



# Updating Under Imprecise Information

BSE Working Paper 1424 | January 2024

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January, 2024

## Abstract

This paper models an agent that ranks actions with uncertain payoffs after observing a signal that could have been generated by multiple objective information structures. Under the assumption that the agent's preferences conform to the multiple priors model (Gilboa and Schmeidler (1989)), we show that a simple behavioral axiom characterizes a generalization of Bayesian Updating. Our axiom requires that whenever all possible sources of information agree that it is more 'likely' for an action with uncertain payoffs to be better than one with certain payoffs, the agent prefers the former. We also provide axiomatizations for several special cases. Finally, we consider the situation where the informational content of a signal is purely subjective. We characterize the existence of a subjective set of information structures under full Bayesian updating for two extreme cases: (i) No ex-ante state ambiguity, and (ii) No signal ambiguity.

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# 1 Introduction

Individuals often acquire information before making a choice. However, in many situations, they are not able to completely pin down the source of the acquired information. Since the same information can have different meanings depending on its source, they may have a hard time processing it if they are not sure which one is the true source. The goal of this paper is to provide a theory of how agents process such information.

To illustrate, we provide an example inspired by the early research conducted during the COVID-19 pandemic.<sup>1</sup> Consider a policy-maker that must choose to adopt a costly novel security measure. The policy-maker is aware that a study has shown the security measure reduces the risk of an accident. However, she is unsure about the quality of the data employed in the study. If the study was done properly, then the results suggest she should in fact adopt the measure. However, if the quality is poor, then the results are meaningless. Indeed, if the sample employed in a study is too small, then it is equally likely to get results that suggest the measure is effective than results that say it has no effect. How should the policy-maker process such information?

We take as primitives and outcome space  $X$ , an objective state space  $\Omega$ , an objective signal space  $S$ , and a set of information structures  $\mathcal{L} \subseteq \{\ell : \Omega \rightarrow \Delta(S)\}$ , where  $\Delta(S)$  is the set of probability measures over  $S$ . We refer to  $\mathcal{L}$  as an imprecise information structure. The idea behind our definition is that  $\mathcal{L}$  is the set of identifiable sources of information. The agent is assumed to have ex-ante preference  $\succeq_0$  and signal-conditional preferences  $(\succeq_s)_{s \in S}$  over actions that have uncertain payoffs. We interpret  $\succeq_s$  to be the agent's preferences conditional on observing signal  $s$  generated by some  $\ell \in \mathcal{L}$ . We assume that ex-ante  $\succeq_0$  and ex-post  $\succeq_s$  conforms to [Gilboa and Schmeidler \(1989\)](#) maxmin model (MEU).

Our theory is built on the premise that if all identifiable information structures recommend choosing an action with uncertain payoffs over one with deterministic payoffs, then the agent should select the action. Imposing this condition as an axiom yields a representation (Theorem 3.1) and consequently, an updating rule. For any action  $f : \Omega \rightarrow X$ , the utility of  $f$  after observing  $s$  is given by:

$$U_s(f) = \min_{q \in \rho(\mathcal{M}_0, \mathcal{L}, s)} \int u(f), dq$$

Here,  $\mathcal{M}_0$  represents the set of priors, and  $\rho(\mathcal{M}_0, \mathcal{L}, s)$  is a subset of the convex hull of all posteriors generated by 'point-wise' Bayesian updating. We term this

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<sup>1</sup>During the early days of the pandemic, over 66 clinical prediction models were proposed in peer-reviewed settings to study the effect of a mask mandate. However, it was later established that all suffered a high risk of bias due to concerns surrounding the data quality, statistical analysis, and reporting. See [Collins et al. \(2020\)](#) for a discussion.

updating rule the Generalized Bayesian Updating (GBU) rule.

In our context, signals are not considered payoff-relevant, meaning that the payoff of actions does not depend on them. This departure from the typical axiomatic literature on updating, where signals are often modeled as subsets of  $\Omega$ , is a distinctive feature of our framework. Unlike the conventional approach, our setting does not assume signals to be subsets of  $\Omega$ . This departure is crucial for our purposes as it allows us to disentangle the agent’s attitude towards imprecise information from her attitude towards uncertainty. If signals were modeled as subsets of  $\Omega$ , this separation would not be feasible. Additionally, our approach facilitates the testing of our axioms in a laboratory setting, as our framework serves as a theoretical representation of experiments conducted in [Epstein and Halevy \(2022\)](#) and [Shishkin et al. \(2021\)](#)

Our updating rule generalizes both *Full Bayesian updating* (FBU) and *Maximum Likelihood updating* (MLU), both of which are widely popular. However, the conceptual rationale behind why an agent might adopt either of these rules is not entirely clear. FBU allows the agent to have a subjective view of the uncertainty about the state space but not about the source of information. On the other hand, MLU utilizes the realized signal to *jointly* discriminate among priors and information structures, evaluating each information structure using the prior that maximizes the likelihood of the observed signal.

A more conservative or intermediate approach would involve evaluating each information structure using *all* priors. In other words, an agent may prefer considering an information structure that has a reasonable likelihood according to all priors rather than one that has maximal likelihood according to a single prior and minimal likelihood according to another. We establish axiomatic foundations for such a rule in [Theorem 7](#).

The psychology literature has consistently demonstrated that individuals, when updating information, can exhibit susceptibility to certain biases. A prevalent finding is the tendency to selectively consider information structures that align with their pre-existing beliefs, a phenomenon commonly referred to as confirmation bias.<sup>2</sup> In our framework, given a signal, the probability that the signal originated in each state induces a likelihood ranking over states. If the agent is able to articulate a likelihood ranking over states and is prone to confirmation bias, she may limit her consideration to information structures that align with her pre-existing likelihood ranking. GBU allows for such updating, and is axiomatized in this paper ([Theorem 4.2](#)).

Our analysis hinges on the assumption that imprecise information is objective. However, in numerous situations, information may be private or lack a reliable description of its informational content. In such cases, assuming observability

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<sup>2</sup>See [Rabin and Schrag \(1999\)](#) for a comprehensive review of evidence in psychology.

of  $\mathcal{L}$  is not appropriate, and its nature should be inferred from behavior. We demonstrate that under FBU,  $\mathcal{L}$  can be recovered in scenarios where there is a single prior belief or when  $\mathcal{L}$  is a singleton.

The paper proceeds as follows: The introduction concludes with a literature review. Axioms and the implied updating rule are described in Sections 2 and 3 respectively. Sections 3.2-3.4 contain a detailed discussion of the properties of the model. Special cases of the Generalized Bayesian Updating (GBU) rule are investigated in Section 4. Section 5 delves into the consideration of subjective information structures. Concluding remarks and discussions on future research questions are presented in Section 6. All proofs are provided in Appendix A.

## 1.1 Related Literature

In recent years, there has been a growing interest in decision-making under uncertainty, particularly in the context of incorporating objective but imprecise information. Our work aligns closely with several recent papers, including [Gul and Pesendorfer \(2021\)](#), [Cheng \(2022\)](#), [Tang \(2022\)](#), and [Kovach \(2023\)](#).

[Gul and Pesendorfer \(2021\)](#) presents a theory of updating for the Choquet Expected Utility model ([Schmeidler \(1989\)](#)). On the other hand, [Cheng \(2022\)](#), [Tang \(2022\)](#), and [Kovach \(2023\)](#) offer distinct axiomatic updating rules for the Maxmin Expected Utility (MEU) model. Notably, the primary distinctions between our work and theirs lie in the framework and the nature of the information considered.

In the frameworks of [Cheng \(2022\)](#), [Tang \(2022\)](#), and [Kovach \(2023\)](#), information is considered payoff-relevant, and the information itself is precise in the sense that it is an event. In contrast, our work focuses on imprecise information that is not necessarily payoff-relevant. The motivation for their work stems from the descriptive shortcomings of the Maximum Likelihood Updating (MLU) and Full Bayesian Updating (FBU) rules, axiomatized by [Gilboa and Schmeidler \(1993\)](#) and [Pacheco Pires \(2002\)](#), respectively.

[Dominiak et al. \(2021\)](#) model imprecise information as sets of probability measures that contain the “true” distribution. They provide an updating rule for the Subjective Expected Utility model that selects the posterior that minimizes the distance between her prior and the set she is provided with. Their updating rule has a somewhat similar flavor to ours due to the fact that they allow the distance to be subjective.

[Jaffray \(1989\)](#), [Ahn \(2008\)](#), [Dumav and Stinchcombe \(2013\)](#), [Olszewski \(2007\)](#), [Gajdos et al. \(2008\)](#), and [Riedel et al. \(2018\)](#) consider settings in which the objects of choice are “merged” with imprecise information. More specifically, [Jaffray](#) considers a preference over belief functions. [Gajdos et al. \(2008\)](#) and [Riedel et al. \(2018\)](#) assume the agent can choose the set of data-generating processes that

contains the true law. The rest of the authors study preferences over sets of lotteries. All of these assumptions imply that imprecise information is payoff-relevant.

Epstein and Schneider (2007,8,10) provide a non-axiomatic updating rule for imprecise information (therein referred to as ambiguous signals). They introduce a thought experiment that highlights its importance. In a dynamic setting, they provide conditions under which their updating rule delivers convergence in beliefs after repeated sampling. Reshidi et al. (2022) further investigate when beliefs converge under a more general data-generating process. Like Epstein and Schneider (2007), Lanzani (2023) also studies a learning problem. However, unlike Epstein and Schneider, he assumes robust-control preferences. He shows how a misspecification concern can lead to different preferences under uncertainty arising in the limit.

Building on Epstein and Schneider’s thought experiment, Epstein and Halevy (2022) provides a definition of aversion towards signal ambiguity and test it in an experimental setting. The key difference between our work and theirs is that they study the attitude towards the information as opposed to *how* to processes it, which is our focus.

Finally, outside the updating literature but within the imprecise information literature, Wang et al. (2023) axiomatize a selection criteria for imprecise information. They provide conditions under which a choice function over “theories” always selects the ones that pass a likelihood ratio test.

## 2 General Model

### 2.1 Preliminary Definitions

We consider a set  $\Omega$  of states of the world, a set  $S$  of signals, and a set  $X$  of consequences. We assume  $\Omega$  and  $S$  are finite and  $X$  is a compact and convex subset of a linear space. This is the case of Anscombe and Aumann (1963) where  $X$  is the set of all lotteries over some finite set of prizes. Let  $n$  denote the cardinality of  $\Omega$ .

An act is a function  $f : \Omega \rightarrow X$ . The set of all acts is denoted by  $\mathcal{F}$ . We write  $x$  for the constant act  $f$  such that  $f(\omega) = x$  for all  $\omega \in \Omega$ . Given  $f, g \in \mathcal{F}$  and  $\alpha \in [0, 1]$ , we denote by  $\alpha f + (1 - \alpha)g$  the act in  $\mathcal{F}$  that takes value  $\alpha f(\omega) + (1 - \alpha)g(\omega)$  in state  $\omega$ .

An information structure is a function  $\ell : \Omega \rightarrow \Delta(S)$  where  $\Delta(S)$  is the set of all probability measures over  $S$ . We only consider information structures that satisfy the following property:

$$\forall s \in S, \text{ there exists } \omega \in \Omega \text{ such that } \ell(s|\omega) > 0. \tag{1}$$

As it will become clear later, this assumption is made purely for exposition purposes. An imprecise information structure is a finite set of such information structures. If the set only contains a single information structure, then we refer to it as a precise information structure. Generic imprecise information structures are denoted by  $\mathcal{L}$ .

