(\bigcap_{k=1}^{\infty} \{Y_k < 0\}\right). \tag{43}$$

Also, let $\tau = \max\{n \in \mathbb{N} : Y_n = 0\}$ and note that

$$\mathbb{P}\left(Y_n \to \infty \mid \tau = 0\right) = \frac{\mathbb{P}\left(Y_n \to \infty, \tau = 0\right)}{\mathbb{P}\left(\tau = 0\right)} = \frac{s_1}{s_1 + s_2}. \tag{44}$$

By Proposition 3.1 we have that $Y_n \to \pm\infty$ a.s., hence $\tau < \infty$ a.s. Therefore,

$$\begin{aligned}
\mathbb{P}\left(Y_n \to \infty\right) &= \mathbb{P}\left(Y_{n+\tau} \to \infty\right) \\
&= \mathbb{P}\left(Y_{n+\tau} \to \infty \mid Y_\tau = 0\right) \\
&= \mathbb{P}\left(Y_{n+\tau} \to \infty \mid Y_\tau = 0, \tau = 0\right) \\
&= \mathbb{P}\left(Y_n \to \infty \mid \tau = 0\right) \\
&= \frac{s_1}{s_1 + s_2},
\end{aligned} \tag{45}$$

where the second and fourth line follow from the fact that $Y_\tau = 0$ a.s. and the third line from the strong Markov property. It remains to calculate s_1 and s_2. For s_1 we have

$$s_1 = \mathbb{P}\left(Y_1 = 1\right) \cdot \mathbb{P}\left(\bigcap_{k=2}^{\infty} \{Y_k > 0\} \mid Y_1 = 1\right). \tag{46}$$

Since at $n = 0$ there is a tie, we get $\mathbb{P}\left(Y_1 = 1\right) = q(a, 0)$. To calculate $\mathbb{P}\left(\bigcap_{k=2}^{\infty} \{Y_k > 0\} \mid Y_1 = 1\right)$, we may assume that Y_n performs a regular biased random walk, starting at $Y_1 = 1$, with probability of stepping to the right equal to $q(a, +)$ and probability of stepping to the left equal to $q(b, +)$. The probability that Y_n never hits 0 is $1 - \frac{q(b,+)}{q(a,+)}$ (see Durrett, 2016, Example 1.54). Therefore,

$$s_1 = q(a, 0) \cdot \left(1 - \frac{q(b, +)}{q(a, +)}\right). \tag{47}$$

Similarly,

$$s_2 = q(b, 0) \cdot \left(1 - \frac{q(a, -)}{q(b, -)}\right). \tag{48}$$

Substituting into Eq. (45) and simplifying, we get Eq. (42). □

A.3 Utility in satisficing model

In this section we show how the utility of the satisficing model can be written in the form of Eq. (9), that is as a function of the agent type, for a suitable set $\mathcal{W}$ of agent types and a distribution Q over it.

We denote the items in X by the numbers 1 through $N = |X|$. Since the agent types are only distinguished by the utility they receive from the various items, we may define $\mathcal{W} = \mathbb{R}^N$ and for any $w = (w_1, \ldots, w_N) \in \mathbb{R}^N$ interpret w_i as the utility that an agent of this type experiences by choosing item i. The utility function $u : \mathcal{X} \times \mathcal{R} \times \mathcal{W} \to \mathbb{R}$ would then look like

$$u(A, r, w) = \begin{cases} w_i, & \text{if } A = \{i\} \\ T, & \text{if } A = \emptyset \\ 0, & \text{if } |A| \geq 2. \end{cases} \tag{49}$$

Note that choosing a set of two or more items is impossible in this model, so the assignment for the case $|A| \geq 2$ is irrelevant. The value T is the threshold that an item's utility should exceed in order to have a chance to be chosen, so it is reasonable to assume that T is the utility of choosing nothing.

