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A Characterization of the Top-Trading-Cycles Mechanism for Housing Markets via Respecting-Improvement*

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Abstract

We provide the first characterization of the prominent top-trading-cycles (TTC) mechanism in the Shapley-Scarf housing market model (Shapley and Scarf, 1974) that uses *respecting-improvement*. Specifically, we show that for strict preferences, the TTC mechanism is the unique mechanism satisfying *pair-efficiency*, *respecting-improvement*, and *strategy-proofness*.

Keywords: housing markets; top-trading-cycles (TTC) mechanism; *respecting improvement*; *pair-efficiency*; *strategy-proofness*; market design.

JEL codes: C78; D47.

1 Introduction

We study Shapley-Scarf housing markets (Shapley and Scarf, 1974) where each agent owns an indivisible object (say, a house); each agent has strict preferences over houses and wishes to consume exactly one house. The objective of the market designer is to reallocate houses among agents. Shapley and Scarf (1974) show that the strong core[1] (defined by a weak blocking notion) has remarkable features: it is non-empty,[2] and can be easily calculated by the so-called top-trading-cycles (TTC) algorithm (due to David Gale). Moreover, the TTC mechanism that

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[1] In the literature, the strong core is sometimes called the strict core.

[2] Roth and Postlewaite (1977) show that the strong core is single-valued.

assigns to each housing market its unique strong core allocation satisfies important incentive properties such as *strategy-proofness* (Roth, 1982) as well as the stronger property of *group strategy-proofness* (Bird, 1984; Sandholtz and Tai, 2024). Furthermore, Ma (1994) shows that the TTC mechanism is the unique mechanism satisfying *Pareto-efficiency*, *individual-rationality*, and *strategy-proofness*.[3]

Ekici (2024) provides a novel axiomatic characterization of the TTC mechanism enhancing its appeal further: the TTC mechanism is the unique mechanism that is *pair-efficient*, *individual-rational*, and *strategy-proof*.[4] Ekici's (2024) characterization uses *pair-efficiency*, which is weaker than *Pareto-efficiency*, employed in Ma's (1994) characterization.[5]

Biró et al. (2024) is the first paper to study the following desirable property in housing markets: if the endowment of a fixed agent improves in the ranking(s) of some other agent(s) while keeping preferences over other objects and other agents' preferences unchanged, then the fixed agent must be weakly better off as the result of the improvement. Biró et al. (2024) dub this property *respecting-improvement* and study it as an incentive property in the context of kidney exchange programs: each patient is incentivized to bring the best possible set of donors to the market. Biró et al. (2024) show that the TTC mechanism *respects improvement*. Biró et al. (2024) and Schlotter et al. (2024) prove several extensions. Ehlers (2023) shows that if a mechanism satisfies *individual-rationality*, *strategy-proofness*, and *non-bossiness*, then it also satisfies *respecting-improvement*.

We provide the first characterization of the TTC mechanism in the Shapley-Scarf housing market model (Shapley and Scarf, 1974) that uses *respecting-improvement*. Specifically, we prove that for strict preferences, the TTC mechanism is the unique mechanism satisfying *pair-efficiency*, *respecting-improvement*, and *strategy-proofness* (Theorem 1). We also show that the three properties are logically independent (Examples 1, 2, and 3).

2 The model

In a Shapley-Scarf housing market (Shapley and Scarf, 1974), there is a finite *set of agents* $N = \{1, 2, \ldots, n\}$ with $n \geq 3$. Each agent $i \in N$ is endowed with exactly one indivisible object, denoted by o_i. The *set of objects* is $O = \{o_1, o_2, \ldots, o_n\}$.

We assume that each agent i has *complete*, *antisymmetric*, and *transitive preferences* R_i over all objects, i.e., R_i is a linear order over O.[6] For two objects o_k and o_l, o_k is *weakly preferred to*

[3] Svensson (1999), Anno (2015), and Sethuraman (2016) provide short proofs of Ma's (1994) characterization.

[4] Ekici and Sethuraman (2024) provide a short proof of Ekici's (2024) characterization.

[5] *Pair-efficiency* rules out gainful trades between pairs of agents, while *Pareto-efficiency* rules out gainful trades among the members of any subset of agents.