$\Delta(\Omega)$  denotes the set of all probability measure on  $\Omega$ . Given a probability measure  $q \in \Delta(\Omega)$ , a information structure  $\ell$  and a signal  $s$ , we denote by  $BU(q, \ell, s)$  the probability measure given by Bayesian Updating:

$$BU(q, \ell, s)(\omega) = \frac{\ell(s|\omega)q(\omega)}{\sum_{\omega} \ell(s|\omega)q(\omega)}$$

whenever it exists. Further, for any set of measures  $\mathcal{M} \subseteq \Delta(\Omega)$ , imprecise information structure  $\mathcal{L}$  and signal  $s \in S$ ,  $BU(\mathcal{M}, \mathcal{L}, s)$  denotes the set of all posteriors generated by *point-wise Bayesian Updating*:

$$BU(\mathcal{M}, \mathcal{L}, s) = \{BU(q, \ell, s) | q \in \mathcal{M}, \ell \in \mathcal{L}\}.$$

Finally, for any set  $C \subseteq \Delta(\Omega)$ ,  $ch(C)$  denotes its convex hull.

## 2.2 Axioms

Our primitive is a family of preferences over acts  $(\succeq_0, (\succeq_{s \in S}))$  and an imprecise information structure  $\mathcal{L}$ . We impose two axioms on  $(\succeq_0, (\succeq_{s \in S}))$  of which the first is that the preferences admit a MEU representation ([Gilboa and Schmeidler \(1989\)](#)).

A utility function  $U : \mathcal{F} \rightarrow \mathbb{R}$  is *MEU* if there exists an affine function  $u : X \rightarrow \mathbb{R}$  and a closed and convex set of probability measures  $\mathcal{M}$  over  $\Omega$  such that

$$U(f) = \min_{q \in \mathcal{M}} \int_{\Omega} u(f(\omega))dq. \tag{2}$$

Say that  $(\mathcal{M}, u)$  represents  $\succeq$  if the utility function given by (2) represents  $\succeq$ .

**MEU Utility**  $\succeq_s$  admits representation by  $(\mathcal{M}_s, u)$  for all  $s \in S \cup \{0\}$ . Moreover, each  $q_0 \in \mathcal{M}_0$  has full support.

This axiom is not stated in terms of behavior which is presumably the only observable. However, its behavioral foundations are widely known. It also imposes that all priors have full support; our model has nothing to say about updating zero-probability events.

Our main axiom is based on the idea that if for a given signal all the identifiable information structures agree that it is more likely that an act is better than a constant act, then then the agent should choose the act. We start by showing that our primitives are rich enough to identify this if the agent satisfies *Reduction*:

She is indifferent between a two-stage lottery and its equivalent one-stage counterpart.

Fix an act  $f$ , a constant act  $x$ , and an information structure  $\ell \in \mathcal{L}$ . Consider an extension of the ex-ante preference to acts in which signals are payoff relevant:  $\{F : \Omega \times S \rightarrow X\}$ . Suppose we told the agent that  $\ell$  is the true information structure and asked her to choose between a constant act  $x$  and a “signal act”

$$F(\omega, s) = \begin{cases} f(\omega) & s = s^* \\ x & s \neq s^* \end{cases}.$$

Figure 1 (a) illustrates this signal act. Note that conditional on a state of the world, there is no ambiguity about the probability of observing  $s^*$ . Indeed, the agent knows it is given by  $\ell(\cdot|\omega) \in \Delta(S)$ . Because both acts give the same payoff if  $\ell$  generates  $s' \neq s^*$ , the agent only needs to compare them assuming  $s^*$  happened. Therefore, she would prefer  $F$  to  $x$  if she thinks that conditional on observing  $s^*$  she would prefer  $f$  to  $x$ .

Notice that for a given state  $\omega$ ,  $F$  gives the same expected payoffs as a lottery that pays  $f(\omega)$  with probability  $\ell(s^*|\omega)$  and  $x$  with probability  $1 - \ell(s^*|\omega)$ . Therefore, under Reduction, she would be indifferent between  $F$  and an act that for each state  $\omega$  pays  $\ell(s^*|\omega)f(\omega) + (1 - \ell(s^*|\omega))x$ . Figure 1 (b) illustrates such act.

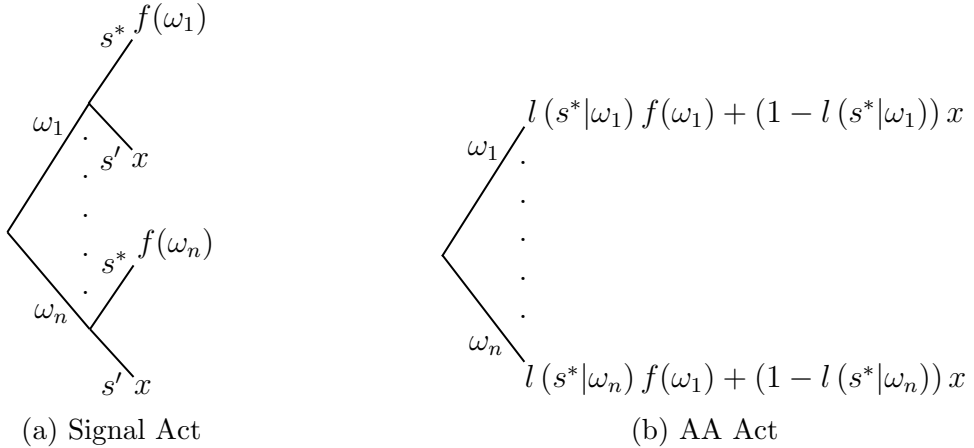


Figure 1

Hence, under Reduction,

$$F \succeq_0 x \iff \begin{bmatrix} \ell(s^*|\omega_1)f(\omega_1) + (1 - \ell(s^*|\omega_1))x \\ \vdots \\ \ell(s^*|\omega_n)f(\omega_n) + (1 - \ell(s^*|\omega_n))x \end{bmatrix} \succeq_0 x$$



which means we can identify when the agent expects to prefer  $f$  over  $x$  after observing  $s$  if she knew  $\ell$  is the true source of information.

To state our main axiom we need some notation. For each  $f, x \in \mathcal{F}, \ell \in \mathcal{L}$  and  $s \in S$ , let  $f_x^{\ell, s}$  be the act that takes value  $\ell(s|\omega)f(\omega) + (1 - \ell(s|\omega))x$  in state  $\omega$ .

**Total Information Agreement** For any  $f, x \in \mathcal{F}$  and  $s \in S$ ,

$$f_x^{\ell, s} \succeq_0 x \text{ for all } \ell \in \mathcal{L} \implies f \succeq_s x.$$

Given our discussion, Total Information Agreement states that if all information structures agree  $f$  is better than  $x$  after observing  $s$ , the agent should choose  $f$  after observing  $s$ .

## 3 Representation

### 3.1 Generalized Bayesian Updating

Our first result is an axiomatization of the Generalized Bayesian Updating described in the introduction.

**Theorem 3.1.** *Let  $(\succeq_0, (\succeq_s)_{s \in S})$  be a family of preferences over  $\mathcal{F}$  that satisfies MEU Utility and  $\mathcal{L}$  an imprecise information structure. Then,  $(\succeq_0, (\succeq_s)_{s \in S})$  satisfies Total Information Agreement if and only if  $\mathcal{M}_s \subseteq ch(BU(\mathcal{M}_0, \mathcal{L}, s))$  for all  $s \in S$ .*

Theorem 3.1 only provides conditions for the set of posteriors to be a subset of the set  $ch(BU(\mathcal{M}_0, \mathcal{L}, s))$ . We now provide the necessary axiom required to replace “ $\subseteq$ ” with “ $=$ ”.

**Default to Certainty** For all  $f, x \in \mathcal{F}$  and  $s \in S$ ,

$$x \succeq_0 f_x^{\ell, s} \text{ for some } \ell \in \mathcal{L} \implies x \succeq_s f.$$

To interpret Default to Certainty note that Total Information Agreement only restricts the agent’s behavior in a very limited sense. Indeed, it only has bite whenever *all* possible sources of information *agree* that an act with uncertain payoffs is better than one with certain payoffs. However, it may be the case that the agent may behave differently depending on which source is the one that generated the signal. For instance, it may be the case that for a given signal the agent would prefer an act  $f$  over a constant act  $x$  if she knew  $\ell$  was the source. However, she may have the opposite preference if she knew  $\ell'$  was the source. Default to Certainty states that whenever such an issue arises, she always chooses the constant act.<sup>3</sup>

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<sup>3</sup>Faro and Lefort (2019) uses a similar axiom to characterize FBU in a precise information context.

**Proposition 3.1.** *Suppose  $(\succeq_0, (\succeq_s)_{s \in S})$  and  $\mathcal{L}$  satisfy the axioms of Theorem 3.1. Then,  $(\succeq_0, (\succeq_s)_{s \in S})$  satisfies Default to Certainty if and only if  $\mathcal{M}_s = ch(BU(\mathcal{M}_0, \mathcal{L}, s))$  for all  $s \in S$ .*

## 3.2 Discussion

The relation between Theorem 3.1 and the standard Bayesian model merits emphasis. The latter is obtained if one strengthens MEU Utility, by requiring that the sets of beliefs be singletons, and assume  $\mathcal{L}$  is a singleton. This has an important implication for the interpretation of Total Information Agreement. Specifically, it is well known that under the SEU axioms, Bayesian updating is characterized by Consequentialism and Dynamic Consistency (see Ghirardato (2002)). Because signals are not payoff-relevant, Consequentialism is implicitly assumed in our setting. Hence, Total Information Agreement reduces to Dynamic Consistency whenever ex-ante and ex-post preferences are SEU. Moreover, it can be interpreted as a weakening of Dynamic Consistency for the case in which ex-ante and ex-post preferences are MEU.

Theorem 3.1 delivers a representation, and thus an updating rule, for imprecise information. Formally, A Generalized Bayesian Updating (GBU) rule is a function  $\rho : (\mathcal{M}, \mathcal{L}, s) \mapsto \mathcal{M}'$  such that  $\mathcal{M}' \subseteq ch(BU(\mathcal{M}, \mathcal{L}, s))$ .

Because the GBU rule imposes little structure on the set of posteriors for the conditional preferences, the model can accommodate a wide range of behavior. On the other hand, some may view the model as “too general” in that it permits the posteriors not to be generated by Bayesian Updating of any of the feasible information sources. Indeed, it may be the case that

$$\rho(\mathcal{M}_0, \mathcal{L}, s) \subseteq ch(BU(\mathcal{M}_0, \mathcal{L}, s)) \setminus BU(\mathcal{M}_0, \mathcal{L}, s). \quad (3)$$

One reason to allow (3) is to nest the case in which the agent has, possibly non-singleton, beliefs over  $\mathcal{L}$ . To illustrate, consider a probability measure  $\lambda$  over  $\mathcal{L}$ . We interpret  $\lambda$  as the agent’s beliefs about which is the true information source. When updating her set of prior  $\mathcal{M}_0$ , such an agent may use her beliefs to determine her set of posteriors:

$$\rho(\mathcal{M}_0, \mathcal{L}, s) = \{p \in \Delta(\Omega) \mid p = \sum_{\ell \in \mathcal{L}} \lambda(\ell) BU(q, \ell, s), q \in \mathcal{M}_0\}.$$

Another potential concern is that the GBU allows the agent to use different  $\ell$ ’s depending on the signal. If one follows a “maximum likelihood” type of rule, then the  $\ell$ ’s that have the maximal likelihood of generating  $s$  may be different than the ones generating  $s'$ . Similarly, an agent who only considers information structures that “confirm” her beliefs may consider different information structures

depending on the realization of the signal. We further investigate these types of rules in Section 4. In the next section, we discuss the necessary conditions needed to rule out this feature of the model.

Next, we discuss the uniqueness properties of the model. Because we assumed the preferences conform to the MEU model,  $(\mathcal{M}_0, (\mathcal{M}_s)_{s \in S})$  are unique.<sup>4</sup> This does not mean that the information structures that were used to construct the posterior set are unique. This is the focus of Section 3.4.

