

# **A General Framework for Growth Models with Non-Competitive Labor and Product Markets and Disequilibrium Unemployment**

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*Barcelona Economics Working Paper Series*

*Working Paper nº 369*

# A General Framework for Growth Models with Non Competitive Labor and Product Markets and Disequilibrium Unemployment

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December 19, 2008

## Abstract

In this paper we derive the general framework for growth models with non competitive labor and output markets and disequilibrium unemployment. For the three standard ways of generating savings, the framework makes clear how capital growth depends on employment and employment on the stock of capital and how both variables depend on the real wage and the markup. Then we analyze the long run relationship between growth and unemployment for some standard wage setting rules: constant real wages, rules that imply a constant unemployment rate and rules with inertia on wages and labor incomes, paying special attention to the neoclassical production function with infinite horizon consumers, the Ramsey model (the canonical model of growth) with unemployment.

*Keywords:* Disequilibrium Unemployment, Growth, Wage Setting.

*JEL number:* E24, O41.

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\*Raurich is grateful to the Spanish Ministry of Education for financial support through grant SEJ2006-05441.

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# 1. Introduction

If you ask to an economist what is the long run relationship between growth and unemployment the most probably answer that you will get is that there is no relationship, meaning that the variables that determine the rate of unemployment are different that the ones that determine the rate of growth. On the contrary, for the short run you can frequently find economic institutions recommending wage moderation since, according to them, a higher wage is going to increase unemployment and decrease capital accumulation and thus lower growth and employment in the future. The purpose of this paper is to set out a general frame of a model in which there is wage setting and disequilibrium unemployment and into which is inserted the three usual ways of generating a savings function: a constant savings ratio (ECSR), an infinite horizon (IH) and overlapping generations (OLG); and then use this framework to analyze the effect of wages on growth and employment and the long run behavior of these variables for different wage setting rules and production functions.

As Aricó (2003) notes in his survey about growth and unemployment, there are not many papers with models that combine both topics. Models that deal with growth and unemployment can be divided according to how unemployment is generated.

On the one hand, we have the disequilibrium approach, the non-Walrasian labor market structure, where there is unemployment because the wage set by somebody implies excess supply of labor in a labor market without frictions. This somebody may be unions, and then we have the union monopoly model (Mc Donald and Solow (1981)), or firms, and then we have the efficiency wage model (Solow (1979)). On the other hand, there is the equilibrium unemployment approach where there are frictions in the labor market due to matching problems and the unemployment rate is the one that equilibrates flows into and out of the labor market. We follow the disequilibrium approach because it is the natural framework for highlighting the effect of wage setting and markups on both unemployment and capital accumulation. Moreover, this approach facilitates the comparison with the standard growth models with full employment that one can find in an advanced textbook of economic growth (Sala-i-Martin (2000), Barro and Sala-i-Martin (2003), Romer (2007)).

In the equilibrium unemployment literature, Pissarides (1990) set out the first model with growth and frictional unemployment where growth is an exogenous variable. Pissarides postulates a positive effect of total factor productivity (TFP) growth (technological progress) on employment via the capitalization effect. Aghion and Howitt (1998 pp. 127) explain the capitalization effect as follows “an increase in growth raises the rate at which the returns for creating a plant (or a firm) will grow and hence increases the capitalized value of those returns, thereby encouraging more entry by new plants and therefore more job creation”. Following this idea several authors (Aghion and Howitt (1994 and 1998), Mortensen and Pissarides (1988), Mortensen (2005)) endogenize the TFP growth in the matching model by introducing Aghion and Howitt’s model of creative destruction that also generates a negative effect on employment due to the creative destruction effect of growth. Bean and Pissarides (1993) with an overlapping generations model (OLG) without either the capitalization or the creative destruction effect analyze the case where employment affects labor income and, hence, growth and Eriksson (1997)

does the same with an infinite horizon model.

The existence of growth due to capital accumulation implies that growth depends on employment and employment depends on growth, this complicates the basic static models used in the disequilibrium approach because of the effect of wages on capital and future employment. Van der Ploeg (1987) and Sorolla (1995) analyze the solution of the program of a union that takes into account this dynamic effect. Models with growth and disequilibrium unemployment use, then, simplifying assumptions on production functions and wage setting rules that make the models tractable.

The first contribution of the paper to this literature is to derive in a transparent way the general set up that illustrates the relationship between growth and unemployment and its dependence on wages and mark ups for the three standard ways of deriving savings. Secondly we analyze the relationship between growth and unemployment for the AK production function and the wage setting rules that appear in some papers, extending the analysis to the neoclassical production function. We pay special attention to the infinite horizon consumers case, the Ramsey model (the canonical model of growth) with unemployment, that is not studied in these papers. The analysis makes clear in which cases there is no relationship between growth and unemployment in the long run.

The general framework of this paper includes, as special cases, the set ups that have been used in the following papers: Raurich, Sala and Sorolla (2006) has an IH model with wage inertia due to past labor income and an AK production function, where different types of fiscal policy generate multiplicity of equilibria and, then, hysteresis. Raurich and Sorolla (2003a) employs an IH model with a wage setting rule that implies a constant unemployment rate and a positive externality on TFP of the ratio of public to private capital, where the effect of taxes on growth and unemployment is analyzed. Raurich and Sorolla (2003b) makes use of an OLG model with wage inertia and an AK production function, where we show that wage inertia implies that growth has a permanent effect on long run unemployment and the effect of different taxes on growth and unemployment is analyzed. Sorolla (1995) sets out a finite horizon model with a Leontieff production function, where the effect of a more proworker government on employment is analyzed. Sorolla (2000) employs an OLG model with wage inertia and neoclassical production function, where the conditions for convergence to an steady state and the effect of a more proworker government on long run employment and growth are analyzed. Daveri and Tabellini (2000) use an OLG model in order to analyze the effect of taxes on both variables with a particular wage setting rule that eliminates the effect of growth on employment and the same do Domenech and García (2008), with a IH model, in a more complex framework with a detailed structure of taxes and social benefits. They use a general production function that, for the AK case, we show, that employment does not have effect on growth. In this case, in order to have a positive effect of employment on growth, they introduce a positive externality of capital per capita, an addition that was criticized by Sala-i-Martin (2000), section 2.8.

Papers with wage setting rules more complex than the ones presented in this paper are: Daveri and Maffezzoli (2000) where they also analyze the effect of taxes on long run growth and unemployment using an infinite horizon model with a CES production function where

the accumulation of knowledge is a by-product of being employed, which implies that growth depends on employment; Kaas and von Thadden (2003) that, using an OLG model with a CES production function, study how the elasticity of substitution between labor and capital affects the dynamics of growth and unemployment; and Galí (1996b) that presents an infinite horizon model with imperfect competition in the product market and labor choice. Models of growth with disequilibrium in the labor market have also been used in the real business cycle literature to analyze the effect of real shocks on unemployment fluctuations (Galí (1995), Benassy (1997) and Maffezzoli (2001)).

The rest of the paper is organized as follows. Section 2 presents the firm behavior. Section 3 analyzes consumer behavior and the fundamental equations for the IH model when there is wage setting and unemployment. Section 4 presents, the special case where growth does not depend on wages and employment. Section 5 describes the constant real wage case. Section 6 the special wage setting rules that imply a constant employment rate and, hence, that employment does not depend on growth. Section 7 adds inertia on wages and labor income to the rules of Section 6 and Section 8 concludes with a summary of the main results. Appendix 1 presents the fundamental equations for the ECSR and OLG models and appendix 2 the special case where growth does not depend on wages and employment for the ECSR and OLG models.

## 2. Firms

### 2.1. Price Taking Firms

We assume the neoclassical production function:

$$Y_t = F(K_t, L_t),$$

with constant returns to scale with respect to  $K$  and  $L$ ,  $F_K > 0$ ,  $F_L > 0$ ,  $F_{KK} < 0$ ,  $F_{LL} < 0$  and the Inada conditions:  $\lim_{K \rightarrow 0} F_K = \infty$ ,  $\lim_{K \rightarrow \infty} F_K = 0$ ,  $\lim_{L \rightarrow 0} F_L = \infty$ ,  $\lim_{L \rightarrow \infty} F_L = 0$ . The production function in terms of output per worker or unit of labor,  $\frac{Y}{L_t} \equiv \hat{y}_t$ , and capital per worker or the capital labor ratio,  $\frac{K}{L_t} \equiv \hat{k}_t$ , that is, in intensive form, is:

$$\hat{y}_t = f(\hat{k}_t),$$

where  $f' > 0$  and  $f'' < 0$ . The firm is price taker and maximizes profits:

$$F(K_t, L_t) - \omega_t L_t - (r_t + \delta)K_t,$$

where  $\omega_t$  is the real wage,  $r_t$  the interest rate and  $\delta$  the constant depreciation rate,  $0 \leq \delta \leq 1$ . As usual, the conditions for the efficient use of factors and zero profits are:

$$F_L(K_t, L_t) = \omega_t,$$

and

$$F_K(K_t, L_t) = r_t + \delta,$$

expressions that we can rewrite in intensive form as:

$$f(\hat{k}_t) - k_t f'(\hat{k}_t) = \omega_t, \quad (2.1)$$

and

$$f'(\hat{k}_t) = r_t + \delta. \quad (2.2)$$

Conditions (2.1) and (2.2) give the combinations between  $\omega_t$ ,  $r_t$  and  $\hat{k}_t$  such that factors are efficiently used and profits are zero and, then, capital and labor demand are any quantity that satisfies  $\hat{k}_t$ <sup>1</sup>.