Since the utilities of different items are chosen independently, the distribution of agent types is the product measure $Q = Q_1 \otimes \ldots \otimes Q_N$, where Q_i is the probability distribution of the utility of item i, which is part of the model specification. The distribution Q_i is related to the probability $p(i)$ that the utility of item i exceeds the threshold T (see Section 5.1) by

$$p(i) = Q_i([T, \infty)). \tag{50}$$

In Section 5.1 we derived the choice probabilities $q(i, r)$ directly from the $p(i)$'s (Eq. (20)). Let us now see how they can alternatively be derived by finding the agent type-specific choice probabilities $q(i, r, w)$ and then integrating over w (see Eq. (8)).

Consider first a ranking $r \in \mathcal{R}$ that contains no ties. Given such a ranking and the agent type w, the agent's choice is actually deterministic; they will choose the most highly ranked item among those whose utility w_i exceeds the threshold T. In other words, an agent of type $w = (w_1, \ldots, w_N)$ will choose i if and only if $w_i \geq T$ and $w_j < T$ for all j that are ranked higher than i in r. That is, defining

$$C_i^r = \{w \in \mathbb{R}^N : w_i \geq T,\ w_j < T \text{ for all } j \succ_r i\}, \tag{51}$$

we have

$$q(i,r,w) = \begin{cases} 1, & \text{if } w \in C_i^r, \\ 0, & \text{otherwise.} \end{cases} \tag{52}$$

To find the choice probabilities for a randomly chosen agent type, we integrate $q(i,r,w)$ over w (see Eq. (8)):

$$\begin{aligned} q(i,r) = \int_{w \in W} q(i,r,w)\, dQ = \int_{C_i^r} dQ = Q\left(C_i^r\right) \\ = Q_i([T,\infty)) \cdot \prod_{j \succ_r i} Q_j((-\infty,T)) \\ = p(i) \cdot \prod_{j \succ_r i} (1 - p(j)), \end{aligned} \tag{53}$$

which is the same as Eq. (20).

For an arbitrary ranking, the same argument that we used in Section 5.1 can be applied to get that

$$q(i,r,w) = \frac{1}{|S_r|} \cdot \sum_{\sigma \in S_r} q(i,\sigma,w), \tag{54}$$

where S_r is the set of all strict rankings σ that satisfy $i \succ_\sigma j$ whenever $i \succ_r j$. Integrating both sides of the last equation over w, we get Eq. (21).

A.4 Proof that the herding model satisfies Assumption 2.2

Here we prove that the herding model of Section 5.5 satisfies Assumption 2.2. Recall that the agents choose from a list of d items, which we have augmented with d virtual items, and we assume that the corresponding real and virtual items (e.g. i and $i+d$) are always chosen together. The virtual items start from popularity -1, hence the equality

$$Z_n(i+d) = Z_n(i) - 1, \quad i = 1, \ldots, d, \tag{55}$$

is satisfied at all times. This implies that among all rankings of the $2d$ elements, many will be impossible to reach. For example, a virtual item can never be ranked higher than its real counterpart. Clearly, the part of Assumption 2.2 that refers to rankings that are not reachable is irrelevant[11], so we may restrict ourselves to the reachable rankings. These are the ones for which $r(i+d) = r(i)+1$ for any $i \leq d$, i.e. each virtual item must be ranked one position below the corresponding real item.

To derive the choice probabilities for the reachable rankings, recall that each agent may have a signal, with probability $p > 0$, indicating an item to be the correct one, but with $p' < 0.5$ this signal might be incorrect. In other words, the correct choice is indicated to the agent with probability $p_T = p \cdot (1-p')$, while each of the incorrect choices is indicated with probability $p_F = p \cdot \frac{p'}{d-1} < p_T$ (T and F stand for *True* and *False*). Also recall that the rational strategy is for the agent to choose the item that their signal indicates if this item is at most one vote behind the top-ranked item, and to choose the top-ranked item otherwise (including if they have no signal).