We conclude the discussion with a few words on the impossibility of providing a measure of aversion to updating imprecise information in the current framework. The main reason is that the current framework does not allow the agent to choose between facing imprecise information and precise information. Indeed, any measure of aversion towards a phenomenon requires a comparison between the phenomenon and an object that does not suffer from the phenomenon.<sup>5</sup> Therefore, in order to provide such a measure, we would need to consider a different setting. One possibility would be pairs of menus of acts and imprecise information structures. Although it is possible to adapt our analysis to such a setting, because most of the experimental literature does not consider menus, we feel it would obscure the message of the paper.

### 3.3 Consistent Updating

As discussed in the previous section, our model allows the agent to use different information structures for different signals. The following axiom makes sure that it can never be the case that  $\ell$  plays a role after observing  $s'$  if it is ignored after observing  $s$ .

**Consistency Across Signals** For any  $f, g, x, x' \in \mathcal{F}$  and  $s, s' \in S$ ,

$$\begin{aligned} &\text{if } x \succeq_0 f_x^{\ell, s} \text{ and } f \succeq_s x, \text{ then} \\ &g_{x'}^{\ell, s'} \succeq_0 x' \text{ and } g \succeq_{s'} x' \implies g_{x'}^{\ell', s'} \succeq_0 x' \text{ for some } \ell' \in \mathcal{L}, \end{aligned}$$

where  $g_{x'}^{\ell', s'} \succeq_0 x'$  holds with indifference if  $g \succeq_{s'} x'$  holds with indifference.

From the discussion in the interpretation of Total Information Agreement,  $x \succeq_0 f_x^{\ell, s}$  and  $f \succeq_s x$  reveal the agent is ignoring the possibility that  $\ell$  generated  $s$ . Therefore, any behavior that considering  $\ell$  would lead to after observing another signal  $s'$ , such as  $x \succeq_0 g_{x'}^{\ell, s'}$  and  $g \succeq_s x'$ , has to be explainable by another information structure  $\ell'$ .

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<sup>4</sup>See the uniqueness result in [Gilboa and Schmeidler \(1989\)](#).

<sup>5</sup>For example, in order to check if an agent is averse to ambiguity one needs to observe a choice between acts and lotteries.

To state the result we need some notation. For any set of priors  $\mathcal{M}$ ,  $\mathcal{E}(\mathcal{M})$  denotes all of its extreme points.

**Proposition 3.2.** *Suppose  $(\succeq_0, (\succeq_s)_{s \in S})$  and  $\mathcal{L}$  satisfy the axioms of Theorem 3.1. Then,  $(\succeq_0, (\succeq_s)_{s \in S})$  satisfies Consistency Across Signals if and only if for all  $s, s' \in S$  and  $\ell \in \mathcal{L}$*

$$BU(q_0, \ell, s) \notin \mathcal{M}_s \text{ for all } q_0 \in \mathcal{M}_0 \implies BU(q'_0, \ell, s') \notin \mathcal{E}(\mathcal{M}_{s'})$$

for all  $q \in \mathcal{M}_0$

Proposition 3.2 states that if  $\ell$  is ignored after observing  $s$ , then it cannot be an extreme point of  $\mathcal{M}_{s'}$ . It is well known that only the extreme points of  $\mathcal{M}_{s'}$  are important in the MEU model. Specifically,

$$\min_{q \in \mathcal{M}_s} \int_{\Omega} u(f) dq = \min_{q \in \mathcal{E}(\mathcal{M}_s)} \int_{\Omega} u(f) dq$$

for all  $f$ . Therefore, our axiom ensures that if the agent does not consider  $\ell$  after  $s$ , she never considers it whenever it could affect her preferences.

### 3.4 Identification

Say that  $(u, \mathcal{M}_0, (\mathcal{M}_s)_{s \in S}, \mathcal{L})$  represents  $(\succeq_0, (\succeq_s)_{s \in S})$  if it satisfies the conditions of Theorem 3.1. Next, we describe up to what degree we can identify which priors and information structures were used to update by the agent.

In general, there is no hope for an identification result. As the following example shows, Bayesian Updating of a set of priors  $\mathcal{M}_0$  and two different subsets of a given  $\mathcal{L}$  may lead to the same set of posteriors.

**Example 3.1.** *Consider a binary state space  $\Omega = \{\omega_1, \omega_2\}$  and a binary signal space  $S = \{s_1, s_2\}$ . The prior belief  $\bar{q}$  is uniform,  $\bar{q} = (1/2, 1/2)$ . Let  $\mathcal{M}_i$  be the set of posteriors given signal  $s_i$ . Since there are only two states, we can identify a belief by its assessment on  $\omega_1$ . Suppose that  $\mathcal{M}_1 = [5/8, 7/8]$  and  $\mathcal{M}_2 = [1/8, 3/8]$ .*

*Consider the following two distinct sets of information structures:*

$$\begin{aligned} \mathcal{L}_1 &= \{\ell : \ell(s_1|\omega_1) \in [5/8, 7/8]; \ell(s_1|\omega_2) = \ell(s_2|\omega_1)\}, \text{ and} \\ \mathcal{L}_2 &= \{\ell : k \in [5/8, 7/8]; \ell(s_1|\omega_1) = 4k(1-k); \ell(s_1|\omega_2) = 4(1-k)^2\}. \end{aligned}$$

*Then under FBU, these two sets of information structures both induce  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . That is, given the unconditional and conditional preferences, from which we can identify  $\bar{q}$ ,  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , we cannot distinguish if the set of information structures adopted by the agent is  $\mathcal{L}_1$  or  $\mathcal{L}_2$ . Hence, if  $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$  there is no hope for identifying the set of information structures used.*

## 4 Specializations

In this section, we discuss several special cases of the GBU. We separate the discussion into two classes, the ones motivated by statistics and the ones motivated by behavioral biases.

### 4.1 Statistical GBU

Taking a statistics point of view, it is natural to only consider pairs  $(q, \ell)$  that pass some statistical test. We model a test as a function  $\phi : (\mathcal{M}, \mathcal{L}, s) \mapsto (\mathcal{M}', \mathcal{L}') \subseteq \mathcal{M} \times \mathcal{L}$ . We refer to such rules as Statistical Generalized Bayesian Updating rules (SGBU):

$$\rho(\mathcal{M}_0, \mathcal{L}, s) = ch(BU(\phi(\mathcal{M}_0, \mathcal{L}, s), s)).$$

It is easy to see that FBU (4) and MLU (5) are special cases of SGBU:

$$\phi(\mathcal{M}_0, \mathcal{L}, s) = (\mathcal{M}_0, \mathcal{L}) \tag{4}$$

$$\phi(\mathcal{M}_0, \mathcal{L}, s) = \{(q_0, \ell) | (q_0, \ell) \in \arg \max_{q_0 \in \mathcal{M}_0, \ell \in \mathcal{L}} \int_{\Omega} \ell(s|\omega) dq_0\}. \tag{5}$$

In the introduction, we discussed how MLU uses the signals to *jointly* discriminate among priors and information structures. Each information structure is evaluated according to the prior that maximizes the likelihood of the observed signal. We propose the following conservative approach: evaluate each information structure using *all* priors. The following special case of SGBU, referred to as Maximum Robust Likelihood Updating (MRLU), formalizes this idea:

$$\phi_{MRLU}(\mathcal{M}_0, \mathcal{L}, s) = \{(q_0, \ell) | q_0 \in \mathcal{M}_0, \ell \in \arg \max_{\ell \in \mathcal{L}} \min_{q \in \mathcal{M}} \int_{\Omega} \ell(s|\omega) dq\}. \tag{6}$$

The axioms that characterize (6) are strengthenings of Total Information Agreement and Default to Certainty. They basically require the agent to only consider information structures she believes are the most likely to generate the observed signal. We now show how this can be identified in our framework if the agent satisfies Reduction.

Fix a signal  $s^* \in S$ , two constant acts  $x, y$  such that  $x \succ y$ . Consider again an extension of  $\succeq_0$  to pairs of signal acts and information structures  $(F, \ell)$ . The idea is that conditional on each state  $\omega$ ,  $\ell(\cdot|\omega)$  describes the probability law on  $S$ .

Suppose we ask the agent to choose between  $(F, \ell_1)$  and  $(F, \ell_2)$  where

$$F(\omega, s) = \begin{cases} x & s = s^* \\ y & s \neq s^*. \end{cases}$$

Figure 2 (a) and Figure 2 (b) illustrate  $(F, \ell_1)$  and  $(F, \ell_2)$  respectively. Then, the agent will choose  $(F, \ell_2)$  over  $(F, \ell_1)$  if and only if she thinks  $\ell_2$  is more likely to generate  $s^*$  than  $\ell_1$ . Further, by an identical argument to the one in the motivation of Total Information Agreement,  $(F, \ell_i) \sim_0 x_y^{\ell_i, s^*}$  for  $i = 1, 2$ . Hence, for a given  $x, y$ , we can identify which information structures the agent believes are more likely to generate a given signal.

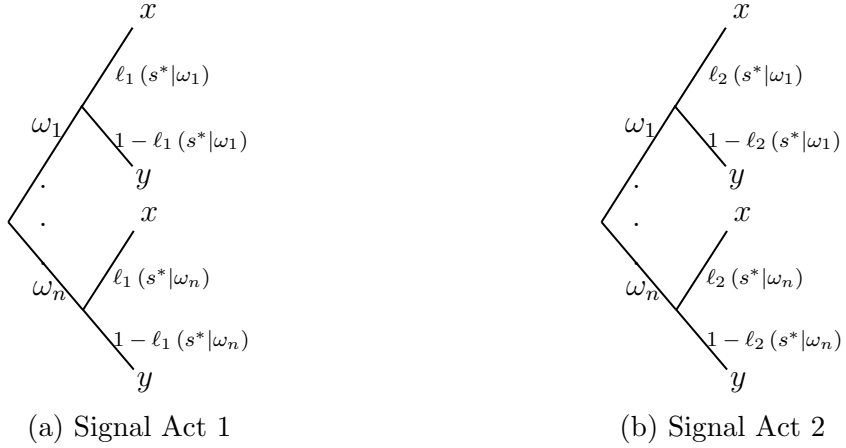


Figure 2

To state the next axiom we need some notation. Let  $\geq_{x,y}^s$  be the binary relation over  $\mathcal{L}$  such that  $l \geq_{x,y}^s l'$  if  $x_y^{\ell, s} \succeq_0 x_y^{\ell', s}$  and let

$$\arg \max(\mathcal{L}, \geq_{x,y}^s) = \{l \in \mathcal{L} | l \geq_{x,y}^s l' \text{ for all } l' \in \mathcal{L}\}.$$

**Likelihood Information Agreement** For any  $x, y, z, f \in \mathcal{F}$  such that  $x \succ_0 y$  and  $s \in S$ ,

$$f_z^{\ell, s} \succeq_0 z \text{ for all } \ell \in \arg \max(\mathcal{L}, \geq_{x,y}^s) \implies f \succeq_s z.$$

Likelihood Information Agreement (LIA) guarantees that the set of posteriors is a subset of the set of the convex hull of the set of posteriors generated by point-wise Bayesian Updating of (6). The following axiom strengthens set contention to equality.

**Likelihood Default to Certainty** For any  $x, y, z, f \in \mathcal{F}$  such that  $x \succ_0 y$  and  $s \in S$ ,

$$z \succeq_0 f_z^{\ell, s} \text{ for some } \ell \in \arg \max(\mathcal{L}, \geq_{x,y}^s) \implies x \succeq_s f.$$

Its interpretation is analogous to the one of Default to Certainty.