We assume that labor supply is equal to population,  $N_t$ , that grows at a constant rate  $n$ . In models with unemployment and population growth, we also need to rewrite the production function in terms of output per capita  $\frac{Y_t}{N_t} \equiv y_t$ , capital per capita  $\frac{K_t}{N_t} \equiv k_t$  and the employment rate  $\frac{L_t}{N_t} \equiv l_t$ . Because of the constant returns to scale assumption, we rewrite the production function as:

$$\frac{Y_t}{N_t} = F\left(\frac{K_t}{N_t}, \frac{L_t}{N_t}\right),$$

that is:

$$y_t = F(k_t, l_t).$$

We also rewrite the conditions for the efficient use of factors and zero profits as:

$$F_l(k_t, l_t) = \omega_t, \quad (2.3)$$

and

$$F_k(k_t, l_t) = r_t + \delta. \quad (2.4)$$

and we rewrite the zero profits condition as:

$$y_t = F(k_t, l_t) = (r_t + \delta)k_t + \omega_t l_t, \quad (2.5)$$

and, it is obvious that:

$$F_K(K_t, L_t) = f'(\hat{k}_t) = F_k(k_t, l_t). \quad (2.6)$$

The main characteristic of models with wage setting is that the real wage  $\omega_t$  is set by somebody<sup>2</sup> and, then, from (2.1) we get the capital labor ratio function:

$$\hat{k}_t = \tilde{k}(\omega_t), \quad (2.7)$$

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<sup>1</sup>See Barro and Sala-i-Martin Section 2.2 for the precise derivation of the conditions and a more extensive discussion on this point.

<sup>2</sup>Thus, we are in economies where labor contracts include indexation of wages.

where  $\tilde{k}' > 0$ . From (2.7) we get the "labor demand" function:

$$L_t^d = \tilde{L}(\omega_t, K_t^d) = \frac{K_t^d}{\tilde{k}(\omega_t)}, \quad (2.8)$$

where  $\tilde{L}_\omega < 0$  and  $\tilde{L}_{K^d} > 0$ .

Also from (2.2) and (2.7) we get the interest rate function (the interest rate-wage frontier):

$$r_t = \tilde{r}(\omega_t) = f'(\tilde{k}(\omega_t)) - \delta \quad (2.9)$$

that gives, for a given wage, the interest rate that implies zero profits. It is easy to check that  $r' < 0$ , which means that, if the wage increases, the interest rate must decrease in order to have zero profits. Of course, if the interest rate is higher than the one given by this function, then profits are negative and then the demands for labor and capital are zero. If it is lower, then profits are positive and capital and labor demand are infinite. In textbook macroeconomic models, with the stock of capital given and a keynesian investment function, it turns out that the relation between the real wage and the real interest rate is positive, this is because an increase in the wage implies less employment and less production and, if we have equilibrium in the goods market (the IS equation), in order for demand to fall, the interest rate must increase.

In models with wage setting in general there is no equilibrium in all markets if the wage is different from the competitive wage. It is only assumed that the interest rate adjusts in order to have equilibrium in one market. This is only possible if the zero profit condition holds, that is, if the interest rate is given by the interest rate function. In this case, as we have seen, the labor demand function is, in fact, only a relationship between capital and labor demand. This property implies that, in models with wage setting, it is the case that one market is not in equilibrium. It is easy to see, that either the capital market is in equilibrium and there is excess supply in the labor market (unemployment) or the labor market is in equilibrium and there is excess supply in the capital market. We will concentrate on the case where there is unemployment and the capital market is in equilibrium, this means that the wage that is set implies that  $K_t^d = K_t^s = K_t$  and  $L_t = L_t^d < N_t$ .<sup>3</sup> Using (2.8), we get the employment function,  $L_t$ , that is:

$$L_t = L_t^d = \tilde{L}(\omega_t, K_t) = \frac{K_t}{\tilde{k}(\omega_t)}. \quad (2.10)$$

Using (2.10) we get the employment rate function:

$$l_t = \tilde{L}(\omega_t, k_t) = \frac{k_t}{\tilde{k}(\omega_t)}. \quad (2.11)$$

Writing the production function per capita:  $y_t = F(k_t, l_t)$  and taking into account the employment rate function (2.11) we get:

$$y_t = F\left(k_t, \frac{k_t}{\tilde{k}(\omega_t)}\right).$$

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<sup>3</sup>That means that in a concrete model we have to check if this is the case for a given  $\omega_t$ .

or the output per unit of capital function<sup>4</sup>:

$$\frac{y_t}{k_t} = F\left(1, \frac{1}{\tilde{k}(\omega_t)}\right) = \tilde{f}\left(\frac{1}{\tilde{k}(\omega_t)}\right).$$

Note, from the last function, that if the wage increases then output per unit of capital decreases, due to the increase in the capital labor ratio, that is,  $\frac{d\tilde{f}}{d\omega} < 0$ . As we will see later, this function plays an important role in some growth models.

Looking at the employment function, (2.10), we see that an increase in the wage implies, directly, less employment, we call this direct effect of the wage on employment channel 1. We see also that an increase in capital increases also employment. We interpret this fact saying that growth affects employment (or employment depends on growth) in the sense that a higher rate of growth of capital implies more capital and, hence, more employment. In a growth model with wage setting, the complication arises from the fact that, with equilibrium in the capital market, there are two indirect channels such that the wage may also affect employment. On the one hand, by the interest rate function (2.9), a change in the wage may change the interest rate and this may change consumption and the saving rate, and then, savings, investment and the supply of capital, if consumption depends on the interest rate. We call this indirect effect of the wage on employment channel 2. On the other hand, a change in the wage affects directly and indirectly, changing employment, aggregate labor income, if it affects the interest rate then also capital income changes. Adding both effects, changes in the wage modify aggregate income and, then, this may change savings, investment and the supply of capital. We call this indirect effect of the wage on employment channel 3. When one of these two channels is active we say, in a general sense, that growth depends on employment (or employment affects growth), meaning that wages affect the accumulation of capital. In general, in a growth model, we have that employment depends on growth and growth depends on employment but, depending on the concrete assumptions made about production functions, households and wage setting, in some cases, these effects and channels are not active.

## 2.2. Extension 1: Monopolistic Competition in the goods market

So far we have seen how wages affect employment when firms are price takers, that is, when there is perfect competition in the goods market. It is interesting to present the extension for price setting firms in order to see that also profits, or markups, affect employment. We introduce the monopolistic competition set up in a growth model (Galí (1996)) having  $S$  sectors with one firm  $i$  per sector that produces product  $Y_{i,t}$  with the production function

$$Y_{i,t} = F(K_{i,t}, L_{i,t}),$$

with the usual properties. The stock of capital is a composite of all produced goods and the firm in sector  $i$  maximizes the wealth of its shareholders subject to the demand function (see Galí (1996) for the complete set up). Assuming that the price elasticity of the demand function

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<sup>4</sup>Bentolila and Saint Paul (2003) use the inverse of output per unit of capital, that is capital per unit of output  $\frac{k}{y} = \frac{k}{F(k,l)} = \frac{1}{f(\frac{l}{k})}$  in order to compute the labor share.

is constant<sup>5</sup> and equal to  $\xi$ , we define  $m \equiv \frac{1}{(1-\frac{1}{\xi})} \geq 1$ , as the monopoly degree or the markup, and in a symmetric equilibrium we get the following two first order conditions for the capital labor ratio for firm  $i$  (see again Galí (1996)):

$$f(\hat{k}_t) - k f'(\hat{k}_t) = m\omega_t, \quad (2.12)$$

and

$$f'(\hat{k}_t) = m(r_t + \delta). \quad (2.13)$$

Thus, we can proceed in the same way that we did in the competitive case, adding the markup. That is from (2.12) we get the capital labor ratio:

$$\hat{k}_t = \tilde{k}(m\omega_t), \quad (2.14)$$

where  $\tilde{k}' > 0$ . and from the last equation we get the "labor demand" function for sector  $i$ :

$$L_{i,t}^d = \tilde{L}(m\omega_t, K_{i,t}^d) = \frac{K_{i,t}^d}{\tilde{k}(m\omega_t)},$$

where  $\tilde{L}_{m\omega} < 0$  and  $\tilde{L}_{K^d} > 0$ . Aggregating for all sectors we have the total "labor demand" function:

$$L_t^d = \tilde{L}(m\omega_t, K_t^d) = \frac{K_t^d}{\tilde{k}(m\omega_t)},$$

where  $K_t^d$  is the aggregate (composite) stock of capital. Also from (2.13) and (2.14) we get the interest rate function (the interest rate-wage frontier):

$$r_t = \tilde{r}(m\omega_t) = \frac{1}{m} f'(\tilde{k}(m\omega_t)) - \delta,$$

where  $\tilde{r}' < 0$ .

The interesting result of this extension is that an increase in market power (in the markup), has the same potential effects on labor demand as an increase in the wage: it decreases labor demand directly (channel 1), it decreases the interest rate (channel 2) and it affects aggregate income by decreasing labor demand and the interest rate and by changing profits (channel 3).