[6] Preferences R_i are *complete* if for any two objects o_k, o_l, $o_k \, R_i \, o_l$ or $o_l \, R_i \, o_k$; they are *antisymmetric* if $o_k \, R_i \, o_l$ and $o_l \, R_i \, o_k$ imply $o_k = o_l$; and they are *transitive* if for any three objects o_k, o_l, o_m, $o_k \, R_i \, o_l$ and $o_l \, R_i \, o_m$ imply $o_k \, R_i \, o_m$.

o_l if $o_k \, R_i \, o_l$, and o_k is *strictly preferred to* o_l if $[o_k \, R_i \, o_l$ and not $o_l \, R_i \, o_k]$, denoted by $o_k \, P_i \, o_l$. Finally, since preferences over objects are strict, agent i is indifferent between o_k and o_l only if $o_k = o_l$. We denote preferences as rankings, e.g., $R_i : o_k, \, o_l, \, o_m$ instead of $o_k \, P_i \, o_l \, P_i \, o_m$. The *set of (all) strict preferences* is denoted by $\mathcal{R}$.

A *preference profile* specifies preferences for all agents and is denoted by a list $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$. We use the standard notation $R_{-i} = (R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n)$ to denote the list of all agents' preferences, except for agent i's preferences.

A *housing market* is a triple (N, O, R) with $R \in \mathcal{R}^N$. When agents and objects remain fixed, a housing market is specified by a preference profile R and $\mathcal{R}^N$ is the *set of housing markets*.

An *allocation* $\mu : N \to O$ is a one-to-one mapping from the set of agents to the set of objects. For each $i \in N$, μ_i denotes agent i's *allotment* at μ. Let $\mathcal{A}$ denote the *set of allocations*.

A *mechanism* $\varphi : \mathcal{R}^N \to \mathcal{A}$ is a mapping from the set of housing markets to the set of allocations. For each housing market $R \in \mathcal{R}^N$ and each $i \in N$, $\varphi_i(R)$ denotes agent i's allotment at allocation $\varphi(R)$.

An allocation μ is *individually-rational* at $R \in \mathcal{R}^N$ if for each $i \in N$, $\mu_i \, R_i \, o_i$. A mechanism φ is *individually-rational* if for each $R \in \mathcal{R}^N$, $\varphi(R)$ is *individually-rational* at R.

An allocation μ is *Pareto-efficient* at $R \in \mathcal{R}$ if there exists no allocation $\bar{\mu}$ such that for each $i \in N$, $\bar{\mu}_i \, R_i \, \mu_i$, and for some $j \in N$, $\bar{\mu}_j \, P_j \, \mu_j$. A mechanism φ is *Pareto-efficient* if for each $R \in \mathcal{R}^N$, $\varphi(R)$ is *Pareto-efficient* at R.

An allocation μ is *pair-efficient* at $R \in \mathcal{R}^N$ if there do not exist $i, j \in N$ with $i \neq j$ such that $\mu_j \, P_i \, \mu_i$ and $\mu_i \, P_j \, \mu_j$. A mechanism φ is *pair-efficient* if for each $R \in \mathcal{R}^N$, $\varphi(R)$ is *pair-efficient* at R. Obviously, *Pareto-efficiency* implies *pair-efficiency*.

A mechanism φ is *strategy-proof* if for each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi_i(R) \, R_i \, \varphi_i(R'_i, R_{-i})$.

Next, we recall the definition of *respecting-improvement* (Biró et al., 2024). Let $i \in N$ and $R, \tilde{R} \in \mathcal{R}^N$. Then, *$\tilde{R}$ is an improvement for i with respect to R* if the only difference between R and $\tilde{R}$ is that, at $\tilde{R}$, object o_i is ranked weakly higher by the other agents than at R. Formally,

(1) $\tilde{R}_i = R_i$;

(2) for all $j, k \neq i$, if $o_i \, P_j \, o_k$, then $o_i \, \tilde{P}_j \, o_k$; and

(3) for all $j, k, l \neq i$, if $o_k \, R_j \, o_l$, then $o_k \, \tilde{R}_j \, o_l$.