**Theorem 4.1.** *Let  $(\succeq_0, (\succeq_s)_{s \in S})$  be a family of preferences over  $\mathcal{F}$  that satisfies MEU Utility and  $\mathcal{L}$  an imprecise information structure. Then,  $(\succeq_0, (\succeq_s)_{s \in S})$  satisfies Likelihood Information Agreement if and only if*

$$\mathcal{M}_s \subseteq ch(BU(\phi_{MRLU}(\mathcal{M}_0, \mathcal{L}, s), s)) \quad (7)$$

for all  $s \in S$ . Moreover, the set contention in (7) is replaced with equality if and only if  $(\succeq_0, (\succeq_s)_{s \in S})$  also satisfies Likelihood Default to Certainty.

We conclude this section by noting that for the case in which the agent holds a single prior  $q_0$ , Theorem 4.1 delivers a characterization of a version of the maximum likelihood updating rule. Specifically, the agent will only consider the information structures  $\ell \in \mathcal{L}$  such that

$$\ell \in \arg \max_{\ell' \in \mathcal{L}} \int_{\Omega} \ell(s|\omega) dq_0.$$

Hence, our result can be viewed as a counterpart of [Gilboa and Schmeidler \(1993\)](#) result for precise information under ambiguity.

## 4.2 Behavioral GBU

One of the more robust findings in the empirical literature on updating is that people tend to only update information that is consistent with their prior beliefs. This phenomenon is called confirmatory bias ([Rabin and Schrag \(1999\)](#)). In this section, we argue our model is well-suited to accommodate it. Because confirmatory bias and state space uncertainty seem like separate phenomena, we focus on the case in which  $\mathcal{M}_0 = \{q_0\}$ .

When the agent's ex-ante prior beliefs can be described by a single probability distribution we can recover the agent's probabilistic ranking among the states of the world. Indeed, for any  $x, y \in \mathcal{F}$  such that  $x \succ_0 y$ , one can recover the ranking by observing the agent's preferences over the following acts:

$$f_{\omega_i}(\omega) = \begin{cases} x & \omega = \omega_i \\ y & \omega \neq \omega_i \end{cases}.$$

Let  $\succeq^*$  be the binary relation over  $\Omega$  induced by the ranking.

Intuitively, we can use  $\succeq^*$  to define whether an information structure  $\ell$  provides information consistent with the agents prior: For a given signal  $s$ ,  $\ell(s|\omega)$  is the likelihood it was generated by state  $\omega$ . Hence, the information will confirm the agent's ex-ante beliefs if the order among the  $\ell(s|\omega)$ 's is the same as her ranking among states. Formally, we say that  $\ell(\cdot|s)$  *s-confirms*  $\succeq^*$  if

$$\omega \succeq^* \omega' \iff \ell(s|\omega) \geq \ell(s|\omega').$$

An agent who suffers from confirmatory bias would only consider information that confirms her beliefs. The following axiom captures this.

**Confirmatory Information Agreement** For any  $f, x \in \mathcal{F}$ ,

$$f_x^{\ell, s} \succeq_0 x \text{ for all } \ell \in \mathcal{L} \text{ that } s\text{-confirms } \succeq^* \implies f \succeq_s x.$$

Confirmatory Information Agreement (CIA) guarantees that the set of posteriors is a subset of the set of the convex hull of the set of posteriors generated by point-wise Bayesian Updating of the set of information structures that confirm the agent's beliefs (whenever it is non-empty). The following axiom strengthens set contention to equality.

**Confirmatory Default to Certainty** For any  $f, x \in \mathcal{F}$  such that  $x \succ_0 y$  and  $s \in S$ ,

$$x \succeq_0 f_x^{\ell, s} \ell \in \mathcal{L} \text{ that } s\text{-confirms } \succeq^* \implies x \succeq_s f.$$

**Theorem 4.2.** *Let  $(\succeq_0, (\succeq_s)_{s \in S})$  be a family of preferences over  $\mathcal{F}$  that satisfies MEU Utility and  $\mathcal{L}$  an imprecise information structure. Assume  $\mathcal{M}_0$  is a singleton and for each  $s \in S$  there exists some  $\ell \in \mathcal{L}$  that confirms  $\succeq^*$ . Then,  $(\succeq_0, (\succeq_s)_{s \in S})$  satisfies Confirmatory Information Agreement if and only if*

$$\mathcal{M}_s \subseteq \text{ch}(BU(\mathcal{M}_0, \{\ell \in \mathcal{L} \mid s\text{-confirms } \succeq^*\}, s)) \tag{8}$$

for all  $s \in S$ . Moreover, the set contention in (8) is replaced with equality if and only if  $(\succeq_0, (\succeq_s)_{s \in S})$  also satisfies Confirmatory Default to Certainty.

## 5 Subjective Information Structures

In some situations, a signal may not come with a reliable description of its informational content. Therefore its interpretation is purely subjective, and can only be inferred from observable behaviors. In this section, we ask when the ex-ante and ex-post preferences are consistent with the existence of a subjective set of information structures. Unlike the previous analyses, here we assume no exogenous imprecise information structure.

Specifically, we are interested in the behavioral implications of a subjective imprecise information structure. Given a family of MEU preferences over acts  $(\succeq_0, (\succeq_s)_{s \in S})$ , when does there exist a set of information structures  $\mathcal{L}$  such that  $\mathcal{M}_s$  is exactly the set of posterior beliefs given  $\mathcal{M}_0$  and  $\mathcal{L}$  under some prespecified updating rule? We provide answers to this question for two special cases under the Full Bayesian updating rule. As the answer for either case already appears complicated, the general case is challenging and left for future research.



In the analyses later, we will also use “Max-Max” preferences. For any MEU preference  $\succeq$  represented by  $(\mathcal{M}, u)$ , let  $\bar{\succeq}$  denote the preference over acts satisfying

$$f \bar{\succeq} g \Leftrightarrow \max_{q \in \mathcal{M}} \sum_{\omega \in \Omega} q(\omega) u(f(\omega)) \geq \max_{q \in \mathcal{M}} \sum_{\omega \in \Omega} q(\omega) u(g(\omega)).$$

These two preferences are behaviorally related as follows. For any acts  $f, g$ , and constant act  $x$  satisfying  $x = \alpha f + (1 - \alpha)g$  for some  $\alpha \in (0, 1)$ , we have

$$f \bar{\succeq} x \text{ if and only if } x \succeq g.$$

Thus  $\bar{\succeq}$  can be understood as a conjugate of  $\succeq$ . Note that  $f \succeq x$  implies  $f \bar{\succeq} x$ , but the converse does not hold in general.

## 5.1 No ex-ante state ambiguity

First, we consider the situation where the state space is ex-ante unambiguous, meaning that the agent possesses a single prior belief, and thus the ex-ante preference  $\succeq_0$  follows the subjective-expected-utility (SEU) theory.

If the signal is also unambiguous, meaning that the agent perceives a single information structure  $\ell$  as the signal distribution, the ex-post preference  $\succeq_s$  is also a SEU. Then the prior belief  $q$  must lie in the convex hull of the posterior beliefs  $\{p^s : s \in S\}$ . This implies a classic dominance property: For any act  $f$  and constant act  $x$ , if  $f \succeq_s x$  for all  $s \in S$ , then  $f \succeq_0 x$ . In words, if  $f$  is preferred to  $x$  under any signal realization, then  $f$  is also preferred to  $x$  ex ante. Conversely, if the ex-ante and ex-post preferences are all SEUs, this dominance property is also sufficient for the existence of an information structure that relates the ex-ante and ex-post beliefs by Bayes’ rule.

However, if the ex-post preferences are MEUs, the aforementioned dominance property is too weak to establish a set  $\mathcal{L}$  of likelihood functions such that the ex-post set of beliefs  $\mathcal{M}_s$  is the full-Bayesian updating of the single prior belief and  $\mathcal{L}$ . The main result in this subsection shows that the following stronger dominance property will work.

**P1.** *Take any act  $f$  and any constant act  $x$ . If  $f \bar{\succeq}_s x$  for some  $s \in S$  and  $f \succeq_{s'} x$  for all  $s' \neq s$ , then  $f \succeq_0 x$ . If, in addition,  $f \bar{\succ}_s x$  or  $f \succ_{s'} x$  for some  $s' \neq s$ , then  $f \succ_0 x$ .*

To see P1 is a stronger property, note that  $f \succeq x$  implies  $f \bar{\succeq} x$ , but the converse fails. The following proposition shows that P1 characterizes the existence of a subjective imprecise information structure  $\mathcal{L}$  under full-Bayesian updating. It is crucial that the ex-ante preference follows SEU.

**Proposition 5.1.** *Suppose that the ex-ante preference  $\succeq_0$  is a SEU represented by  $(\bar{q}, u)$ , and the ex-post preferences  $\succeq_1, \dots, \succeq_S$  are MEUs represented by  $(\mathcal{M}_1, u), \dots, (\mathcal{M}_S, u)$  respectively. The following two statements are equivalent:*

1. *There exists a set  $\mathcal{L}$  of information structures such that  $\mathcal{M}_s = BU(\bar{q}, \mathcal{L}, s)$  for all  $s$ .*
2.  *$(\succeq_0, \{\succeq_s\}_{s \in S})$  satisfies P1.*

Below we explain the necessity of P1. Under MEU,  $f \bar{\succeq}_s x$  requires the existence of a belief  $p^s$  in  $\mathcal{M}_s$  under which the expected utility of  $f$  is no less than the utility of  $x$ . Under full Bayesian updating,  $p^s$  comes from Bayesian updating of prior  $\bar{q}$  and some information structure  $\ell \in \mathcal{L}$ . Moreover, for any other signal  $s'$ , the Bayesian updating of  $\bar{q}$  and  $\ell$ , say  $p^{s'}$ , also lies in  $\mathcal{M}_{s'}$ . Thus,  $f \succeq_{s'} x$  implies that the expected utility of  $f$  under  $p^{s'}$  is no less than the utility of  $x$ . Since the prior  $\bar{q}$  must lie in the convex hull of the posteriors  $\{p^s : s \in S\}$ , we obtain  $f \succeq_0 x$ .

## 5.2 No signal ambiguity

Now we allow the ex-ante preference to be MEU, but focus on the characterization of the existence of a single subjective information structure, meaning that the signal is perceived unambiguous.

For any outcomes  $x, y, z \in X$  and any states  $\omega, \omega'$ , let  $[x, \omega | y, \omega' | z]$  denote the act that yields  $x$  in state  $\omega$ ,  $y$  in state  $\omega'$ , and  $z$  in any other state. Consider the following two properties on the preferences.

**P2.** *Take any act  $f$  and fix any state  $\omega_i$ . Take  $y_j^0, y_j^s \in X$  for each  $s \in S$  and  $j \in \{1, \dots, |\Omega|\} \setminus \{i\}$  such that  $f(\omega_i) = \sum_{j \neq i} \frac{y_j^s}{|\Omega|-1} = \sum_{j \neq i} \frac{y_j^0}{|\Omega|-1}$  for all  $s \in S$ . Suppose that for all  $s \in S$  and all  $j \neq i$ ,*

$$\begin{cases} x \succeq_s [f(\omega_j), \omega_j | y_j^s, \omega_i | x]; \\ x \bar{\succeq}_s [f(\omega_j), \omega_j | y_j^s, \omega_i | x] \text{ if } f(\omega_j) \succeq_s x. \end{cases}$$

*Then there exists  $j \neq i$  such that*

$$\begin{cases} x \succeq_0 [f(\omega_j), \omega_j | y_j^s, \omega_i | x]; \\ x \bar{\succeq}_0 [f(\omega_j), \omega_j | y_j^s, \omega_i | x] \text{ if } f(\omega_j) \succeq_0 x. \end{cases}$$

**P3.** *Take any acts  $f, g$  and fix any state  $\omega_i$ . Suppose that  $f(\omega_i) = g(\omega_i)$ . Take any  $y_j \in X$  for each  $j \in \{1, \dots, |\Omega|\} \setminus \{i\}$ . Then the following two statements are true:*

(i) For any  $x \in X$ , if for all  $j \neq i$ ,

$$\begin{cases} [y_j, \omega_j | g(\omega_j), \omega_i | x] \bar{\succeq}_s x; \\ [y_j, \omega_j | g(\omega_j), \omega_i | x] \succeq_s x \text{ if } x \succeq_s y_j \end{cases} \quad \text{and} \quad \begin{cases} x \succeq_0 [y_j, \omega_j | f(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_0 [y_j, \omega_j | f(\omega_j), \omega_i | x] \text{ if } y_j \succeq_0 x \end{cases}$$

then  $x \succeq_0 g$  implies  $x \succeq_s f$ .