### 2.3. Extension 2: Exogenous Technological Progress

It is also interesting to examine this extension in order to understand the relationship between wages and productivity growth. Now we use the production function, with labor-augmenting technological progress:

$$Y_t = F(K_t, A_t L_t),$$

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<sup>5</sup>The complication of the monopolistic competition set up in a growth model arises from the fact that both consumers and firms demand product  $i$  due to the demand of capital of each firm. In principle the price elasticity of both types of demand may be different, this is the point of Galí's paper, and this opens the door for multiplicity of equilibria. The assumption that  $\xi$  is constant is the  $\sigma = \mu$  case in Galí's paper.

with constant returns to scale and  $F_K > 0$ ,  $F_{AL} > 0$ ,  $F_{KK} < 0$ ,  $F_{ALAL} < 0$  and we assume that  $\frac{\dot{A}}{A} = x > 0$ , that means that there is technological progress. The production function in terms of output per efficiency unit of labor,  $\frac{Y}{AL_t} \equiv \hat{y}_{e,t}$ , and capital per efficient unit of labor or the capital efficient labor ratio,  $\frac{K}{AL_t} \equiv \hat{k}_{e,t}$ , that is, in intensive form, is:

$$\hat{y}_{e,t} = f(\hat{k}_{e,t}).$$

The firm is price taker and maximizes profits:

$$F(K_t, A_t L_t) - \omega_t L_t - (r_t + \delta)K_t,$$

where  $\omega_t$  is the real wage,  $r_t$  the interest rate and  $\delta$  the constant depreciation rate,  $0 \leq \delta \leq 1$ . As usual, the conditions for profit maximization and zero profits are:

$$F_L(K_t, A_t L_t) = \frac{\omega_t}{A_t},$$

and

$$F_K(K_t, A_t L_t) = r_t + \delta,$$

expressions that we can rewrite in intensive form as:

$$f(\hat{k}_{e,t}) - \hat{k}_{e,t} f'(\hat{k}_{e,t}) = \frac{\omega_t}{A_t}, \quad (2.15)$$

and

$$f'(\hat{k}_{e,t}) = r_t + \delta. \quad (2.16)$$

It is useful to rewrite the production function in terms of output per efficiency person  $\frac{Y_t}{A_t N_t} \equiv y_{e,t}$ , capital per efficiency person  $\frac{K_t}{A_t N_t} \equiv k_{e,t}$  and the employment rate  $l_t = \frac{L_t}{N_t}$ . Because of the constant returns to scale assumption, we can rewrite the production function as:

$$\frac{Y_t}{A_t N_t} = F\left(\frac{K_t}{A_t N_t}, A_t \frac{L_t}{A_t N_t}\right),$$

that is:

$$y_{e,t} = F(k_{e,t}, l_t).$$

In this case the first order conditions for profit maximization and zero profits are:

$$F_l(k_{e,t}, l_t) = \frac{\omega_t}{A_t}, \quad (2.17)$$

and

$$F_{k_e}(k_{e,t}, l_t) = r_t + \delta. \quad (2.18)$$

We can rewrite the zero profits condition as:

$$y_t = (r_t + \delta)k_{e,t} + \frac{\omega_t}{A_t}l_t, \quad (2.19)$$

and, it is obvious that:

$$F_K(K_t, A_t L_t) = f'(\hat{k}_{e,t}) = F_{k_e}(k_{e,t}, l_t). \quad (2.20)$$

Now we can proceed in the same way that we did without technical progress. From (2.15) we get the capital effective labor ratio is

$$\hat{k}_{e,t} = \tilde{k}_e \left( \frac{\omega_t}{A_t} \right), \quad (2.21)$$

where  $\tilde{k}'_e > 0$ . From (2.21) we get the "labor demand" function:

$$L_t^d = \tilde{L}(\omega_t, K_t^d) = \frac{K_t^d}{A_t \tilde{k}_e \left( \frac{\omega_t}{A_t} \right)},$$

where  $\tilde{L}_\omega < 0$  and  $\tilde{L}_{K^d} > 0$ . Assuming equilibrium in the capital market, we get the employment rate function:

$$l_t = \tilde{L}(\omega_t, k_{e,t}) = \frac{k_{e,t}}{\tilde{k}_e \left( \frac{\omega_t}{A_t} \right)}. \quad (2.22)$$

Also from (2.21) and (2.16) we get the interest rate function (the interest rate-wage frontier):

$$r_t = \tilde{r} \left( \frac{\omega_t}{A_t} \right) = f' \left( \tilde{k}_e \left( \frac{\omega_t}{A_t} \right) \right) - \delta.$$

Now we can discuss the popular idea that in order to have a constant employment rate we have to link the real wage growth with productivity growth. This result comes from the use of the Cobb-Douglas production function  $Y_t = A_t K^\alpha (L_t)^{1-\alpha}$  with a given level of capital,  $\bar{K}$ <sup>6</sup>. In this case employment is given by:

$$L_t = \left[ \frac{1-\alpha}{\frac{\omega_t}{A_t}} \right]^{\frac{1}{\alpha}} \bar{K}.$$

Assuming the stock of capital given, we have:  $\frac{\dot{L}_t}{L_t} = -\frac{1}{\alpha} \left( \frac{\dot{\omega}_t}{\omega_t} - \frac{\dot{A}_t}{A_t} \right)$  so that, in order to have employment constant, we need  $\frac{\omega_t}{A_t}$  constant, that is the rate of growth of wages must be equal to the productivity rate ( $\frac{\dot{\omega}_t}{\omega_t} = x$ ). The popular idea comes from this result.

First remark: if there is population growth then the employment rate is:

$$l_t = \frac{1}{N_t} \left[ \frac{1-\alpha}{\frac{\omega_t}{A_t}} \right]^{\frac{1}{\alpha}} \bar{K}$$

and then, taking logs and differentiating the employment function with respect to time, we get

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<sup>6</sup>Or  $Y_t = A_t (L_t)^{1-\alpha}$ , as it is usual in an static model.

$\frac{\dot{l}_t}{l_t} = -\frac{1}{\alpha}\left(\frac{\dot{\omega}_t}{\omega_t} - \frac{\dot{A}_t}{A_t}\right) - n$ , and in order to have a constant employment rate we need  $\frac{\dot{\omega}_t}{\omega_t} = x - \alpha n$ , that is the rate of growth of wages must be less than the productivity growth.

Second remark: Note that if capital changes, taking logs and differentiating the employment function with respect to time, we have that the rate of growth of employment is given by:

$$\frac{\dot{l}_t}{l_t} = \frac{\dot{K}_t}{K_t} + \frac{1}{\alpha}\left(\frac{\dot{A}_t}{A_t} - \frac{\dot{\omega}_t}{\omega_t}\right) - n$$

that means than in order to keep employment constant  $\frac{\dot{\omega}_t}{\omega_t} = x + \alpha\left(\frac{\dot{K}_t}{K_t} - n\right) = x - \alpha n + \alpha\frac{\dot{K}_t}{K_t}$ .

Third remark: if the production function is  $Y_t = K_t^\alpha(A_t L_t)^{1-\alpha}$  then, in order to keep employment constant, we need:  $\frac{\dot{\omega}_t}{\omega_t} = x + \alpha\left(\frac{\dot{K}_t}{K_t} - x - n\right)$ . If it turns out that in the long run capital grows at the rate  $n + x$ , which is the case in the full employment models, then it must be that  $\frac{\dot{\omega}_t}{\omega_t} = x$ . But now we are in models with unemployment and in general, as we will see, the rate of growth of capital depends on the wage.

### 3. Households and equilibrium

The key equation for a model with growth and unemployment is the capital accumulation equation but capital accumulation depends on how savings are generated. There are three standard models of generating savings: the exogenous saving rate, the infinite horizon and the overlapping generations. It turns out that the unemployment version of the three models generates the same type of results. For this reason we present in the main text the most popular model: the IH and the other two models are presented and discussed in appendix 1.

In the IH model we have a representative family that chooses consumption per capita,  $c_t$ , in order to maximize:

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} \left[ \frac{c_t^{1-\theta} - 1}{1-\theta} \right]$$

subject to:

$$\dot{k}_t^s = r_t k_t + \omega_t l_t - c_t - n k_t.$$

and  $k_0 = \bar{k}_0 > 0$ .

Note that the revenues of this family accrue from total labor income because we assume the family is so big that it considers all workers, employed and unemployed, Daveri and Maffezzoli (2000), Eriksson (1997) and Raurich and Sorolla (2006) follow also the big family assumption. If unemployed workers get an unemployment benefit,  $b_t$ , then the budget constraint changes to:  $\dot{k}_t^s = r_t k_t + (1 - \tau)\omega_t l_t + b_t(1 - l_t) - c_t - n k_t$ , where  $\tau$  is the tax rate on employed workers. As long as the government budget constraint for covering the unemployment benefit is balanced, this change does not affect the equilibrium conditions. If instead of a big family we have heterogeneous agents the solution does not change as long as we assume complete competitive insurance markets for unemployment or that the union pursues a redistributive goal, acting as a substitute for the insurance markets (Maffezzoli (2001) and Benassy (1997)).

As usual, the first order conditions for optimal consumption are:

$$\frac{\dot{c}}{c} = \frac{1}{\theta} (r_t - \rho),$$

$$\dot{k}_t^s = r_t k_t + \omega_t l_t - c_t - n k_t,$$

$$\lim_{t \rightarrow \infty} \lambda_t k_t^s = 0$$

Note that in this model channel two is active because the interest rate affects the rate of growth of consumption and channel three is also active because a change in the wage affects aggregate income,  $r_t k_t + \omega_t l_t$ , changing  $\omega$  and, indirectly,  $l$  and  $r$ . Assuming equilibrium in the capital market and using the condition (2.4), and the zero profits condition, (2.5), we get:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta} (F_K(k_t, l_t) - (\rho + \delta)), \quad (3.1)$$

$$\dot{k}_t = f(k_t, l_t) - c_t - (n + \delta)k_t. \quad (3.2)$$

Note that the difference between these equations and the equations of the standard solution with equilibrium in the labor market, that are:

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &= \frac{1}{\theta} (f'(k_t) - (\rho + \delta)), \\ \dot{k}_t &= f(k_t) - c_t - (n + \delta)k_t; \end{aligned}$$

is in the production function<sup>7</sup>. Note also that employment affects growth meaning that a higher level of  $l_t$  increases the growth of capital.

Dividing by  $k$  the second equation, substituting in  $f'(\hat{k}_t) = F_K(k_t, l_t)$  and using the capital labor ratio function (2.7), we get the fundamental equations of the infinite horizon model with wage setting:

$$\gamma_{c,IH} = \frac{\dot{c}_t}{c_t} = \frac{1}{\theta} \left( f'(\tilde{k}(\omega_t)) - (\rho + \delta) \right), \quad (3.3)$$

$$\gamma_{k,IH} = \frac{\dot{k}_t}{k_t} = \tilde{f}\left(\frac{1}{\tilde{k}(\omega_t)}\right) - \frac{c_t}{k_t} - (n + \delta). \quad (3.4)$$

Note that, from equation (3.3), an increase in the wage increases the capital labor ratio (decreases the interest rate) and then reduces the rate of growth of the optimal consumption per capita, that is channel 2 is active in this model. On the other hand, from equation (3.4), an increase in the wage reduces the rate of growth of capital per capita, given  $c_t$ , via channel 3. These two channels make the effect of an increase of the wage on the accumulation of capital ambiguous because of the decrease in consumption via channel 3.