A mechanism φ *respects improvement* (or satisfies *respecting-improvement*) if for each $i \in N$ and each pair of housing markets $R, \tilde{R} \in \mathcal{R}^N$ such that $\tilde{R}$ is an improvement for i with respect to R, we have that $\varphi_i(\tilde{R}) \, R_i \, \varphi_i(R)$.

The so-called *top-trading-cycles (TTC) allocation* of a housing market can be obtained through the *TTC algorithm*: in each round, each agent points to her most preferred (remaining) object, and each object points to its owner. As there are a finite number of agents, there must exist a cycle. All agents that are part of a cycle are assigned the object they point to and all agents and objects in cycles are removed. The *TTC mechanism* assigns to each market $R \in \mathcal{R}^N$ its *TTC allocation* $\tau(R)$.

3 Characterizing the TTC mechanism via respecting-improvement

The following lemma will be instrumental in proving our main result.

Lemma 1. *If a mechanism is pair-efficient and respects improvement, then it is individually-rational.*

Proof. Let φ be a mechanism that is *pair-efficient* and *respects improvement*. Let $\tilde{R} \in \mathcal{R}^N$ and $i \in N$. We show *individual-rationality* of φ by showing that $\varphi_i(\tilde{R})\,\tilde{R}_i\,o_i$.

Let $R \in \mathcal{R}^N$ be the housing market obtained from $\tilde{R}$ by making agent i's object o_i the least preferred (worst) object for all other agents. In other words, o_i is pushed to the bottom of the ranking of each agent $j \neq i$; the relative ranking of all other objects remains the same. Then, $\tilde{R}$ is an improvement for agent i with respect to R, in particular, $\tilde{R}_i = R_i$. Since φ *respects improvement*, $\varphi_i(\tilde{R})\,R_i\,\varphi_i(R)$. Hence, it only remains to show that $\varphi_i(R)\,R_i\,o_i$.

If $\varphi_i(R) = o_i$, then $\varphi_i(R)\,R_i\,o_i$. Suppose that for some $j \neq i$, $\varphi_i(R) = o_j$. Let $k \in N$, $k \neq i$, be the agent that receives object o_i at $\varphi(R)$, i.e., $\varphi_k(R) = o_i$. By definition of R, object o_i is the worst object for agent k at R. In particular,

$$o_j\,P_k\,o_i = \varphi_k(R). \tag{1}$$

Pair-efficiency of φ rules out a gainful trade between agents i and k at $\varphi(R)$. Thus, (1) implies that $\varphi_i(R) = o_j\,R_i\,o_i$. □

Theorem 1. *The TTC mechanism is the unique mechanism that satisfies pair-efficiency, respecting-improvement, and strategy-proofness.*

Proof. Note that

(a) the TTC mechanism *respects improvement* (Biró et al., 2024, Theorem 1);

(b) the TTC mechanism is the unique mechanism that satisfies *pair-efficiency*, *individual-rationality*, and *strategy-proofness* (Ekici, 2024, Theorem 1).

Thus, (a) and (b) show that the TTC mechanism satisfies the three properties. Lemma 1 and (b) show that there is no other mechanism that satisfies the three properties. □

The following examples establish the logical independence of the properties in Theorem 1. We label the examples by the property that is not satisfied.

Example 1 (*Pair-efficiency*).

The no-trade mechanism that assigns the endowment allocation to each housing market satisfies *respecting-improvement* and *strategy-proofness*, but not *pair-efficiency*. ⋄

Example 2 (*Respecting-improvement*).

A serial dictatorship mechanism ζ^π, based on a fixed order of the agents π, for any housing market, lets agents sequentially pick their most preferred allotments from the available objects.

Serial dictatorship mechanisms are *Pareto-efficient*, hence *pair-efficient*, and *strategy-proof*. The next example shows that serial-dictatorship mechanisms do not *respect improvement*.

Let $N = \{1, 2\}$ and the order of agents is $\pi = 1, 2$. Let $R, \tilde{R} \in \mathcal{R}^N$ be the preferences given in Table 1.

R_1	R_2	$\tilde{R}_1$	$\tilde{R}_2$
$\boldsymbol{o_1}$	$\boldsymbol{o_2}$	$\underline{\boldsymbol{o_2}}$	o_2
$\underline{o_2}$	o_1	o_1	$\boldsymbol{o_1}$

Table 1: Preferences R and $\tilde{R}$; allocations $\zeta^\pi(R)$ and $\zeta^\pi(\tilde{R})$ in boldface.