(ii) For any  $x \in X$ , if for all  $j \neq i$ ,

$$\begin{cases} x \succeq_s [y_j, \omega_j | g(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_s [y_j, \omega_j | g(\omega_j), \omega_i | x] \text{ if } y_j \succeq_s x \end{cases} \quad \text{and} \quad \begin{cases} [y_j, \omega_j | f(\omega_j), \omega_i | x] \bar{\succeq}_0 x; \\ [y_j, \omega_j | f(\omega_j), \omega_i | x] \succeq_0 x \text{ if } x \succeq_0 y_j \end{cases}$$

then  $g \succeq_0 x$  implies  $f \succeq_s x$ .

The existence of a single information structure under MEU preferences and full-Bayesian updating is characterized by P2 and P3, as shown in the following proposition.

**Proposition 5.2.** *Suppose that the ex-ante preference  $\succeq_0$  is a MEU represented by  $(\mathcal{M}_0, u)$ , and the ex-post preferences  $\succeq_1, \dots, \succeq_S$  are MEUs represented by  $(\mathcal{M}_1, u), \dots, (\mathcal{M}_S, u)$  respectively. Suppose that every belief in  $\mathcal{M}_0$  or in  $\mathcal{M}_s$  has full support. The following statements are equivalent:*

1. *There exists an information structure  $\ell : \mathcal{S} \times \Omega \rightarrow (0, 1)$  such that for all  $s \in \mathcal{S}$ ,  $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$ .*
2.  *$(\succeq_0, \{\succeq_s\}_{s \in \mathcal{S}})$  satisfies P2 and P3.*

To understand P2, consider utility acts (i.e.  $X \subset \mathbb{R}$  and  $u(x) = x$ ) for simplicity. Suppose that  $|\Omega| = 3$ . Thus an act is an element of  $\mathbb{R}^3$ . Suppose that, for instance, we have for all  $s \in S$ ,

$$x \succeq_s (x - a_1, x, x + b_1^s) \text{ and } x \succeq_s (x, x - a_2, x + b_2^s),$$

where  $a_1, a_2 > 0$  and  $b_1^s + b_2^s = k$  for all  $s$ . We can interpret these rankings as follows. Given a constant act  $x$ , we lower its payoff in  $\omega_1$  by  $a_1$ , and we increase its payoff in  $\omega_3$  by  $b_1^s$  as compensation. However, the compensation is not large enough to fully compensate for the loss in  $\omega_1$ . Thus the first ranking follows. Similarly, if we lower the payoff in  $\omega_2$  by  $a_2$ , then we increase the payoff in  $\omega_3$  by  $b_2^s$  as compensation. The second-ranking suggests that this compensation is not large enough. As  $b_1^s + b_2^s = k$  for all  $s$ , we can say that  $k$  is too small as a total stake for compensation for any signal realization. Then P2 says that  $k$  is also too small from the ex-ante perspective. We can never split  $k$  into  $b_1^0$  and  $b_2^0$  such that

$$(x - a_1, x, x + b_1^0) \succeq_0 x \text{ and } (x, x - a_2, x + b_2^0) \succeq_0 x.$$

P2 captures the following “convex hull” implication of a single subjective information structure. Pick a state  $\omega_i$ . As we show in the proof, if  $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$  for all  $s \in S$ , then for all  $j \neq i$ ,

$$\sum_s \ell(s|\omega_i) \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} = \sum_s \ell(s|\omega_i) \times \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \times \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} = \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)}.$$

This means that the vector  $(\max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)})_{j \neq i}$  lies in the convex hull of the set  $\{(\max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)})_{j \neq i}\}_{s \in S}$ . An implication is that for all reals  $k$  and  $(r_j)_{j \neq i}$ ,

$$\left[ \sum_{j \neq i} r_j \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} \geq k \quad \forall s \in S \right] \Rightarrow \sum_{j \neq i} r_j \times \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} \geq k.$$

P2 follows from this dominance property. Specifically, the necessity of P2 follows from that for any MEU preference  $\succeq$  represented by  $(\mathcal{M}, u)$ ,

$$\begin{cases} x \succeq [f(\omega_j), \omega_j | y_j, \omega_i | x]; \\ x \succeq [f(\omega_j), \omega_j | y_j, \omega_i | x] \text{ if } f(\omega_j) \succeq x \end{cases}$$

if and only if

$$[u(x) - u(f(\omega_j))] \max_{p \in \mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \geq u(y_j) - u(x).$$

So P2 allows us to find an information structure  $l$  such that

$$\max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} = \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \times \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)}$$

for all  $i, j$  and all  $s \in S$ . Yet, this is not enough to establish  $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$ . By Proposition 3.1, we need that  $f_x^{\ell, s} \succeq_0 x \Leftrightarrow f \succeq_s x$ . This implication is captured by P3.

To illustrate P3, again we consider utility acts for simplicity and assume  $|\Omega| = 3$ . Consider acts  $f = (f_1, f_2, f_3)$  and  $g = (g_1, g_2, g_3)$  where  $f_3 = g_3$ . Suppose that

$$(a_1, x, g_1) \succeq_s x \text{ and } (x, a_2, g_2) \succeq_s x$$

where  $x > a_1$  and  $x > a_2$ . The former suggests that  $g_1$ , under signal  $s$ , is good enough in the following sense: if we lower the payoff of a constant act  $x$  in  $\omega_2$  to  $a_2$  and replace its payoff in  $\omega_3$  by  $g_1$  as a compensation, we improve it. The latter ranking also suggests that  $g_2$  is good enough under signal realization  $s$ . Suppose also that

$$x \succeq_0 (a_1, x, f_1) \text{ and } x \succeq_0 (x, a_2, f_2).$$

Again, they suggest that  $f_1$  and  $f_2$  are not good enough from ex-ante perspective. Now P3 requires that we cannot have  $g$  worse than  $x$  ex ante but  $f$  better than  $x$  ex post.

## 6 Concluding Remarks

In this paper, we have provided a theory of updating under objective imprecise information that generalizes FBU and MLU. Although both of these rules are widely popular, we argue that there is no conceptual reason to adopt either. To do so we draw a parallel with the 3 color Ellsberg paradox.

In the 3 colors-Ellsberg paradox provides an example of intuitive behavior that cannot be rationalized if one assumes the agent has a single probabilistic belief. However, these choices can be rationalized if one assumes the agent holds multiple probabilistic beliefs. [Gajdos et al. \(2008\)](#) formalizes the intuition that there is no reason for the agent to consider *all* the objectively possible probabilistic beliefs (priors). They argue that the beliefs that the agent actually considers are part of her *subjective* view of the situation and represent the agent’s attitude towards the information provided. The same intuition applies to our setting, there is no reason for the agent to consider all objectively possible information structures.

Allowing for different attitudes towards updating may be especially important in applications. For instance [Beauchêne et al. \(2019\)](#) shows that under FBU a sender can extract full surplus from a receiver in a Bayesian Persuasion style game.<sup>6</sup> This result relies heavily on the FBU assumption. Indeed, once we allow for a more general updating rule as in our paper, such a result would not hold. This opens the door for a richer theory of persuasion under ambiguity.

The framework we employed was inspired by the experimental literature. Indeed, in the lab, one at most can observe ex-ante and conditional on signal preferences. Further, it is standard for signals to be payoff irrelevant. It is our hope that the “construction” of our axioms provides some guidance in how to test behavior when information is imprecise.

We have shown that one can recover the set of information structures an agent uses to update from observed behavior when one knows all the possible sources of information. A natural question to ask is if this is feasible when the analyst does not know the imprecise information structure. To be precise, Suppose  $(\succeq_0, (\succeq_s)_{s \in S})$  admit a MEU representation and assume  $\mathcal{M}_s = ch(BU(\mathcal{M}_0, \mathcal{L}^*, s))$ . Is it the case that we can recover  $\mathcal{L}^*$  from  $(\succeq_0, (\succeq_s)_{s \in S})$ ? If so, to what degree is  $\mathcal{L}^*$  unique? We leave such questions for future research.

Finally, the analysis in this paper has been mostly normative. Indeed, Total Information Agreement has been presented as an appealing property that delivers a generalization of Bayesian Updating. Therefore, it can be viewed as a test for any theory of updating that challenges Bayesianism in the context of the MEU model. Specifically, any updating rule that violates it must then be motivated by an example in which such an axiom is not reasonable. We view this as a separate

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<sup>6</sup>An equally striking result holds in a cheap talk game with FBU. See [Kellner and Le Quement \(2018\)](#).

contribution of the paper as it can help guide future research.

## Appendix A Proofs

The necessity of the axioms is obvious in each of our representation theorems. Therefore, we only prove sufficiency.

Throughout, it is assumed that  $\succeq_s$  is represented by  $(u, \mathcal{M}_s)$  for all  $s \in S \cup \{0\}$ . Finally, each proof of sufficiency makes use of the following lemma.

**Lemma A.1.** *Assume  $\succeq$  and  $\succeq'$  admit MEU representations  $(u, \mathcal{M})$ ,  $(u, \mathcal{M}')$  respectively such that  $\mathcal{M} \not\subseteq \mathcal{M}'$ . Then there exists an act  $f$  and a constant act  $x$  such that  $f \sim x$  and  $f \succ' x$ .*

The proof follows from the uniqueness result in [Gilboa and Schmeidler \(1989\)](#).

### A.1 Proof of Theorem 3.1

Let  $(\mathcal{M}_s)_{s \in S \cup \{0\}}$  be the sets of probability measures described by MEU Utility. Suppose  $(\succeq_s)_{s \in S \cup \{0\}}$  satisfies Total Information Agreement (TIA) and assume by way of contradiction that  $\mathcal{M}_s \not\subseteq ch(BU(\mathcal{M}_0, \mathcal{L}, s))$  for some  $s \in S$ .

By Lemma A.1, there exists an act  $f$  and a constant act  $x$  such that

$$u(x) < \int_{\Omega} u(f) dBU(q', \ell, s) \text{ for all } (q', \ell) \in \mathcal{M}_0 \times \mathcal{L}.$$

Fix  $(q, \ell) \in \mathcal{M}_0 \times \mathcal{L}$ , then

$$\begin{aligned} u(x) &< \int_{\Omega} u(f) dBU(q, \ell, s) \\ \left( \int_{\Omega} \ell(s|\omega) dq \right) u(x) &< \int_{\Omega} u(f) \ell(s|\omega) dq \\ \left( \int_{\Omega} \ell(s|\omega) dq \right) u(x) + \left( 1 - \int_{\Omega} \ell(s|\omega) dq \right) u(x) &< \int_{\Omega} u(f) \ell(s|\omega) dq + \left( 1 - \int_{\Omega} \ell(s|\omega) dq \right) u(x) \\ u(x) &< \int_{\Omega} u(\ell(s|\omega) f(\omega) + (1 - \ell(s|\omega))x) dq. \end{aligned}$$

Thus,  $u(x) < \int_{\Omega} u(\ell(s|\omega) f(\omega) + (1 - \ell(s|\omega))x) dq$  for all  $q \in \mathcal{M}_0$  and  $\ell \in \mathcal{L}$ . Therefore,  $u(x) < \min_{q \in \mathcal{M}_0} \int_{\Omega} u(\ell(s|\omega) f(\omega) + (1 - \ell(s|\omega))x) dq$  for all  $\ell \in \mathcal{L}$ . Hence,

$f_x^{\ell, s} \succ_0 x$  for all  $\ell \in \mathcal{L}$  and  $x \succeq_s f$ , a contradiction of TIA.