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<sup>7</sup>Again, when the labor market is in equilibrium,  $k_t = \hat{k}_t$ .

Note that in principle, we can not solve these equations, this means that we can not compute  $k_t$  and hence to obtain the employment rate  $l_t$  from this equation, but we can compute the evolution of the employment rate. Taking logs of the employment rate function (2.11) and differentiating with respect to time we get:

$$\gamma_l = \frac{\dot{l}_t}{l_t} = \frac{\dot{k}_t}{k_t} - \frac{(\dot{\tilde{k}}(\omega_t))}{\tilde{k}(\omega_t)} = \frac{\dot{k}_t}{k_t} - \frac{d\tilde{k}}{d\omega_t} \frac{\omega_t \dot{\omega}_t}{\tilde{k} \omega_t} = \gamma_k - \frac{d\tilde{k}}{d\omega_t} \frac{\omega_t}{\tilde{k}} \gamma_\omega. \quad (3.5)$$

that is a differential equation in terms of the employment rate and the wage. For closing the model given by the three differential equations (3.3), (3.4) and (3.5) in terms of consumption, capital, employment and the wage we need to specify the wage setting equation.

With monopolistic competition in the output market the fundamental equations become (Galí (1996), section 3):

$$\gamma_{c,IHMP} = \frac{\dot{c}_t}{c_t} = \frac{1}{\theta} \left( f'(\tilde{k}(m\omega_t)) - (\rho + \delta) \right), \quad (3.6)$$

$$\gamma_{k,IHMP} = \frac{\dot{k}_t}{k_t} = \tilde{f}\left(\frac{1}{\tilde{k}(m\omega_t)}\right) - \frac{c_t}{k_t} - (n + \delta). \quad (3.7)$$

With exogenous technological progress, where now  $c_{e,t} = \frac{C_t}{A_t N_t}$  (and  $k_{e,t} = \frac{K_t}{A_t N_t}$ ) these fundamental equations become:

$$\gamma_{c_e,IHTP} = \frac{\dot{c}_{e,t}}{c_{e,t}} = \frac{1}{\theta} \left( f'(\tilde{k}\left(\frac{\omega_t}{A_t}\right)) - (\rho + \delta + x) \right), \quad (3.8)$$

$$\gamma_{k_e,IHTP} = \frac{\dot{k}_{e,t}}{k_{e,t}} = \tilde{f}\left(\frac{1}{\tilde{k}\left(\frac{\omega_t}{A_t}\right)}\right) - \frac{c_{e,t}}{k_{e,t}} - (n + \delta + x). \quad (3.9)$$

Finally, from the employment rate function (2.22) we get

$$\gamma_l = \frac{\dot{l}_t}{l_t} = \frac{\dot{k}_{e,t}}{k_{e,t}} - \frac{(\dot{\tilde{k}}\left(\frac{\omega_t}{A_t}\right))}{\tilde{k}\left(\frac{\omega_t}{A_t}\right)} = \frac{\dot{k}_{e,t}}{k_{e,t}} - \frac{d\tilde{k}}{d\left(\frac{\omega_t}{A_t}\right)} \frac{\frac{\omega_t}{A_t} \dot{\omega}_t}{\tilde{k} \left(\frac{\omega_t}{A_t}\right)} = \gamma_{k_e} - \frac{d\tilde{k}}{d\left(\frac{\omega_t}{A_t}\right)} \frac{\frac{\omega_t}{A_t}}{\tilde{k}} (\gamma_\omega - x). \quad (3.10)$$

#### 4. Growth does not depend on employment: the AK production function

From the fundamental equations obtained, in a typical model with wage setting and unemployment we have that the rate of growth of capital depends on the wage via channels 2 and 3, that is, growth depends on employment; but it is also the case that employment rate function depends on capital, that is, employment depends on growth. The equations derived make transparent that the growth and unemployment rates depend on the wage and markups and, in order to analyze the relationship between growth and unemployment, we need to specify the wage equation. As we have seen, for the IH model, this implies a system of three differential equations plus the wage equation that seems difficult to solve. However, there are special cases

considered in the literature where one of these routes is not active, in these cases the dynamics are less complex. The first special case is when growth does not depend on employment. This happens when there is an  $AK$  production function derived, for example, from a Cobb-Douglas with capital per unit of labor externalities<sup>8</sup>:  $Y_t = AK_t^\alpha L_t^{1-\alpha} \bar{K}_t^{1-\alpha}$  (Raurich, Sala and Sorolla (2006)<sup>9</sup>). From (2.1), we get:

$$(1 - \alpha)A\hat{k}_t^\alpha \bar{K}_t^{1-\alpha} = \omega_t,$$

and if the externality is the capital labor ratio  $\bar{K}_t = \hat{k}_t$  we get:

$$(1 - \alpha)A\hat{k}_t = \omega_t,$$

or:

$$\hat{k}_t = \frac{\omega_t}{A(1 - \alpha)},$$

and the employment rate function is:

$$l_t = \frac{A(1 - \alpha)k_t}{\omega_t}. \quad (4.1)$$

and, then, employment depends on growth.

In this case the fundamental equations of the IH model (3.3) and (3.4) become:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta} (\alpha A - (\rho + \delta)) = \gamma_{IHAK}^*,$$

$$\frac{\dot{k}_t}{k_t} = A - \frac{c_t}{k_t} - (n + \delta).$$

Because the rate of growth of consumption is constant, one can prove<sup>10</sup> that in this set up the rate of growth of capital is also constant and equal to the consumption growth rate which implies:

$$\gamma_k = \gamma_{IHAK}^* = \frac{1}{\theta} (\alpha A - (\rho + \delta)),$$

which means that growth does not depend on employment. From (4.1) we get:

$$\gamma_l = \gamma_k - \gamma_\omega = \gamma_{IHAK}^* - \gamma_\omega. \quad (4.2)$$

That is, the growth rate of the employment rate depend on capital growth and wage growth, and the wage dynamics affects only the dynamics of employment. For the same reason if the output market is not competitive a change in the markup does not affect growth and the labor demand is given by:

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<sup>8</sup>Or, alternatively, the case where the  $AK$  function is obtained via productive government expenditures  $Y_t = AK_t^\alpha L_t^{1-\alpha} g_t^{1-\alpha}$  (Raurich and Sorolla (2003b)).

<sup>9</sup>In Raurich and Sorolla (2003a) we also have this function but  $A$  depends on average employment and public capital.

<sup>10</sup>See, for example, Sala-i-Martin (2000) section 5.3.

$$l_t = \frac{A(1-\alpha)k_t}{m\omega_t}. \quad (4.3)$$

As shown in appendix 2, the same kind of result holds for the ECSR and OLG models and then, when the production function is  $AK$ , the rate of growth of capital per capita does not depend on the wage and the level of wages only affects employment. Summarizing, when the production function is  $AK$  then growth does not depend on wage setting and growth will affect employment depending on the dynamics of wages, that is, on the wage setting rule.

## 5. A particular wage setting rule: Constant real wages

If the production function is not  $AK$ , then the rate of growth of capital depends on wages and in order to close the model we need to specify the wage setting rule. The simplest rule is to consider an exogenous constant real wage,  $\omega_t = \bar{\omega}$ . This case is interesting because a constant real wage is usually recommended by governments, central banks and other international institutions in order to have "stability", whatever that means.

With this real wage rigidity the rate of growth of capital per capita is constant for any of the ECSR, IH or OLG models. For the IH model this is easy to see because now equation (3.3) becomes:

$$\gamma_{c,IH} = \frac{\dot{c}_t}{c_t} = \frac{1}{\theta} \left( f'(\tilde{k}(\bar{\omega})) - (\rho + \delta) \right),$$

that is constant and then we get this unique fundamental equation:

$$\gamma_{k,IH} = \gamma_{c,IH} = \frac{1}{\theta} \left( f'(\tilde{k}(\bar{\omega})) - (\rho + \delta) \right).$$

Looking at this equation it is easy to see that a negative productivity shock implies a decrease in the rate of growth and for this rule, for any of the ECSR, IH or OLG models one can prove that there exists  $\bar{\omega}^*$  such that  $\gamma_k = \gamma_l = 0$ . If  $\bar{\omega} < \bar{\omega}^*$  then  $\gamma_k = \gamma_l > 0$  until full employment is achieved. If  $\bar{\omega} > \bar{\omega}^*$  the  $\gamma_k = \gamma_l < 0$  and the employment rate decreases to zero.

The proof for the IH model is easy to see looking at the fundamental equation, where  $\bar{\omega}^*$  implies  $f'(\tilde{k}(\bar{\omega}^*)) = (\rho + \delta)$ . Then using equation (3.5) we get  $\gamma_l = \gamma_k$ . The same type of proof applies to the ECSR or the OLG models, using the fundamental equations. With a constant real wage, there is a relationship between the rate of growth and the unemployment rate in the long run: a constant positive rate of growth of capital per capita implies full employment in the long run.

Moreover, it turns out that  $\bar{\omega}^*$  is the equilibrium wage of the full employment model in the long run,  $\omega_{c,lr}$ . This is easy to see because the long run value of capital per capita in the full employment model,  $k_{c,lr}$  is given by  $f'(k_{c,lr}) = (\rho + \delta)$ , and, using the employment function, the competitive wage in the long run implies  $f'(\tilde{k}(\omega_{c,lr})) = (\rho + \delta)$ . This means that if the real wage set is the long run competitive wage, then there is no growth and the employment rate is constant and given by:

$$l^* = \frac{k_o}{\tilde{k}(\omega_{c,lr})}.$$

The fact that the employment rate depends on the initial level of the capital stock opens a way for hysteresis on employment. The story is the following: assume that the wage is the long run competitive wage,  $\bar{\omega} = \omega_{c,lr}$ , then there is no growth and the employment rate is constant. If there is a negative technological shock, the value of the new long run competitive wage,  $\omega'_{c,lr}$ , decreases, but, if the set wage does not change, then we have negative growth and increasing unemployment rates,  $\bar{\omega} > \omega'_{c,lr}$ . When the shock disappears the rate of growth of capital per capita and employment is again zero but the employment rate is  $\frac{k'_o}{\tilde{k}(\omega_{c,lr})}$ , that is lower than the initial employment rate, because the stock of capital has decreased during the negative shock.