To find the probabilities of choice for the various items, note first that if $r(i) = 1$ for some real item i, meaning that this item maximizes $Z_n(i)$, then its virtual counterpart $i+d$, for which we have $Z_n(i+d) = Z_n(i) - 1$, must necessarily satisfy $r(i+d) = 2$. Therefore, if $r(j) = 2$ for some different, real item j, then $Z_n(j) = Z_n(i+d) = Z_n(i) - 1$, meaning that this item is just one vote behind the top, so it must be chosen by the agent if and only if their signal indicates that it's the correct one. The same is true for a *virtual* item with $r(j) = 3$, since this is equivalent to its real counterpart $j - d$ satisfying $r(j-d) = 2$. On the other hand, if $r(i) = 1$ for a real item i, or if $r(i) = 2$ and i is virtual, then this item may be chosen either because the signal indicates it or because there is no signal. In other words, we have the following (recall that i^* denotes the correct choice):

- If i is a real item with $r(i) = 1$ or a virtual item with $r(i) = 2$, then

$$q(i,r) = \begin{cases} p_T + \frac{1-p}{|r^{-1}(1)|}, & \text{if } i \in \{i^*, i^*+d\} \\ p_F + \frac{1-p}{|r^{-1}(1)|}, & \text{otherwise.} \end{cases} \tag{56}$$

[11]All results continue to hold if we restrict Assumption 2.2 to rankings that have positive probability of ever taking place.

- If i is a real item with $r(i) = 2$ or a virtual item with $r(i) = 3$, then

$$q(i,r) = \begin{cases} p_T, & \text{if } i \in \{i^*, i^* + d\} \\ p_F, & \text{otherwise.} \end{cases} \tag{57}$$

- If i is a real item with $r(i) \geq 3$ or a virtual item with $r(i) \geq 4$, then

$$q(i,r) = 0. \tag{58}$$

The reason that in Eq. (56) we divide by $|r^{-1}(1)|$, the number of items tied in the first place, is that when there is no signal, the agent chooses one of the top-ranked items at random.

We now proceed to show that Assumption 2.2 is satisfied. Recall that we have defined the relation P as follows:

- iPj for any virtual item i and real item $j \neq i^*$, and
- i^*Pj and $(i^* + d)P(j + d)$ for any real item $j \neq i^*$.

Let $r \in \mathcal{R}$ and $i, j \in X$ such that $i \succ_r j$. For the first part of Assumption 2.2 to hold, we have to show that either of the three cases $q(ji', r) = 0$, jPi, or $q(i,r) > q(j,r)$ holds. We have the following cases:

- If i and j are both real or both virtual and $j \notin \{i^*, i^* + d\}$, then from Eqs. (56) to (58) it follows that either $q(j,r) = 0$ or $q(i,r) > q(j,r)$ (recall we are assuming that $i \succ_r j$, i.e. $r(i) < r(j)$).
- If i and j are both real or both virtual and $j \in \{i^*, i^* + d\}$, then jPi holds by the definition of P (the fact that $i \neq j$ follows from $i \succ_r j$).
- If i is virtual and j is real, then $r(j) > r(i) \geq 2$, hence $q(j,r) = 0$ by Eq. (58).
- If i is real, $i \neq i^*$, and j is virtual, then jPi holds by the definition of P.
- If $i = i^*$ and j is virtual, then $q(i,r) > q(j,r)$ whenever $q(j,r) > 0$, by Eqs. (56) to (58).

For the second part of Assumption 2.2, suppose $i \approx_r j$. Clearly, i and j cannot be real and virtual item counterparts, hence $q(ji', r) = q(j, r)$ and $q(ij', r) = q(i, r)$. We need to show that either of the three cases $q(j, r) = 0$, $q(i, r) > 0$, or $i \not P j$ holds. We have the following cases:

- If j is a real item or i is a virtual item, then by Eqs. (56) to (58), either $q(j, r) = 0$ or $q(i, r) > 0$.
- If j is a virtual item and i is a real item, then iPj is impossible, by the definition of P.

This concludes the proof that the choice probabilities defined by Eqs. (56) to (58) satisfy Assumption 2.2.