## A.2 Proof of Proposition 3.1

Given Theorem 3.1, we only need to show that if  $(\succeq, (\succeq_s)_{s \in S})$  also satisfies Default to Certainty (DTC), then  $ch(BU(\mathcal{M}_0, \mathcal{L}, s)) \subseteq \mathcal{M}_s$ .

Fix  $s \in S$  and assume by way of contradiction there exists  $p^* \in ch(BU(\mathcal{M}_0, \mathcal{L}, s))$  such that  $p^* \notin \mathcal{M}_s$ . By Lemma A.1, there exists  $f$  and  $x$  such that  $\int_{\Omega} u(f)p^* < \min_{q' \in \mathcal{M}_s} \int_{\Omega} u(f)dq' = u(x)$ .

Since  $p^* \in ch(BU(\mathcal{M}_0, \mathcal{L}, s))$ , then there exist  $(q_1, \ell_1), \dots, (q_n, \ell_n) \in \mathcal{M}_0 \times \mathcal{L}$  such that  $p^* = \sum_i \alpha_i BU(q_i, \ell_i, s)$  and

$$\int_{\Omega} u(f)p^* = \sum_i \alpha_i \int_{\Omega} u(f)dBU(q_i, \ell_i, s).$$

Hence, there exists  $i$  such that  $\int_{\Omega} u(f)dBU(q_i, \ell_i, s) \leq \int_{\Omega} u(f)dp^*$ . Let  $q = q_i$  and  $\ell = \ell_i$ . Then,

$$\begin{aligned} & \int_{\Omega} u(f)dBU(q, \ell, s) < u(x) \\ & \int_{\Omega} u(f)\ell(s|\omega)dq + \left(1 - \int_{\Omega} \ell(s|\omega)dq\right) u(x) < u(x) \left(\int_{\Omega} \ell(s|\omega)dq\right) + \left(1 - \int_{\Omega} \ell(s|\omega)dq\right) u(x) \\ & \int_{\Omega} u(f(\omega)\ell(s|\omega) + (1 - \ell(s|\omega))x)dq < u(x). \end{aligned}$$

Hence,  $\min_{q \in \mathcal{M}_0} \int_{\Omega} u(f(\omega)\ell(s|\omega) + (1 - \ell(s|\omega))x)dq < u(x)$ . This implies that there exists  $\ell \in \mathcal{L}$  such that  $x \succ_0 f_x^{\ell, s}$ . Thus, by DTC,  $x \succ_s f$ , a contradiction.

## A.3 Proof of Proposition 3.2

Assume by way of contradiction there there exists  $\ell \in \mathcal{L}$ ,  $s, s' \in S$  and  $q'_0 \in \mathcal{M}_0$  such that  $BU(q_0, \ell, s) \notin \mathcal{M}_s$  for all  $q_0$  and  $BU(q'_0, \ell, s') \in \mathcal{E}(\mathcal{M}_{s'})$ . The fact that  $BU(q'_0, \ell, s') \in \mathcal{E}(\mathcal{M}_{s'})$  implies there exists an act  $g$  such that

$$\min_{q \in \mathcal{M}_{s'}} \int_{\Omega} u(g)dq = \int_{\Omega} u(g)dBU(q'_0, \ell, s') < \int_{\Omega} u(g)dq \text{ for all } q \in \mathcal{M}_{s'} \setminus \{BU(q'_0, \ell, s')\}.$$

Let  $y$  be the constant act such that  $u(y) = \int_{\Omega} u(g)dBU(q'_0, \ell, s')$ . Define  $U_{\ell, s}(f) = \min_{q \in BU(\mathcal{M}_0, \ell, s)} \int_{\Omega} f(f)dq$ . Then, by Lemma A.1 there exists  $f, x$  such that  $u(x) = U_{\ell, s}(f) > \min_{q \in \mathcal{M}_s} \int_{\Omega} u(f)dq$ . By an identical argument to the one in Theorem 3.1, we can show that  $x \sim_0 f_x^{\ell, s}, x \succ_s f, y \sim_0 g_y^{\ell, s'}$  and  $y \sim_{s'} g$ . Hence, by Consistency Across Signals, there exists  $\ell'$  such that  $y \sim_0 g_y^{\ell', s'}$ , an impossibility.

## A.4 Proof of Theorem 4.1

Suppose  $(\succeq_0, (\succeq_s)_{s \in S})$  satisfies Likelihood Information Agreement (LIA) and assume by way of contradiction that

$$\mathcal{M}_s \not\subseteq \text{ch}(\{BU(q, \ell, s) | q \in \mathcal{M}_0, \ell \in \arg \max_{\ell \in \mathcal{L}} \max_{q \in \mathcal{M}_0} \int_{\Omega} \ell(s|\omega) dq\})$$

for some  $s \in S$ .

Let  $\mathcal{L}_s^{**} = \arg \max_{\ell \in \mathcal{L}} \min_{q \in \mathcal{M}_0} \int_{\Omega} \ell(s|\omega) dq$ . By Lemma A.1, there exists  $f$  and  $x$  such that

$$u(x) = \min_{q \in \text{ch}(BU(\mathcal{M}_0, \mathcal{L}_s^{**}, s))} \int_{\Omega} u(f) dq \leq \min_{(q, \ell) \in \mathcal{M}_0 \times \mathcal{L}_s^{**}} \int_{\Omega} u(f) dBU(q, \ell, s)$$

Hence, for all  $q \in \mathcal{M}_0$  and  $\ell \in \mathcal{L}_s^{**}$

$$\begin{aligned} u(x) &< \int_{\Omega} u(f) dBU(q, \ell, s) \\ u(x) &< \int_{\Omega} u(f) \ell(s|\omega) dq + u(p) \left(1 - \int_{\Omega} \ell(s|\omega) dq\right). \end{aligned}$$

Thus,

$$u(x) < \min_{q \in \mathcal{M}_0} \int_{\Omega} u(f) \ell(s|\omega) dq + u(x) \left(1 - \int_{\Omega} \ell(s|\omega) dq\right).$$

If we can show that  $\ell \in \arg \max(\mathcal{L}, \succeq_{x,y}^s)$  implies  $\ell \in \mathcal{L}_s^{**}$  for any  $x, y$  such that  $u(x) > u(y)$ , we will have a contradiction of LIA. To see that  $\ell \in \arg \max(\mathcal{L}, \succeq_{x,y}^s)$  implies  $\ell \in \mathcal{L}_s^{**}$  note that  $\ell \succeq_{x,y}^s \ell'$  if and only if

$$\min_{q \in \mathcal{M}_0} [u(x) \int_{\Omega} \ell(s|\omega) dq + u(y) \left(1 - \int_{\Omega} \ell(s|\omega) dq\right)] \geq \min_{q \in \mathcal{M}_0} [u(x) \int_{\Omega} \ell'(s|\omega) dq + u(y) \left(1 - \int_{\Omega} \ell'(s|\omega) dq\right)]$$

which holds if and only if

$$\min_{q \in \mathcal{M}_0} \int_{\Omega} \ell(s|\omega) dq \geq \min_{q \in \mathcal{M}_0} \int_{\Omega} \ell'(s|\omega) dq.$$

Hence,  $\ell \in \arg \max(\mathcal{L}, \succeq_{x,y}^s)$  implies  $\ell \in \arg \max_{\ell \in \mathcal{L}} \min_{q \in \mathcal{M}_0} \int_{\Omega} \ell(s|\omega) dq = \mathcal{L}_s^{**}$ .

Next, we prove that Likelihood Default to Certainty (LDC) implies equality of the sets described in the Theorem.

Let  $\text{ch}(\{BU(q, \ell, s) | \ell \in \arg \max_{\ell \in \mathcal{L}} \min_{q \in \mathcal{M}_0} \int_{\Omega} \ell(s|\omega) dq \text{ and } q \in \mathcal{M}_0\}) \equiv \mathcal{M}_s^{MM}$ . We only need to show that  $\mathcal{M}_s^{MM} \subseteq \mathcal{M}_s$ .



Assume by way of contradiction that  $\mathcal{M}_s^{MM} \not\subseteq \mathcal{M}_s$ . Then, by Lemma A.1, there exists  $f$  and  $x$  such that

$$\min_{q \in \mathcal{M}_s^{MM}} \int_{\Omega} u(f) dq < \min_{q \in \mathcal{M}_s} \int_{\Omega} u(f) dq = u(x).$$

Fix  $q' \in \arg \min_{q \in \mathcal{M}_s^{MM}} \int_{\Omega} u(f) dq$ . Then, by an analogous argument as in the proof of Proposition 3.1, there exists  $(q, \ell) \in \mathcal{M} \times \mathcal{L}^{**}$  such that  $\int_{\Omega} u(f) dBU(q, \ell, s) \leq \int_{\Omega} u(f) dq'$ . Hence,

$$\begin{aligned} \int_{\Omega} u(f) dBU(q, \ell, s) &< u(x) \\ \int_{\Omega} \ell(s|\omega) u(f) dq + (1 - \int_{\Omega} \ell(s|\omega) dq) u(x) &< u(x) \\ \min_{q \in \mathcal{M}} \int_{\Omega} \ell(s|\omega) u(f) dq + (1 - \int_{\Omega} \ell(s|\omega) dq) u(x) &< u(x). \end{aligned}$$

Thus,  $x \succ_0 f_x^{\ell, s}$ . Moreover,  $\ell \in \arg \max_{\ell \in \mathcal{L}} \min_{q \in \mathcal{M}_0} \int_{\Omega} \ell(s|\omega) dq$ , thus,  $\ell \geq_{x, y}^s \ell'$  for all  $\ell' \in \mathcal{L}$  and  $x, y$  such that  $u(x) > u(y)$ . By LDC,  $x \succ_s f$ , a contradiction.

## A.5 Proof of Theorem 4.2

Suppose  $(\succeq_s)_{s \in S \cup \{0\}}$  the satisfies Confirmatory Information agreement (CIA) and assume by way of contradiction  $\mathcal{M}_s \not\subseteq ch(BU(\mathcal{M}_0, \{\ell \in \mathcal{L} | \ell \text{ confirms } \succeq^*\}, s)))$ . Then, by Lemma A.1, there exists  $f$  and  $x$  such that

$$u(x) = \min_{q \in \mathcal{M}_s} \int_{\Omega} u(f) dq < \int_{\Omega} u(f) dBU(q, \ell, s) \text{ for all } (q, \ell) \in \mathcal{M}_0 \times \{\ell \in \mathcal{L} | \ell \text{ confirms } \succeq^*\}.$$

An identical argument as in Theorem 3.1 shows that  $f_x^{\ell, s} \succ_0 x$  for all  $\ell \in \{\ell \in \mathcal{L} | \ell \text{ confirms } \succeq^*\}$ . This contradicts CIA as  $x \sim_s f$ .