For the monopolistic competition case the fundamental equation for the IH model is:

$$\gamma_{k,IH} = \frac{1}{\theta} \left( f'(\tilde{k}(m\bar{\omega})) - (\rho + \delta) \right),$$

which means that the higher the markup the lower the wage that implies no growth.

It is very interesting to consider the case with exogenous technological progress. Assuming  $\frac{\omega_t}{A_t} = \bar{\omega}$ , we get also the same type of result, that is, there exists  $\bar{\omega}^*$  such that  $\gamma_{k_e} = \gamma_l = 0$ . If  $\bar{\omega} < \bar{\omega}^*$  then  $\gamma_{k_e} = \gamma_l > 0$  until full employment is achieved. If  $\bar{\omega} > \bar{\omega}^*$  then  $\gamma_{k_e} = \gamma_l < 0$  and the rate of employment decreases to zero, where  $\bar{\omega}^*$  is the equilibrium wage per efficiency unit of the full employment model in the long run,  $\omega_{c,lr}$ . We get the proof for the IH model using equations (3.8) and (3.10). We say that this case is interesting because with real wages growing at the TFP productivity rate we can have situations,  $\bar{\omega} < \omega_{c,lr}$ , with positive growth and a reduction of the unemployment rate contrary to the popular belief, discussed in section 2.3, that says that in this case the rate of employment will remain constant. On the other hand, when  $\bar{\omega} = \omega_{c,lr}$ , then the rate of growth of capital per capita is positive and equal to the exogenous productivity rate and the rate of employment is constant and given by:

$$l_t = \frac{k_{e,o}}{\tilde{k}_e(\omega_{c,lr})}.$$

When the production function is  $AK$ , it is clear, from (4.2), that for this specific wage setting rule, that the rate of growth of the employment rate is equal to the rate of growth, that is employment depends on growth and for a positive rate of growth, the rate of employment increases until full employment is achieved.

## 6. Some particular wage setting rules: A constant employment rate

As we have said, the fact that, in general, employment depends directly, via channel 1, and indirectly, via channels 2 and 3, on the wage implies that the IH model is a model with three differential equations plus the wage setting equation<sup>11</sup>. But it turns out that for some wage setting rules we can directly derive the employment rate from these rules. We interpret this result as saying that employment does not depend on growth, because in order to determine the employment rate we do not use the employment rate function, that depends on capital. In this case the analysis of the model becomes also more simple because, with a constant

<sup>11</sup>Compared with the IH the ECSR and the OLG models are more simple because they become a system of two differential equations one for unemployment and one for the wage (see appendix 1).

unemployment rate, we close the model using only the fundamental equations. To obtain these cases we need to have, first, that the wage set is a constant markup,  $m_\omega > 1$ , over an alternative reservation wage,  $\varrho_t$ , that is  $\omega_t = m_\omega \varrho_t$ . This is a typical result for static wage setting models but when there is growth to obtain this result is more complicated because the wage set affects the accumulation of capital via channels 2 and 3. In order to avoid this problem in models with unions it is assumed that, for some reason, "the union fails to internalize the dynamic consequences of today's wage setting on capital accumulation (Maffezzoli (2001), p.869)" and, then, on future employment. This may happen if the union does not care about future employment or it is sufficiently small so that, changing wages, it does not affect the accumulation of capital via channels 2 and 3. Van der Ploeg (1987) and Sorolla (1995) present models where these dynamic consequences are taken into account.

In models with unions, because the mark up depends on the elasticity of labor demand in order to have a constant mark up we need a constant elasticity of the labor demand, that is a Cobb-Douglas production function when the product market is competitive. With monopolistic competition and linear production function,  $Y_i = L_i$ , the elasticity of the labor demand depends only on the markup,  $m^{12}$ . In the efficiency wage models where in order to have a constant mark up we need an special effort function<sup>13</sup> but not a Cobb-Douglas production function.<sup>14</sup>

The second step consists in specifying a convenient reservation wage, with the following cases:

Case 1 (Raurich and Sorolla (2003b), Sorolla (2000)). If the alternative wage is the unemployment benefit,  $b_t$ ,<sup>15</sup>  $\varrho_t = b_t$ , that is:

$$\omega_t = m_\omega b_t,$$

and the unemployment benefit is financed by taxes on employed workers, where  $\tau$  is the tax rate, i.e.

$$b_t = \frac{\tau \omega_t L_t}{N_t - L_t},$$

combining these we have:

$$\omega_t = m_\omega \frac{\tau \omega_t L_t}{N_t - L_t},$$

and, in terms of the employment rate:

$$1 = m_\omega \frac{\tau l_t}{1 - l_t},$$

so that the employment rate is:

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<sup>12</sup>See Sorensen and Whitta-Jacobsen (2005), section 3.

<sup>13</sup>The effort function needs to be  $e(\omega) = (\frac{\omega-r}{r})^\beta$  if  $\omega > r$  where  $0 < \beta < 1$ .

<sup>14</sup>Raurich and Sorolla (2003a) and Kaas and von Thadden (2003) are models with a non constant labor demand elasticity.

<sup>15</sup>This assumption makes sense when wages are set at a national level, that is, for the overall economy.

$$l_t = \frac{1}{(1 + \tau m_\omega)} = l^* < 1,$$

if  $\tau > 0$ . Hence, the employment rate is constant over time and does not depend on growth.

In this case the dynamics of models with unemployment are of the same type as those of the full employment models. Consider for example, the IH model, when growth depends on employment, that is, when the production function is not an  $AK$  function. From equations (3.1) and (3.2) we get:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta} (F_K(k_t, l^*) - (\rho + \delta)), \quad (6.1)$$

$$\dot{k}_t = f(k_t, l^*) - c_t - (n + \delta)k_t. \quad (6.2)$$

and in the full employment model the equations are:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta} (F_K(k_t, 1) - (\rho + \delta)),$$

$$\dot{k}_t = f(k_t, 1) - c_t - (n + \delta)k_t.$$

Because  $l^* < 1$ , it is clear that the rate of growth of capital per capita is lower for a given level of  $c$  and  $k$  in a model with unemployment. Then economies with a constant rate of unemployment grow less than economies with full employment. It is also clear, from (6.1) and (6.2), that consumption and capital per capita converge to a steady state with zero rate of growth of capital per capita and consumption per capita. In this case, with a neoclassical production function and this particular wage setting rule, there is no relationship between growth and unemployment in the long run: the constant rate of unemployment is given by  $l^*$  and the rate of growth of income per capita is zero, or  $x$ , if there is technological progress. The version of the Ramsey model with non competitive labor and product markets and this particular wage setting rule implies, then, that there is no relationship between growth and unemployment in the long run. As shown in the appendix the same result holds for the two other ways of generating savings. It is also easy to see, drawing the phase diagram, that a decrease in  $l^*$  decreases the long run level of consumption, capital and income per capita, that is, there is a positive relationship between income, capital and consumption per capita and employment in the long run. In other words, all other parameters equal, economies with full employment have a higher level of income per capita in the long run than economies with unemployment.

Using equation (3.5), the rate of growth of wages is given by:

$$\gamma_\omega = \frac{1}{\frac{d\tilde{k}}{d\omega_t} \frac{\omega_t}{\tilde{k}}} \gamma_k.$$

so that it also converges to zero. When there is productivity growth, using equation (3.10), the rate of growth of wages is given by:

$$\gamma_\omega = \frac{1}{\frac{d\tilde{k}}{d\left(\frac{\omega_t}{A_t}\right)} \frac{\omega_t}{A_t}} \gamma_k + x.$$

that converges to  $x$ .

When employment does not depend on growth any shock that affects growth is translated immediately to wages leaving the unemployment rate unaffected, that is, there is no (short or long run) effect of productivity on the rate of unemployment. Blanchard and Katz (1997) say that any model with unemployment should satisfy the condition that there is no long run effect of the level of productivity on unemployment (p.56).

If the production function is an  $AK$  production function, with this wage rule it turns out that growth does not depend on employment and employment does not depend on growth and we have constant growth and employment rates. For the IH model we have obtained  $l_t = l^* = \frac{1}{(1+\tau m_\omega)}$  and  $\gamma_k = \gamma_k = \gamma_{IHAK}^* = \frac{1}{\theta}(\alpha A - (\rho + \delta))$ . In this case, then, there is no relationship between growth and unemployment because the parameters that affect  $l^*$  and  $\gamma^*$  are different. This does not mean that in general with an  $AK$  production and a wage setting rule that implies a constant employment rate, there is no relationship between growth and unemployment because it can be the case that an exogenous parameter affects both rates. We present below an example where this happens.

Case 2: The other alternative assumption is that the alternative wage is the average labor income<sup>16</sup>, that is,  $r_t = l_t(1 - \tau)\omega_t + (1 - l_t)b_t$ <sup>17</sup>. Assuming again that the unemployment benefit is financed by taxes on employed workers:

$$b_t = \frac{\tau\omega_t L_t}{N_t - L_t} = \frac{\tau\omega_t l_t}{1 - l_t}$$

combining these we get:

$$\omega_t = m_\omega \omega_t l_t,$$

so that the employment rate is:

$$l_t = \frac{1}{m_\omega} = l^* < 1.$$

Alternatively, one can assume that the unemployment benefit is proportional to the wage that is  $b_t = \varphi\omega_t$ <sup>18</sup>. In this case we have:

$$\omega_t = m_\omega (l_t(1 - \tau)\omega_t + (1 - l_t)\varphi\omega_t),$$

and the employment rate is:

$$l_t = \frac{1 - \varphi m_\omega}{(1 - \tau)m_\omega - \varphi m_\omega} = l^*.$$

<sup>16</sup>This assumption makes sense if wages are set in a decentralized way.