Note that $\tilde{R}$ is an improvement for agent 2 with respect to R: the only difference of $\tilde{R}$ relative to R is that o_2 moves up in the preferences of agent 1 (see underlined o_2 in Table 1).

Then, $\zeta_2^\pi(R) = o_2 \, P_2 \, o_1 = \zeta_2^\pi(\tilde{R})$. So, even though $\tilde{R}$ is an improvement for agent 2 with respect to R, agent 2 is strictly worse off at $\zeta^\pi(\tilde{R})$ relative to $\zeta^\pi(R)$. Hence, ζ^π does not satisfy *respecting-improvement*. $\diamond$

Example 3 (***Strategy-proofness***)**.**
Let $N = \{1, 2, 3\}$. Let $\bar{R} \in \mathcal{R}^N$ be the preferences given in Table 2.

$\bar{R}_1$	$\bar{R}_2$	$\bar{R}_3$
o_2	o_3	o_1
$\boldsymbol{o_1}$	$\boldsymbol{o_2}$	$\boldsymbol{o_3}$
o_3	o_1	o_2

Table 2: Preferences $\bar{R}$; allocation $\beta(\bar{R})$ in boldface.

Define mechanism β such that $\beta(\bar{R}) \equiv (o_1, o_2, o_3)$, i.e., each agent receives his own endowment at $\bar{R}$. For each $R \in \mathcal{R}^N \backslash \{\bar{R}\}$, mechanism β yields the TTC allocation, i.e., $\beta(R) \equiv \tau(R)$.

Since $\beta(\bar{R})$ is *pair-efficient* at $\bar{R}$ and since the TTC mechanism is *Pareto-efficient*, β is *pair-efficient*.

Next, we show that β *respects improvement*. Since τ *respects improvement*, we only have to verify the property for $R, \tilde{R} \in \mathcal{R}^N$ with $R \neq \tilde{R}$ such that (a) $\tilde{R}$ is an improvement for some agent i with respect to R and (b) $R = \bar{R}$ or $\tilde{R} = \bar{R}$. Given the symmetry of preferences at profile $\bar{R}$, we can assume, without loss of generality, that $i = 1$. Note that at $\bar{R}$, o_1 is the least preferred object for agent 2 and o_1 is the most preferred object for agent 3. Hence, if (b) $R = \bar{R}$, then only agent 2 can rank o_1 higher at $\tilde{R}$ (Case I) and if (b) $\tilde{R} = \bar{R}$, then only agent 3 can rank o_1 lower at R (Case II).

Case I: $R = \bar{R}$ and $\tilde{R} = (\bar{R}_1, \tilde{R}_2, \bar{R}_3)$. Then, $\tilde{R}_2 : o_3,\, o_1,\, o_2$ or $\tilde{R}_2 : o_1,\, o_3,\, o_2$.
In either case, $\beta_1(\tilde{R}) = \tau_1(\tilde{R}) = o_2$, which implies $\beta_1(\tilde{R}) = o_2 \, R_1 \, o_1 = \beta_1(R)$.

Case II: $\tilde{R} = \bar{R}$ and $R = (\bar{R}_1, \bar{R}_2, R_3)$. Then, $R_3 : o_3,\, o_1,\, o_2$ or $R_3 : o_3,\, o_2,\, o_1$.
In either case, $\beta(R) = \tau(R) = (o_1, o_2, o_3)$, which implies that $\beta_1(\tilde{R}) = o_1 \, R_1 \, o_1 = \beta_1(R)$.

Finally, β is not *strategy-proof*. To see this, let $R'_1 : o_2, o_3, o_1$. One easily verifies that $\beta(R'_1, \bar{R}_{-1}) = \tau(R'_1, \bar{R}_{-1}) = (o_2, o_3, o_1)$. Since $\beta_1(R'_1, \bar{R}_{-1}) = o_2 \, \bar{P}_1 \, o_1 = \beta_1(\bar{R})$, reporting preferences R'_1 constitutes a profitable deviation for agent 1 at $\bar{R}$, i.e., β is not *strategy-proof*. $\diamond$

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