Next, we show that  $\mathcal{M}_s^{CM} \equiv ch(BU(\mathcal{M}_0, \{\ell \in \mathcal{L} | \ell \text{ confirms } \succeq^*\}, s))) \subseteq \mathcal{M}_s$  under Confirmatory Default to Certainty (CDC). Assume by way of contradiction that this is not the case. Then, by Lemma A.1, there exists  $f, x$  such that

$$\min_{p \in \mathcal{M}_s^{CM}} \int_{\Omega} u(f) dp < \min_{q \in \mathcal{M}_s} \int_{\Omega} u(f) dq = u(x).$$

Fix  $p \in \arg \min_{p \in \mathcal{M}_s^{CM}} \int_{\Omega} u(f) dp$ . Then an analogous argument as in the proof of Proposition 3.1, establishes that there exists  $(q, \ell) \in \mathcal{M}_0 \times \{\ell \in \mathcal{L} | \ell \text{ confirms } \succeq^*\}$  such

that  $\int_{\Omega} u(f)dB U(q, \ell, s) \leq \int_{\Omega} u(f)dp$ . Hence,

$$\begin{aligned} \int_{\Omega} u(f)dB U(q, \ell, s) &< u(x) \\ \int_{\Omega} \ell(s|\omega)u(f)dq + (1 - \int_{\Omega} \ell(s|\omega)dq)u(x) &< u(x) \\ \min_{q \in \mathcal{M}_0} \int_{\Omega} \ell(s|\omega)u(f)dq + (1 - \int_{\Omega} \ell(s|\omega)dq)u(x) &< u(x). \end{aligned}$$

Thus,  $x \succ_0 f_x^{\ell, s}$ . Moreover,  $\ell \in \{\ell \in \mathcal{L} | \ell \text{ confirms } \succeq^*\}$ . Therefore, CDC,  $x \succ_s f$ , a contradiction.

## A.6 Proof of Proposition 5.1

We first prove the necessity of statement 1. Take any  $s \in S$ . Suppose that  $f \bar{\succeq}_s x$  and  $f \succeq_{s'} x$  for all  $s' \neq s$ . Since  $f \bar{\succeq}_s x$ , there exists  $p^s \in \mathcal{M}_s$  such that  $p^s \cdot (u \circ f) \geq u(x)$ . Since  $\mathcal{M}_s = BU(\bar{q}, \mathcal{L}, s)$ , there exists  $\ell \in \mathcal{L}$  such that  $p^s = BU(\bar{q}, \ell, s)$ . Let  $p^{s'} = BU(\bar{q}, \ell, s')$  for all  $s' \neq s$ . Since  $\mathcal{M}_{s'} = BU(\bar{q}, \mathcal{L}, s')$ ,  $p^{s'} \in \mathcal{M}_{s'}$ . Now for all  $s' \neq s$ , because  $f \succeq_{s'} x$ , we have  $p^{s'} \cdot (u \circ f) \geq u(x)$ . Since  $\bar{q}$  lies in the convex hull of  $p^s, p^{s'}, \dots$ , we have  $\bar{q} \cdot (u \circ f) \geq u(x)$ . In addition, if  $f \bar{\succeq}_s x$  or  $f \succ_{s'} x$  for some  $s' \neq s$ , we obtain  $\bar{q} \cdot (u \circ f) > u(x)$ . This proves the necessity of statement 1.

Next, we prove the sufficiency of statement 1. The following claim will be useful.

**Claim 1.** *Suppose that  $A_1, \dots, A_I$  are closed and convex sets in  $\mathbb{R}^N$ . Then*

$$ri(ch(\cup_i A_i)) \subset \left\{ v \in \mathbb{R}^N : \exists w_i \in A_i, \lambda_i > 0 \forall i \text{ s.t. } \sum_i \lambda_i w_i = v, \sum_i \lambda_i = 1 \right\}.$$

*Proof.* Suppose  $v \in ri(ch(\cup_i A_i))$ . Take any  $v' = \sum_i \lambda_i w_i$  with  $w_i \in A_i$ ,  $\lambda_i > 0$ , and  $\sum_i \lambda_i = 1$ . Since  $v \in ri(ch(\cup_i A_i))$ , there exists  $v'' \in ch(\cup_i A_i)$  and  $k \in (0, 1)$  such that  $v = kv' + (1 - k)v''$ . Since  $A_i$  is convex for all  $i$ ,  $v''$  can be expressed as  $v'' = \sum_i \lambda'_i w'_i$  with  $w'_i \in A_i$ ,  $\lambda'_i \geq 0$ , and  $\sum_i \lambda'_i = 1$ . Thus,

$$v = kv' + (1 - k)v'' = \sum_i (k\lambda_i + (1 - k)\lambda'_i) \frac{k\lambda_i w_i + (1 - k)\lambda'_i w'_i}{k\lambda_i + (1 - k)\lambda'_i}.$$

Since  $\lambda_i > 0$ ,  $k\lambda_i + (1 - k)\lambda'_i > 0$ . Since  $A_i$  is convex,  $\frac{k\lambda_i w_i + (1 - k)\lambda'_i w'_i}{k\lambda_i + (1 - k)\lambda'_i} \in A_i$ . Therefore  $v$  belongs to the set on the right-hand side.  $\square$

We will also use the following separating hyperplane theorem: Two non-empty convex sets  $A$  and  $B$  can be separated properly if and only if their relative interiors

do not intersect. Here, proper separation means that there is a hyperplane  $H$  such that  $A$  and  $B$  lie in opposite closed half-spaces with respect to  $H$ , and at least one of the sets  $A, B$  is not contained in  $H$ .

Suppose that statement 2 fails. There exists a signal realization  $s$  and  $q^s \in \mathcal{M}_s$  such that there exist no  $q^{s'} \in \mathcal{M}_{s'}$  for each  $s' \neq s$  such that  $\bar{q}$  equals a convex combination of  $q^s, q^{s'}, \dots$ . We want to establish a violation of statement 1. Consider two cases:  $q^s = \bar{q}$  and  $q^s \neq \bar{q}$ .

Assume  $q^s = \bar{q}$ . By the claim,  $\bar{q} \notin \text{ri}(\text{ch}(\cup_{s' \neq s} \mathcal{M}_{s'}))$ . By the aforementioned separating hyperplane theorem, there exist a vector  $(v_\omega)_{\omega \in \Omega} \equiv v$  and a real  $r$  such that  $\bar{q} \cdot v = r \leq q \cdot v$  for all  $q \in \text{ch}(\cup_{s' \neq s} \mathcal{M}_{s'})$  where the inequality holds strictly for some  $q$ . Note that we are free to take a positive linear transformation on  $v$  and  $r$ , which means that we can choose  $v$  and  $r$  such that  $r$  and  $v_\omega$  all lie in the range of  $u$ . Therefore, there exist an act  $f$  and a constant act  $x$  such that  $\bar{q} \cdot (u \circ f) = u(x) \leq q \cdot (u \circ f)$  for all  $q \in \text{ch}(\cup_{s' \neq s} \mathcal{M}_{s'})$  where the inequality holds strictly for some  $q$ . Now we have  $f \succeq_{s'} x$  for all  $s' \neq s$  and  $f \succ_{s'} x$  for some  $s' \neq s$ . Since  $\bar{q} \cdot (u \circ f) = u(x)$ ,  $f \sim_0 x$ . Since  $q_s = \bar{q} \in \mathcal{M}_s$  by assumption,  $f \bar{\succeq}_s x$ . Hence statement 1 fails.

Assume instead  $q^s \neq \bar{q}$ . Consider the convex set

$$\{q \in \Delta(S) : \exists \lambda > 1 \text{ s.t. } q - q_s = \lambda(\bar{q} - q_s)\} \equiv C.$$

By the claim,  $C$  is disjoint with  $\text{ri}(\text{ch}(\cup_{s' \neq s} \mathcal{M}_{s'}))$ . By a separating hyperplane theorem, there exist an act  $f$  and an constant act  $x$  such that  $\bar{q} \cdot (u \circ f) = u(x) > q \cdot (u \circ f)$  for all  $q \in C$ , and  $\bar{q} \cdot (u \circ f) = u(x) \leq q \cdot (u \circ f)$  for all  $q \in \text{ch}(\cup_{s' \neq s} \mathcal{M}_{s'})$ . It follows that  $f \sim_0 x$  and  $f \succeq_{s'} x$  for all  $s' \neq s$ . Notice that  $C$  and  $\{q_s\}$  are also separated by  $f$  and  $x$ . So  $q_s \cdot (u \circ f) > u(x)$ , implying  $f \bar{\succ}_s x$ . Hence statement 1 fails.

## A.7 Proof of Proposition 5.2

Take any MEU preference  $\succeq$  represented by  $(\mathcal{M}, u)$ . Take any  $x, y, z \in X$  and any distinct states  $\omega_i, \omega_j \in \Omega$ . We have

$$\begin{aligned} x &\succeq [y, \omega_j | z, \omega_i | x] \\ &\Leftrightarrow \exists p \in \mathcal{M}, u(x) \geq p(\omega_j)u(y) + p(\omega_i)u(z) + [1 - p(\omega_i) - p(\omega_j)]u(x) \\ &\Leftrightarrow \exists p \in \mathcal{M}, [u(x) - u(y)] \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)] \\ &\Leftrightarrow [u(x) - u(y)] \max_{p \in \mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)]. \end{aligned}$$

When  $u(x) - u(y) \geq 0$ ,

$$x \succeq [y, \omega_j | z, \omega_i | x] \Leftrightarrow [u(x) - u(y)] \max_{p \in \mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)].$$

Similarly, we have

$$\begin{aligned}
& x \bar{\succeq} [y, \omega_j | z, \omega_i | x] \\
& \Leftrightarrow \forall p \in \mathcal{M}, u(x) \geq p(\omega_j)u(y) + p(\omega_i)u(z) + [1 - p(\omega_i) - p(\omega_j)]u(x) \\
& \Leftrightarrow \forall p \in \mathcal{M}, [u(x) - u(y)] \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)] \\
& \Rightarrow [u(x) - u(y)] \max_{p \in \mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)].
\end{aligned}$$

When  $u(x) - u(y) \leq 0$ ,

$$x \bar{\succeq} [y, \omega_j | z, \omega_i | x] \Leftrightarrow [u(x) - u(y)] \max_{p \in \mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)].$$

Consequently,

$$\begin{cases} x \succeq [y, \omega_j | z, \omega_i | x]; \\ x \bar{\succeq} [y, \omega_j | z, \omega_i | x] \text{ if } y \succeq x \end{cases} \Leftrightarrow [u(x) - u(y)] \max_{p \in \mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \geq u(z) - u(x),$$

and

$$\begin{cases} [y, \omega_j | z, \omega_i | x] \bar{\succeq} x; \\ [y, \omega_j | z, \omega_i | x] \succeq x \text{ if } x \succeq y \end{cases} \Leftrightarrow [u(x) - u(y)] \max_{p \in \mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \leq u(z) - u(x).$$

Now we prove the necessity of P2 and P3. Consider P2 first. For any belief  $q \in \Delta(\Omega)$ , if  $p \in \Delta(\Omega)$  is the Bayesian updating of  $q$  given signal  $s$  and likelihood function  $\ell$ , then for any states  $\omega_i, \omega_j$ ,

$$\frac{p(\omega_j)}{p(\omega_i)} = \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \times \frac{q(\omega_j)}{q(\omega_i)}.$$

Hence  $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$  implies

$$\max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} = \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \times \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)}.$$

Fixing any  $i$ , we have

$$\sum_{s \in \mathcal{S}} \ell(s|\omega_i) \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} = \sum_{s \in \mathcal{S}} \ell(s|\omega_i) \times \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \times \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} = \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)}$$

for any  $j \neq i$ . Hence the vector  $\left( \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} \right)_{j \neq i} \in \mathbb{R}^{|\Omega|-1}$  is in the convex hull of  $\left\{ \left( \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} \right)_{j \neq i} : s \in \mathcal{S} \right\}$ .