<sup>17</sup>In a similar way Romer (2006) p.454 assumes:  $r_t = (1 - bu_t)\omega_t$

<sup>18</sup>For example Pissarides (1990) or Raurich, Sala and Sorolla (2006).

so that  $l^* < 1$  if  $1 < (1 - \tau)m_\omega$ .

Daveri and Tabellini (2000), Raurich and Sorolla (2003a) consider a particular case of Case 2. They assume a Cobb-Douglas production function and that the wage setting equation is:  $\omega_t = v y_t$ , a "natural assumption for a growth model", i.e., the rate of growth of real wage is equal to the rate of growth of income per capita<sup>19</sup>. We can derive this wage setting equation assuming, for example,  $\omega_t = m_\omega b_t$  and  $b_t = \xi y_t$ . With a Cobb-Douglas production function and this wage setting rule, the employment rate function becomes:

$$l_t = \frac{(A(1 - \alpha))^{\frac{1}{\alpha}} k_t}{[v A k_t^\alpha l_t^{1-\alpha}]^{\frac{1}{\alpha}}} = \frac{(A(1 - \alpha))^{\frac{1}{\alpha}} k_t}{v^{\frac{1}{\alpha}} A^{\frac{1}{\alpha}} k_t^{\frac{1-\alpha}{\alpha}} l_t^{\frac{1-\alpha}{\alpha}}} = \frac{(1 - \alpha)^{\frac{1}{\alpha}}}{v^{\frac{1}{\alpha}} l_t^{\frac{1-\alpha}{\alpha}}},$$

that is:

$$l_t = \frac{1 - \alpha}{v} = l^*,$$

so that the unemployment rate is constant. In fact, this result comes directly from the constant labor share property of the Cobb-Douglas production function. As follows:

$$\frac{\omega_t L_t}{Y_t} = 1 - \alpha,$$

or:

$$\frac{\omega_t l_t}{y_t} = 1 - \alpha.$$

Assuming  $\omega_t = v y_t$ , we get directly:

$$l_t = \frac{1 - \alpha}{v} = l^*.$$

Note that the constant labor share property of the Cobb-Douglas production function implies  $y_t = (1 - \alpha)\omega_t l_t$  so that  $\omega_t = v(1 - \alpha)\omega_t l_t$ , which is the assumption of case 2.

It is easy to check that the same result holds in the Cobb-Douglas case with externalities or public capital. Moreover, as we have seen in the section 5, if the production function is  $AK$ , growth does not depend on employment. For a OLG model (the one used by Daveri and Tabellini) the rate of growth of capital is (see appendix 2):

$$\frac{k_{t+1}}{k_t} - 1 = \frac{s(1 - \alpha)A}{1 + n} - 1,$$

so that a decrease in  $\alpha$  implies more growth and employment. Daveri and Tabellini consider this special case in an OLG model with the general production function  $y_t = \phi(k_t)l_t^{1-\alpha}$ . In this last case: OLG model,  $AK$  production function and this particular wage setting rule, there is some relationship between growth and unemployment because a decrease in  $\alpha$  implies more growth and employment, but an increase in  $s$  implies more growth and leaves the rate of unemployment unaffected and a decrease in  $v$  increases the rate of employment and growth remains the same. In order to have a positive effect of employment on growth, Daveri and Tabellini introduce a positive externality of capital per capita, an addition that was criticized

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<sup>19</sup>Taking logs and differentiating  $\omega_t = v y_t$  we obtain:  $\gamma_\omega \equiv \frac{\dot{\omega}}{\omega} = \frac{\dot{y}}{y} \equiv \gamma_y$ .

by Sala-i-Martin (2000), section 2.8.

## 7. Wage setting rules with inertia

If we add inertia (persistence) to the wage setting rules of Section 6, so that wage setting depends also on past values of wages and employment rates then the IH model yields a system of four differential equations with dynamics for both employment rates and wages. In section 6 the unemployment rate was constant and in section 5 the wage was constant so that one of the four differential equations was eliminated.

There are several ways of introducing inertia, the usual are the following:

Case 1: Wage inertia (Raurich and Sorolla (2003b), Sorolla (2000)). In this case the set wage is, as usual, a mark-up over the reservation wage,  $\omega_t = m_\omega \varrho_t$ , but the reservation wage is a weighted average of the current unemployment benefit and past wages. In the OLG model where workers work for one period past wages are the wages of parents.

Blanchard and Katz (1997) justify wage inertia on the reservation wage saying that "models based on fairness suggest that the reservation wage may depend on factors such as the level and the rate of growth of wages in the past, if workers have come to consider such wage increases as fair. Perhaps a better word than "reservation" wage in that context is an "aspiration" wage (p.54)". Wage inertia is derived in an efficient wage model where workers' disutility depends on the comparison between current and past wages (see Collard and de la Croix. (2000), de la Croix et al. (2000) and Danthine and Kursmann (2004)). Empirical evidence shows that there is wage persistence in the wage setting process (see Blanchard and Katz (1997) and (1999) and Hogan (2004)).

We introduce wage persistence in the discrete time version considering:  $\varrho_t = \lambda\omega_{t-1} + (1-\lambda)b_t$ , where  $0 \leq \lambda \leq 1$ . Combining with the wage setting equation we get:

$$\omega_t = m_\omega(\lambda\omega_{t-1} + (1-\lambda)b_t).$$

In order to obtain the continuous time version we subtract  $\omega_{t-1}$  and, making  $\omega_{t-1} = \omega_t$ , we obtain:

$$\dot{\omega}_t = (\lambda m_\omega - 1)\omega_t + (1-\lambda)m_\omega b_t$$

If the unemployment benefit is financed by taxes, i.e.

$$b_t = \frac{\tau\omega_t L_t}{N_t - L_t} = \frac{\tau\omega_t l_t}{1 - l_t},$$

we get for the continuous time version:

$$\dot{\omega}_t = (\lambda m_\omega - 1)\omega_t + (1-\lambda)m_\omega \frac{\tau\omega_t l_t}{1 - l_t}. \quad (7.1)$$

Assuming that the model converges<sup>20</sup> to an steady state where  $\gamma_\omega = \gamma_l = 0$ , from equation

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<sup>20</sup>With this wage setting rule Sorolla (2000) studies the conditions for convergence in an OLG model with constant saving rate, neoclassical production function and exogenous technological progress. Raurich and Sorolla

(7.1) we get:

$$l^* = \frac{1 - \lambda m_\omega}{1 - \lambda m_\omega + (1 - \lambda)\tau m_\omega}.$$

Assuming  $0 \leq \lambda$  then  $0 < l^* < 1$ . Moreover, if  $\gamma_\omega = \gamma_l = 0$  then, by (3.5),  $\gamma_k = 0$ , which is only possible with a neoclassical production function. This means that if the model converges to an steady state it turns out that growth stops. This should not be a surprise because this is Case 1 of section 6, where in the long run growth stops, plus wage inertia. The novelty here is that the long run employment rate depends on the degree of wage inertia,  $\lambda$ , having  $\frac{\partial l^*}{\partial \lambda} < 0$ , that is, a higher degree of wage inertia reduces the long run employment rate. The introduction of wage inertia means that a productivity shock affects the employment rate in the short run but not in the long run. With wage inertia, then, there is also no relationship between growth and unemployment in the long run without technological progress. If the production function is  $AK$ , Raurich and Sorolla (2003b) show that, with wage inertia, growth affects employment in the long run.

With exogenous technological progress, equation (7.1) becomes:

$$\frac{\dot{\omega}_t}{A_t} = (\lambda m_\omega - 1)\frac{\omega_t}{A_t} + (1 - \lambda)m_\omega \frac{\tau \frac{\omega_t}{A_t} l_t}{1 - l_t},$$

or:

$$\left(\frac{\dot{\omega}_t}{A_t}\right) = (\lambda m_\omega - 1)\frac{\omega_t}{A_t} + (1 - \lambda)m_\omega \frac{\tau \frac{\omega_t}{A_t} l_t}{1 - l_t} - x \frac{\omega_t}{A_t}, \quad (7.2)$$

and hence in an steady state where  $\gamma_{\frac{\omega}{A}} = \gamma_l = 0$ , using (7.2) we obtain:

$$l^* = \frac{1 - \lambda m_\omega + x}{1 - \lambda m_\omega + x + (1 - \lambda)\tau m_\omega}.$$

Here the novelty is that the rate of employment in the long run depends on the degree of wage inertia,  $\lambda$  and the productivity rate  $x$ , having  $\frac{\partial l^*}{\partial \lambda} < 0$  and  $\frac{\partial l^*}{\partial x} > 0$ , that is, a higher degree of wage inertia reduces the long run employment rate. Because in an steady state  $\gamma_{\frac{\omega}{A}} = \gamma_l = 0$ , with a neoclassical production function, then  $\gamma_{k_e} = 0$ . So that the rate of growth of wages is equal to the rate of growth of capital per capita and they are equal to the productivity growth rate. This means that with wage inertia and technological progress there is some relationship between growth and unemployment in the long run in the sense that an increase in  $x$  implies more growth and more employment in the long run<sup>21</sup>. It follows that in the version of the Ramsey model with non competitive labor and product markets and wage inertia there is some relationship between growth and unemployment in the long run. The same result holds for the two other ways of generating savings. This is easy to see looking at the fundamental equations

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(2003b) study the convergence conditions when the production function is an AK production function due to productive government expenditures (and without exogenous technological progress).

<sup>21</sup>The wage equation obtained in the continuous case, assuming  $\omega_{t-1} = \omega_t$  in the discrete time case, produces the weird result that when  $\lambda = 0$  then  $l^* = \frac{1+x}{1+x+\tau m_\omega}$ , whereas for this case in the previous section we have obtained  $l^* = \frac{1}{1+\tau m_\omega}$ . This is because the "forced" version of wage inertia in continuous time. For the discrete time version everything works, but we maintain here the continuous version in order to have the model in the same type of time.

derived in appendix.

Case 2: Labor income inertia (Raurich, Sala and Sorolla (2006)). As in the previous case the set wage is a mark-up over the reservation wage,  $\omega_t = m_\omega \varrho_t$ , but the reservation wage is a weighted average of past average labor income, that is, there is labor income inertia. In the discrete time version we have:  $\varrho_t = \lambda(l_{t-1}(1-\tau)\omega_{t-1} + (1-l_{t-1})b_{t-1}) + \lambda(1-\lambda)(l_{t-2}(1-\tau)\omega_{t-2} + (1-l_{t-2})b_{t-2}) + \lambda(1-\lambda)^2(l_{t-3}(1-\tau)\omega_{t-3} + (1-l_{t-3})b_{t-3}) + \dots$  where  $0 < \lambda \leq 1$ <sup>22</sup>. We transform last equation into a difference equation of the form:

$$r_t = \lambda [l_{t-1}(1-\tau)\omega_{t-1} + (1-l_{t-1})b_{t-1}] + (1-\lambda)r_{t-1}.$$

And, hence, the continuous time version is:

$$\dot{r}_t = \lambda([l_t(1-\tau)\omega_t + (1-l_{t-1})b_t] - r_t).$$

If the unemployment benefit is financed by taxes<sup>23</sup>, i.e.:

$$b_t = \frac{\tau\omega_t L_t}{N_t - L_t} = \frac{\tau\omega_t l_t}{1-l_t},$$

we get for the continuous time version:

$$\dot{r}_t = \lambda(l_t\omega_t - r_t).$$

From  $\omega_t = m_\omega r_t$  we get  $\dot{\omega}_t = \dot{r}_t$  and also  $r_t = \frac{\omega_t}{m_\omega}$ , expressions that, when substituted to in the last equation, give:

$$\dot{\omega}_t = \lambda(l_t\omega_t - \frac{\omega_t}{m_\omega}). \quad (7.3)$$

Hence, in an steady state:

$$l^* = \frac{1}{m_\omega} < 1.$$

We have the same result in the long run that the we obtained in Case 2 of the constant employment rate section. Thus there is no relationship between growth and unemployment in the long run when the production function is neoclassical.

With exogenous technological progress (7.3) becomes:

$$\frac{\dot{\omega}_t}{A_t} = \lambda(l_t \frac{\omega_t}{A_t} - \frac{\frac{\omega_t}{A_t}}{m_\omega}),$$

or:

$$\left(\frac{\dot{\omega}_t}{A_t}\right) = \lambda(l_t \frac{\omega_t}{A_t} - \frac{\frac{\omega_t}{A_t}}{m_\omega}) - x \frac{\omega_t}{A_t},$$

and hence, in the steady state:

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<sup>22</sup>The case  $\lambda = 0$  makes no sense. If  $\lambda = 1$  then  $r_t = l_{t-1}(1-\tau)\omega_{t-1} + (1-l_{t-1})b_{t-1}$ . A similar result holds if we link the wage with past income per capita, that is,  $\omega_t = \lambda y_{t-1} + \lambda(1-\lambda)y_{t-2} + \lambda(1-\lambda)^2 y_{t-3} + \dots$  and have a Cobb-Douglas production function.

<sup>23</sup>Or  $b_t = \sigma\omega_t$  as in Raurich Sala and Sorolla (2006).

$$l^* = \frac{1}{m_\omega} + \frac{x}{\lambda}.$$

Here the novelty is that the rate of employment in the long run depends on the degree of past labor income inertia,  $\lambda$  and the productivity growth rate  $x$ , that is, in the version of the Ramsey model with non competitive labor and product markets and labor income inertia there is some relationship between growth and unemployment in the long run. As shown in the appendix the same result holds for the two other ways of generating savings.

## 8. Conclusions

In this paper we have derived the set up that shows clearly that, in growth models with disequilibrium unemployment, in general the growth rate of capital depends on the employment rate (growth depends on employment) and the employment rate depends on the stock of capital (employment depends on growth) and how both variables depend on the real wage and, with monopolistic competition in the output market, on markups. The first relationship gives rise to the fundamental equations of a growth model with unemployment that we derive explicitly for the standard growth models: exogenous constant saving rate, infinite horizon and overlapping generations, and we compare them with the ones obtained in the models with full employment.

Once we have made explicit the fundamental equations, in order to close the models, we need to specify a wage equation. Then, it turns out that the relationship between growth and unemployment will depend on the specific wage setting rule and on the form of the production function.

We show that if the production function is *AK* growth does not depend on employment and, hence, not on wages.

When the production function is neoclassical, we analyze the behavior of the models with the standard wage setting rules. When a constant real wage is set, if this wage is lower than the long competitive real wage then there is always a constant positive rate of growth of capital and full employment in the long run, that is, there is a relationship between them in the long run. If this wage is the long competitive real wage then there is no growth of capital per capita and employment remains constant and depends on the initial level of the stock capital, result that opens the door for hysteresis on unemployment.

If the rule implies a constant employment rate (employment does not depend on growth) then economies with a constant rate of unemployment grow less than economies with full employment and the long run rate of growth of income per capita is equal to the productivity rate, that is, there is no relationship between growth and unemployment in the long run for the three standard ways of generating saving, in particular for the Ramsey model (the canonical model of growth) with unemployment.

If inertia, which depends on past wages or past labor incomes, is present in the wage setting process, we show that, a smaller degree of inertia or a faster productivity growth implies a smaller long run rate of unemployment. Thus, since the long run rate of income per capita is equal to the productivity rate, there is some relationship between growth and unemployment

in the long run for the three standard ways of generating saving, in particular, again, for the Ramsey model with unemployment.

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## 10. Appendix 1

### 10.1. The ECSR model

We consider the case of a constant saving rate,  $s$ , exogenous and independent of the interest rate which means that channel 2 is inactive. Consider the following equation for the supply of capital  $K_t^s$  in continuous time<sup>25</sup>:

$$\dot{K}_t^s = r_t K_t + \omega_t L_t - C_t$$

where  $C_t$  is total consumption. In per capita terms we have:

$$\dot{k}_t^s = r_t k_t + \omega_t l_t - c_t - n k_t.$$

In the last equation channel three becomes transparent, a change in the wage affects aggregate income,  $r_t k_t + \omega_t l_t$ , changing  $\omega$  and, indirectly,  $l$  and  $r$ , that is, the wage affects growth.

Equilibrium in capital market,  $k_t^s = k_t^d = k_t$ , and the zero profits condition, (2.5), imply:

$$\dot{k}_t = F(k_t, l_t) - c_t - (n + \delta)k_t.$$

Assuming that savings are a constant proportion of output, that is, there is a constant rate of saving  $s$ , we have  $c_t = (1 - s)F(k_t, l_t)$ , which implies:

$$\dot{k}_t = sF(k_t, l_t) - (n + \delta)k_t. \quad (10.1)$$

Note that the difference between this equation and the fundamental equation of the exogenous constant saving rate model with a competitive labor market with full employment:  $\dot{k}_t = sf(k_t) - (n + \delta)k_t$  is in the production function<sup>26</sup> The production function in (10.1) depends on the employment rate, because there is no full employment. Also note that, in this model, employment affects growth because a higher level of  $l_t$  increases the growth of capital per capita.

Dividing by  $k_t$  we get the equation in terms of the capital per capita growth rate,  $\gamma_k \equiv \frac{\dot{k}_t}{k_t}$ <sup>27</sup>:

$$\gamma_k = \frac{sF(k_t, l_t)}{k_t} - (n + \delta) = sF\left(1, \frac{l_t}{k_t}\right) - (n + \delta) = s\tilde{f}\left(\frac{l_t}{k_t}\right) - (n + \delta) = s\tilde{f}\left(\frac{1}{\tilde{k}}\right) - (n + \delta).$$

Now substituting in the last equation the capital labor ratio we get:

$$\gamma_{k,ECSR} \equiv \frac{\dot{k}_t}{k_t} = s\tilde{f}\left(\frac{1}{\tilde{k}(\omega_t)}\right) - (n + \delta). \quad (10.2)$$

We call to this equation the fundamental equation of a model with an exogenous constant savings rate and wage setting. Note that, in equilibrium, channel three is transformed into the effect of the wage on output per unit of capital, via the capital labor ratio. If the wage increases, then output per unit of capital decreases, due to the increase in the capital labor ratio, and,

<sup>24</sup>Alternatively, we can use the notation  $K_t^s = B_t$ , where the  $B_t$  are financial assets or bonds.

<sup>25</sup>Because there is unemployment it may be the case that the unemployed people get some unemployment benefit, when they do, we assume that this unemployment benefit is financed by taxes on capital and wage income. If the government budget constraint is balanced then we get this equation.

<sup>26</sup>In the full employment equation we also have  $k_t = \hat{k}_t$  because  $N_t = L_t$ .

<sup>27</sup>From now on we will define the rate of growth of a variable by the letter  $\gamma$ .

then, the rate of growth of capital decreases. We can say that if the production function has the usual properties, without ambiguities, via channel three, an increase in the wage implies a decrease in aggregate income per unit of capital and a decrease in the rate of growth of capital per capita and, then, less capital per capita and a lower employment rate in the future.

Note that in principle, we can not solve the fundamental equation, this means that we can not compute  $k_t$  and hence to obtain the employment rate  $l_t$  from this equation, but we can compute the evolution of the employment rate using (3.5).

Finally substituting (10.2) into (3.5) we get:

$$\gamma_{l,ECSR} = s\tilde{f}\left(\frac{1}{\tilde{k}(\omega_t)}\right) - (n + \delta) - \frac{d\tilde{k}}{d\omega_t} \frac{\omega_t \dot{\omega}_t}{\tilde{k} \omega_t}, \quad (10.3)$$

which is a differential equation in terms of the employment rate and the wage. Once we have the fundamental equation, the closing of the model depends on how the wage is set.