For all  $s \in \mathcal{S}$  and all  $j \neq i$ , because

$$\begin{cases} x \succeq_s [f(\omega_j), \omega_j | y_j^s, \omega_i | x]; \\ x \bar{\succeq}_s [f(\omega_j), \omega_j | y_j^s, \omega_i | x] \text{ if } f(\omega_j) \succeq_s x \end{cases},$$

we have

$$[u(x) - u(f(\omega_j))] \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} \geq u(y_j^s) - u(x).$$

Thus, for all  $s \in \mathcal{S}$ ,

$$\sum_{j \neq i} [u(x) - u(f(\omega_j))] \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} \geq \sum_{j \neq i} [u(y_j^s) - u(x)].$$

Because  $\sum_{j \neq i} \frac{y_j^s}{|\Omega|-1} = \sum_{j \neq i} \frac{y_j^0}{|\Omega|-1}$ , we have  $\sum_{j \neq i} u(y_j^s) = \sum_{j \neq i} u(y_j^0)$ . Consequently,

$$\sum_{j \neq i} [u(x) - u(f(\omega_j))] \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} \geq \sum_{j \neq i} [u(y_j^0) - u(x)].$$

This implies that

$$\exists j \neq i, [u(x) - u(f(\omega_j))] \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} \geq u(y_j^0) - u(x).$$

Therefore,

$$\exists j \neq i, \begin{cases} x \succeq_0 [f(\omega_j), \omega_j | y_j^s, \omega_i | x]; \\ x \bar{\succeq}_0 [f(\omega_j), \omega_j | y_j^s, \omega_i | x] \text{ if } f(\omega_j) \succeq_0 x. \end{cases}$$

We have established P2.

Now we check P3. By Proposition 3.1,  $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$  implies that  $f_x^{\ell, s} \succeq_0 x \Leftrightarrow f \succeq_s x$ . Fix any  $i$ . Suppose that

$$\begin{cases} [y_j, \omega_j | g(\omega_j), \omega_i | x] \bar{\succeq}_s x; \\ [y_j, \omega_j | g(\omega_j), \omega_i | x] \succeq_s x \text{ if } x \succeq_s y_j \end{cases} \quad \text{and} \quad \begin{cases} x \succeq_0 [y_j, \omega_j | f(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_0 [y_j, \omega_j | f(\omega_j), \omega_i | x] \text{ if } y_j \succeq_0 x \end{cases}$$

for all  $j \neq i$ . Then

$$\begin{aligned} u(g(\omega_j)) - u(x) &\geq [u(x) - u(y_j)] \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} = [u(x) - u(y_j)] \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} \\ &\geq \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} [u(f(\omega_j)) - u(x)]. \end{aligned}$$

Hence

$$u(g(\omega_j)) \geq \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} u(f(\omega_j)) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] u(x) \quad \forall j \neq i.$$

Moreover,  $f(\omega_i) = g(\omega_i)$  by assumption. Thus,  $g$  state-by-state dominates the act  $\left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right]x\right)_{j=1}^{|\Omega|}$ . Note that

$$\ell(s|\omega_i) \left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right]x\right)_{j=1}^{|\Omega|} + (1 - \ell(s|\omega_i))x = f_x^{\ell,s}.$$

Hence

$$x \succeq_0 g \Rightarrow x \succeq_0 \left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right]x\right)_{j=1}^{|\Omega|} \Rightarrow x \succeq_0 f_x^{\ell,s} \Rightarrow x \succeq_s f.$$

Suppose instead

$$\begin{cases} x \succeq_s [y_j, \omega_j | g(\omega_j), \omega_i | x]; \\ x \succeq_s [y_j, \omega_j | g(\omega_j), \omega_i | x] \text{ if } y_j \succeq_s x \end{cases} \quad \text{and} \quad \begin{cases} [y_j, \omega_j | f(\omega_j), \omega_i | x] \succeq_0 x; \\ [y_j, \omega_j | f(\omega_j), \omega_i | x] \succeq_0 x \text{ if } x \succeq_0 y_j \end{cases}$$

for all  $j \neq i$ . Then

$$\begin{aligned} u(g(\omega_j)) - u(x) &\leq [u(x) - u(y_j)] \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} = [u(x) - u(y_j)] \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} \\ &\leq \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} [u(f(\omega_j)) - u(x)]. \end{aligned}$$

Hence

$$u(g(\omega_j)) \leq \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} u(f(\omega_j)) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] u(x) \quad \forall j \neq i.$$

Thus, the act  $\left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right]x\right)_{j=1}^{|\Omega|}$  state-by-state dominates  $g$ . We have

$$g \succeq_0 x \Rightarrow \left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right]x\right)_{j=1}^{|\Omega|} \succeq_0 x \Rightarrow f \succeq_s x.$$

We have established P3.

Now we prove the sufficiency of P2 and P3. Let  $|\Omega| = N$ . For each  $j = 1, \dots, N-1$ , let

$$\phi_j^0 = \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_N)} \quad \text{and} \quad \phi_j^s = \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_N)} \quad \forall s \in \mathcal{S}.$$

Let  $\phi^0 = (\phi_j^0)_{j=1}^{N-1} \in \mathbb{R}^{N-1}$  and  $\phi^s = (\phi_j^s)_{j=1}^{N-1} \in \mathbb{R}^{N-1}$ .

Claim that P2 implies that  $\phi^0$  lies in the relative interior of the convex hull of  $\{\phi^s : s \in \mathcal{S}\}$ . If not, then by a separating hyperplane theorem, there exists

$r \in \mathbb{R}^{N-1}$  and  $k \in \mathbb{R}$  such that (i)  $\phi^s \cdot r \geq k$  for all  $s \in \mathcal{S}$ , (ii)  $\phi^s \cdot r > k$  for some  $s \in \mathcal{S}$ , and (iii)  $\phi^0 \cdot r \leq k$ .

For any  $s \in \mathcal{S}$ , take  $(k_j^s)_{j=1}^{N-1} \in \mathbb{R}^{N-1}$  such that (i)  $\phi_j^s r_j \geq k_j^s$  for all  $j = 1, \dots, N-1$ , and (ii)  $\sum_{j=1}^{N-1} k_j^s = k$ .

Take  $(k_j^0)_{j=1}^{N-1} \in \mathbb{R}^{N-1}$  such that (i)  $\phi_j^0 r_j \leq k_j^0$  for all  $j = 1, \dots, N-1$ , and (ii)  $\sum_{j=1}^{N-1} k_j^0 = k$ .

Fix  $x \in X$ . Consider any  $s \in \mathcal{S}$ . Let  $f$  be an act such that  $u(x) - u(f(\omega_j)) = r_j$  for all  $j = 1, \dots, N-1$ . Let  $y_j^s \in X$  satisfying  $u(y_j^s) - u(x) = k_j^s$  for all  $j = 1, \dots, N-1$ . We have

$$\begin{aligned} & \forall j \in \{1, \dots, N-1\}, \phi_j^s r_j \geq k_j^s \\ \Leftrightarrow & \forall j \in \{1, \dots, N-1\}, [u(x) - u(f(\omega_j))] \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_N)} \geq u(y_j^s) - u(x) \\ \Rightarrow & \forall j \in \{1, \dots, N-1\}, \begin{cases} x \succeq_s [f(\omega_j), \omega_j | y_j^s, \omega_N | x]; \\ x \bar{\succeq}_s [f(\omega_j), \omega_j | y_j^s, \omega_N | x] \text{ if } f(\omega_j) \succeq_s x. \end{cases} \end{aligned}$$

Let  $y_j^0 \in X$  satisfying  $u(y_j^0) - u(x) = k_j^0$  for all  $j = 1, \dots, N-1$ . We have

$$\begin{aligned} & \forall j \in \{1, \dots, N-1\}, \phi_j^0 r_j \leq k_j^0 \\ \Leftrightarrow & \forall j \in \{1, \dots, N-1\}, [u(x) - u(f(\omega_j))] \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_N)} \leq u(y_j^0) - u(x) \\ \Rightarrow & \forall j \in \{1, \dots, N-1\}, \begin{cases} [f(\omega_j), \omega_j | y_j^0, \omega_N | x] \bar{\succeq}_0 x; \\ [f(\omega_j), \omega_j | y_j^0, \omega_N | x] \succeq_0 x \text{ if } x \succeq_0 f(\omega_j). \end{cases} \end{aligned}$$

We have found a violation of P2. Therefore, we must have  $\phi^0$  in the relative interior of the convex hull of  $\{\phi^s : s \in \mathcal{S}\}$ .

Let  $(\lambda_s)_{s \in \mathcal{S}}$  be such that (i)  $\lambda_s \in (0, 1)$  for all  $s \in \mathcal{S}$ , (ii)  $\sum_{s \in \mathcal{S}} \lambda_s = 1$ , and (iii)  $\phi^0 = \sum_{s \in \mathcal{S}} \lambda_s \phi^s$ . Let  $\ell(s|\omega_N) = \lambda_s$  and  $\ell(s|\omega_j) = \lambda_s \phi_j^s / \phi_j^0$  for each  $s \in \mathcal{S}$  and each  $j \in \{1, \dots, N-1\}$ . Observe that  $\ell(s|\omega) > 0$  and  $\sum_{s \in \mathcal{S}} \ell(s|\omega) = 1$  for all  $s \in \mathcal{S}$  and  $\omega \in \Omega$ .

Now we verify that  $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$ . By Proposition 3.1, it is equivalent to show  $f_x^{\ell, s} \succeq_0 x \Leftrightarrow f \succeq_s x$ . Take any act  $f$  and constant act  $x$ . Let  $g$  be such that

$$g(\omega) := \frac{\ell(s|\omega)}{\ell(s|\omega_N)} f(\omega) + \left[ 1 - \frac{\ell(s|\omega)}{\ell(s|\omega_N)} \right] x \quad \forall \omega \in \Omega.$$

For every  $j < N$ , pick  $y_j$  such that

$$u(x) = \frac{u(y_j) + \frac{1}{\phi_j^0} u(f(\omega_j))}{1 + \frac{1}{\phi_j^0}}.$$

Then we have

$$\frac{u(f(\omega_j)) - u(x)}{u(x) - u(y_j)} = \phi_0^j$$

and

$$\frac{\frac{\ell(s|\omega_j)}{\ell(s|\omega_N)}u(f(\omega_j)) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_N)}\right]u(x) - u(x)}{u(x) - u(y_j)} = \frac{\ell(s|\omega_j)}{\ell(s|\omega_N)} \times \phi_0^j = \phi_j^s.$$

Thus,

$$u(f(\omega_j)) - u(x) = [u(x) - u(y_j)] \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_N)}$$

and

$$u(g(\omega_j)) - u(x) = [u(x) - u(y_j)] \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_N)}.$$

Therefore, we have

$$\left\{ \begin{array}{l} [y_j, \omega_j | g(\omega_j), \omega_i | x] \bar{\succeq}_s x; \\ [y_j, \omega_j | g(\omega_j), \omega_i | x] \succeq_s x \text{ if } x \succeq_s y_j \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} x \succeq_0 [y_j, \omega_j | f(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_0 [y_j, \omega_j | f(\omega_j), \omega_i | x] \text{ if } y_j \succeq_0 x \end{array} \right. .$$

We also have

$$\left\{ \begin{array}{l} x \succeq_s [y_j, \omega_j | g(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_s [y_j, \omega_j | g(\omega_j), \omega_i | x] \text{ if } y_j \succeq_s x \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} [y_j, \omega_j | f(\omega_j), \omega_i | x] \bar{\succeq}_0 x; \\ [y_j, \omega_j | f(\omega_j), \omega_i | x] \succeq_0 x \text{ if } x \succeq_0 y_j \end{array} \right. .$$

By P3, we have  $g \succeq_0 x$  if and only if  $f \succeq_s x$ . Since  $\ell(s|\omega_N)g + (1 - \ell(s|\omega_N))x = f_x^{\ell, s}$ , we obtain  $f_x^{\ell, s} \succeq_0 x \Leftrightarrow f \succeq_s x$ . This completes the proof.

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