For the monopolistic competition case it also turns out (see Galí (1996) section three) that the equation for the evolution of the composite stock of capital is also:

$$\dot{k}_t = F(k_t, l_t) - c_t - (n + \delta)k_t.$$

Assuming a constant savings rate and substituting in the capital labor ratio we get the fundamental equation:

$$\gamma_{k,ECSRMC} \equiv \frac{\dot{k}_t}{k_t} = s\tilde{f}\left(\frac{1}{\tilde{k}(m\omega_t)}\right) - (n + \delta), \quad (10.4)$$

where an increase in the mark up also decreases the rate of growth of capital per capita so that there is less capital per capita and a lower employment rate in the future.

It is also useful to derive the fundamental equation with technical progress in continuous time. Rewriting the supply of capital in terms of the supply of capital per efficiency person and assuming equilibrium we get :

$$\dot{k}_{e,t} = sF(k_{e,t}, l_t) - (n + \delta + x)k_{e,t},$$

or, in terms of the growth rate of capital per efficiency person,  $\gamma_{k_e}$  :

$$\gamma_{k_e} = \frac{sF(k_{e,t}, l_t)}{k_{e,t}} - (n + \delta + x) = sF\left(1, \frac{l_t}{k_{e,t}}\right) - (n + \delta + x) = s\tilde{f}\left(\frac{l_t}{k_{e,t}}\right) - (n + \delta + x) = s\tilde{f}\left(\frac{1}{\hat{k}_{e,t}}\right) - (n + \delta + x).$$

Substituting into the last equation the capital efficiency labor ratio function we get:

$$\gamma_{k_e,ECSSRTP} \equiv \frac{\dot{k}_{e,t}}{k_{e,t}} = s\tilde{f}\left(\frac{1}{\tilde{k}\left(\frac{\omega_t}{A_t}\right)}\right) - (n + \delta + x). \quad (10.5)$$

Note that depending on the value of  $\frac{\omega_t}{A_t}$  we can have a positive rate of growth of capital per efficiency person, this case is discussed for the IH model in section 6.

## 10.2. OLG

We consider an OLG model where the agents live two periods and have the utility function:

$$U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\varrho} \frac{C_{2t+1}^{1-\theta}}{1-\theta},$$

where  $\theta > 0$ . When  $\theta = 1$  utility is logarithmic. If  $\theta \neq 1$  it turns out that channel two is active because the saving rate depends on the interest rate. The savings rate function  $\tilde{s}(r_{t+1})$  is given by (Romer (2006), section 2.9) :

$$s_t = \tilde{s}(r_{t+1}) = \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\varrho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} = \frac{1}{\frac{(1+\varrho)^{\frac{1}{\theta}}}{(1+r_{t+1})^{\frac{1-\theta}{\theta}}} + 1}.$$

If  $\theta < 1$  then  $\tilde{s}' > 0$  and if  $\theta > 1$  then  $\tilde{s}' < 0$ . Using the interest rate function (2.9) we get:

$$s_t = \hat{s}(\omega_{t+1}) = \tilde{s}(\tilde{r}(\omega_{t+1})) = \frac{1}{\frac{(1+\varrho)^{\frac{1}{\theta}}}{(1+f'(\tilde{k}(\omega_{t+1}))-\delta)^{\frac{1-\theta}{\theta}}} + 1}$$

with  $\hat{s}' < 0$  if  $\theta < 1$  and  $\hat{s}' > 0$  if  $\theta > 1$ .

The accumulation of capital is given by the equation (see also Romer (2006) ):

$$K_{t+1} = \hat{s}(\omega_{t+1})\omega_t L_t,$$

that is, in per capita terms:

$$k_{t+1} = \frac{1}{1+n} \hat{s}(\omega_{t+1})\omega_t l_t,$$

where it is also evident that channel three is active because a change in the wage modifies labor income. Using the employment rate function (2.11), we get:

$$\frac{k_{t+1}}{k_t} = \frac{1}{1+n} \frac{\hat{s}(\omega_{t+1})\omega_t}{\tilde{k}(\omega_t)}.$$

Then the rate of growth of capital per capita, in discrete terms, the fundamental equation, is given by:

$$\gamma_{k,OLG} = \frac{k_{t+1}}{k_t} - 1 = \frac{1}{1+n} \frac{\hat{s}(\omega_{t+1})\omega_t}{\tilde{k}(\omega_t)} - 1 \quad (10.6)$$

Using the employment rate function (2.11), the employment rate is given by the difference equation:

$$l_{t+1} = \frac{k_{t+1}}{\tilde{k}(\omega_{t+1})} = \frac{1}{1+n} \frac{\hat{s}(\omega_{t+1})\omega_t l_t}{\tilde{k}(\omega_{t+1})}, \quad (10.7)$$

and the rate of growth of the employment rate is:

$$\gamma_{l,OLG} = \frac{l_{t+1}}{l_t} - 1 = \frac{k_{t+1}}{k_t} \frac{\tilde{k}(\omega_t)}{\tilde{k}(\omega_{t+1})} - 1 = \frac{1}{1+n} \frac{\hat{s}(\omega_{t+1})\omega_t}{\tilde{k}(\omega_{t+1})} - 1, \quad (10.8)$$

a difference equation in terms of employment and the wage.

In the monopolistic competition case the equations become:

$$\gamma_{k,OLGMC} = \frac{k_{t+1}}{k_t} - 1 = \frac{1}{1+n} \frac{\hat{s}(m\omega_{t+1})\omega_t}{\tilde{k}(m\omega_t)} - 1, \quad (10.9)$$

and

$$\gamma_{l,OLGMC} = \frac{l_{t+1}}{l_t} - 1 = \frac{k_{t+1}}{k_t} \frac{\tilde{k}(m\omega_t)}{\tilde{k}(m\omega_{t+1})} - 1 = \frac{1}{1+n} \frac{\hat{s}(m\omega_{t+1})\omega_t}{\tilde{k}(m\omega_{t+1})} - 1. \quad (10.10)$$

With exogenous technological progress (now  $k_{e,t} = \frac{K_t}{A_t N_t}$ ) they are:

$$\gamma_{k_e,OLGETP} = \frac{k_{e,t+1}}{k_{e,t}} - 1 = \frac{1}{(1+n)(1+x)} \frac{\hat{s}(\frac{\omega_{t+1}}{A_{t+1}})\frac{\omega_t}{A_t}}{\tilde{k}(\frac{\omega_t}{A_t})} - 1 \quad (10.11)$$

and

$$\gamma_{l,OLGETP} = \frac{l_{t+1}}{l_t} - 1 = \frac{k_{e,t+1}}{k_{e,t}} \frac{\tilde{k}(\frac{\omega_t}{A_t})}{\tilde{k}(\frac{\omega_{t+1}}{A_{t+1}})} - 1 = \frac{1}{(1+n)(1+x)} \frac{\hat{s}(\frac{\omega_{t+1}}{A_{t+1}})\frac{\omega_t}{A_t}}{\tilde{k}(\frac{\omega_{t+1}}{A_{t+1}})} - 1. \quad (10.12)$$

With a logarithmic utility function it turns out that the savings rate does not depend on the interest rate<sup>28</sup>, that is channel two is inactive and, in this case, we only have to substitute  $s$  for the savings function in the previous fundamental equations (10.6), (10.9) and (10.11).

## 11. Appendix 2 The AK model

In this case the fundamental equation of a model with an exogenous constant saving rate (10.2) becomes:

$$\gamma_{k,ECSSR} = sA \left[ \frac{1}{\hat{k}_t} \right]^{1-\alpha} \bar{K}_t^{1-\alpha} - (n+\delta) = sA \left[ \frac{1}{\hat{k}_t} \right]^{1-\alpha} \hat{k}_t^{1-\alpha} - (n+\delta) = sA - (n+\delta) = \gamma_{ECSRAK}^*,$$

and, then, channel 3 is inactive<sup>29</sup>, and growth does not depend on employment. This means that the rate of growth of capital and income per capita does not depend on the wage setting.

From (4.1) we get:

$$\gamma_l = \gamma_k - \gamma_\omega = \gamma_{ECSRAK}^* - \gamma_\omega.$$

<sup>28</sup>We get  $s = \frac{1}{2+\rho}$ .

<sup>29</sup>Channel 2 is also inactive because of the assumption of a constant savings rate independent of the interest rate.

That is, the growth rate of the employment rate depends on capital growth and wage growth, and the wage dynamics affect only the dynamics of employment. For the same reason if the output market is not competitive a change in the markup does not affect growth and the labor demand is given by:

$$l_t = \frac{A(1-\alpha)k_t}{m\omega_t}. \quad (11.1)$$

Finally, we consider the OLG model with a constant saving rate  $s$ . In this case, it is easy to check that:

$$\omega_t L_t = (1-\alpha)AK_t$$

and, because,

$$K_{t+1} = s\omega_t L_t,$$

we have:

$$\frac{K_{t+1}}{K_t} = s(1-\alpha)A.$$

The fundamental equation is given by:

$$\frac{k_{t+1}}{k_t} - 1 = \frac{s(1-\alpha)A}{1+n} - 1 = \gamma_{OLGAK}^*,$$

which shows that growth does not depend on employment<sup>30</sup>. In this case the rate of growth of the employment rate is given by:

$$\frac{l_{t+1}}{l_t} - 1 = \left[ \frac{s(1-\alpha)A}{1+n} \frac{\omega_t}{\omega_{t+1}} \right] - 1.$$

Daveri and Tabellini (2000) present an OLG model where they consider as special case the production function  $Y_t = AK_t^\alpha L_t^{1-\alpha} \bar{K}_t^{1-\alpha}$  where the externality, instead of being  $\bar{K}_t = \hat{k}_t$ , is capital per person  $\bar{K}_t = k_t$ . With this production function it turns out that:

$$\omega_t L_t = (1-\alpha)AK_t l_t$$

so that:

$$\frac{K_{t+1}}{K_t} = s(1-\alpha)A l_t.$$

Thus growth depends on employment. This particular externality is criticized in Sala-i-Martin (2000), section 2.8.

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<sup>30</sup>This is the set up of Raurich and Sorolla (2003b), where for a specific real wage dynamics, the effect of taxes on growth and long run employment is studied.